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2013 Eur. J. Phys. 34 689

(http://iopscience.iop.org/0143-0807/34/3/689)

IOP PUBLISHING EUROPEAN JOURNAL OF PHYSICS
Eur. J. Phys. **34** (2013) 689–693 doi:10.1088/0143-0807/34/3/689

A simple demonstration of the central limit theorem by dropping balls onto a grid of pins

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Received 5 January 2013, in final form 6 March 2013
Published 28 March 2013
Online at stacks.iop.org/EJP/34/689

Abstract
We present a simple procedure for demonstrating the central limit theorem by dropping stainless steel balls on a grid of pins. The experiment is part of a laboratory course on statistics for physics students that emphasizes the application of statistics in data analysis.

(Some figures may appear in colour only in the online journal)

1. Introduction

Many observations from daily life and physical experiments give rise to a bell-shaped, Gaussian frequency distribution. This is a consequence of the central limit theorem (CLT) of probability. In this paper, we present a procedure for demonstrating the CLT by dropping stainless steel balls on a grid of pins. The experiment is part of a laboratory course on statistics for physics students that emphasizes the application of statistics in data analysis.

The CLT may be stated as follows. Let $Y_1, Y_2, \ldots, Y_n$ be a sequence of n independent random variables [1], each with the same probability distribution. Suppose that the mean (μ) and variance (σ^2) of this distribution are both finite. Then the probability P for the normalized difference between the sum of the random variables and $n\mu$ to be between two numbers a and b is given by a unit Gaussian with mean at $y = 0$:

$$\lim_{n\to\infty} P\left[a < \frac{Y_1 + Y_2 + \cdots Y_n - n\mu}{\sigma\sqrt{n}} < b\right] = \frac{1}{\sqrt{2\pi}} \int_a^b \mathrm{e}^{-\frac{1}{2}y^2}\mathrm{d}y. \quad (1)$$

The theorem is still valid if the Y_is are from different probability distributions, provided each distribution has a finite mean and variance and no one term in the sum dominates. The theorem implies that under a wide range of circumstances the probability distribution that describes the sum of random variables tends toward a Gaussian distribution as the number of terms in the sum approaches infinity.

To demonstrate the CLT result that the probability distribution is consistent with a Gaussian, two experimental conditions must be satisfied:

- each measurement must be the result of the sum of a large number of random variables ($n \to \infty$);
- and the number of measurements must be large to smooth out the fluctuations in the measured distribution.

It is difficult to satisfy both conditions in a classroom experiment. Fortunately, in practice a small number of random variables is adequate for the first requirement and, for the second requirement, a finite number of measurements (30 or more) will produce a histogram that a student can recognize as Gaussian in shape.

The CLT class includes both a computer simulation and a hands-on experiment. The computer simulation shows that the distribution of the sum of 12 random numbers is consistent with a Gaussian. The hands-on experiment shows that a measured quantity is Gaussian distributed as a natural consequence of the CLT. We believe that it is important for a student to be confronted with such unavoidable experimental limitations.

2. Experimental demonstrations of CLT

For almost 20 years, we have searched continuously for a good CLT experiment that is not too tedious and yet allows a student to collect a large enough data sample within a reasonable time to obtain a Gaussian distribution with not too much fluctuation. Four of the experiments were reported in our previous communication [2] and we briefly summarize the experiments here.

(i) Measure the length of a 3–4 m table using a 30 cm (1 foot) ruler. This experiment was quite tedious and time-consuming and we therefore required each student to perform a total of only 30 measurements.
(ii) Measure the mass of 100 ml of water in a graduated 400 ml beaker. The experiment was performed using two students, one filling the beaker while the other measured the mass, without revealing the value to the other student to eliminate any potential bias. The experiment was still somewhat tedious, so each student was required to perform only 30 measurements.
(iii) Measure the mass of cans of small steel balls. The experiment was somewhat less tedious and each student was required to perform 100 measurements in order to obtain a distribution with less fluctuation.
(iv) Measure the mass of a tray with nine small holes filled with steel balls. This was a modified version of the third experiment with more random variables (n), which helped the students to appreciate the CLT concept.

We have used the fourth experiment in the statistics class since 2001. Each experiment takes about 1 h to accumulate 100 data points. In addition, the pouring of the balls makes a lot of noise. These limitations led to the search for a more pleasant experiment. We believe that we have found the ultimate experiment in which stainless steel balls are dropped on a grid of pins, resulting in the accumulation of balls with a Gaussian shape.

The experiment is a practical implementation of the invention by Sir Francis Galton [3] more than 100 years ago. A sketch of the apparatus we designed is shown in figure 1. The apparatus is fabricated using acrylic glass (polycarbonate) with a nominal thickness of 3.2 mm, except those noted below. The various pieces of acrylic glass are solvent-welded together with methyl ethyl ketone. The apparatus consists of a grid of stainless steel pins

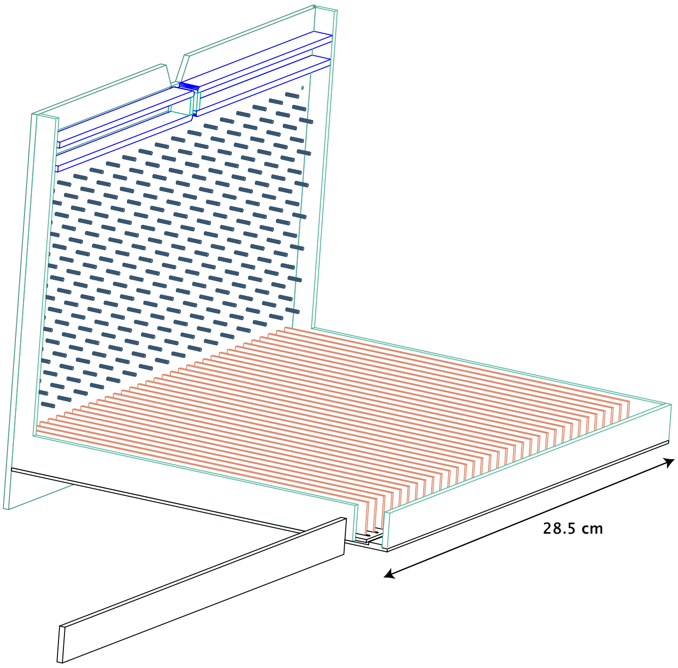

Figure 1. A schematic view of the apparatus for demonstrating the CLT. The vertical front cover and horizontal top cover have been removed for clarity. The notch at the top is for placement of the funnel.

(1.6 mm in diameter) sandwiched between two sheets of acrylic glass. The back sheet has a grid of holes drilled and the pins are press fit into the holes (no glue). The two sheets are separated by 8 mm, somewhat larger than the diameter of a ball, which is 6.4 mm. The evenly spaced pins are arranged in a 14×15 matrix, with a space of 8.4 mm between two pins in a row. Two adjacent rows of pins are separated by half the spacing and offset by the same distance.

At the top of the grid is an opening for pouring down the balls. There is a V-shaped notch on both acrylic glass sheets for the placement of a funnel for the pour. At the bottom of the notch, there is a narrow channel constructed from two small parallel pieces (15 mm in height and 8 mm in width) of acrylic glass to ensure that the balls fall down vertically. Students should pour the balls down the funnel very slowly as the balls can easily pack themselves and become stuck in the funnel.

At the base of the apparatus is a row of 17 dividers sandwiched between two sheets of acrylic glass. The dimension of a divider is 21 cm $\times$ 1 cm $\times$ 1.6 mm. These dividers form slots to collect the balls for counting. The slots therefore function as a 'histogram machine', with each slot representing a bin in a histogram. The students can just count the number of balls in each slot to produce a histogram of the ball distribution. The bottom sheet is somewhat thicker, 8 mm, so that grooves can be cut to hold the dividers. The space between the dividers is 8 mm, somewhat larger than the diameter of a ball. The apparatus is tilted by about 10°, so that when a ball reaches the bottom of the grid, it will roll down the incline and accumulate at the bottom of a slot for easy counting at the completion of the experiment.

At the bottom of the row of dividers there is a removable stopper that can be inserted to prevent the balls from rolling into the drain used for collecting the balls at the completion of

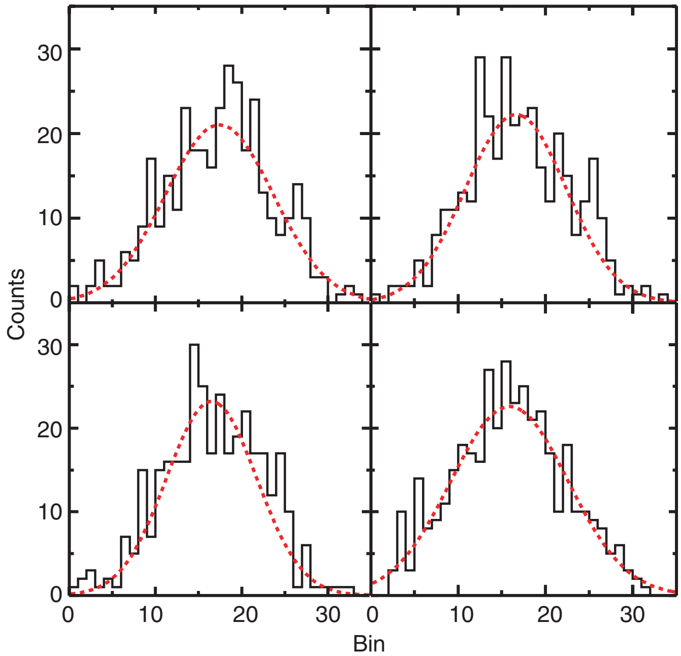

Figure 2. The distributions of the landing locations of the balls as measured in four experiments. The dashed curves show the result of a Gaussian fit.

the pour. The removable stopper slides along the groove on the bottom plate of the histogram machine. After completing the pour, the stopper is removed and the balls are collected in the drain and poured into a beaker. The balls can easily pack themselves in such a way and get stuck in the drain. Students are advised to shake the apparatus as they pour the balls into the beaker.

The apparatus can be readily constructed by a machine shop. Due to the simplicity and low cost of the construction, we fabricated 25 for a class of 20 students per session. We require each student to collect ∼70 ml of balls in a beaker and then pour the balls slowly into the funnel. This corresponds to ∼300 balls, enough to almost fill up the central slot occasionally.

In the experiment, a stainless steel ball dropped into the grid can scatter to the left or right on the first pin it encounters. The scattered ball then hits the next pin and rescatters either to the left or right. The process continues until it reaches the bottom of the grid, where there are evenly spaced slots to receive the ball. The final location where a ball lands is determined by the number of right and left scatterings. There are 14 rows of staggered pins. This is the n in equation (1). Each scattering contributes a deviation, Y_n, from the center horizontal location where the ball is released. The deviation can have positive or negative sign and its value depends on the particular angle of the scattering. The final location is then the sum of all Y_ns. CLT predicts that the location should be distributed approximately like a Gaussian probability distribution.

Figure 2 shows the histograms of four experiments, each fitted with a Gaussian with normalization floating. The χ^2 of the fits are 29.1, 34.5, 42.7 and 20.0 for 29, 29, 30, and 27 degrees of freedom, respectively. The χ^2 per degree of freedom of each fit is close to unity and hence the distributions are consistent with Gaussian, as expected from the CLT. The observed distributions are also reasonably smooth, indicating that the statistics of ∼300 data points is quite adequate. Students find that it is fun to watch the scattering of the balls and the formation of a bell-shaped distribution.

The sophistication of the data analysis depends on the level of the course and the time allocated to the experiment. In the above data analysis, the data are fitted with a Gaussian with floating normalization and then the goodness of fit is used to verify the consistency of the data with the CLT expectation. This requires taking the statistical uncertainty of each data point into account in order for the fitting program to compute the χ^2. A less sophisticated analysis is to ask the students to superimpose the Gaussian expectation as a histogram on the data distribution. The students can then comment on whether the data distribution is Gaussian-like, as expected from the CLT.

In summary, we have a quick and simple procedure for demonstrating the central limit theorem for a laboratory experiment.

Acknowledgments

The author wishes to thank Kent Lugwig and Bob Wells for their tireless efforts in the prototyping, Josh Gueth of the machine shop for the excellent machining job, Yi Yang for help with plotting the Gaussian distributions and Harris Kagan for suggesting the use of the 'bean machine'.

References

[1] Devore J L 1991 A random variable is any function that associates a number with each possible outcome. See, for example *Probability and Statistics for Engineering and the Sciences* 3rd edn (Pacific Grove, CA: Brooks/Cole) pp 80–83

[2] Gan K K, Kagan H P and Kass R D 2001 Simple demonstration of the central limit theorem using mass measurements *Am. J. Phys.* **69** 1014–6

[3] http://en.wikipedia.org/wiki/Bean_machine