APPARATUS AND DEMONSTRATION NOTES

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This department welcomes brief communications reporting new demonstrations, laboratory equipment, techniques, or materials of interest to teachers of physics. Notes on new applications of older apparatus, measurements supplementing data supplied by manufacturers, information which, while not new, is not generally known, procurement information, and news about apparatus under development may be suitable for publication in this section. Neither the *American Journal of Physics* nor the Editors assume responsibility for the correctness of the information presented. Submit materials to Jeffrey S. Dunham, *Editor.*

Simple demonstration of the central limit theorem using mass measurements

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Many observations from daily life and physical experiments give rise to a bell-shaped, Gaussian frequency distribution. This is a consequence of the central limit theorem (CLT) of probability. In this paper, we present a procedure for demonstrating the CLT by repeatedly measuring the mass of trays containing small steel balls. The experiment is part of a laboratory course for physics majors that emphasizes the application of statistics to data analysis.

The CLT may be stated as follows: Let $Y_1, Y_2, \ldots, Y_n$ be a sequence of n independent random variables [1] each with the same probability distribution. Suppose that the mean (μ) and variance (σ^2) of this distribution are both finite. Then the probability P for the normalized difference between the sum of the random variables and $n\mu$ to be between two numbers a and b is given by a unit Gaussian with mean at $y=0$:

$$\lim_{n\to\infty} P\left[a < \frac{Y_1+Y_2+\cdots+Y_n-n\mu}{\sigma\sqrt{n}} < b\right]$$

$$=\frac{1}{\sqrt{2\pi}}\int_a^b e^{-(1/2)y^2}\,dy.$$

The theorem is still valid if the Y_i's are from different probability distributions, provided each distribution has a finite mean and variance and no one term in the sum dominates. The theorem implies that under a wide range of circumstances the probability distribution that describes the sum of random variables tends toward a Gaussian distribution as the number of terms in the sum approaches infinity.

To demonstrate the CLT result that the probability distribution is consistent with a Gaussian, two experimental conditions must be satisfied:

(i) each measurement must be the result of the sum of a large number of random variables ($n\to\infty$); and

(ii) the number of measurements must be large to smooth out the fluctuations in the measured distribution.

It is difficult to satisfy both conditions in a classroom experiment. Fortunately, in practice a small number of random variables is adequate for the first requirement and, for the second requirement, about 30 measurements will produce a histogram that a student can recognize is Gaussian in shape. A classic demonstration of the CLT using a computer is the use of the sum of 12 random numbers to generate a Gaussian distribution. Each measurement is a sum of 12 random variables ($n=12$) with uniform probability distribution between 0 and 1. A histogram of 30 or more such measurements (sums) looks like a Gaussian frequency distribution.

For our laboratory experiment, it is convenient to write the probability for the sum to be between two numbers α and β:[2]

$$P[\alpha < Y_1+Y_2+\cdots+Y_n < \beta]$$

$$\simeq \frac{1}{\sqrt{2\pi}\sigma_s}\int_\alpha^\beta e^{-(1/2)((y-\mu_s)/\sigma_s)^2}\,dy,$$

where $\mu_s=n\mu$ and $\sigma_s^2=n\sigma^2$ are the mean and variance of the probability distribution for the sum. The new limits α and β are related to a and b by

$$\alpha=\mu_s+a\sigma_s, \quad \beta=\mu_s+b\sigma_s.$$

We have tested four CLT experiments in the last few years in a laboratory course for physics undergraduates in their junior year. The course also includes a computer experiment that shows that the distribution of the sum of 12 random numbers is consistent with a Gaussian; however, we believe that it is important that students perform hands-on experiments in addition to the computer simulation.

The first CLT experiment that was tried involved measuring the length of a 3 to 4 m long table using a 30 cm (1 foot)

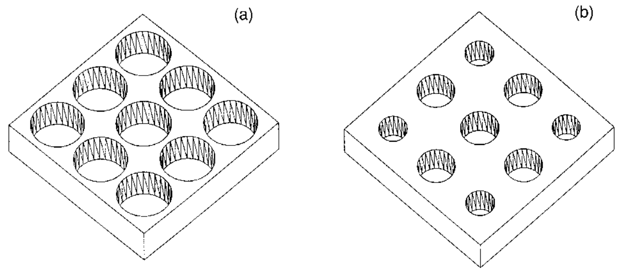

Fig. 1. Trays with nine holes of (a) same diameter (23 mm) and (b) different diameters (12 and 16 mm). The depth of the holes is 16 mm and the thickness of the trays is 19 mm.

ruler. After collecting data, a student histogrammed the table length measurements in bins of 1 cm. The measured distribution was expected to be Gaussian-like because each measurement of the table length consisted of the sum of 10 to 12 measurements with a ruler. Unfortunately, each measurement of the table length was quite tedious and time-consuming and we therefore required the student to perform a total of only 30 measurements.

The experiment was therefore replaced by a second experiment which measured the mass of 100 ml of water in a graduated 400 ml beaker. The experiment was performed using two students. A and B. Student A used two 400 ml beakers and filled one of them with water. The student carefully poured the water into the other beaker, stopping when the water level reached the 100 ml mark. After finishing the pouring, student B measured the mass of the beaker (with its 100 ml of water) using a digital scale[3] with a graduation of 0.1 g. Student B did not reveal the value to student A to eliminate any potential bias. Student A then emptied the beaker, dried it off with a paper towel and repeated the process a total of 30 times. The students then switched roles so that a second set of measurements is obtained for student B. The students histogramed the mass measurements in bins of 0.5 g. In this experiment, there were two sources of uncertainties. First, it was impossible to control the exact amount of water poured, resulting in under or over pour. Second, it was difficult to tell if the water level was exactly at the 100 ml mark due to the capillary effect. These two (desirable, for the purposes of this experiment) effects yielded a Gaussian-like distribution for the measured mass. This experiment was less tedious than the first experiment but was still too time-consuming. To save time, we required the students to perform only 30 measurements. However, the problem with an experiment with only 30 measurements is that some students will see large fluctuations, resulting in a distribution that does not look like a Gaussian although the distribution is statistically consistent with a Gaussian. For the untrained eyes of a student, this is not good evidence for the CLT. An experiment that allows the collection of approximately 100 measurements in a reasonable amount of time would give a better demonstration of the theorem.

We therefore tried a third experiment, involving measurement of the mass of cans containing very small steel balls.[4] Each ball had a diameter of 3.2 mm and mass of 0.13 g. We used as a can the plastic container that typically comes with 35 mm film. This had a diameter of 3 cm and height of 5 cm. A student filled a small plastic beaker with balls and then poured the balls into a can. The student then used a wood stick to wipe across the top of the can to remove excess balls. The student should always wipe in the same manner to ensure that approximately the same number of balls remain in the can. All the manipulations were done above a large container to prevent the loss of balls. The process was repeated a total of 100 times, which took a total of about 30 min. The student then histogrammed the mass measurements in bins of 0.5 g. In this experiment, there were two contributions to the uncertainty in the number of balls in each can (and hence the total mass). One was the packing of the balls which was slightly different for each pouring. The other was the slightly different number of balls being removed by each wipe. Due to the two uncertainties, a slightly different mass was obtained each time. The measured mass distribution looked Gaussian.

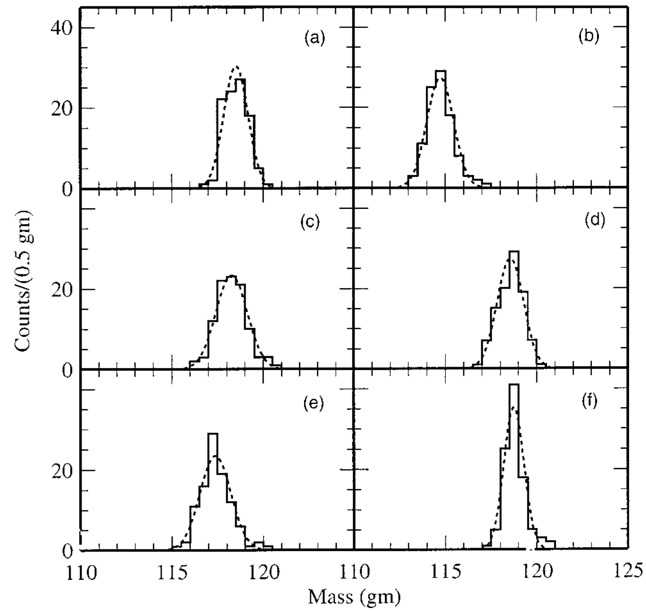

Fig. 2. The mass distributions of trays containing steel balls as measured by six students using the tray of Fig. 1(b). The dashed curves show the result of a Gaussian fit.

It is difficult to explain the observed Gaussian distribution using the CLT because we do not know the number of random variables (n) in the experiment. We have therefore modified the experiment to measure the total mass of the steel balls in a tray which contains nine small holes ($n=9$) as shown in Fig. 1(a), with the mass of balls in each hole representing a random variable. The tray is made of a sturdy but machinable foam.[5] The procedure for this experiment is similar to the previous experiments; however, it takes a

Table I. χ^2 per degree of freedom (DOF) and confidence level (CL) for a Gaussian distribution fitted to the mass distributions shown in Figs. 2 and 3.

Figure	χ^2/DOF	CL (%)
2(a)	5.3/5	39
2(b)	5.9/6	44
2(c)	5.3/8	72
2(d)	2.3/5	81
2(e)	8.8/7	27
2(f)	8.3/4	8
3(a)	45/9	0.0001
3(b)	33/13	0.2

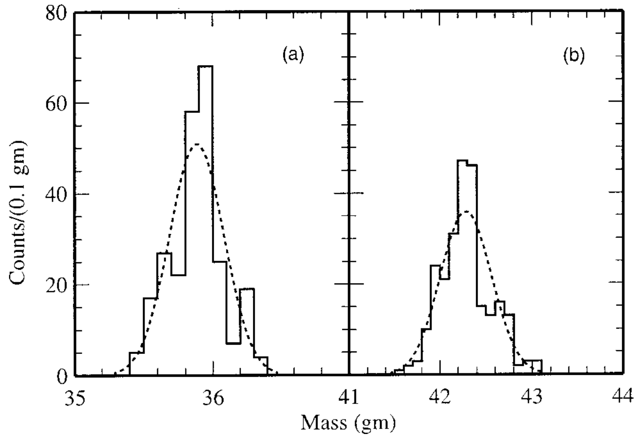

Fig. 3. The mass distributions of the steel balls in the small (a) and big (b) holes in the tray of Fig. 1(b). The dashed curves show the result of a Gaussian fit.

longer time (1 h) to accumulate 100 data points. The measured mass distribution looks Gaussian, as expected from the CLT.

We can also modify this experiment to illustrate the CLT when the random variables have different probability distributions. Thus we have another version of the experiment in which the tray has holes of two different diameters (four of 12 mm and five of 16 mm) as shown in Fig. 1(b). The results from six students are shown in Fig. 2. The means of the distributions are slightly different for each student because students wipe the excess balls off differently. Each distribution is fitted to a Gaussian and the χ^2 per degree of freedom and confidence level are summarized in Table I. It is evident that the distributions are consistent with a Gaussian distribution and the statistics are quite adequate. To verify that the Gaussian distribution is not the result of the sum of nine Gaussian distributions,[6] we measure the mass distributions of the balls in the individual holes of different diameters and the results are shown in Fig. 3. The distributions have a more pronounced peak and fit poorly to a Gaussian distribution as evident from the large χ^2 per degree of freedom and low confidence level given in Table I.

In summary, we have a simple procedure for demonstrating the central limit theorem in an acceptable length of time for a laboratory experiment.

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[1]A random variable is any function that associates a number with each possible outcome. See, for example, J. L. Devore, *Probability and Statistics for Engineering and the Sciences* (Brooks/Cole, Pacific Grove, CA, 1991), 3rd ed., pp. 80–83.
[2]See, for example, G. Cowan, *Statistical Data Analysis* (Oxford U. P., Oxford, 1998), pp. 147–149.
[3]Acculab digital scale, model V-333.
[4]The steel balls were purchased from McMaster-Carr.
[5]Last-A-Foam FR6725, General Plastics Mfg. Co.
[6]The contribution to the width of the Gaussian from the spread in the mass of the balls is small since the ball mass is measured to be uniform to within 0.3% (standard deviation).

Magnetically driven chaotic pendulum

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I. INTRODUCTION

A recent article in this journal compared several commercial chaotic pendulum systems.[1] The data obtained are impressive, but the units are rather expensive. This project describes a simple, robust, and inexpensive way to demonstrate and analyze chaotic motion quantitatively in the lab. We use a relatively inexpensive physical pendulum in conjunction with typical data acquisition equipment (rotary motion sensor, photogate, and computer). The pendulum is driven by a rotating permanent magnet. The data are analyzed by plotting them in phase space, looking at time-delay plots, finding the Poincaré section, and taking the Fourier transform. Examples of both periodic and chaotic motion are illustrated.

II. EQUIPMENT

The physical pendulum was purchased from Team Labs[2] and is shown in Fig. 1 attached to a rotary motion probe. It can be described as a "triple" pendulum. The three longer arms, fixed at 120° from each other, are 12 cm long, and each has a 10-cm-long pendulum attached to it with a precision roller bearing. Each of the three pendulums has a 40-g disk that can be adjusted along the length of the pendulum to change its natural frequency. The mass of the complete apparatus is 470 g. It has four degrees of freedom and oscillates, at least visually, in a complex and seemingly haphazard manner. To enable the pendulum to be driven, a permanent magnet is attached to its front. Two rectangular ($1\times0.5\times0.125$ in.) rare earth magnets purchased from Edmund Scientific[3] were placed adjacent to each other and taped to the pendulum. The body of the pendulum is attached to the rotary motion probe, which reads the angle of displacement forty times per second with a resolution of 0.25°. To keep the angular displacement between $-\pi$ and $+\pi$ rad, the point of attachment of the pendulum to the rotary probe was offset 2 cm from the geometrical center. The data acquisition system sends the measurements directly to a spreadsheet. Typically, 12 000 data points are acquired for each run.

To drive the pendulum, a rotating permanent magnet is placed 1–2 cm in front of the magnets attached to the pen-

