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BIFURCATION INVOLVING THE HEXAGONAL LATTICE

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1. Introduction. This article is an outline of the results obtained in our joint paper [**1**] on the problem of pattern formation as it relates to bifurcation with respect to the hexagonal lattice. The reader should refer to [**1**] for the details and the proofs.

Our work is motivated by the very interesting ideas of D. H. Sattinger [**5**] concerning a classical problem in fluid mechanics known as the planar Bénard problem (which we describe below). We note however that our results like Sattinger's are proved for bifurcation problems involving the hexagonal lattice only and thus may be useful in other areas where the hexagonal lattice appears. See, for example, the study of visual hallucination patterns by Ermentrout and Cowan [**2**].

Loosely stated, the planar Bénard problem is the study of thermal conduction and convection for a fluid contained between two parallel infinite planes. The convective motion is driven by a temperature gradient λ between the upper and lower planes. In experiments (performed in a finite box, of course) what is observed is that for small λ the fluid remains at rest while the heat is conducted from one plane to the other. However as λ is increased there is a certain critical value λ_0 after which the pure conduction solution looses stability and convective motion begins. What makes this problem so interesting is that under different experimental conditions the equilibrium motion can evolve into a variety of spatial patterns. For example in Bénard's original experiment the convective motion splits into hexagonal cells, arranged on a hexagonal lattice, where the motion is up-welling at the center of the hexagon and down-welling at the boundaries. Another configuration often observed is rolls (see Figure 1). Other possibilities are cross-rolls and wavy-rolls.

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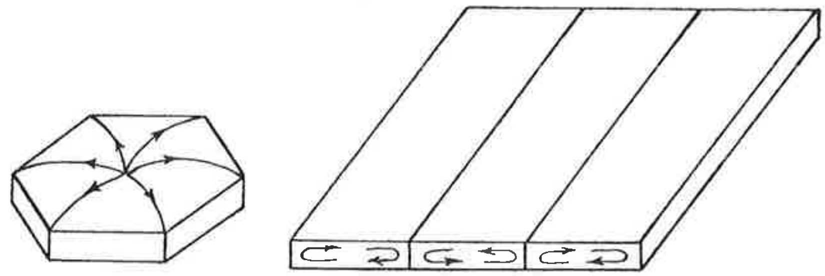

FIGURE 1. Convective patterns

The mathematical analysis of the planar Bènard problem involves postulating rules for the motion of the fluid and rules for the time evolution of the temperature distribution. A well-accepted formulation for this problem is given by the Navier-Stokes equations in the Boussinesq approximation. Our analysis will *not* involve the explicit use of these equations; we make no attempt to describe them here. We just note here that one of the reasons why the mathematical analysis of these equations is so difficult is that the associated operator commutes with the full group of rigid motion of the plane and this group is *not* compact.

Sattinger's analysis of the problem begins with the simplification that one should first look for solutions which are doubly periodic. This simplification has the effect of reducing the group of symmetries to a compact group, the translations being identified with the 2-torus T^2. However, now one is forced to choose the exact class of doubly periodic functions which one wants to study; that is, one must choose a lattice in the plane to support the double periodicity. The most natural choice, given the experimental results, is the hexagonal lattice for both rolls and hexagons respect this lattice. (We note, however, that cross-rolls can be found on a rectangular lattice but not on a hexagonal lattice, so this restriction does have undesirable effects.)

Once one has restricted the Boussinesq equations to act on functions which are doubly periodic with respect to the hexagonal lattice, then there is a well-known classical method—called the Liapunov-Schmidt procedure—which allows one to find equilibrium solutions to the Boussinesq equations near the pure conduction solution by finding the zeroes of a mapping g between finite-dimensional spaces. The explicit construction of this mapping is quite difficult as the Liapunov-Schmidt procedure only identifies g implicitly. However, with some effort, it is possible in certain cases to obtain the beginnings of the Taylor expansion of g. Moreover, it can be shown **[1]** that g must commute with the symmetry group of the full problem which in this case is $T^2 + D_6$, D_6 being the symmetry group of the hexagon. Without formal justification we observe that (after making some nondegeneracy assumptions) one can give the setting that the reduced mapping g must have. The idea is: suppose that the linearized Boussinesq equations have plane waves solutions in a certain direction θ. Then the group action demands that there exist plane waves in the directions $\theta + 2\pi/3$ and $\theta + 4\pi/3$ as solutions. Since each set of plane waves is two dimensional (sin and cos) one has a 6-dimensional space which we can identify with $\mathbf{C}^3$. Moreover, no other solutions are forced by the symmetry group. So generically there are just 6 independent solutions to the linearized problem. One can then show that the reduced bifurcation equations must have the form

$$g \colon \mathbf{C}^3 \times \mathbf{R} \to \mathbf{C}^3$$

where the real parameter represents the temperature gradient λ. Moreover, $g(z, \lambda)$ must commute with $T^2 + D_6$.

The remainder of the paper is structured as follows. In §2 we describe the relevant group theory ending with a description of g. In §3 we describe the singularity theory analysis of the bifurcations of g. Here we use the results of [3] as a guide. Finally, in §4 we give an interpretation of our results for the Bénard problem.

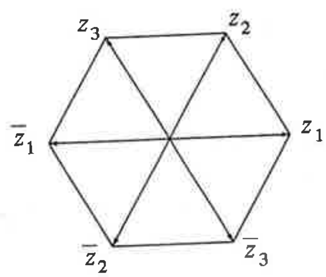

FIGURE 2

2. The group theory. Consider the hexagon of Figure 2. The action of D_6 on $\mathbf{C}^3$ is given by the standard action of D_6 on the hexagon. For example, flipping the hexagon about the horizontal axis leads to the group element $(z_1, z_2, z_3) \to (z_1, \bar{z}_3, \bar{z}_2)$. The action of the 2-torus is induced on $\mathbf{C}^3$ by its action as translations of plane waves and may be given explicitly by $(s, t)(z_1, z_2, z_3) = (e^{is}z_1, e^{i(s+t)}z_2, e^{it}z_3)$ where $(s, t) \in T^2$.

We now describe the ring of germs of smooth real-valued invariant functions on $\mathbf{C}^3 \times \mathbf{R}$, which we denote by $\mathcal{E}(T^2 + D_6)$. Note that f is *invariant* if $f(\gamma z, \lambda) = f(z, \lambda)\ \forall \gamma \in T^2 + D_6$.

Let $u_j = z_j \bar{z}_j$ for $j = 1, 2, 3$, let $\sigma_1 = u_1 + u_2 + u_3$, $\sigma_2 = u_1 u_2 + u_1 u_3 + u_2 u_3$, $\sigma_3 = u_1 u_2 u_3$ and let $q = 2 \operatorname{Re} z_1 \bar{z}_2 z_3$.

LEMMA 1. *Let f be in $\mathcal{E}(T^3 + D_6)$. Then there exists a smooth function h: $\mathbf{R}^4 \times \mathbf{R} \to \mathbf{R}$ such that*

$$f(z, \lambda) = h(\sigma, q, \lambda).$$

Define $E(T^2 + D_6)$ to be the module of equivariant mappings $g\colon \mathbf{C}^3 \times \mathbf{R} \to \mathbf{C}^3$ over the ring $\mathcal{E}(T^2 + D_6)$. So $g \in E(T^2 + D_6)$ if $g(\gamma z, \lambda) = \gamma g(z, \lambda)\ \forall \gamma \in T^2 + D_6$. This module may be computed explicitly using:

LEMMA 2. *$E(T^2 + D_6)$ is a free module over $\mathcal{E}(T^2 + D_6)$ with generators*

$$\begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \quad \begin{pmatrix} u_1 z_1 \\ u_2 z_2 \\ u_3 z_3 \end{pmatrix} \quad \begin{pmatrix} u_1^2 z_1 \\ u_2^2 z_2 \\ u_3^2 z_3 \end{pmatrix} \quad \begin{pmatrix} z_2 \bar{z}_3 \\ z_1 z_3 \\ \bar{z}_1 z_2 \end{pmatrix} \quad \begin{pmatrix} u_1 z_2 \bar{z}_3 \\ u_2 z_1 z_3 \\ u_3 \bar{z}_1 z_2 \end{pmatrix} \quad \begin{pmatrix} u_1^2 z_2 \bar{z}_3 \\ u_2^2 z_1 z_3 \\ u_3^2 \bar{z}_1 z_2 \end{pmatrix}.$$

It follows from the above lemma that the assumption of symmetry gives g a very special form, namely

$$g(z, \lambda) = \begin{pmatrix} H_1 z_1 + K_1 z_2 \bar{z}_3 \\ H_2 z_2 + K_2 z_1 z_3 \\ H_3 z_3 + K_3 \bar{z}_1 z_2 \end{pmatrix} \tag{2.1}$$

where $H_j = h_1 + h_3 u_j + h_5 u_j^2$ and $K_j = k_2 + k_4 u_j + k_6 u_j^2$ for $j = 1, 2, 3$ and h_1, h_3, h_5, k_2, k_4 and k_6 are invariant functions. In fact (2.1) allows us to identify g with the 6-tuple

$$g \sim (h_1, h_3, h_5, k_2, k_4, k_6).$$

One should observe that the only linear term in the Taylor expansion of g is $h_1(0)$ and the only quadratic term in the Taylor expansion of g is $k_2(0)$. The Liapunov-Schmidt procedure guarantees that $h_1(0) = 0$ for the g coming from the Bénard problem. A fact which depends on the explicit form of the Boussinesq equations is that $k_2(0) = 0$ for the g of the standard Bénard problem. However, if one changes the Bénard problem slightly to admit, say, a temperature dependent viscosity term the $k_2(0)$ will be nonzero.

3. The singularity theory. Using the ideas of **[3]** one begins to classify bifurcation problems in $E(T^2 + D_6)$ by codimension. The equivalence we use here is Γ-equivalence, an equivalent form of contact equivalence, which treats λ as a distinguished parameter. See **[4]**.

Assume that $h_1(0) = 0$ as discussed in the last section.

THEOREM 3. (a) *Suppose $k_2(0) \neq 0$ and that other nondegeneracy conditions on higher order terms (see* **[1]** *for details) hold. Then g is $(T^2 + D_6)$-equivalent to* $(-\lambda, 1, 0, 1, 0, 0)$ *and* codim $g = 0$.

(b) *Assume $k_2(0) = 0$ and that certain nondegeneracy conditions hold (see* **[1]** *for details). Then g is $(T^2 + D_6)$-equivalent to*

$$\left(-\lambda + a\sigma_1 + d\sigma_1^2, 1, 0, b\sigma_1 + cq, 1, 0\right)$$

where $a \neq -1, -\frac{1}{2}, -\frac{1}{3}$, $b \neq -\frac{1}{3}$, and $c \neq 0$.

The codimension of g is 5 and a universal unfolding is

$$\left(-\lambda + a\sigma_1 + d\sigma_1^2, 1, 0, b\sigma_1 + cq - e, 1, 0\right).$$

By explicit calculation of the zero set for this normal form one can show that the topological codimension is 1; that is, there are 4 modal parameters.

We note here that in both normal forms each coefficient which is $+1$ could just as well have been -1 and vice-versa. We have made a single choice of signs to reduce the complication in the statement of the theorem. In any physical interpretation these signs will be important.

4. The interpretation of the singularity theory results. The first step in recovering physically meaningful information from Theorem 3 is to have the ability to graph the zero sets of the normal forms. This is nontrivial since the zero set sits in $\mathbf{C}^3 \times \mathbf{R}$. However, one can simplify the problem by noting that since g commutes with $T^2 + D_6$ it must be zero on entire orbits. It is now possible using the explicit presentation of g in (2.1) to find for each orbit of zeros a unique point on the orbit. Moreover, one can classify the types of solutions to $g = 0$ which may occur by listing the isotropy subgroup which corresponds to the given solution. The results of these calculations yield:

PROPOSITION 4. *There are 7 types of solutions given by:*

	Nomenclature	Isotropy Subgroup
I	*Trivial Solution*	$T^2 + D_6$
II	*Rolls*	$S^1 + \mathbf{Z}_2 + \mathbf{Z}_2$
III^+, III^-	*Hexagons*	D_6
IV_R	*Wavy-Rolls*	$\mathbf{Z}_2 + \mathbf{Z}_2$
IV_H	*False Hexagons*	$\mathbf{Z}_2 + \mathbf{Z}_2$
V	*False Hexagons*	$\mathbf{Z}_2$
VI		$\mathbf{Z}_2$
VII		$\{1\}$

We arrived at the terminology in the following way. Each solution corresponds to a superposition of plane waves defined on $\mathbf{R}^2$, the actual plane of the Bénard problem. If one plots the zero set of the associated superposition of plane waves one can see the isotropy subgroups arising (see Figure 3). The difference between Hexagons and False Hexagons involves the symmetry of the "circlelike" figures in the zero sets. For Hexagons they have D_6 symmetry while for False Hexagons they have either 4-sided or 2-sided symmetry, depending on the isotropy subgroups. We note that solution types VI and VII do not occur in the singularities we consider in Theorem 3; they will however appear in more degenerate (higher codimension) bifurcation problems.

Now how does this relate to the fluid flow in the Bénard problem? Roughly speaking the superposition of plane waves we described above corresponds to (a linearization of) the vertical component of the velocity field in the Bénard problem. Observe that the negative flow is not necessarily the same physically as the original flow. For example, in our discussion of the hexagonal solutions in

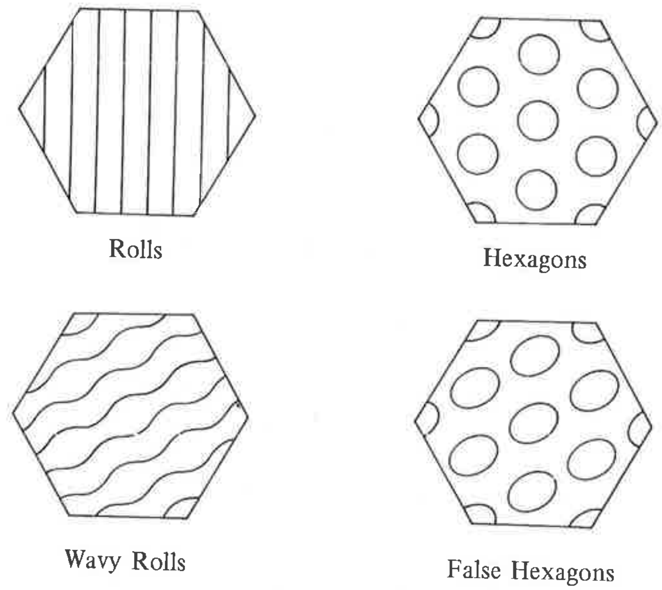

FIGURE 3. Zeroes of the superposition of plane waves

Bénard's experiments we noted that up-welling occurs at the center of the Hexagons. This is quite different physically from down-welling at the center which would be represented by the negative flow. This is the reason that one finds two types of Hexagonal solutions, III^+ and III^-. On the other hand, reversing the flow for rolls is equivalent to a phase shift and no new behavior is observed.

One more issue must be addressed before we can interpret the results of Theorem 3. From a physical point of view no equilibrium solution can be observed if it is a saddle or a source for the time-dependent problem. The way that one computes the stability of a given solution is through the notion of linearized stability; one computes the eigenvalues of $d_z g$ (the Jacobian of g in the $\mathbf{C}^3$ directions) and if they are in the correct half-plane for stability then the solution represents a stable equilibrium. There are two remarks needed at this time. First, if g commutes with the action of a continuous group, then $d_z g$ must have zero eigenvalues. Thus the best one can hope for is a form of orbital stability where all the eigenvalues of $d_z g$ which may be nonzero actually sit in the correct half-plane for stability. Second, in general, contact equivalence does not preserve the eigenvalues of $d_z g$ at solutions; however, in some cases Γ-equivalence does preserve the half-plane in which the eigenvalues sit (see [**4**]). With regard to the specific group discussed in the paper, one can prove

PROPOSITION 5. *The signs of the real parts of the eigenvalues of $d_z g$ are invariants of $(T^2 + D_6)$-equivalence at solutions of types* I, II, *and* III. *Sufficient information about solutions of type* IV_R *can be obtained to show that these solutions are unstable for the bifurcation problems considered here.*

We give a schematic diagram of the zero set for the normal form of codimension zero bifurcation problem of Theorem 3 in Figure 4. Observe that both Rolls and (both types of) Hexagons appear as solutions. Since this problem appears in the universal unfolding of all degenerate bifurcation problems there is an indication that Rolls and Hexagons should be the most observed solutions. However, each of these solutions is unstable, suggesting that this bifurcation problem cannot describe by itself physically interesting phenomena. (One should observe that different choices of signs in the normal form will interchange III^+ and III^- as well as allow the possibility that solutions II occur for $\lambda < 0$ rather then $\lambda > 0$.)

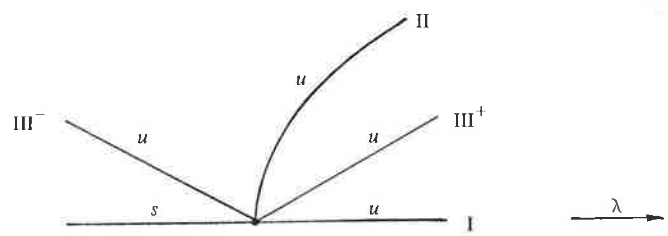

FIGURE 4. The codimension 0 bifurcation problem

However, if one looks at the next degeneracy (case (b) of Theorem 3) one finds much more interesting behavior. Moreover, this is the situation which *must* occur for the Bènard problem. (The reason for this fact involves looking more carefully at the Boussinesq equations.) There are many cases depending on the exact values of the modal parameters. (The interested reader may look at [**1**] where all the possibilities are discussed.) We give in Figure 5 the diagram which is perhaps the most interesting physically.

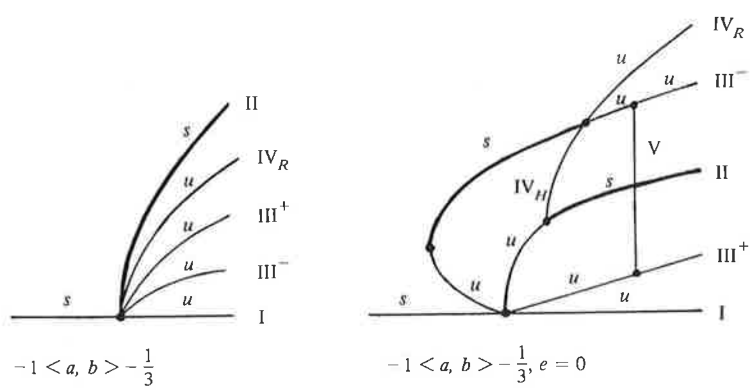

FIGURE 5. The topological codimension 1 problem

Here we find stable Roll solutions in the unperturbed problem along with the existence of (unstable) Wavy-Rolls solutions. When this problem is unfolded one finds, in addition, stable Hexagons and unstable False Hexagons of two different types.

There is still substantial work to be done in relating these results to the original Bénard problem. What is interesting here is that even with the restriction to the hexagonal lattice one finds a solution structure which is more complicated than might be suspected; moreover, one can classify all the likely cases in an organized fashion. For example the case shown in Figure 6 admits the possibility that both types of Hexagons are stable (for different ranges of λ). An equilibrium theory would predict a jump from hexagonal solutions of one type (III^+) to hexagonal solutions of the other type (III^-) as the temperature gradient is increased. Admittedly this jump has not been observed; nevertheless it is intriguing that the mathematical analysis admits this possibility as equally likely with the more physical diagram in Figure 5.

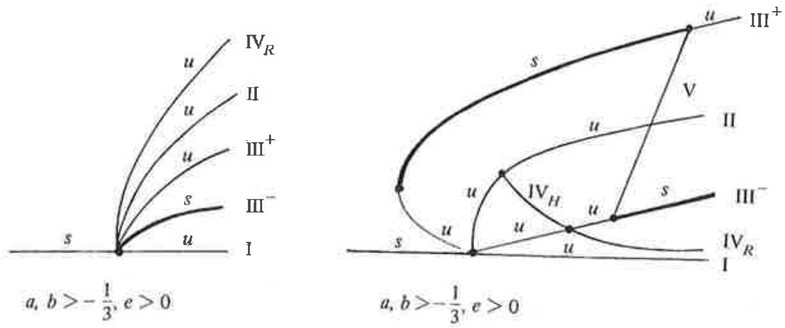

FIGURE 6. Another topological dimension 1 problem

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