Demonstration of the propagation of errors using resistors

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We present a simple experiment to demonstrate the concept of propagation of errors by measuring resistors on printed circuit boards (PCBs). We devise a method for connecting the resistors on a PCB that allows a relatively quick measurement of 200 resistor pairs to show that the distributions of the measurement of individual resistances, as well as the total resistance, are Gaussian-distributed, as expected from the central limit theorem. The measurement demonstrates how individual uncertainties propagate to the total uncertainty. The experiment is part of a laboratory course on statistics for physics students that emphasizes the application of statistics in data analysis.
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I. INTRODUCTION

The propagation of errors[1] is a statistical tool used for estimating the uncertainty of a quantity that is inferred from measurements rather than measured directly. If this quantity is represented by a function dependent on variables with known uncertainties, these uncertainties can be propagated to calculate the total uncertainty in the function.

Consider an example function, Q, that depends on two variables, x and y. The uncertainty of Q is given by

$$\sigma_Q^2 = \sigma_x^2\left(\frac{\partial Q}{\partial x}\right)^2 + \sigma_y^2\left(\frac{\partial Q}{\partial y}\right)^2 + 2\sigma_{xy}\left(\frac{\partial Q}{\partial x}\right)\left(\frac{\partial Q}{\partial y}\right). \quad (1)$$

If x and y are uncorrelated, the formula is simplified to

$$\sigma_Q^2 = \sigma_x^2\left(\frac{\partial Q}{\partial x}\right)^2 + \sigma_y^2\left(\frac{\partial Q}{\partial y}\right)^2. \quad (2)$$

A simple method to demonstrate the concept of propagation of errors is through Monte Carlo simulations.[2] However, since physics is an experimental science, it is more instructive to demonstrate the concept experimentally, but collecting the few hundred measurements necessary to do so is a major challenge. Ideally, for a student laboratory class, the data should be collected in class in less than one hour without being too tedious and the experimental setup should not be costly.

In this paper, we present such an experiment for physics students in a laboratory course that emphasizes the application of statistics in data analysis. In this lab, a large set of two resistors in series are available. Students are asked to measure the resistance of each resistor as well as the total resistance. The uncertainty of the total resistance is then compared with the expectation based on propagation of error.

For each pair, three quantities are measured, R_1 and R_2 and the total resistance,

$$R_{12} = R_1 + R_2. \quad (3)$$

By propagation of errors, the uncertainty of R_{12} is related to the uncertainties of the individual resistors by

$$\sigma_{12}^2 = \sigma_1^2 + \sigma_2^2. \quad (4)$$

In our experiment, a large sample of nominally identical resistor pairs are measured. The uncertainties of R_1 and R_2 correspond both to the variability of manufactured resistances and to the measurement itself. The measurements of R_1, R_2, and R_{12} are then plotted so that each distribution can be fitted with a Gaussian to extract σ_1, σ_2, and σ_{12} for comparison with Eq. (4).

Using resistors is highly advantageous because of their low cost and availability. This allows the fabrication of a large number of setups to achieve the precision needed to demonstrate the principle of propagation of errors.

II. EXPERIMENTAL SETUP

We tried three kinds of resistors before deciding to use the traditional axial resistors. The first setup used the modern surface mount resistors. Unfortunately, the distributions of R_1, R_2, and R_{12} were asymmetric, i.e., not Gaussian, probably due to the manufacturing process. Consequently, the difference between the measured and predicted σ_{12} was measured to be as large as 35% in some samples. Next we tried resistors fabricated using long copper traces etched on a PCB, eliminating the need to solder resistors. The resistance used were $R_1 = 6.5\ \Omega$ (238 cm) and $R_2 = 8.1\ \Omega$ (295 cm). The measured σ_{12}'s were consistent with the predictions to the level of 20%–40%. This poor agreement was probably due to the fact that the errors on R_1 and R_2 were correlated: the long copper traces that provided the resistances were either under- or over-etched.

In the third setup, a PCB was designed with axial-lead resistors using a connection scheme that allows for relatively quick and methodical measurements of many resistor pairs. This alleviates the need to use a pair of probes to contact the two leads of each resistor. Figure 1 shows a rendition of the PCB with 40 pairs of resistors. It takes about an hour to solder the components on a PCB. The leads of each pair of resistors are connected to a 4-pin male connector[3] soldered to the PCB using the trace layout shown in Fig. 2. R_1, R_2, and R_{12} are measured by plugging successively a 2-pin female connector[4] on a cable to two adjacent pins of the 4-pin male connector: R_1 is measured using pins 1 and 2, R_2 using pins 2 and 3, and R_{12} using pins 3 and 4. The other side of the cable is connected to a BK Precision 2831E multimeter via two banana plugs.

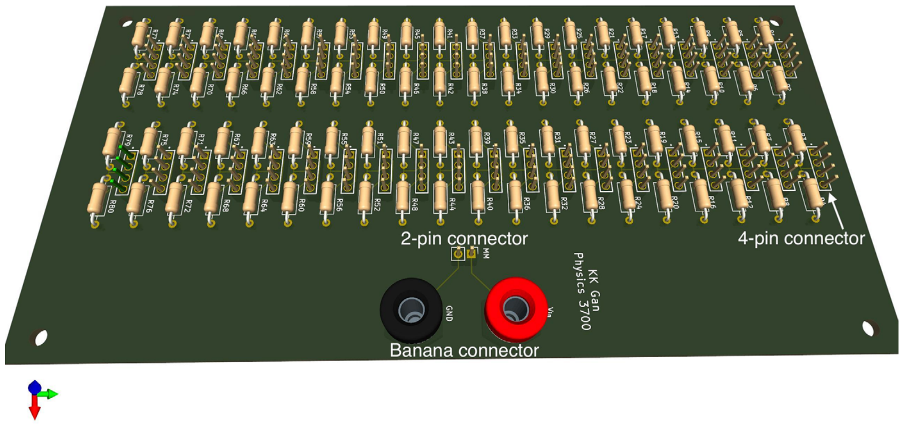

Fig. 1. A rendition of the PCB showing two rows, each containing 20 resistor pairs. Each pair is connected to a 4-pin connector (see Fig. 2). The two banana connectors are connected to the 2-pin connector.

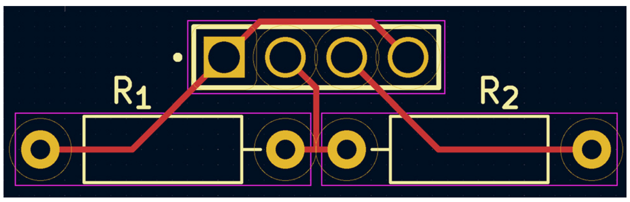

Fig. 2. The PCB layout for a pair of resistors R_1 and R_2. The yellow square denotes pin 1 of the 4-pin connector. The left lead of R_1 is connected to pins 1 and 4. The right lead of R_1 is connected to pin 2 and also to the left lead of R_2. The right lead of R_2 is connected to pin 3. A student would plug a 2-pin connector onto pins 1 and 2 to measure R_1, onto pins 2 and 3 to measure R_2, and onto pins 3 and 4 to measure R_{12}, R_1 and R_2 in series.

Table I. Summary of the width (σ) of the resistance distributions of 28 PCBs. The PCBs were divided into six groups with the first five groups containing 5 PCBs and the last group containing 3 PCBs. The difference, Δ, between the measured and expected σ_{12} is also given.

PCB	1–5	6–10	11–15	16–20	21–25	26–28
$\sigma_1(\Omega)$	4.5	4.7	3.8	4.5	4.2	5.0
$\sigma_2(\Omega)$	5.4	5.8	6.0	5.6	5.8	6.1
$\sigma_{12}(\Omega)$ (expected)	7.0	7.4	7.0	7.2	7.1	7.9
$\sigma_{12}(\Omega)$ (measured)	7.7	7.4	7.0	6.8	7.7	8.2
$\Delta(\%)$	9.9	−0.2	−0.8	−5.8	8.0	3.1

A LabView program was written to automatically record the measurement of the multimeter. The program allows two seconds for each measurement. The measurements are displayed in a three-column array. We chose $R_1 \sim 1.1$ kΩ and $R_2 \sim 1.3$ kΩ. The difference in the nominal resistances of about 20% is chosen, so that if the resistance is not within 10% of the expected value because of bad contacts or the 2-pin connector is plugged onto the wrong pair of pins, the measured value will not be displayed by the program. In addition, there is an indicator light which will turn from red to green if the measured resistance is within the expected range. With this setup, a student can reasonably measure a PCB with 40 pairs of resistors in less than 10 min. A total of 28 PCBs was fabricated. Our students were asked to randomly select five PCBs to accumulate 200 data points for each type of resistor.

In this study, the collected data were divided into six sets, five with measurements of 200 resistor pairs (5 PCBs)

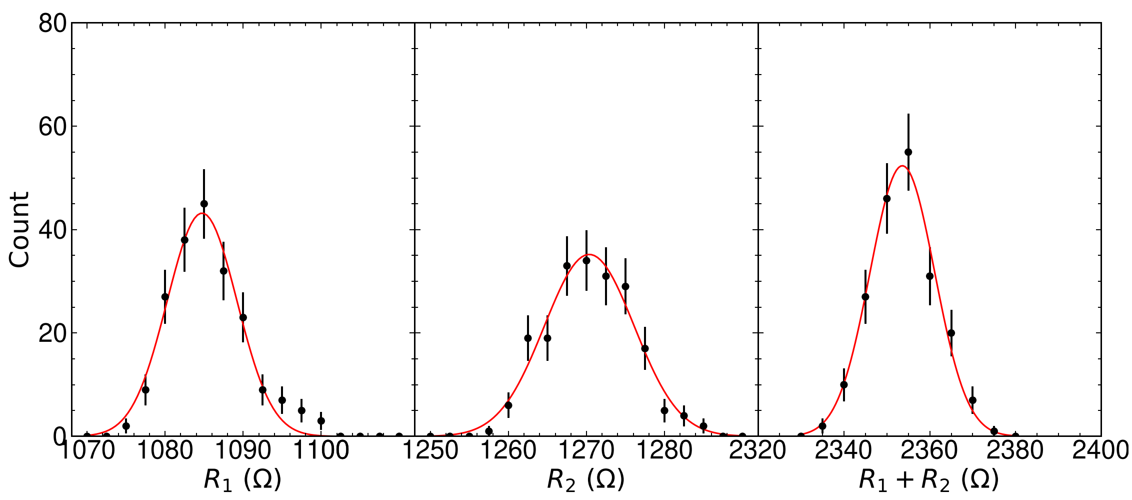

Fig. 3. The distributions of the resistance of R_1, R_2, and R_{12} from a sample of five PCBs, containing a total of 200 resistor pairs. Each error bar represents the statistical uncertainty of the number of counts in each bin. Each curve is a fit to a Gaussian used to extract the width (σ).

and one with 120 pairs (3 PCBs). The distributions of the different resistances for a representative set of 5 PCBs are shown in Fig. 3. Each distribution is well described by a Gaussian as expected from the central limit theorem, an educational bonus for the students. Each distribution is fitted with a Gaussian to extract the width (standard deviation or σ) of the distribution. The results for the six sets of PCBs are summarized in Table I. There is good agreement between the measured σ_{12} and the value expected from Eq. (4), with an average difference of 4.6%. Therefore, this is a satisfactory demonstration of the principle of propagation of errors.

III. SUMMARY

In summary, we have devised an experimental procedure for demonstrating the principle of propagation of errors. The low cost test setup uses PCBs mounted with axial-lead resistors. Each pair of resistors are connected in such a way that the resistances can be readily measured. This allows the collection of a large data set to demonstrate the principle of propagation of errors.

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AUTHOR DECLARATIONS

Conflict of Interest

The author has no conflicts to disclose.

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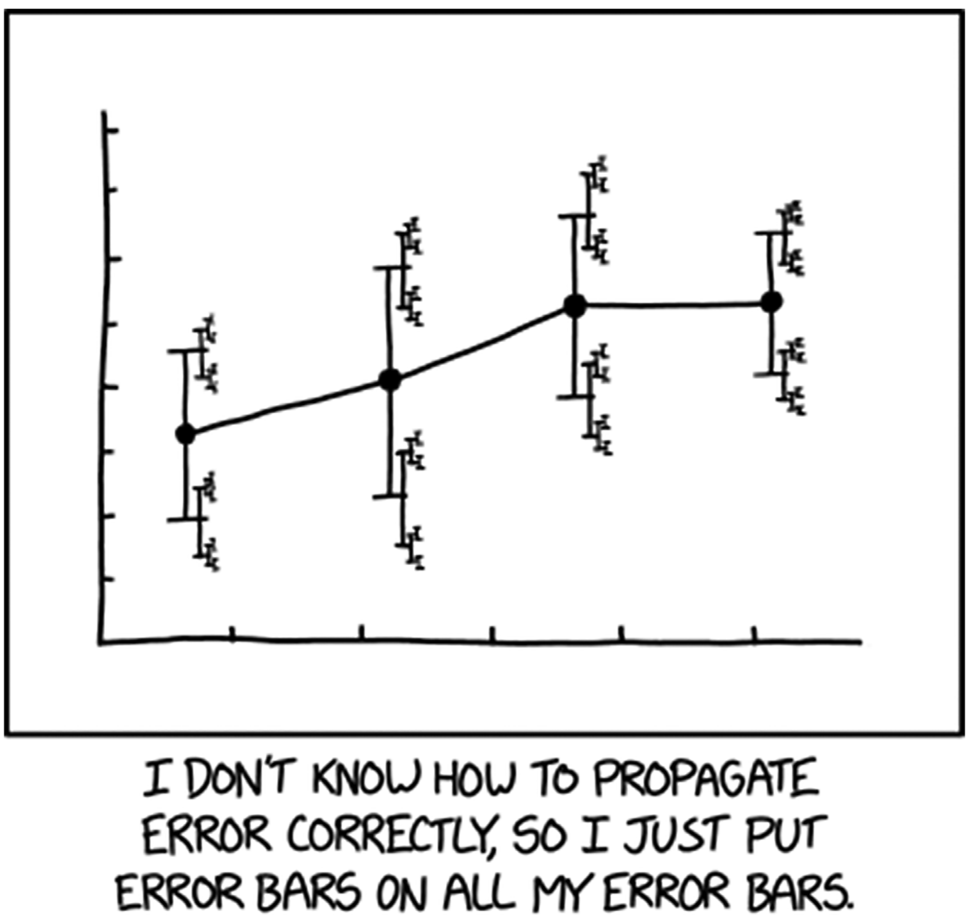
[1]See, for example, John Taylor, *An Introduction to Error Analysis*, 2nd ed. (University Science Books, Sausalito, California, 1939), pp. 73–79.

[2]For example, one could readily simulate a large number of resistor pairs in series. In each pair, the resistances would be (randomly) selected from two Gaussian distributions that represent the variances of the two resistances. The sum of the two resistances should also be Gaussian with the variance given by the sum of the two variances.

[3]Precision Screw Machined Strips, Samtec, <https://www.samtec.com/connectors/standard-board-to-board/screw-machine/strips>

[4]2-pin female wire connector with 2.54 mm pitch.



(Source: https://xkcd.com/2110/)