30

THE BÉNARD PROBLEM, SYMMETRY AND THE LATTICE OF ISOTROPY SUBGROUPS

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Published in:

Bifurcation Theory, Mechanics and Physics: Mathematical Developments and Applications

C.P. Bruter, A. Aragnol, and A. Lichnerowicz (Eds)

Mathematics and Its Applications
D. Reidel, Dordrecht (1983) pp. 225-256

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INTRODUCTION

In this lecture I would like to describe some of the effects that are forced on steady state bifurcation problems by the existence of a group of symmetries. I shall discuss this relationship between symmetry and bifurcation by describing several mathematical problems which are motivated by the Bénard problem.

The Bénard problem in its simplest form is the study of the transition from pure conduction to convective motion in a contained fluid heated on (part of) its boundary. The model equations which lie behind the analysis are the Navier-Stokes equations in the Boussinesq approximation. The purpose of this exposition is to indicate the type of information that can be obtained through the use of singularity theory and group theory. For this reason the exact form of the Boussinesq equations is not needed and they will not be presented. The interested reader can consult the paper by Fauvre and Libchaber [1983] in this volume or the extensive and very interesting work of Busse [1962, 1975, 1978].

The specific results outlined below rely for their proofs on the machinery of group theory and singularity theory plus very extensive calculations. What is remarkable is that after this effort has been expended the final answer has a delightful and compelling organization based on the lattice of isotropy subgroups of the given group representation. It is our intention to emphasize this relationship throughout.

The paper is divided into four sections. The first three sections concern specific realizations of the Bénard problem while the last section presents certain general results concerning bifurcation with symmetry.

C. P. Bruter et al. (eds.), Bifurcation Theory, Mechanics and Physics, 225–256.

A. The spherical Bénard problem:

The fluid is contained between two fixed concentric spheres and is heated along the inner sphere. This is a model for convection of the molten layer of the Earth contained between the solid inner core and the mantle. Since the Earth rotates this is a simplified model. The symmetry group for this problem is O(3).

B. The planar Bénard problem with non-symmetric boundary conditions:

The fluid is contained between two parallel planes and is heated from below. Moreover, the boundary conditions on the upper plane are assumed to be different from those on the lower plane. This is the situation found in Bénard's original experiment as there one finds a free boundary on top and a rigid boundary below. One should note that we consider an infinite plane while all experiments are - of course - performed on a finite plane. So the relationship between the mathematics presented here and any given experiment is, at best, arguable.

In addition, the fact that this form of the Bénard problem is posed on the infinite plane makes the mathematical analysis - even a local one near the pure conduction solution - extremely difficult. A popular restriction (Busse [1962], Sattinger [1978]) which alleviates some of the technical difficulties is to look only for solutions which are doubly periodic in the plane. There are, however, several different types of double periodicity possible and it is here that Bénard's experiment serves as a useful guide. In his experiment, Bénard found that convection patterns in the shape of hexagons occur and, moreover, that these hexagons are arranged on the hexagonal lattice - at least away from the boundary. Given this fact, it seems reasonable to look first for solutions which are doubly periodic with respect to the hexagonal lattice. We note, however, that among the many patterns of convection found by experiment there are some which are not doubly periodic with respect to the hexagonal lattice.

We now describe the symmetry group for this problem. The Boussinesq equations have the symmetry of the Euclidean group in the plane generated by rotations, reflections and translations. The assumption of double periodicity implies that the translations act periodically; that is, the 2-torus

T^2 is part of the group of symmetries. The assumption of double periodicity with respect to the hexagonal lattice implies that the Boussinesq equations are left invariant only by those rotations and reflections which preserve the hexagonal lattice. Let D_6 denote the dihedral group of symmetries of the regular hexagon in the plane. Then the group of symmetries for this form of the Bénard problem is $T^2 + D_6$.

C. The planar Bénard problem with symmetric boundary conditions:

This problem is posed like the preceding one with a single exception. We assume that the boundary conditions on the top plane are identical to those on the bottom. This is the case found in experiments where the fluid is contained between two bounding planes of the same type and, moreover, such experiments are performed frequently.

For this form of the Bénard problem the symmetries include $T^2 + D_6$ as above and, in addition, a reflectional symmetry obtained by reflecting the fluid layer about it's midplane. This reflection interchanges top and bottom and - with the assumption on the boundary conditions - leaves the Boussinesq equations invariant. Thus the symmetry group is $T^2 + D_6 + Z_2$.

Each of the first three sections is devoted, in order, to a description of the bifurcation behavior found in the above problems. The exposition involves, in each case, describing how one reduces these problems to the problem of finding the zeroes of a mapping $g: \mathbf{R}^n \to \mathbf{R}^n$ depending on a parameter where g commutes with (a given representation of) the groups indicated above. Specifically we describe how group theory restricts the form of g and how singularity theory allows one to find the zero set of g. We shall describe, in each case, an a postiori relationship between the bifurcation structure found and the lattice of isotropy subgroups of the group.

In the last section we present several general results - some new - which indicate that it may be possible in the future to obtain much of the structure of bifurcation problems directly from a knowledge of the group of symmetries of the given problem. Such a result would give a good beginning to the resolution of the problem of spontaneous symmetry breaking - which is the ultimate mathematical goal of this line of research.

The specific results described in the sections below have been obtained in collaboration with the following individuals: David Schaeffer, Ernesto Buzano, Jim Swift and Edgar Knobloch, and Ian Stewart.

1. THE BÉNARD PROBLEM IN SPHERICAL GEOMETRY

The details of the results outlined in this section are contained in Busse [1975], Chossat [1979], Golubitsky and Schaeffer [1982] and the Thèse D'État of Chossat [1982].

Given a fluid contained between two concentric spheres, let τ denote the temperature difference between the inner and outer spheres. If τ is increased there is a first τ_0 at which point the pure conduction solution looses stability and convective motion begins. In the mathematical formulation of this problem τ_0 is the first value of τ where the linearization of the Boussinesq equations about the pure conduction solution, $L(\tau)$, has a zero eigenvalue. Let $V = \ker L(\tau_0)$. It is known that V is the space of spherical harmonics of order p and that p depends on the aspect ratio η where η is the ratio of the radius of the inner sphere to the radius of the outer sphere. See Chossat [1979,1982].

In particular, if one views the spherical Bénard problem as a model for convection in the Earth's molten inner core then $.25 < \eta < .5$ and V is either the spherical harmonics of order 2 or 4. We consider here the case $\eta \sim .3$ and $V = Y_2$ the spherical harmonics of order 2. So dim V = 5. Moreover SO(3) acts irreducibly on Y_2.

Let $\lambda=\tau-\tau_0$. The Liapunov-Schmidt procedure (or the implicit function theorem) shows that finding all solutions to the Boussinesq equations near the pure conduction solution and $\lambda = 0$ is equivalent to finding the zeroes of a C^∞ mapping

$$g: V \times \mathbf{R} \to V; \quad g(A,\lambda) = 0$$

which is defined on some neighborhood of $(0,0) \in V \times \mathbf{R}$ and satisfies

(i) $g(0,\lambda) = 0$ where $0 \in V$ corresponds to the pure conduction solution

(ii) $(d_A g)_{0,0} \equiv 0$ (1.1)

(iii) $g(\gamma \cdot A,\lambda) = \gamma \cdot g(A,\lambda)$ for all $\gamma \in O(3)$.

The important point here is that one is looking for the zeroes of an equivariant mapping near a singular point. We analyse such mappings first by group theory and then using singularity theory.

Another realization of the five dimensional irreducible representation of $O(3)$ is as follows: Let V be the vector space of all 3×3 symmetric trace 0 matrices. Note that $\dim V = 5$. Let $\gamma \in O(3)$ act on $A \in V$ by $\gamma A = \gamma A \gamma^t$. Via this linear action $O(3)$ acts irreducibly on V. We prefer using this presentation of the representation as opposed to the presentation on spherical harmonics.

Let E be the space of C^∞ equivariant (germs of) mappings $g: V \times \mathbf{R} \to V$; i.e. g satisfies (1.1)(iii) above. Let I denote the ring of (germs of) invariant C^∞ functions $f: V \times \mathbf{R} \to \mathbf{R}$; i.e., $f(\gamma \cdot A, \lambda) = f(A, \lambda)$ for all $\gamma \in O(3)$. Then E is a module over I and one can describe this module structure explicitly.

<u>Proposition 1.1</u>: (a) Let f be in I. Then there is a smooth function $p: \mathbf{R}^3 \to \mathbf{R}$ such that

$$f(A,\lambda) = p(u,v,\lambda)$$

where $u = \mathrm{tr}(A^2)$ and $v = \det A$.
(b) Let g be in E. Then there exist invariant functions p and q such that

$$g(A,\lambda) = p(u,v,\lambda)A + q(u,v,\lambda)(A^2 - \tfrac{1}{3}\mathrm{tr}(A^2)I).$$

For details see Golubitsky and Schaeffer [1982].

<u>Remarks</u>: (a) If g comes from a Liapunov-Schmidt reduction then (1.1)(i) implies that $p(0,0,0) = 0$.
(b) In the Bénard problem the pure conduction solution looses stability at τ_0 (which we have identified in the Liapunov-Schmidt reduction with $\lambda = 0$). Observe that the chain rule applied to (1.1)(iii) implies

$$(dg)_{\gamma \cdot A,\lambda}\gamma = \gamma(dg)_{A,\lambda}.$$

In particular $(dg)_{0,A}\gamma = \gamma(dg)_{0,A}$ for all γ. Now this representation of $O(3)$ on V is absolutely irreducible, the only linear maps commuting with $O(3)$ are scalar multiples of the identity. Thus

$$(dg)_{0,\lambda} = p(0,0,\lambda)I.$$

Since the stability of the pure conduction solution is assumed to change at τ_0 as τ is varied, it is reasonable to assume - and may be computed for the Boussinesq equations - that $f_\lambda(0,0,0) \neq 0$.

(c) In general one might assume that $q(0,0,0) \neq 0$. Chossat [1979] has shown that in one important case $q(0,0,0) = 0$ and this is the basis for his analysis. More precisely, let β_i and β_0 be the thermal conductivities of the materials in the inner sphere and the outer shell respectively. Let ρ_i and ρ_0 denote the respective densities of these materials. One finds that the linearization $L(\lambda_0)$ of the Boussinesq equations is self-adjoint if the constants β and ρ satisfy

$$\beta_i/\beta_0 = \rho_i/\rho_0 \tag{1.2}$$

The identity (1.2) defines the self-adjoint case. Chossat shows that in the self-adjoint case $q(0,0,0) = 0$.

Using singularity theory one can analyse two questions.

(1) (Recognition Problem) When does one have enough information about the Taylor expansion of g to ignore higher order terms?

(2) (Imperfect Bifurcation) Determine all possible perturbations of the given bifurcation problem by finding the universal unfolding.

We give the answers to these questions in two cases.

<u>Proposition 1.2</u>: Let $g \in E$ satisfy $p(0,0,0) = 0$ and $p_\lambda(0,0,0) \neq 0$. Then

(a) if $g(0,0,0) \neq 0$ then g is O(3)-equivalent to

$$h(A,\lambda) = \varepsilon_1 \lambda A + \varepsilon_2 (A^2 - \frac{1}{3}\operatorname{tr}(A^2)I)$$

where $\varepsilon_1 = \operatorname{sgn}(p_\lambda(0,0,0))$ and $\varepsilon_2 = \operatorname{sgn}(q(0,0,0))$.

(b) if $q(0,0,0) = 0$ and

(i) $p_u(0,0,0) \neq 0$

(ii) $C \equiv p_\lambda q_u - p_u q_\lambda \neq 0$ at $(0,0,0)$

(iii) $D \equiv p_u q_v - p_v q_u \neq 0$ at $(0,0,0)$

then g is O(3)-equivalent to

$$h(A,\lambda) = (\varepsilon_1 u + \varepsilon_2\lambda)A + (\varepsilon_3 u + Dv)(A^2 - \frac{1}{3}\operatorname{tr}(A^2)I).$$

where $\varepsilon_1 = \operatorname{sgn} p_u(0,0,0)$, $\varepsilon_2 = \operatorname{sgn} p_\lambda(0,0,0)$ and $\varepsilon_3 = \varepsilon_1 \operatorname{sgn} C$.

Moreover the universal unfolding of h is

$$H(A,\lambda,\alpha) = (\varepsilon_1 u+\varepsilon_2\lambda)A + (\varepsilon_3 u+Dv+\alpha)(A^2 - \frac{1}{3}\operatorname{tr}(A^2)I).$$

Remark: Chossat [1979] has computed analytically ε_1 and ε_2 showing, in particular, that $\varepsilon_1\varepsilon_2 = -1$. Chossat [1982] has computed numerically the sign ε_3 about which we shall say more later.

For a discussion of O(3)-equivalence see Golubitsky and Schaeffer [1979,1982]. The main attribute we use here is the observation that O(3)-equivalence does not change, in a precise qualitative way, the zeroes of g.

We now present a schematic rendering of the bifurcation diagrams, $h(A,\lambda) = 0$, occurring in the normal forms of Proposition 1.2. Note that the equivariance properties imply that if $h(A,\lambda) = 0$ then $h(\gamma\cdot A,\lambda) = \gamma h(A,\lambda) = 0$. So h is zero on orbits of the action of O(3). We wish to identify all solutions which are on the same orbit as one. See Figure 1.1 for the unperturbed bifurcation diagrams.

We note that two pieces of information have been added to the diagrams in Figure 1.1. The first is the stability assignments. We have used the notation "s" for stable and "u" for unstable. These stability assignments refer to linearized orbital stability. To determine linearized stability we ask, "what are the signs of the real parts of the eigenvalues of dh along a given solution branch". Note, however, that equivariance implies that at least one eigenvalue of dh evaluated at a non-trivial solution to h = 0 will be zero. To determine linearized orbital stability we must ask "what are the signs of the real part of those eigenvalues of dh which are not forced by symmetry considerations to be zero?" Thus linearized orbital stability is a kind of conditional stability.

Remarks: (a) In the first bifurcation diagram of Figure 1.1 none of the bifurcating solutions are stable. Thus, in this situation, the non-self-adjoint case in the Bénard problem, no physically reasonable information would be gained by a local analysis.

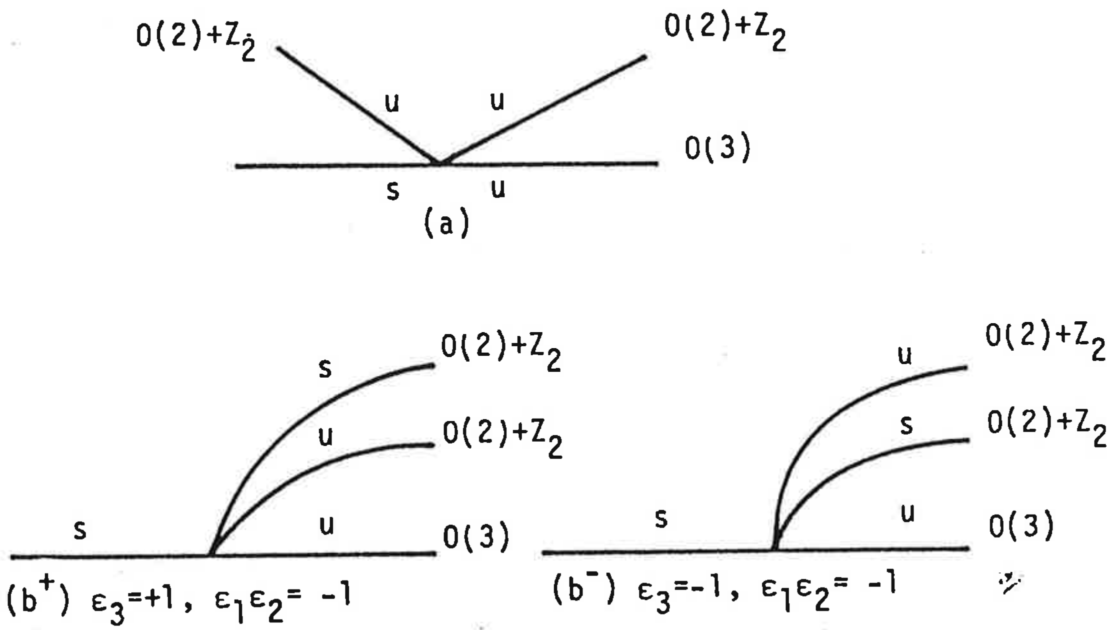


Figure 1.1 Unperturbed bifurcation diagrams.

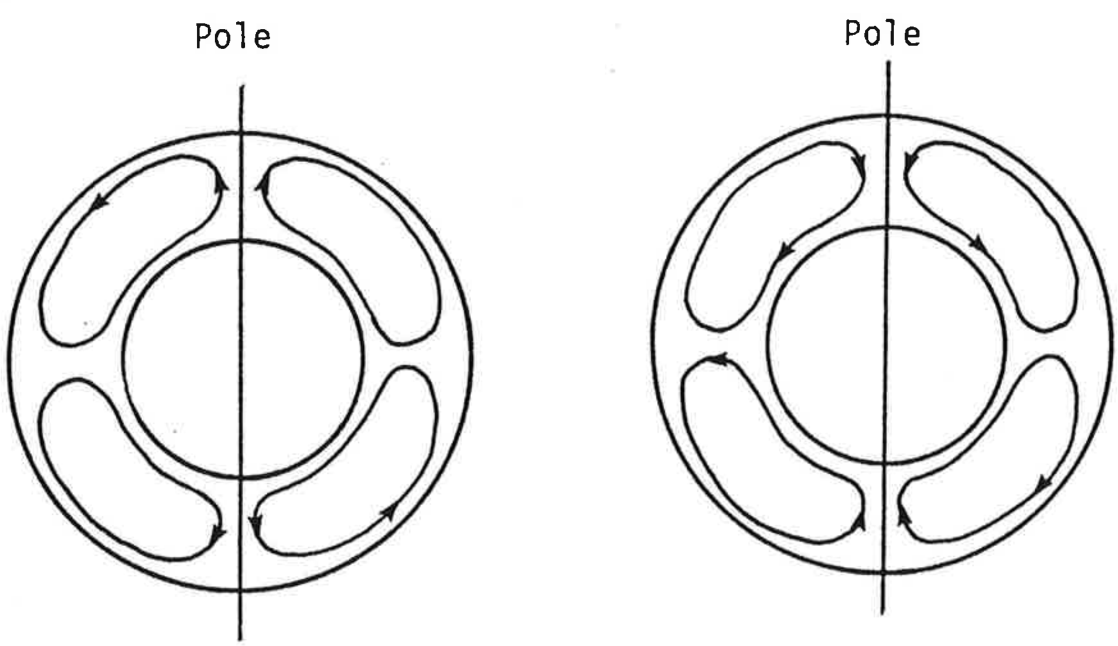
(b) Note that in the self-adjoint case there is a stable solution which bifurcates from the pure conduction solution; however, which solution branch is stable depends on the information in the higher order term ε_3.

The second piece of extra information which is given on the bifurcation diagrams in Figure 1.1 is the isotropy subgroup of solutions on the given branch. Let A be in V then

$$\Sigma_A = \{\gamma \in O(3) \mid \gamma \cdot A = A\}$$

is the <u>isotropy subgroup</u> corresponding to A. For a solution A to $h(A,\lambda) = 0$, Σ_A is the set of symmetries that the solution A has and these symmetries are usually observable. For example if $\Sigma_A \cong O(2) \oplus Z_2$ then A has an axis of rotation given by the rotation group $SO(2) \subset O(2)$. Such solutions are called <u>axisymmetric</u>.

Remarks: (a) The only solutions which appear in the unperturbed normal forms of Proposition 1.2 are axisymmetric. The isotropy subgroups are preserved up to isomorphism (in fact, inner automorphism) by O(3) equivalences.
(b) One of the main points in Chossat's analysis is that there are two families of axisymmetric solutions which bifurcate supercritically in the self-adjoint case. One can ask how these two families of solutions differ physically. Both represent flows with two cells as shown in Figure 1.2. The difference is that in one case the flow has upwelling at the poles (defined by the axis of symmetry) and downwelling at the equator with the reverse flow the situation for solutions in the other family. (These descriptions are obtained using spherical harmonics.)



(b^+) upwelling at poles $\quad$ (b^-) downwelling at poles

Figure 1.2. Two cell axisymmetric solutions

(c) Both families of axisymmetric solutions appear in the non-self-adjoint case. One appears supercritically and one subcritically, the choice depending on the sign of $\varepsilon_1\varepsilon_2$.

Next we discuss the perturbed bifurcation diagrams associated with Figure 1.1(b^+), the case (b^-) is similar. The analytic expression for these zero sets is given by the universal unfolding H in Proposition 1.2 (b). These diagrams are presented in Figure 1.3.

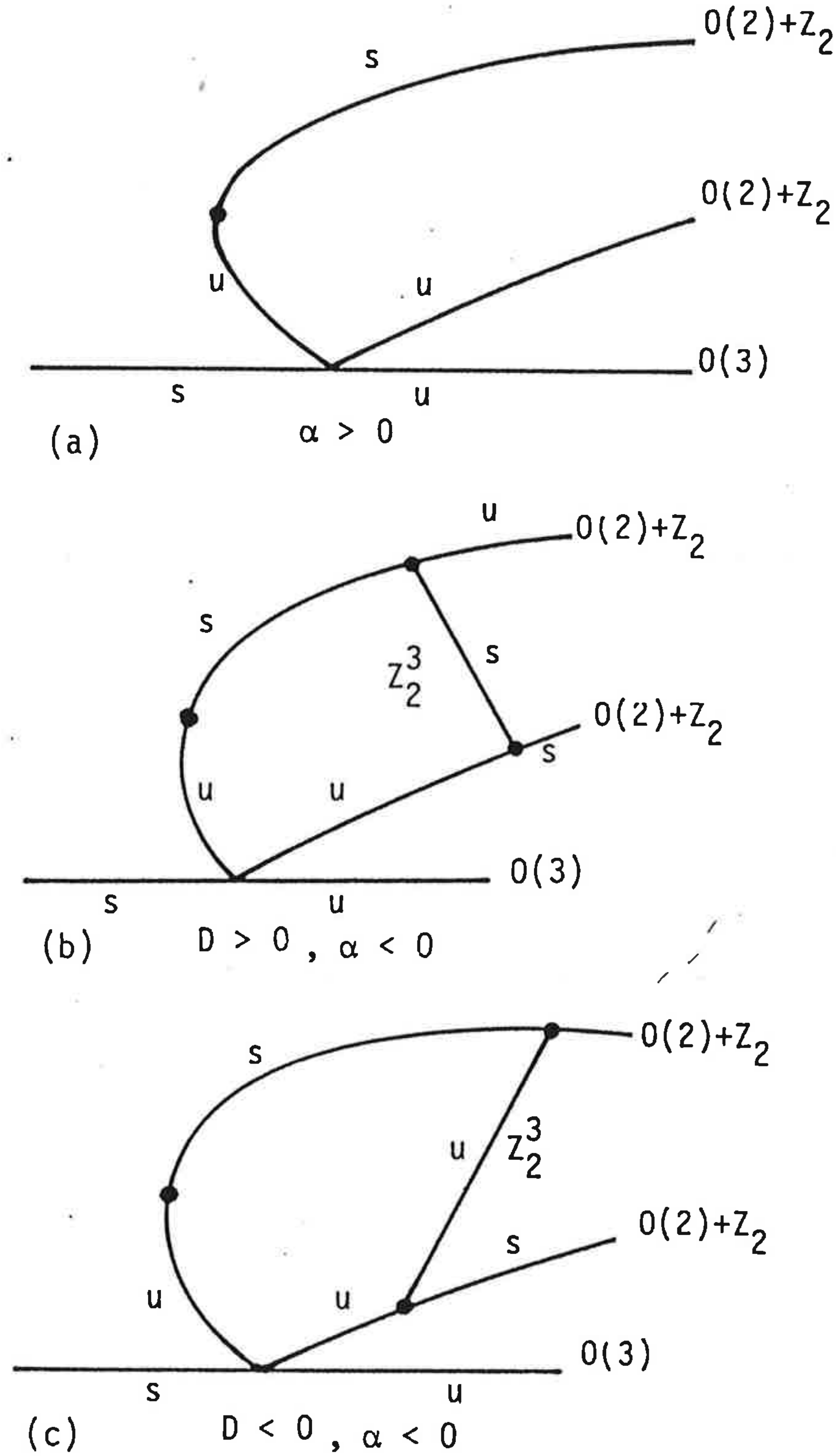


Figure 1.3. Perturbations of Figure 1.1(b^+).

Remarks: (a) When $\alpha < 0$ there exists a new branch of solutions whose isotropy subgroup is the eight element subgroup Z_2^3; these solutions are non-axisymmetric.

(b) Non-axisymmetric solutions may be stable depending on the sign of D.

(c) In the unperturbed problem only axisymmetric solutions exist and only one of the two families of axisymmetric solutions are stable. In the perturbed case both families may be stable, depending on the sign of the perturbation α.

(d) Physically the unfolding parameter α corresponds to making the Bénard problem slightly non-self-adjoint, that is, violating (1.2) by a small amount.

We end this section with a discussion of the lattice of isotropy subgroups. The isotropy subgroups of O(3) - corresponding to the five dimensional irreducible representation - are all isomorphic to $O(3), O(2)+Z_2, Z_2^3$. If Σ_1 and Σ_2 are two isotropy subgroups we define $\Sigma_1 < \Sigma_2$ if some conjugate of Σ_1 is contained in Σ_2. This definition of $<$ makes the set of isotropy subgroups a lattice. In Figure 1.4 we give this lattice structure, the arrows indicating inclusion.

$$\begin{array}{c} O(3) \\ \uparrow \\ O(2) + Z_2 \\ \uparrow \\ Z_2^3 \end{array}$$

Figure 1.4: Lattice of Isotropy Subgroups of O(3).

<u>Remarks</u>: (a) The least degenerate bifurcation problem has non-trivial solution branches whose isotropy subgroups are maximal in the lattice of isotropy subgroups.

(b) The universal unfolding of the next degenerate bifurcation problem has solution branches which have maximal and submaximal isotropy subgroups. Since the lattice has only three subgroups this includes all the isotropy subgroups.

(c) Secondary bifurcation branches, see Figure 1.3, correspond to submaximal isotropy subgroups and these branches connect branches with maximal isotropy subgroups.

2. THE PLANAR BÉNARD PROBLEM WITH NONSYMMETRIC BOUNDARY CONDITIONS

The reader should recall from the Introduction that we study only solutions to the Boussinesq equations which are doubly periodic with respect to the hexagonal lattice in the plane. Moreover, the assumption that the boundary conditions on the upper and lower planes are different implies that the group of symmetries for this problem is $\Gamma = T^2 + D_6$.

Busse [1962] has shown that ker $L(\tau_0)$, where L is the linearization of the Boussinesq equations about the pure conduction solution and τ_0 is the first eigenvalue, is six-dimensional. The basic idea is that the eigenfunctions of $L(\tau_0)$ are plane waves. Once one has one plane wave then translation gives a second (e.g., sine and cosine). Rotation by 120° and 240° yields four more independent plane waves. Thus dim $L(\tau_0) \geq 6$ and in the case we consider, it is exactly 6. As above, set $\lambda = \tau - \tau_0$.

The Liapunov-Schmidt procedure implies that finding solutions to the Boussinesq equations (which are doubly periodic in the respect to the hexagonal lattice) reduces to finding the zeroes of a mapping $g: \mathbf{R}^6 \times \mathbf{R} \to \mathbf{R}^6$; i.e., solving $g(x,\lambda) = 0$, where

(1) $(d_x g)_{0,0} \equiv 0$

(2) $g(\gamma x,\lambda) = \gamma g(x,\lambda)$ for all $\gamma \in \Gamma$.

The description of the group theory and singularity theory is much more complicated in this case than in the case of Propositions 1.1 and 1.2. The reader is referred to Buzano and Golubitsky [1983] for details. Our interest centers in the bifurcation diagrams and the lattice of isotropy subgroups which we describe below.

We make several remarks about the structure of g as it relates to the planar Bénard problem.

Remarks: (a) The action of $\Gamma = T^2 + D_6$ on $\mathbf{R}^6$ is absolutely irreducible. Therefore $(d_x g)_{0,\lambda} = p(\lambda)I$ where I is the 6×6 identity matrix. The assumption that the pure conduction solution looses stability at the bifurcation point indicates that $p_\lambda(0) \neq 0$, which we assume.

(b) Symmetry implies that only one quadratic term, the sum of the squares of the coordinates which we denote by Q, can be non-zero. For an idealized Boussinesq fluid (i.e., no surface tension, no temperature dependent viscosity, etc.) Q is, in fact, zero. See Busse [1962].

(c) After symmetry considerations there are two cubic terms which are permitted to be non-zero. We denote the ratio of the coefficients of these cubic terms by a. The value of a will enter our discussion later.

We now describe part of the lattice of isotropy subgroups of Γ acting on R^6. See Figure 2.1. This lattice has two maximal subgroups and two submaximal subgroups. The notation used is S^1 for the rotation group, D_3 for the dihedral group of symmetries of the equilateral triangle and Z_2 for a reflectional symmetry.

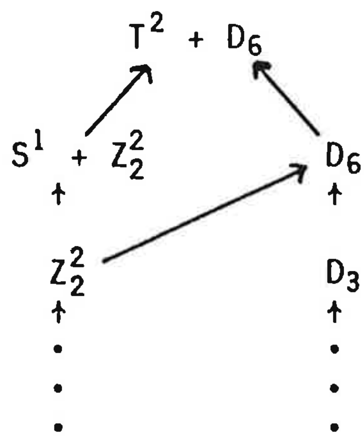
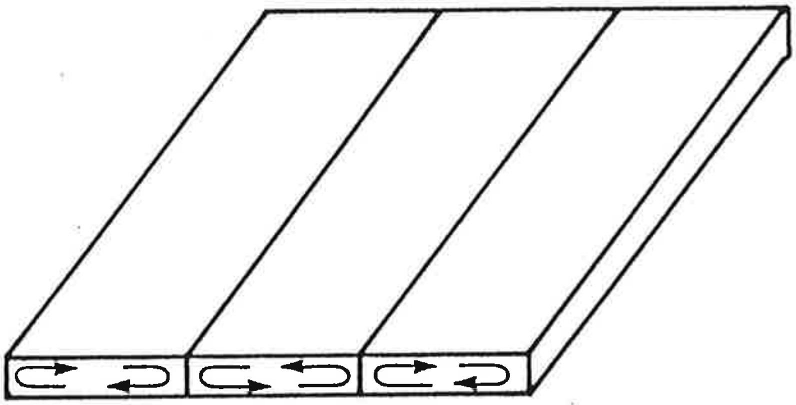


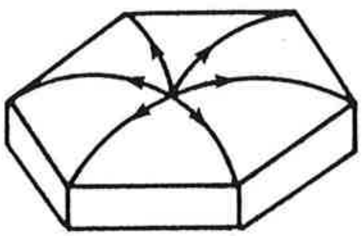
Figure 2.1: The lattice of isotropy subgroups of Γ acting on R^6

Next we describe what solutions with the various isotropy subgroups look like when related to the planar Bénard problem. This should give a better understanding of the effects of the isotropy subgroups. The reader should beware that the results are more complicated than one might think as each isotropy subgroup has several physical realizations. The easiest case is $T^2 + D_6$ which correspond to the pure conduction solution; there is no convective motion. The next simplest is the isotropy subgroup $S^1 + Z_2^2$ which corresponds to <u>rolls</u> as pictured in Figure 2.2(a).

The isotropy subgroup D_6 has two physical realizations. In Figure 2.2(b) one sees a fluid flow which is upwelling at the center and downwelling along the edges of the hexagon.



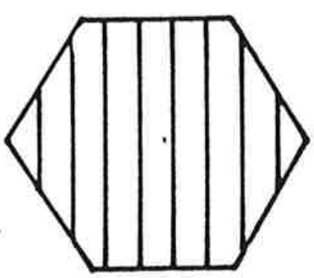
(a) Rolls



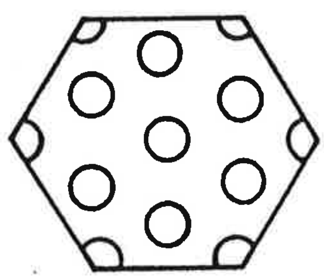
(b) Hexagons

Figure 2.2 Convective motion patterns

Of course the reverse flow with downwelling at the center is possible; this observation is analogous to the existence of two families of axisymmetric solutions in the spherical Bénard problem. Such solutions are called hexagons. The isotropy subgroups for these solutions are easier to visualize if one lets ψ be the vertical velocity component of the (linearized) fluid flow evaluated halfway between the bounding planes and the graphs $\psi = 0$. The results for rolls and hexagons are given in Figure 2.3.



Rolls



Hexagons

Figure 2.3 $\psi = 0$ for Rolls and Hexagons.

Note that the oval-like closed curve in Figure 2.3(b) is really a smoothed-out hexagon with D_6 symmetry.

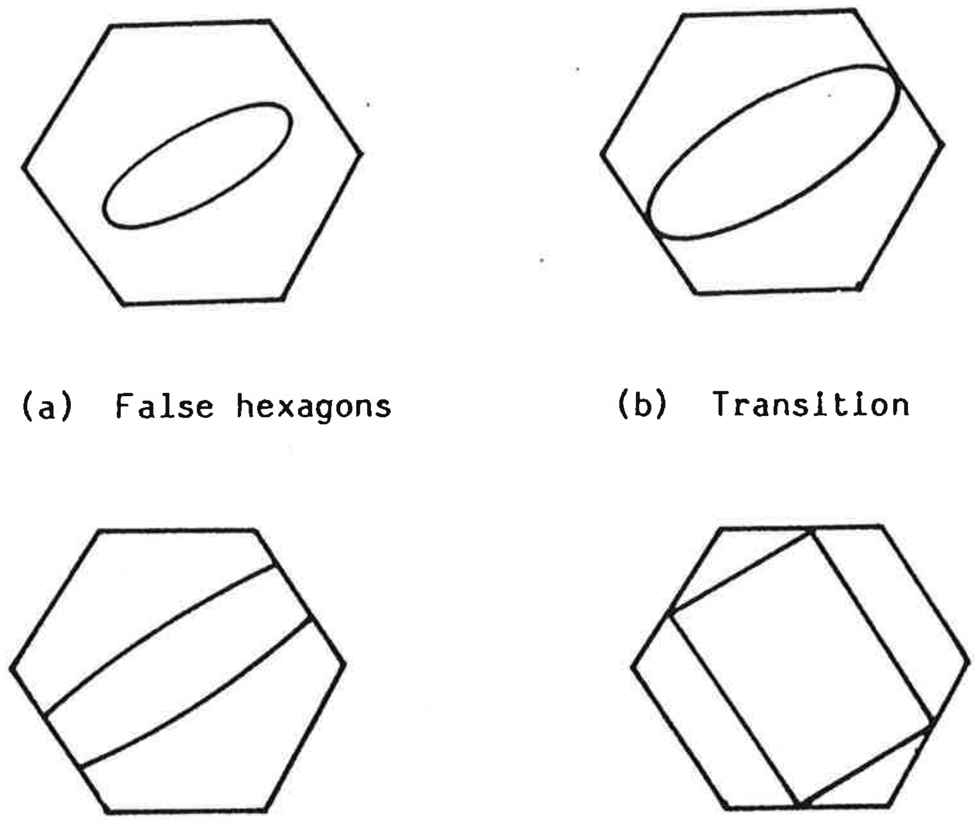
For D_3, the triangles, the zero set of ψ comes in two types as shown in Figure 2.4. The flow corresponding to triangles (a) has two realizations either upwelling or

(a) Triangles (b) Regular triangles

Figure 2.4: $\psi = 0$ for D_3 solutions.

downwelling at the center of the triangle-like curve in the figure. This is analogous to the situation for hexagons. The regular triangles have only one realization as the periodicity implies that upwelling at the center of one triangle implies downwelling in the adjacent triangles.

There are four types of zero sets of ψ for solutions with Z_2^2 isotropy subgroups. They are pictured in Figure 2.5. In the first case (a) the zero set of ψ is a smoothed out rectangle-like figure with Z_2^2 symmetry. These solutions we call false hexagons and they come with the two standard physical realizations given by upwelling or downwelling at the center. As the false hexagons could easily be confused with hexagons in an experimental situation. These rectanglar-like figures can grow until they touch the sides of the hexagon (b) and break through the boundary (c). If (c) is continued periodically one gets a zero set for ψ which resembles those of rolls Figure 2.3 (a) except for the periodic behavior along the axis of the rolls; hence the term wavy rolls.



(a) False hexagons (b) Transition

(c) Wavy rolls (d) Patchwork quilt

Figure 2.5 $\psi = 0$ for Z_2^2 solutons.

Finally Patchwork quilt is obtained by letting the rectangle like figure actually approach a rectangle. This case (d) is analogous to the regular triangles described above.

"Theorem 2.1"(a): If the quadratic term $Q \neq 0$ and certain non-degeneracy conditions on the higher order terms hold then the associated bifurcation diagram for g is given in Figure 2.6

(b) If $Q = 0$ and certain non-degeneracy conditions on higher order terms hold then the associated bifurcation diagrams for g depend on the cubic term a. There are four possibilities two of which are given in Figure 2.7.

Typical perturbed bifurcation diagrams for the cases of Figure 2.7 are given in Figure 2.8.

Remarks: (a) Busse [1962] has shown that for an ideal Boussinesq fluid $a < -1$. So the bifurcation diagram for the

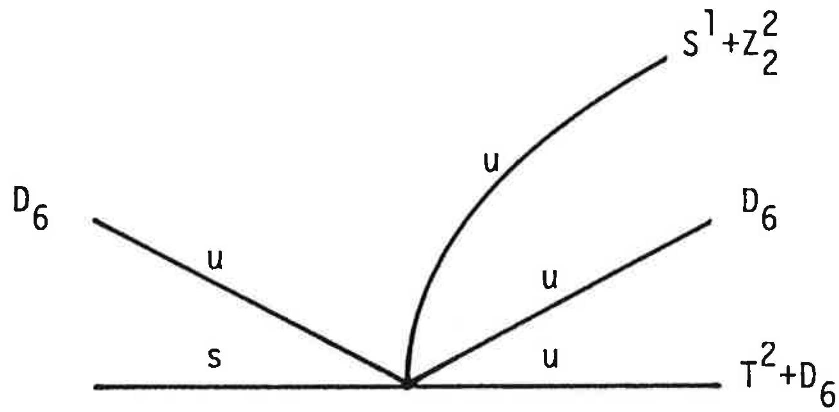


Figure 2.6: Simplest bifurcation diagram with $T^2 + D_6$ symmetry

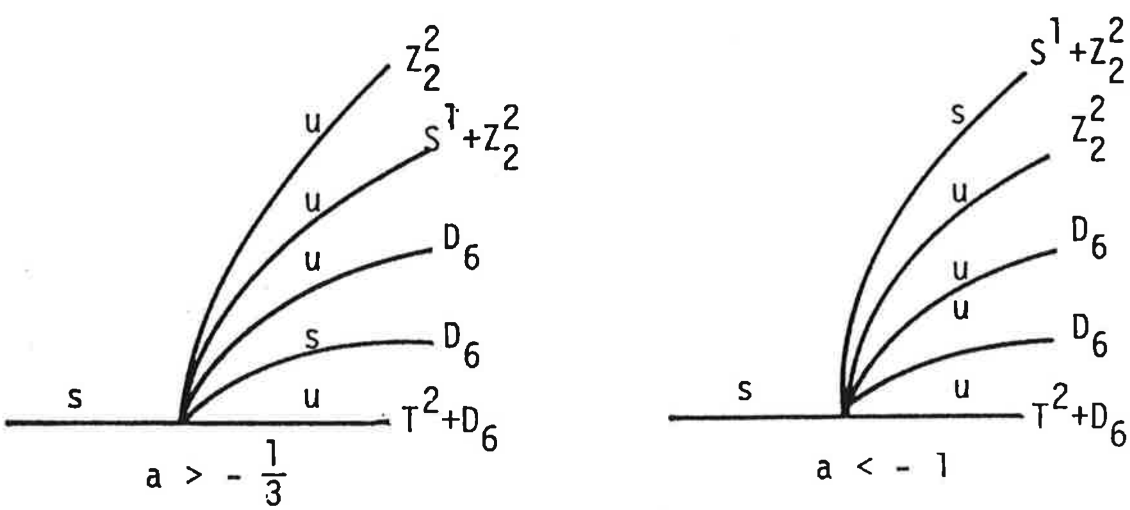


Figure 2.7: Some of the unperturbed bifurcation diagrams for the second least degenerate bifurcation problem with $T^2 + D_6$ symmetry.

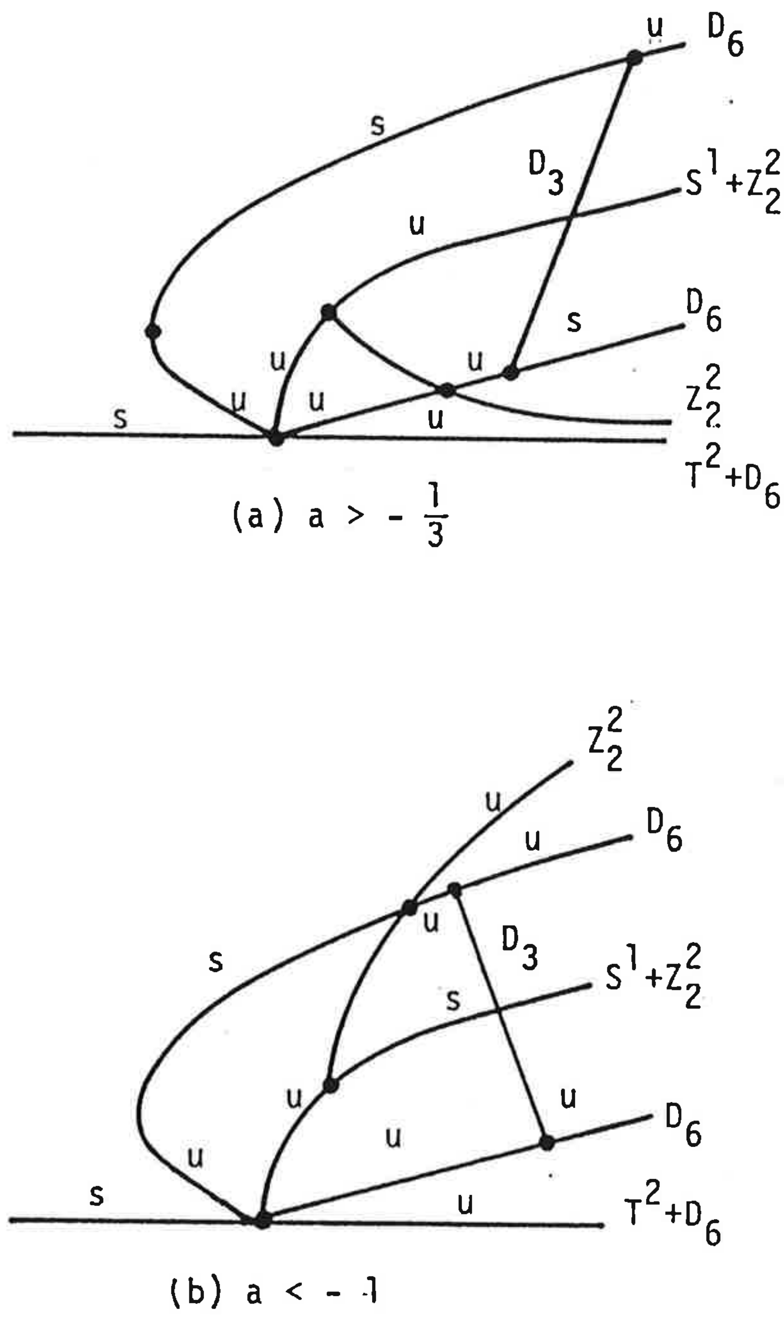


Figure 2.8: Perturbed bifurcation diagrams: Intersection of branches only occur at black dots.

planar Bénard problem has a stable family of rolls bifurcating supercritically. If the fluid is slightly non-idealized then one expects to see stable hexagon solutions with a jump to rolls as the temperature gradient is increased.

(b) For other convection situations stable hexagons may be the idealized case (as with $a > -1/3$). This is a mathematical possibility consistent with the symmetry. In such a case one may observe stable triangles. The stability of the solutions along the branch of triangle solutions has not been rigorously established; there are indications - using symmetry - that in the case given in Figure 2.8(b) they will be stable.

(c) There are two branches of D_6 solutions in each figure; they correspond to upwelling and downwelling at the center of the hexagon. Which family occurs supercritically and which subcritically depends on higher order terms. In Figure 2.7 (b) either branch could be stable depending on higher order terms.

The most important point in this description is given by comparing the solution branches which actually occur with the lattice of isotropy subgroups given in Figure 2.1. Note that only the maximal isotropy subgroups occur in the least degenerate bifurcation problem while both the maximal and submaximal isotropy subgroups appear in the universal unfolding of the second least degenerate bifurcation problems. Observe that when a submaximal isotropy subgroup appears, the associated solution branch connects branches of solutions corresponding to maximal isotropy subgroups containing that submaximal isotropy subgroup.

3. THE PLANAR BÉNARD PROBLEM WITH SYMMETRIC BOUNDARY CONDITIONS

I shall describe here joint work with Jim Swift and Edgar Knobloch. One changes the formulation of the planar Bénard problem given in the last section by assuming that the boundary conditions on the bounding planes of the fluid are the same. Typically one might assume that the fluid is contained between two identical surfaces - say glass - so that rigid boundary conditions above and below are reasonable. The effect of this change is to add a symmetry to the problem.

We shall still look for solutions to the Boussinesq equations near the pure conduction solution which are doubly periodic with respect to the hexagonal lattice, so that the symmetry group includes $\Gamma = T^2 + D_6$. The added assumption on the boundary conditions implies that reflection about the midplane (in the vertical direction) commutes with the

Boussinesq equations. Thus the symmetry group for this problem is $\Gamma' = \Gamma + Z_2$. Under the same assumption of the six-dimensional kernel of $L(\tau_0)$ as in Section 2, one applies the Liapunov-Schmidt method to obtain $g: \mathbf{R}^6 \times \mathbf{R} \to \mathbf{R}^6$ such that

(a) $(d_x g)_{0,0} \equiv 0$

(b) $g(\gamma x, \lambda) = \gamma g(x, \lambda)$ for all $\gamma \in \Gamma$

(c) $g(-x, \lambda) = -g(x, \lambda)$,

the zeroes of g corresponding to solutions to the Boussinesq equations.

Remark: This new symmetry implies that the reduced bifurcation mapping g is odd in x. See (c) above. It follows that the quadratic term Q of g described in Section 2 must be zero; one might be tempted to conclude that the analysis outlined in "Theorem 2.1(b)" is applicable. One should note however that the hypotheses in that theorem included "certain non-degeneracy conditions on higher order terms". These hypotheses fail when g is odd in x so that the analysis of "Theorem 2.1(b)" is definitely not applicable. As we shall see below the introduction of even a single reflectional symmetry into a bifurcation problem can alter quite substantially the resulting bifurcation pattern.

The beginning of the lattice of isotropy subgroups for the action of Γ' on $\mathbf{R}^6$ is given in Figure 3.1. Note that even though we give only the maximal isotropy subgroups of Γ' that the lattice structure is quite different from that of Γ given in Figure 2.1. This difference will be made more apparent in "Theorem 3.1" below.

$$T^2 + D_6 + Z_2$$

$$\uparrow \qquad \uparrow$$

$$S + Z_2^3 \qquad D_6 \qquad D_3 + Z_2 \qquad Z_2^3$$

$$\uparrow \qquad \uparrow \qquad \uparrow \qquad \uparrow$$

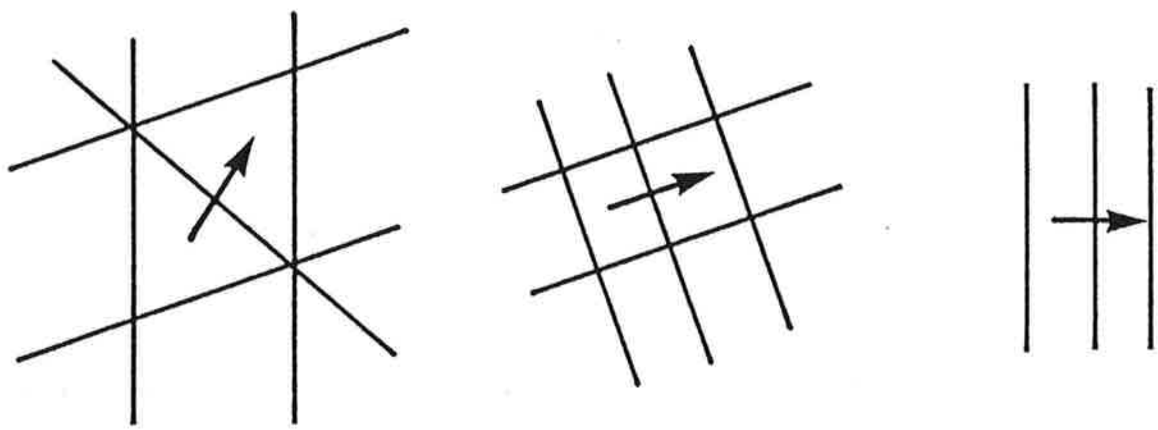
$$\vdots \qquad \vdots \qquad \vdots \qquad \vdots$$

Figure 3.1: The maximal isotropy subgroups of Γ' acting on $\mathbf{R}^6$.

Before describing the bifurcation structure we indicate the physical structures of the convection solutions corresponding to each maximal isotropy subgroup. The first subgroup $S^1 + Z_2^3$ corresponding to rolls. Note that flipping the rolls solution about the midplane and translating by one cell perpendicular to the axis of the rolls gives a symmetry for rolls in Γ' which was not present in Γ. See Figure 2.2(a). This observation explains the extra factor of Z_2 which appears in the isotropy subgroup for rolls in Γ'. Reflection about the midplane for hexagon solutions, D_6, takes a hexagon with upwelling at the center to a hexagon with downwelling at the center. Thus there is only one type of orbit of solutions of hexagons in this formulation, that is, if upwelling occurs as a solution so must downwelling. Note that the new symmetry is not included in the isotropy subgroup for hexagons in Γ'.

The triangle solutions, D_3, intertwine with the new symmetry in a more complicated way. As in the case of hexagons the non-regular triangles have both upwelling and downwelling solutions which are identified by the new symmetry. Moreover the isotropy subgroup for these solutions remains the same D_3 in Γ'. However for the regular triangle solutions a new symmetry is added to the isotropy subgroup. This symmetry is obtained by flipping about the midplane and translating from the center of one triangular cell to the center of an adjacent triangular cell as shown by the arrow in Figure 3.2(a). Thus the isotropy subgroup for regular triangles is $D_3 + Z_2 < D_6$ and we have a new maximal isotropy subgroup.

The situation for the Z_2^2 solutions is similar to the case of triangles. For wavy rolls and false hexagons there is an identification made by upwellings and downwellings so that the isotropy subgroup remains Z_2^2. However, in the single case of patchwork quilt (see Figure 2.5(d)) the new symmetry does add a reflectional symmetry to the isotropy subgroup. This symmetry is obtained by flipping about the midplane and then translating the cells as in the case of regular triangles. See the arrow indicating a relevant translation in Figure 3.2(b). We claim that the isotropy subgroup Z_2^3 obtained this way for patchwork quilt is also a maximal isotropy subgroup. Recall from Figure 2.1 that Z_2^2 is contained (up to conjugacy) in $S^1 + Z_2^2$ and D_6. As with regular triangles $Z_2^3 \not< D_6$ since a flip type symmetry has



(a) Regular triangles (b) Patchwork quilt (c) Rolls

Figure 3.2. Convection patterns with extra symmetry

been added to the isotropy subgroup. Since the isotropy subgroup for rolls is now $S^1 + Z_2^3$, a flip type symmetry having been added, one might question why Z_2^3 is a maximal isotropy subgroup is Γ'. The answer lies in the direction of the translation which appears in the flip-type symmetry added to the isotropy subgroups of rolls and patchwork quilt. For rolls the translation points from the center of the basic hexagon in the hexagonal lattice to one of the vertices in that hexagon. See Figure 3.2(c). In the case of patchwork quilt it does not. So $Z_2^3 \not< S^1 + Z_2^3$ and we have the fourth maximal isotropy subgroup.

We now describe the simplest bifurcation problem commuting in the action of Γ' or $\mathbf{R}^6$.

<u>"Theorem 3.1"</u> Let g be the reduced bifurcation mapping obtained by the Liapunov-Schmidt reduction. Assuming certain non-degeneracy assumptions on g one finds the possibilities for the bifurcation diagrams as shown in Figure 3.3. Recall that a is the ratio of cubic terms. Other possibilities with no stable bifurcating branches exist for $-1 < a < -1/3$.

<u>Remarks</u>: (a) The interesting observation here is the mathematical possibility of the existence of stable regular triangle solutions.

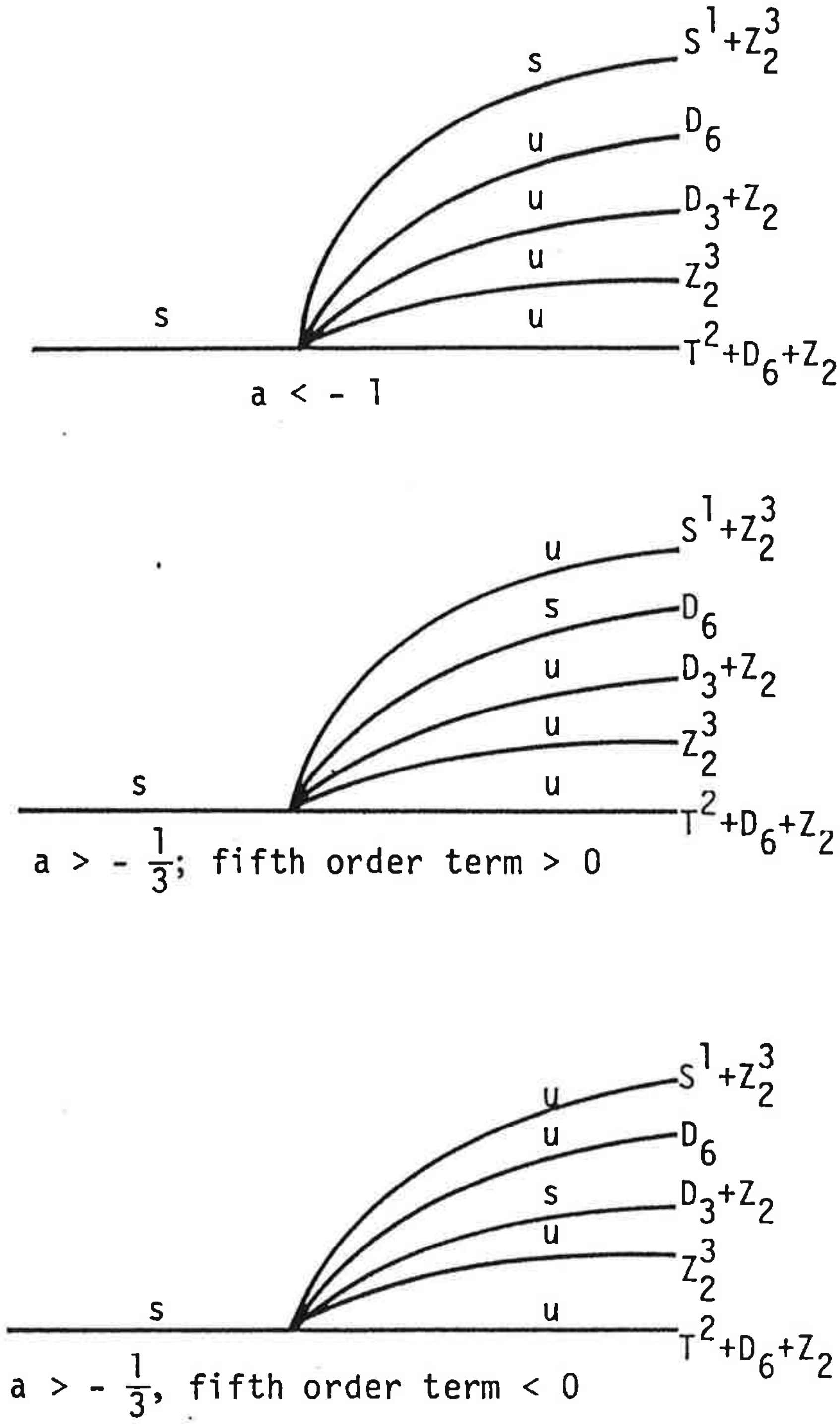


Figure 3.3. Least degenerate bifurcation diagrams with symmetric boundary conditions.

(b) Note once again that there is a one to one correspondence between maximal isotropy subgroups and branches of bifurcating solutions in the least degenerate symmetry preserving bifurcation problem.

4. REMARKS CONCERNING MAXIMAL ISOTROPY SUBGROUPS

A number of people have considered the problem of spontaneous symmetry breaking, yet there still does not exist a completely satisfactory resolution of this problem. There is however one general result by L. Michel [1976] (as described by D. Sattinger [1982]) which is noteworthy. The object of this section is to explain how this result relates to the examples given in the previous section and to maximal isotropy subgroups. This section is a continuation of a discussion of symmetry breaking given in Golubitsky and Schaeffer [1983].

Let $\Gamma \subset O(n)$ be a compact group acting linearly on R^n. For $x \in R^n$, let Σ be the isotropy subgroup of Γ corresponding to x and let F be the subspace of R^n fixed by Σ; i.e.,

$$F = \{y \in R^n | \Sigma y = y\}.$$

Note that for $x = 0$, $\Sigma = \Gamma$ since the action is linear. Moreover if Γ acts irreducibly then $\Sigma = \Gamma$ implies $x = 0$. (Proof: $\Sigma = \Gamma$ implies that F is an invariant subspace under Γ. Irreducibility implies that either $F = R^n$ or that $F = \{0\}$. In the first case $\Gamma \subset O(n)$ is just the group $\{I\}$ since every $\gamma \in \Gamma$ fixes each $y \in R^n$. In the second case one notes that $x \in F$ and concludes that $x = 0$.)

Using F one has a simple condition describing when Σ is a maximal isotropy subgroup.

Lemma 4.1: Assume that Γ acts irreducibly on R^n then Σ is a maximal isotropy subgroup of Γ if $\dim F = 1$.

Proof: Note that when $x = 0$ irreducibility implies $\dim F = 0$. So $x \neq 0$. Let Ξ be the isotropy subgroup corresponding to y and assume that $\Sigma \subset \Xi$. Since $\dim F = 1$, either $y = 0$ or y is a non-zero multiple of x. In the first case $\Xi = \Gamma$. In the second case $\Xi = \Sigma$ since the action is linear.

Note: The converse of Lemma 4.1 is not true. As pointed out to us by George Bergman the six-dimensional irreducible representation of the permutation group S_5 has the cycle group Z_5 as a maximal isotropy subgroup. For this example $\dim F = 2$. We shall give other examples below.

Michel and Sattinger have used the condition $\dim F = 1$ to good avail in studying solution branches of bifurcation problems. We describe their result. Let $g: R^n \times R \to R^n$ be a bifurcation problem equivariant with respect to Γ, i.e. $g(\gamma x,\lambda) = \gamma g(x,\lambda)$ for all $\gamma \in \Gamma$. The following lemma summarizes some basic results concerning the relationship between g and isotropy subgroups.

In the following lemma we use the notation $\Sigma(y)$ to indicate the isotropy subgroup corresponding to y. Thus $\Sigma(x) = \Sigma$.

Lemma 4.2: Let $g: R^n \times R \to R^n$ commute with Γ. Then
(a) $\Sigma \subset \Sigma(g(x,\lambda))$

(b) $g: F \times R \to F$

(c) Let $N(\Sigma)$ be the normalizer of Σ in Γ. Then $\gamma(F) = F$ for all $\gamma \in N(\Sigma)$.

(d) $g|F \times R$ commutes with the group $D = N(\Sigma)/\Sigma$.

Proof: (a) If $\gamma x = x$ then $g(x,\lambda) = g(\gamma x,\lambda) = \gamma g(x,\lambda)$. Thus $\gamma \in \Sigma(g(x,\lambda))$.

(b) Apply (a) to $y \in F$ to see that $\Sigma(y) \subset \Sigma(g(y,\lambda))$. Since $y \in F$ it follows that $\Sigma \subset \Sigma(y)$. Hence $g(y,\lambda) \in F$.

(c) Let δ be in Γ. Observe that

$$\delta\Sigma\delta^{-1} = \Sigma(\delta x). \tag{4.1}$$

For if $\gamma x = x$, then $\delta\gamma\delta^{-1}(\delta x) = \delta\gamma x = \delta x$. So $\delta\gamma\delta^{-1} \in \Sigma(\delta x)$. Now if $\delta \in N(\Sigma)$ then (4.1) implies that $\Sigma = \Sigma(\delta x)$.

Suppose $y \in F$. Then $\Sigma \subset \Sigma(y)$. For $\delta \in N(\Sigma)$ one has

$$\Sigma = \delta\Sigma\delta^{-1} \subset \delta\Sigma(y)\delta^{-1} = \Sigma(\delta y).$$

Thus $\Sigma \subset \Sigma(\delta y)$ and $\delta y \in F$. (Note that $N(\Sigma)$ is the largest subgroup of Γ which leaves the subspace F invariant.)

(d) Since g commutes with Γ one has that $g|F \times R$ commutes with $N(\Sigma)$ using (c). But Σ acts as the identity on F so D acts on F and $g|F \times R$ commutes with D.

The following proposition, due to Michel [1976], is the first general result about the existence of bifurcating branches corresponding to maximal isotropy subgroups. First note that if Γ acts absolutely irreducibly, (i.e., the only matrices on R^n which commute with Γ are multiples of the identity matrix), then $(dg)_{0,\lambda} = c(\lambda)I$ since the chain rule implies that $(dg)_{0,\lambda}$ commutes with Γ. We shall say that the trivial solution $x = 0$ changes stability non degenerately if $c(0) = 0$ and $c'(0) \neq 0$. Note that if Γ acts irreducibly then $g(0,\lambda) = 0$ for all λ and $x = 0$ is a solution.

Proposition 4.3: Let g commute with Γ. Assume

(i) Γ acts absolutely irreducibly

(ii) $\dim F = 1$, so Σ is a maximal isotropy subgroup

(iii) the trivial solution changes stability non-degerately.

Then there exists a solution branch bifurcating from the origin whose solutions have isotropy subgroup Σ.

Proof: By Lemma 4.2(b) $g: F \times \mathbf{R} \to F$. Let y be the single coordinate in F. Then $g(0,\lambda) \equiv 0$ since the trivial solution persists for all λ. Since g is assumed to change stability at (0,0) one sees that $g_y(0,0) = 0$ and $g_{y\lambda}(0,0) \neq 0$.

Since $g(0,\lambda) \equiv 0$ one may write, using Taylor's theorem, $g(y,\lambda) = yh(y,\lambda)$. The assumptions $g_y(0,0) = 0$ and $g_{y\lambda}(0,0) \neq 0$ imply $h(0,0) = 0$ and $h_y(0,0) \neq 0$. Now one can solve the equation $h(y,\lambda) = 0$ by the implicit function theorem for a unique smooth function $\lambda = \Lambda(y)$ so that $\Lambda(0) = 0$ and $h(y,\Lambda(y)) \equiv 0$. The curve $\lambda = \Lambda(y)$ is the desired branch of solutions.

Remarks: (a) From the point of view of bifurcation theory the information given by Proposition 4.3 is insufficient in several ways. First of all no information is given about the stability of the solutions on the new branch. Michel's interest in this problem came from assuming that $g = \nabla f$ where $f: \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}$ is invariant under Γ. Once one has a potential function the problem of linearized stability is easier. Second, Proposition 4.3 gives no information about how many solutions y exist for each λ. For example, it is possible, though quite improbable, that $\Lambda(y) \equiv 0$. So all the new solutions occur at $\lambda = 0$, a rather unreasonable eventuality. Third, no information is given about the existence or non-existence of other branches.

(b) Sattinger [1982] has used this proposition along with some standard though sophisticated techniques from group representation theory to make statements about the existence of solution branches for bifurcation problems commuting with the higher (than five) dimensional irreducible representations of O(3).

(c) All of the examples in the first three sections satisfy the assumptions of Proposition 4.3. In particular, all of the maximal isotropy subgrups Σ have dim F = 1. Thus the existence of each of the branches of solutions corresponding to maximal isotropy subgroups is guaranteed by this proposition.

Michel [1976] claimed that a partial converse to this theorem is also true. More precisely, Michel claimed that if a solution exists for every g satisfying (iii), Γ is assumed to act absolutely irreducibly (i), then dim F = 1. Dancer and Sattinger noted that an extra hypothesis is needed to prove this converse. One has:

Proposition 4.4: Assume that Γ acts absolutely irreducibly on R^n. Let Σ be an isotropy subgroup. Assume

(i) The group $D = N(\Sigma)/\Sigma$ is finite.

(ii) For every $g: R^n \times \mathbf{R} \to R^n$ commuting with Γ such that the trivial solution changes stability non-degenerately there is a solution branch of g = 0 with isotropy subgroup Σ bifurcating from the origin.

Then dim F = 1.

Proof: See Sattinger [1982], §4.

This proposition states, in a sense, that if one wants to find a solution branch corresponding to Σ for every bifurcation problem then one needs to know that dim F = 1. This seems to us to be a misplaced emphasis. Perhaps one really wants to know which conditions on Σ imply that for almost every g there is a solution branch corresponding to Σ. We suggest that the appropriate set of g's to investigate are those with (topological) Γ-codimension equal to zero, that is, those g's whose singularities are the simplest possible consistent with Γ-equivariance. These g's are the equivalent of Morse functions in the Γ-equivariant bifurcation theory context. For these g's one might conjecture that there is a one to one correspondence between maximal isotropy subgroups and solution branches. This conjecture is not true as stated and will be refined in the discussion below.

We now return to the extra hypothesis to Proposition 4.4. As observed in Lemma 4.2 the group D acts on F. We can make a further observation about this action. The following observations were made jointly with Ian Stewart.

Lemma 4.5: Assume that Σ is a maximal isotropy subgroup of Γ and that Γ acts irreducibly on R^n. Then the action of D on F is fixed point free.

Proof: Suppose the action of D on F is not fixed point free. Then there is a $y \neq 0$ in F and $\delta \in N(\Sigma) \sim \Sigma$ which satisfies $\delta y = y$. It follows that $\Sigma(y) \underset{\neq}{\supset} \Sigma$ since $\delta \not\in \Sigma$. By the maximality of Σ it follows that $\Sigma(y) = \Gamma$. Since Γ acts irreducibly $y = 0$ contradicting our assumption and the lemma is proved.

Proposition 4.6: Let Γ act irreducibly on R^n and let Σ be a maximal isotropy subgroup of Γ. Let D^0 be the connected component of the identity of $D = N(\Sigma)/\Sigma$. (D is compact since Γ is assumed to be compact.) Then either

(a) $D^0 = \{1\}$,

or (b) $D^0 = S^1$ and F is the direct sum of irreducible subspaces under D^0, $\cong C$, where S^1 is identified with the unit complex numbers and the action of S^1 on C is given by complex multiplication

or (c) $D^0 = SU(2)$ and F is the direct sum of irreducible subspaces under D^0, $\cong \mathbf{Q}$, where $\mathbf{Q}$ is the skew field of quaternions, SU(2) is identified with the unit quaternions and the action of D^0 on $\mathbf{Q}$ is given by quaternionic multiplication.

Definition 4.7: We call a maximal isotropy subgroup of a compact group Γ acting irreducibly on R^n either real, complex, or quaternionic depending on whether D^0 is $\{1\}$, S^1, or SU(2).

Proof: The basic observation is the one given in Lemma 4.5

that D and hence D^0 acts fixed point free. The result then following from Theorem 8.5 in Bredon [1972]. We include a proof here as it is short and it does not appear in the bifurcation theory literature. Assume that $\dim D^0 \geq 1$. Since D^0 is compact it has a maximal torus T^ℓ. We claim that if $\ell \geq 2$ then the action cannot be fixed point free. First note that if S^1 acts on F fixed point free then it acts fixed point free in each irreducible subspace V. Irreducibility implies that dim V = 1 or dim V = 2 and S^1 cannot act in a fixed point free way on **R**. Thus dim V = 2. Moreover the irreducible actions of S^1 on $\mathbf{R}^2 \simeq \mathbf{C}$ are enumerated by $\theta \to \exp(m\theta i)$ for some positive integer m. If $m > 1$ then this action is not fixed point free, take $\theta = 2\pi/m$. So we may assume that $m = 1$.

Now suppose that $T^2 = S^1 \times S^1$ acts on F. Let V be an irreducible subspace of F under this action of T^2. Again irreducibility implies that dim V = 1 or 2 with dim V = 1 and a fixed point free action being incompatible. So we identify V with **C**. The result above states that $(\theta,0)$ acts on **C** by $(\theta,0) \to \exp(i\theta)$ and $(0,\theta)$ acts on **C** by $(0,\theta) \to \exp(i\theta)$. It follows that the diagonal (θ,θ) of T^2 acts on **C** by $\exp(2i\theta)$. However the diagonal of T^2 is S^1 and such an action of the diagonal is not fixed point free. The claim is proved.

Using the classification theorem for compact, connected Lie groups of positive dimension one sees that there are only three whose maximal torus is one dimensional, namely, S^1, SU(2), and SO(3). Suppose D^0 = SO(3). Then write F as a sum of irreducible subspaces. All of the irreducible actions of SO(3) are odd-dimensional. As a rotation matrix acting on an odd-dimensional space always has an axis of rotation, such actions cannot be fixed point free. So D^0 is either 1, S^1 or SU(2).

As discussed above the irreducible decomposition of S^1 acting on F is as stated in the proposition. Finally one checks that the only fixed point free, irreducible action of SU(2) is given by SU(2) acting as the unit quaternions on the quaternions.

Remarks: (a) In the complex case dim F = 0 mod 2 and in the quaternionic case dim F = 0 mod 4. Here one can obtain more examples of cases where dim F > 1 and Σ is a maximal isotropy subgroup.

(b) Recall from Lemma 4.2(d) that $g|F \times \mathbf{R}$ commutes with D.

Suppose that $D = S^1$ and F is two dimensional. Identify F with C and note that if g commutes with S^1 then g has the form

$$g(z,\lambda) = p(z\overline{z},\lambda)z + q(z\overline{z},\lambda)iz$$

where p and q are real valued. (Cf. Golubitsky and Langford [1981].) If g has a singularity at the origin then $p(0,0) = 0 = q(0,0)$. One can show easily that $g = 0$ reduces to $z = 0$ or

$$p(z\overline{z},\lambda) = 0 = q(z\overline{z},\lambda)$$

since z and iz are independent if $z \neq 0$. Generally the solution to a system of two equations in two unknowns is a discrete set of points, so generically $z = 0$ is the only solution. No branch of solution bifurcates in $F \times \mathbf{R}$ from the origin. (Aside: if g depends on an extra parameter τ then one can obtain a curve of solutions. This happens in Hopf bifurcation where τ is the perturbed period. See Golubitsky and Langford [1981].)

A similar situation occurs when $D = SU(2)$ and $F = Q$ only there one needs to add three additional parameters in order to find a solution branch. I know of no interesting situation (such as Hopf bifurcation) where this phenomenon occurs. It is an interesting question!

Proposition 4.6 puts the extra assumption in Proposition 4.4 into perspective. The remarks above suggest the following:

Conjecture: Let Γ act absolutely irreducibly on $\mathbf{R}^n$. Let $g\colon \mathbf{R}^n \times \mathbf{R} \to \mathbf{R}^n$ commute with Γ, have a singularity at (0,0) and have (topological) codimension 0. (In particular, this implies that there is a non-degenerate change of stability along the trivial solution at $\lambda = 0$.) Then each non-trivial branch of solutions to $g = 0$ corresponds to a real, maximal isotropy subgroup. Moreover, each real maximal isotropy subgroup corresponds to a branch of solutions to $g = 0$ for some g with (topological) codimension 0.

I feel confident that this conjecture is true if all of the real, maximal isotropy subgroups also satisfy $\dim F = 1$. This is the case for the examples in the first three sections. In fact, more is true.

Recall that D acts fixed point free on F. If $\dim D = 1$ then either $D = \{1\}$ or $D = Z_2$. If $D = Z_2$ then $g\colon F \times \mathbf{R} \to F$

is odd in y by Lemma 4.2(d). The simplest such bifurcations for odd functions is the pitchfork bifurcation, $y^3 \pm \lambda y$. See Golubitsky and Schaeffer [1979]. The following fact is true for the examples given in Figure 1.1(a), Figure 2.6 and Figure 3.3. If $D = Z_2$ then the branch of solutions corresponding to Σ is parabola-like as in the pitchfork. If $D_2 = \{1\}$ then the branch of solutions is transcritical, $y^2-\lambda y$, and has two components, one supercritical and one subcritical.

We are in the situation where by abstract techniques one can recover much of the information in the bifurcation diagrams associated to the simplest, least degenerate cases of the examples in the previous sections. Those results were obtained by long, tedious calculations and to be able to replace them by only group theoretic considerations would be a major accomplishment. We are not there yet but the project seems feasible. Finally we note that we havenot yet considered here, in a coherent way, the problem of linearized orbital stability nor have we considered the problem of submaximal isotropy subgroups from an abstract point of view. The examples in Sections 1 and 2, in particular, show that such considerations are absolutely necessary if the abstract theory is to be truely applicable.

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Research supported in part by ARO Contract DAAG-79C-0086 and by NSF Grant MCS-8101580.

REFERENCES

Bredon, G.E., 1972. Introduction to Compact Transformation Groups. Academic Press, New York.

Busse, F.H., 1962. Das Stabilitätsverhalten der Zellarkonvektion bei end licker amplitude, Dissertation, University of Munich. (Engl. Transl. by S.H. Davis, Rand Rep. LT-66-19, Rand Corp. Santa Monica, CA.)

Busse, F.H., 1975. Pattern of convection in spherical shells, J. Fluid Mech. 72, 65-85.

Busse, F.H., 1978. Nonlinear properties of thermal convection. Rep. Prog. in Phys. 41, 1929-1967.

E. Buzano and M. Golubitsky, 1983. Bifurcation on the hexagonal lattice and the planar Bénard problem. Phil. Trans. Roy. Soc. London. To appear.

Chossat, P., 1979. Bifurcation and stability of convective flows in a rotating or not rotating spherical shell, SIAM J. Appl. Math. 37, 624-647.

Chossat, P., 1982. Le probleme de Bénard dans une couche sphérique. Thése d'État. Université de Nice.

S. Fauve and A. Libchaber, 1983. Rayleigh-Bénard experiments and dynamical systems. This volume.

M. Golubitsky and D. Schaeffer, 1979. Imperfect bifurcation in the presence of symmetry. Commun. Math. Phys. 67, 205-232.

M. Golubitsky and D. Schaeffer, 1982. Bifurcation with O(3) symmetry including applications to the Bénard problem. Commun. Pure and Appl. Math. 35, 81-111.

M. Golubitsky and D. Schaeffer, 1983. A discussion of symmetry and symmetry breaking. Proc. Symposia in Pure Mathematics, 40, A.M.S. To appear.

M. Golubitsky, J. Swift and E. Knobloch, 1983. Boussinesq convection with symmetric boundary conditions. In preparation.

Michel, L., 1976. Simple mathematical models of symmetry breaking, application to particle physics. Mathematical Physics and Physical Mathematics, Polish Scientific Publishers, Warsaw, 251-262.

Sattinger, D.H., 1978. Group representation theory, bifurcation theory, and pattern formation. J. Func. Anal. 28, 58-101.

Sattinger, D.H., 1979. Group Theoretic Methods in Bifurcation Theory. Lec. Notes in Math 762, Springer-Verlag, Berlin.

Sattinger, D.H., 1982. Branching in the presence of symmetry CBMS Lectures, Gainsville, FL. Dec. 1981. To appear.