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Incorporation of the statistical uncertainty in the background estimate into the upper limit on the signal

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Abstract

We present a procedure for calculating an upper limit on the number of signal events which incorporates the Poisson uncertainty in the background, estimated from control regions of one or two dimensions. For a small number of signal events, the upper limit obtained is more stringent than that extracted without including the Poisson uncertainty. This trend continues until the number of background events is comparable with the signal. When the number of background events is comparable or larger than the signal, the upper limit obtained is less stringent than that extracted without including the Poisson uncertainty. It is therefore important to incorporate the Poisson uncertainty into the upper limit; otherwise the upper limit obtained could be too stringent.

1 Introduction

In the search for a rare or forbidden process, an experiment usually observes zero or few candidate events and sets an upper limit on the number of signal events. The background is often estimated from control regions (sidebands) of one or two dimensions. In the case of a small number of candidates events, it is important to incorporate the uncertainty in the background estimate due to Poisson statistics into the upper limit.

For N_{ob} observed events with an expected background of λ_{bg} events, the upper limit on the number of signal events λ_0 at a confidence level δ is given by:

$$\frac{e^{-(\lambda_0+\lambda_{bg})}\sum_{n=0}^{N_{ob}}\frac{(\lambda_0+\lambda_{bg})^n}{n!}}{e^{-\lambda_{bg}}\sum_{n=0}^{N_{ob}}\frac{(\lambda_{bg})^n}{n!}} = 1-\delta, \qquad (1)$$

assuming that there is no uncertainty in the background estimate [1]. The uncertainty in the background estimate can be included by expanding the equation to sum (integrate) over all possible background values weighted by the background probability distribution $g(\lambda_{bg})$:

$$\frac{\int_0^\infty g(\lambda_{bg}) e^{-(\lambda_0+\lambda_{bg})} \sum_{n=0}^{N_{ob}} \frac{(\lambda_0+\lambda_{bg})^n}{n!} d\lambda_{bg}}{\int_0^\infty g(\lambda_{bg}) e^{-\lambda_{bg}} \sum_{n=0}^{N_{ob}} \frac{(\lambda_{bg})^n}{n!} d\lambda_{bg}} = 1 - \delta. \tag{2}$$

In this paper, we present a procedure for incorporating the statistical uncertainty in the background estimate in which $g(\lambda_{bg})$ is given by a Poisson distribution.

2 Incorporation of the statistical uncertainty

In experimental high energy and nuclear physics, the background is often estimated from the sideband regions because of the difficulty in predicting the distribution of background events in the signal region. This estimation assumes that the background is linear in the vicinity of the signal region, a good approximation when the sidebands are chosen close to the signal region. Figure 1(a) shows an example of a signal and two sideband regions in the distribution of a physical variable for the one-dimensional case. The number of signal events N_0 is given by:

$$N_0 = N_{ob} - N_{bg} \equiv N_{ob} - \frac{1}{2} N_{sb}, \tag{3}$$

where N_{ob} is the number of events in the signal region and N_{sb} is the total number of events in the sideband regions.

For the two-dimensional case, the background is estimated using four sideband and four corner-band regions as shown in Fig. 1(b). The number of signal events N_0 is given by:

$$N_0 = N_{ob} - N_{bg} \equiv N_{ob} - \frac{1}{2} N_{sb} + \frac{1}{4} N_{cb}, \tag{4}$$

where N_{ob} is the number of events in the signal region and N_{sb} (N_{cb}) is the total number of events in the sideband (corner-band) regions. To understand the coefficients of $-\frac{1}{2}$ and $+\frac{1}{4}$, we consider the specific example of a data sample with four photons. The sample contains the signal from two $\pi^0 \to \gamma\gamma$ decays as well as background events with one $\pi^0 \to \gamma\gamma$ decay and two photons not associated with a π^0 decay, and events where all four photons are not

π^0 decay products. In order to reconstruct the two π^0's one would plot in Fig. 1(b) the invariant mass of one $\gamma\gamma$ combination vs the other combination for all exclusive four-photon combinations. The signal region includes three sources of background: (i) real π^0 signal for $\gamma\gamma$ combinations on the X-axis together with fake π^0 signal for $\gamma\gamma$ combinations on the Y-axis, (ii) real π^0 signal for $\gamma\gamma$ combinations on the Y-axis together with fake π^0 signal for $\gamma\gamma$ combinations on the X-axis, (iii) random $\gamma\gamma$ combinations (fake π^0) on both axes. We estimate background (i) by taking the average of the number of events in the sideband regions above and below the signal region in Fig. 1(b). Similarly, background (ii) is estimated by taking the average of the number of events in the sideband regions on the left and right of the signal region. These two background corrections give the factor of $-\frac{1}{2}$ in Eq. (4). Since background (iii) is also present in the sideband regions above and below the signal region, the estimate for background (i) also includes this background. Similarly background (iii) is also present in the sideband regions on the left and right of the signal region. To correct for the over subtraction of this background, we take the average of the number of events in the four corner-band regions and add them to N_{ob}. This account for the factor of $+\frac{1}{4}$ in Eq. (4).

The general formula for the number of signal events for arbitrary sideband and corner-band size is given by:

$$N_0 = N_{ob} - N_{bg} \equiv N_{ob} - \alpha N_{sb} + \beta N_{cb}. \tag{5}$$

For the one-dimensional case, α is the ratio of the width of the signal region to the total width of the sideband regions and β is zero. In the two-dimensional case, α is two times the ratio of the area of the signal region to the total area of the sideband regions and β is the ratio of the area of the signal region to the total area of the corner-band regions.

The extraction of the upper limit on the signal must account for the uncertainty in the background estimate due to limited statistics in the control regions. For the one-dimensional case, this is implemented by replacing $g(\lambda_{sb})$ in Eq. (2) by a Poisson probability distribution for observing the N_{sb} background events given an expected number of events λ_{sb} in the sideband regions. The upper limit on the number of signal events λ at a confidence level δ is given by:

$$\frac{\int_0^\infty d\lambda_{sb} \frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!} e^{-\lambda_{sb}} e^{-(\lambda+\alpha\lambda_{sb})} \sum_{n=0}^{N_{ob}} \frac{(\lambda+\alpha\lambda_{sb})^n}{n!}}{\int_0^\infty d\lambda_{sb} \frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!} e^{-\lambda_{sb}} e^{-\alpha\lambda_{sb}} \sum_{n=0}^{N_{ob}} \frac{(\alpha\lambda_{sb})^n}{n!}}$$
$$= \frac{\sum_{n=0}^{N_{ob}} \sum_{k=0}^{n} \frac{\alpha^k \lambda^{n-k} e^{-\lambda} (N_{sb}+k)!}{(1+\alpha)^k k! (n-k)!}}{\sum_{k=0}^{N_{ob}} \frac{\alpha^k (N_{sb}+k)!}{(1+\alpha)^k k!}}$$

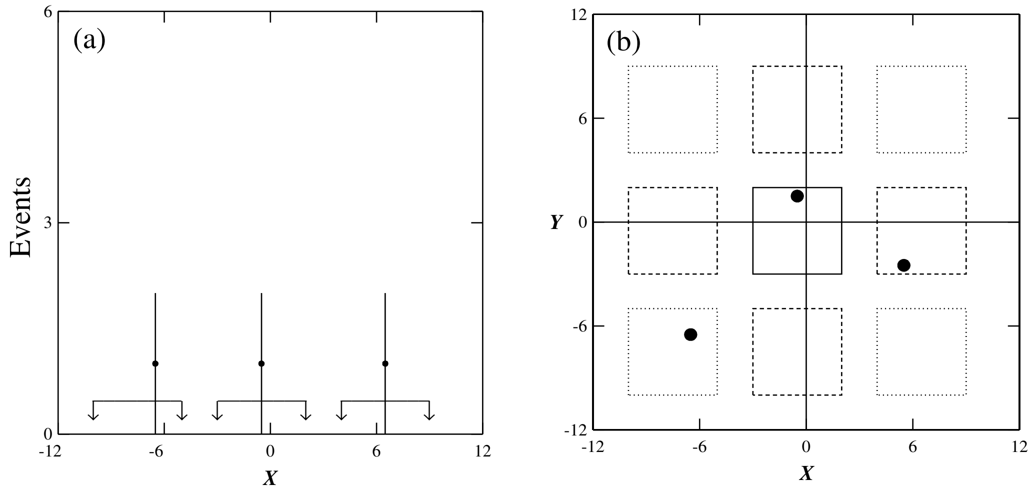

Fig. 1. (a) Definition of signal and two sideband regions (arrowed brackets) in the X distribution for the one-dimensional case. The width of each sideband region is the same as the signal region. (b) Definition of signal, sideband (dashed), and corner-band (dotted) regions in the X vs. Y distribution for the two-dimensional case. The area of a sideband or corner-band region is the same as the signal region.

$$= 1 - \delta. \tag{6}$$

The integral of λ_{sb} is performed with $\int_0^\infty dy e^{-y} y^k = k!$. Note that this formula can also be used to calculate the upper limit on the number of signal events for the case in which the background is estimated from a Monte Carlo simulation. In this case, N_{sb} is the number of background events that satisfied the event selection criteria in a Monte Carlo sample that is α^{-1} times larger than the data.

For control regions of two dimensions, the formula for the upper limit is much more complicated. However, it can be simplified depending on the value of β. In the case of $\beta = 1$ which corresponds a total corner-band area that is equal to the area of the signal region, the upper limit is given by:

$$\begin{aligned}
&\frac{\int_0^\infty d\lambda_{sb} \int_0^{\alpha\lambda_{sb}} d\lambda_{cb} \frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!} e^{-\lambda_{sb}} \frac{\lambda_{cb}^{N_{cb}}}{N_{cb}!} e^{-\lambda_{cb}} e^{-(\lambda+\alpha\lambda_{sb}-\lambda_{cb})} \sum_{n=0}^{N_{ob}} \frac{(\lambda+\alpha\lambda_{sb}-\lambda_{cb})^n}{n!}}{\int_0^\infty d\lambda_{sb} \int_0^{\alpha\lambda_{sb}} d\lambda_{cb} \frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!} e^{-\lambda_{sb}} \frac{\lambda_{cb}^{N_{cb}}}{N_{cb}!} e^{-\lambda_{cb}} e^{-(\alpha\lambda_{sb}-\lambda_{cb})} \sum_{n=0}^{N_{ob}} \frac{(\alpha\lambda_{sb}-\lambda_{cb})^n}{n!}} \\
&= \frac{\sum_{n=0}^{N_{ob}} \sum_{k=0}^{n} \frac{\alpha^k \lambda^{n-k} e^{-\lambda}}{(1+\alpha)^k (n-k)!} \sum_{j=0}^{k} \frac{(-1)^{k-j} (N_{sb}+N_{cb}+k+1)!}{j!(k-j)!(N_{cb}+k-j+1)}}{\sum_{k=0}^{N_{ob}} \frac{\alpha^k}{(1+\alpha)^k} \sum_{j=0}^{k} \frac{(-1)^{k-j} (N_{sb}+N_{cb}+k+1)!}{j!(k-j)!(N_{cb}+k-j+1)}} \\
&= 1 - \delta,
\end{aligned} \tag{7}$$

where λ_{cb} is the expected number of events in the corner-band regions. Note that the unphysical region ($\lambda_{cb} > \alpha\lambda_{sb}$) is excluded from the integral.

For a larger corner-band area, $0 < \beta < 1$, the upper limit is given by:

$$
\begin{aligned}
&\frac{\int_0^\infty d\lambda_{sb}\int_0^{\gamma\lambda_{sb}} d\lambda_{cb}\frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!}e^{-\lambda_{sb}}\frac{\lambda_{cb}^{N_{cb}}}{N_{cb}!}e^{-\lambda_{cb}}e^{-(\lambda+\alpha\lambda_{sb}-\beta\lambda_{cb})}\sum_{n=0}^{N_{ob}}\frac{(\lambda+\alpha\lambda_{sb}-\beta\lambda_{cb})^n}{n!}}{\int_0^\infty d\lambda_{sb}\int_0^{\gamma\lambda_{sb}} d\lambda_{cb}\frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!}e^{-\lambda_{sb}}\frac{\lambda_{cb}^{N_{cb}}}{N_{cb}!}e^{-\lambda_{cb}}e^{-(\alpha\lambda_{sb}-\beta\lambda_{cb})}\sum_{n=0}^{N_{ob}}\frac{(\alpha\lambda_{sb}-\beta\lambda_{cb})^n}{n!}}\\
&=\frac{\sum_{n=0}^{N_{ob}}\sum_{k=0}^{n}\alpha^k\frac{\lambda^{n-k}e^{-\lambda}}{(n-k)!}\sum_{j=0}^{k}\frac{(-1)^{k-j}(N_{cb}+k-j)!}{(\gamma-\alpha)^{k-j}j!(k-j)!}\left[\frac{(N_{sb}+j)!}{(1+\alpha)^{N_{sb}+j+1}}-\sum_{i=0}^{N_{cb}+k-j}\frac{(\gamma-\alpha)^i(N_{sb}+j+i)!}{(1+\gamma)^{N_{sb}+j+i+1}i!}\right]}{\sum_{k=0}^{N_{ob}}\alpha^k\sum_{j=0}^{k}\frac{(-1)^{k-j}(N_{cb}+k-j)!}{(\gamma-\alpha)^{k-j}j!(k-j)!}\left[\frac{(N_{sb}+j)!}{(1+\alpha)^{N_{sb}+j+1}}-\sum_{i=0}^{N_{cb}+k-j}\frac{(\gamma-\alpha)^i(N_{sb}+j+i)!}{(1+\gamma)^{N_{sb}+j+i+1}i!}\right]}\\
&=1-\delta,
\end{aligned}\tag{8}
$$

where the unphysical region ($\lambda_{cb} > \frac{\alpha}{\beta}\lambda_{sb} \equiv \gamma\lambda_{sb}$) is excluded from the integral. The integral of λ_{cb} is obtained with $\int_0^x dy e^{-y}y^k = k!(1-\sum_{i=0}^{k}\frac{x^i}{i!}e^{-x})$.

In the following sections, we consider two widely used relative dimensions between the signal and control regions.

2.1 Case I: $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$

We first consider the one-dimensional case in which the total width of the sideband regions is twice that of the signal region as shown in Fig. 1(a). This corresponds to $\alpha = \frac{1}{2}$ and Eq. (6) for the upper limit on the signal is simplified to:

$$
\begin{aligned}
&\frac{\int_0^\infty d\lambda_{sb}\frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!}e^{-\lambda_{sb}}e^{-(\lambda+\frac{\lambda_{sb}}{2})}\sum_{n=0}^{N_{ob}}\frac{(\lambda+\frac{\lambda_{sb}}{2})^n}{n!}}{\int_0^\infty d\lambda_{sb}\frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!}e^{-\lambda_{sb}}e^{-\frac{\lambda_{sb}}{2}}\sum_{n=0}^{N_{ob}}\frac{(\frac{\lambda_{sb}}{2})^n}{n!}}\\
&=\frac{\sum_{n=0}^{N_{ob}}\sum_{k=0}^{n}\frac{\lambda^{n-k}e^{-\lambda}(N_{sb}+k)!}{3^k k!(n-k)!}}{\sum_{k=0}^{N_{ob}}\frac{(N_{sb}+k)!}{3^k k!}}\\
&=1-\delta.
\end{aligned}\tag{9}
$$

We now consider the two-dimensional case in which the total area of both the sideband and corner-band regions is four times that of the signal region as shown in Fig. 1(b). This corresponds to $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$. From Eq. (8), the upper limit on the signal is given by:

$$
\begin{aligned}
&\frac{\int_0^\infty d\lambda_{sb}\int_0^{2\lambda_{sb}} d\lambda_{cb}\frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!}e^{-\lambda_{sb}}\frac{\lambda_{cb}^{N_{cb}}}{N_{cb}!}e^{-\lambda_{cb}}e^{-(\lambda+\frac{\lambda_{sb}}{2}-\frac{\lambda_{cb}}{4})}\sum_{n=0}^{N_{ob}}\frac{(\lambda+\frac{\lambda_{sb}}{2}-\frac{\lambda_{cb}}{4})^n}{n!}}{\int_0^\infty d\lambda_{sb}\int_0^{2\lambda_{sb}} d\lambda_{cb}\frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!}e^{-\lambda_{sb}}\frac{\lambda_{cb}^{N_{cb}}}{N_{cb}!}e^{-\lambda_{cb}}e^{-(\frac{\lambda_{sb}}{2}-\frac{\lambda_{cb}}{4})}\sum_{n=0}^{N_{ob}}\frac{(\frac{\lambda_{sb}}{2}-\frac{\lambda_{cb}}{4})^n}{n!}}\\
&=\frac{\sum_{n=0}^{N_{ob}}\sum_{k=0}^{n}\frac{\lambda^{n-k}e^{-\lambda}}{3^k(n-k)!}\sum_{j=0}^{k}\frac{(-1)^{k-j}(N_{cb}+k-j)!}{j!(k-j)!}[(N_{sb}+j)!-\sum_{i=0}^{N_{cb}+k-j}\frac{(N_{sb}+j+i)!}{2^{N_{sb}+j+i+1}i!}]}{\sum_{k=0}^{N_{ob}}\frac{1}{3^k}\sum_{j=0}^{k}\frac{(-1)^{k-j}(N_{cb}+k-j)!}{j!(k-j)!}[(N_{sb}+j)!-\sum_{i=0}^{N_{cb}+k-j}\frac{(N_{sb}+j+i)!}{2^{N_{sb}+j+i+1}i!}]}
\end{aligned}
$$

$$= 1 - \delta. \tag{10}$$

2.2 Case II: $\alpha = \beta = 1$

We first consider the one-dimensional case in which the total width of the sideband regions is the same as that of the signal region (Fig. 2(a)); an experimenter may choose this kind of smaller background control regions so that the background is more linear in the vicinity of the signal region. Substituting for $\alpha = 1$ in Eq. (6), the upper limit of the signal is given by:

$$\frac{\int_0^\infty d\lambda_{sb} \frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!} e^{-\lambda_{sb}} e^{-(\lambda+\lambda_{sb})} \sum_{n=0}^{N_{ob}} \frac{(\lambda+\lambda_{sb})^n}{n!}}{\int_0^\infty d\lambda_{sb} \frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!} e^{-\lambda_{sb}} e^{-\lambda_{sb}} \sum_{n=0}^{N_{ob}} \frac{(\lambda_{sb})^n}{n!}}$$
$$= \frac{\sum_{n=0}^{N_{ob}} \sum_{k=0}^{n} \frac{\lambda^{n-k} e^{-\lambda} (N_{sb}+k)!}{2^k k! (n-k)!}}{\sum_{k=0}^{N_{ob}} \frac{(N_{sb}+k)!}{2^k k!}}$$
$$= 1 - \delta. \tag{11}$$

We now consider the two-dimensional case in which the total area of the sideband regions is twice that of the signal region and the total area of cornerband regions is the same as that of the signal region (Fig. 2(b)). The upper limit is given by Eq. (7) through substitution of $\alpha = \beta = 1$:

$$\frac{\int_0^\infty d\lambda_{sb} \int_0^{\lambda_{sb}} d\lambda_{cb} \frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!} e^{-\lambda_{sb}} \frac{\lambda_{cb}^{N_{cb}}}{N_{cb}!} e^{-\lambda_{cb}} e^{-(\lambda+\lambda_{sb}-\lambda_{cb})} \sum_{n=0}^{N_{ob}} \frac{(\lambda+\lambda_{sb}-\lambda_{cb})^n}{n!}}{\int_0^\infty d\lambda_{sb} \int_0^{\lambda_{sb}} d\lambda_{cb} \frac{\lambda_{sb}^{N_{sb}}}{N_{sb}!} e^{-\lambda_{sb}} \frac{\lambda_{cb}^{N_{cb}}}{N_{cb}!} e^{-\lambda_{cb}} e^{-(\lambda_{sb}-\lambda_{cb})} \sum_{n=0}^{N_{ob}} \frac{(\lambda_{sb}-\lambda_{cb})^n}{n!}}$$
$$= \frac{\sum_{n=0}^{N_{ob}} \sum_{k=0}^{n} \frac{\lambda^{n-k} e^{-\lambda}}{2^k (n-k)!} \sum_{j=0}^{k} \frac{(-1)^{k-j} (N_{sb}+N_{cb}+k+1)!}{j!(k-j)!(N_{cb}+k-j+1)}}{\sum_{k=0}^{N_{ob}} \frac{1}{2^k} \sum_{j=0}^{k} \frac{(-1)^{k-j} (N_{sb}+N_{cb}+k+1)!}{j!(k-j)!(N_{cb}+k-j+1)}}$$
$$= 1 - \delta. \tag{12}$$

3 Results

We have investigated the implication of including the Poisson uncertainty in the estimated background in the calculation of upper limits. In Tables 1-2, we give the 90% and 95% confidence level upper limits on the signal with and without including the Poisson uncertainty for several cases of small numbers of observed events and various backgrounds. The control regions have been

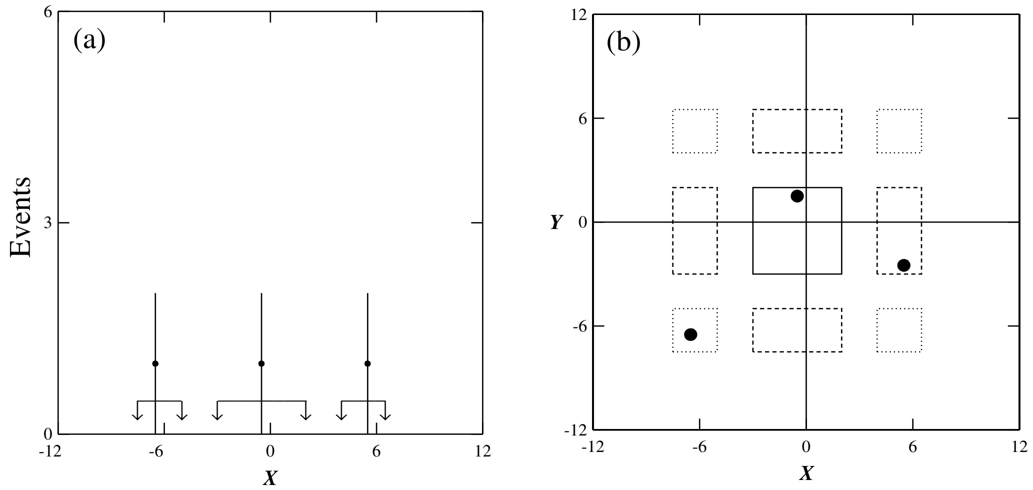

Fig. 2. (a) Definition of signal and two sideband regions (arrowed brackets) in the X distribution for the one-dimensional case. The width of each sideband region is half that of the signal region. (b) Definition of signal, sideband (dashed), and corner-band (dotted) regions in the X vs. Y distribution for the two-dimensional case. The area of each sideband (corner-band) region is $\frac{1}{2}$ ($\frac{1}{4}$) that of the signal region.

chosen such that $\alpha = \frac{1}{2}$ and $\beta = \frac{1}{4}$. In the calculation of λ_0 with Eq. (1), the background is set to zero if $N_{cb} > 2N_{sb}$. As expected, λ is identical to λ_0 when the number of events in the signal region is zero, regardless of the number of estimated background events. However, λ is smaller than λ_0 when the number of events in the signal region is non-zero and the estimated background is zero. This is not unexpected because Eqs. (9) and (10) allow for the estimated zero background to fluctuate up, in contrast to Eq. (1) in which the background is estimated to be zero with no uncertainty. This trend continues until the number of background events is comparable with the signal. When the number of background events is comparable or larger than the signal, the upper limit obtained is less stringent than that extracted without including the Poisson uncertainty in the estimated background. It is therefore important to incorporate the Poisson uncertainty into the upper limit; otherwise the upper limit obtained could be too stringent.

For a large number of observed events, we can use the common assumption that the signal N_0 in Eq. (4) is normally distributed (Gaussian) with the variance given by

$$\sigma^2 = N_{ob} + \frac{1}{4}N_{sb} + \frac{1}{16}N_{cb}. \tag{13}$$

The upper limit on the number of signal events λ_G is obtained by integrating from N_0 to λ_G so that the integrated area is δ of the area integrated to $+\infty$. For an unphysical signal, $N_0 < 0$, the integration should be renormalized by setting $N_0 = 0$ to obtain a more conservative limit [2]. Tables 3-4 show a comparison

Table 1
Upper limits on the signal for a few observed events and various backgrounds in the one-dimensional case. The total width of the sideband regions is twice that of the signal region.

	90% C.L.		95% C.L.	
$(N_{ob}, \frac{N_{sb}}{2})$	λ_0	λ	λ_0	λ
$(0, \frac{N_{sb}}{2})$	2.30	2.30	3.00	3.00
(1,0)	3.89	3.61	4.74	4.47
$(1, \frac{1}{2})$	3.51	3.42	4.36	4.27
(1,1)	3.27	3.27	4.11	4.11
$(1, 1\frac{1}{2})$	3.11	3.16	3.94	3.99
(1,2)	3.00	3.07	3.82	3.90
(2,0)	5.32	4.94	6.30	5.92
(2,1)	4.44	4.36	5.41	5.33
(2,2)	3.88	3.96	4.82	4.91
(2,3)	3.52	3.68	4.44	4.61
(2,4)	3.29	3.47	4.18	4.39
(3,0)	6.68	6.26	7.75	7.34
(3,1)	5.71	5.53	6.78	6.61
(3,2)	4.93	4.96	5.98	6.03
(3,3)	4.36	4.53	5.40	5.58

of λ_G and λ for the case of $N_{ob} = N_{bg}$. Due to the longer tail of the Poisson distribution, λ is larger than λ_G. The significance of the difference diminishes with larger N_{ob}. For example, in the one-dimensional case, it decreases from $\sim 10\%$ for $N_{ob} = 10$ events to $\sim 5\%$ for $N_{ob} = 50$ events. For an experiment with small systematic error, Eq. (9) or (10) should be used to compute the upper limit instead of using the Gaussian approximation.

For completeness, we also listed in Tables 5-6 the 90% and 95% confidence level upper limits on the signal for $\alpha = \beta = 1$ with and without including the Poisson uncertainty in the estimated background for a small number of observed events and various backgrounds.

As noted in the previous section, Eq. (6) can also be used to calculate the upper limit on the signal for the case in which the background is estimated from a Monte Carlo simulation. The impact of including the Poisson uncertainty in the estimated background into the upper limit can be investigated

Table 2
Upper limits on the signal for a few observed events and various backgrounds in the two-dimensional case. The total area of both the sideband and corner-band regions is four times that of the signal region.

	90% C.L.		95% C.L.	
$(N_{ob}, \frac{N_{sb}}{2}, \frac{N_{cb}}{4})$	λ_0	λ	λ_0	λ
$(0, \frac{N_{sb}}{2}, \frac{N_{cb}}{4})$	2.30	2.30	3.00	3.00
$(1, 0, \frac{N_{cb}}{4})$	3.89	3.61	4.74	4.47
$(1, \frac{1}{2}, 0)$	3.51	3.48	4.36	4.33
$(1, \frac{1}{2}, \frac{1}{4})$	3.67	3.51	4.53	4.36
$(1, \frac{1}{2}, \frac{1}{2})$	3.89	3.53	4.74	4.38
$(1, \frac{1}{2}, 1)$	3.89	3.55	4.74	4.40
(1,1,0)	3.27	3.35	4.11	4.20
$(1, 1, \frac{1}{4})$	3.38	3.40	4.22	4.25
$(1, 1, \frac{1}{2})$	3.51	3.44	4.36	4.29
(1,1,1)	3.89	3.49	4.74	4.33
(1,2,0)	3.00	3.14	3.82	3.98
$(1, 2, \frac{1}{4})$	3.05	3.21	3.88	4.05
$(1, 2, \frac{1}{2})$	3.11	3.26	3.94	4.11
(1,2,1)	3.27	3.35	4.11	4.19
$(2, 0, \frac{N_{cb}}{4})$	5.32	4.94	6.30	5.92
(2,1,0)	4.44	4.49	5.41	5.46
$(2, 1, \frac{1}{2})$	4.84	4.64	5.82	5.61
(2,1,1)	5.32	4.72	6.30	5.69
(2,1,2)	5.32	4.80	6.30	5.77
(2,2,0)	3.88	4.09	4.82	5.05
$(2, 2, \frac{1}{2})$	4.13	4.31	5.08	5.27
(2,2,1)	4.44	4.45	5.41	5.43
(2,2,2)	5.32	4.62	6.30	5.60

by comparing the limit with that obtained without including the uncertainty. Table 7 lists the 90% confidence level upper limits obtained with and without including the Poisson uncertainty for several cases of small numbers of observed events and various backgrounds. As in the previous examples, λ is smaller than λ_0 for a small number of observed events with comparable or

Table 3
Upper limits on the signal for a large number of observed events with large background in the one-dimensional case. The total width of the sideband regions is twice that of the signal region.

	90% C.L.			95% C.L.		
$(N_{ob}, \frac{N_{sb}}{2})$	λ_0	λ	λ_G	λ_0	λ	λ_G
(10,10)	6.63	7.24	6.35	8.08	8.78	7.59
(20,20)	8.75	9.82	8.98	10.60	11.83	10.74
(35,35)	11.11	12.69	11.88	13.41	15.24	14.20
(50,50)	13.00	15.00	14.20	15.66	17.98	16.97

Table 4
Upper limits on the signal for a large number of observed events with large background in the two-dimensional case. The total area of both the sideband and cornerband regions is four times that of the signal region.

	90% C.L.			95% C.L.		
$(N_{ob}, \frac{N_{sb}}{2}, \frac{N_{cb}}{4})$	λ_0	λ	λ_G	λ_0	λ	λ_G
(10,20,10)	6.63	8.61	7.78	8.08	10.31	9.30
(15,25,10)	7.78	9.86	8.98	9.45	11.82	10.74
(20,30,10)	8.75	10.93	10.04	10.60	13.10	12.00

smaller background. However, λ is larger than λ_0 for larger background. Not including the Poisson uncertainty leads to a too stringent limit in this case.

4 Conclusion

We have presented a procedure for calculating an upper limit on the number of signal events which incorporates the Poisson uncertainty in the background estimated from control regions of one or two dimensions. For a small number of observed events in the signal region, the limit obtained is more stringent than that extracted assuming no uncertainty in the estimated background. This trend continues until the number of background events is comparable with the signal. When the number of background events is comparable or larger than the signal, the upper limit obtained is less stringent than that extracted without including the Poisson uncertainty in the estimated background. It is therefore important to incorporate the Poisson uncertainty into the upper limit; otherwise the upper limit obtained could be too stringent.

Table 5
Upper limits on the signal for a few observed events and various backgrounds in the one-dimensional case. The total width of the sideband regions is the same as that of the signal region.

	90% C.L.		95% C.L.	
(N_{ob}, N_{sb})	λ_0	λ	λ_0	λ
$(0, N_{sb})$	2.30	2.30	3.00	3.00
(1,0)	3.89	3.51	4.74	4.36
(1,1)	3.27	3.27	4.11	4.11
(1,2)	3.00	3.11	3.82	3.94
(1,3)	2.84	3.00	3.64	3.82
(1,4)	2.74	2.91	3.53	3.72
(2,0)	5.32	4.75	6.30	5.72
(2,1)	4.44	4.32	5.41	5.29
(2,2)	3.88	4.01	4.82	4.97
(2,3)	3.52	3.77	4.44	4.72
(2,4)	3.29	3.59	4.18	4.52
(3,0)	6.68	5.99	7.75	7.08
(3,1)	5.71	5.43	6.78	6.52
(3,2)	4.93	4.99	5.98	6.06
(3,3)	4.36	4.63	5.40	5.69
(3,4)	3.97	4.35	4.97	5.39

5 Acknowledgment

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Table 6
Upper limits on the signal for a few observed events and various backgrounds in the two-dimensional case. The total area of the sideband regions is twice that of the signal region while the total area of the corner-band regions is the same as that of the signal region.

	90% C.L.		95% C.L.	
(N_{ob},N_{sb},N_{cb})	λ_0	λ	λ_0	λ
$(0,N_{sb},N_{cb})$	2.30	2.30	3.00	3.00
$(1,0,N_{cb})$	3.89	3.51	4.74	4.36
(1,1,0)	3.27	3.38	4.11	4.22
(1,1,1)	3.89	3.42	4.74	4.27
(1,1,2)	3.89	3.44	4.74	4.29
(1,1,3)	3.89	3.45	4.74	4.30
(1,2,0)	3.00	3.27	3.82	4.11
(1,2,1)	3.27	3.34	4.11	4.18
(1,2,2)	3.89	3.38	4.74	4.22
(1,2,3)	3.89	3.40	4.74	4.25
(1,3,0)	2.84	3.18	3.64	4.02
(1,3,1)	2.99	3.27	3.82	4.11
(1,3,2)	3.27	3.32	4.11	4.17
(1,3,3)	3.89	3.36	4.74	4.20
$(2,0,N_{cb})$	5.32	4.75	6.30	5.72
(2,1,0)	4.44	4.50	5.41	5.47
(2,1,1)	5.32	4.57	6.30	5.55
(2,1,2)	5.32	4.61	6.30	5.59
(2,1,3)	5.32	4.64	6.30	5.61
(2,2,0)	3.88	4.29	4.82	5.26
(2,2,1)	4.44	4.41	5.41	5.39
(2,2,2)	5.32	4.49	6.30	5.46
(2,2,3)	5.32	4.53	6.30	5.51

Table 7
90% confidence level upper limits on the signal for a small number of observed events and various backgrounds estimated from a Monte Carlo Simulation.

α^{-1}	3		6		9	
(N_{ob},N_{sb})	λ_0	λ	λ_0	λ	λ_0	λ
(1,0)	3.89	3.67	3.89	3.76	3.89	3.80
(1,1)	3.61	3.51	3.74	3.65	3.79	3.71
(1,2)	3.42	3.38	3.61	3.55	3.70	3.64
(1,3)	3.27	3.27	3.51	3.47	3.61	3.57
(1,4)	3.16	3.18	3.42	3.40	3.54	3.51
(1,5)	3.07	3.11	3.34	3.33	3.48	3.45
(1,6)	3.00	3.05	3.27	3.27	3.42	3.40
(1,7)	2.93	3.00	3.21	3.22	3.37	3.36
(1,8)	2.88	2.95	3.16	3.17	3.32	3.31
(1,9)	2.84	2.91	3.11	3.13	3.27	3.27
(2,0)	5.32	5.04	5.32	5.17	5.32	5.22
(2,1)	5.00	4.79	5.16	5.02	5.21	5.11
(2,2)	4.70	4.57	5.00	4.88	5.10	5.01
(2,3)	4.44	4.38	4.84	4.75	5.00	4.92
(2,4)	4.22	4.21	4.70	4.63	4.89	4.82
(2,5)	4.04	4.07	4.57	4.51	4.79	4.73
(2,6)	3.88	3.94	4.44	4.41	4.70	4.65
(2,7)	3.74	3.82	4.33	4.31	4.61	4.57
(2,8)	3.62	3.72	4.22	4.22	4.52	4.49
(2,9)	3.52	3.63	4.13	4.13	4.44	4.42