Contemporary Mathematics
Volume 56, 1986

Hopf Bifurcation with Dihedral Group Symmetry: Coupled Nonlinear Oscillators

Martin Golubitsky [1]
Ian Stewart [2]

ABSTRACT We apply the theory of Hopf bifurcation with symmetry developed in Golubitsky and Stewart (1985) to systems of ODEs having the symmetries of a regular polygon, that is, whose symmetry group is dihedral. We consider the existence and stability of symmetry-breaking branches of periodic solutions. In particular we apply these results to a general system of n nonlinear oscillators, coupled symmetrically in a ring, and describe the generic oscillation patterns. We find, for example, that the symmetry can force some oscillators to have twice the frequency of others. The case of four oscillators has exceptional features.

0. Introduction

Systems of differential equations with symmetry can undergo an analogue of Hopf bifurcation, whereby a symmetric steady state loses stability and throws off a number of branches of symmetry-breaking periodic states. A general theory of symmetric Hopf bifurcation was developed in Golubitsky and Stewart (1985); some of the results have also been found independently by Sattinger (1984). In this paper we shall apply that theory to systems whose symmetries are those of a regular n-sided polygon. More precisely, consider the system of ODEs

$$dx/dt + f(x,\lambda) = 0 \qquad (0.1)$$

where $x(t) \in \mathbb{R}^p$ and $\lambda \in \mathbb{R}$ is the bifurcation parameter. We suppose that the dihedral group $\mathbf{D}_n$ of order 2n acts on $\mathbb{R}^p$ and that $f:\mathbb{R}^p\times\mathbb{R} \to \mathbb{R}^p$ is a smooth (C^∞) mapping commuting with this action of $\mathbf{D}_n$, so that

1980 *Mathematics Subject Classification* (1985 *Revision*) 58 F 22, 58 C 27, 58 F 14.
[1] Partially supported by NSF grant DMS-8402604 and NASA-Ames grant NAG 2-279.
[2] Partially supported by NASA-Ames grant NAG 2-279 and SERC Grant GR/D/33359.

$$f(\gamma x,\lambda) = \gamma f(x,\lambda) \qquad (\gamma \in \mathbf{D}_n). \tag{0.2}$$

Further assume that $f(0,\lambda) \equiv 0$, so that there is a trivial solution; and that the Jacobian $(df)_{(0,0)}$ has some purely imaginary eigenvalues. Generically we may assume that there is only one pair of such eigenvalues, and after rescaling t in (0.1) we may assume that these eigenvalues are $\pm i$. Unlike standard Hopf bifurcation, these eigenvalues need not be simple. As was shown in Golubitsky and Stewart (1985), one standard situation is that the (real) generalized eigenspace corresponding to the eigenvalues $\pm i$ has the form $W \oplus W$ where the group acts absolutely irreducibly on W. In the specific context of $\mathbf{D}_n$, we assume that $W \cong \mathbf{R}^2$ and that $\mathbf{D}_n$ acts by its *standard* representation on $\mathbf{R}^2$. Finally, we assume that the critical eigenvalues cross the imaginary axis with nonzero speed.

In §3 we use a Liapunov-Schmidt reduction to find periodic solutions to (0.1). This reduction allows us to find periodic trajectories of (0.1) by finding zeros of a mapping $g:\mathbf{C}^2 \times \mathbf{R} \to \mathbf{C}^2$ which commutes with an action of $\mathbf{D}_n \times \mathbf{S}^1$ (where $\mathbf{S}^1$ is the circle group of phase shift symmetries). The details of this reduction process are simplified by assuming that $p = 4$, so that the vector field itself may be thought of as a mapping from $\mathbf{C}^2 \times \mathbf{R} \to \mathbf{C}^2$. In §§1-4 we make this assumption; but it must be relaxed in the later sections when we discuss coupled oscillators.

Note that although we assert the existence of various types of periodic solution, we do *not* assert that no other solutions exist. On the contrary, we would expect quite complicated dynamics to be possible, especially in systems such as the coupled oscillators where the state space has dimension greater than 4. We do not even claim that we have found all possible periodic solutions of period near 2π. Further, in numerical simulations we have observed what appear to be subharmonic oscillations and quasiperiodic solutions. Chaotic behaviour might conceivably occur in suitable parameter ranges. The dynamics of these symmetric systems deserves further attention.

In §1 we show that generically there will be three branches of periodic solutions of period near 2π, bifurcating from the trivial solution at $\lambda = 0$. Similar results have been obtained independently by van Gils and Valkering (1985). Each branch has its own symmetry group, combining spatial symmetries (from $\mathbf{D}_n$) with temporal (phase shift) symmetries (from $\mathbf{S}^1$).

The action of the symmetry group $\mathbf{D}_n \times \mathbf{S}^1$ places strong restrictions on the form of the reduced mapping g, and in §2 we describe the most general possible form of g. In §3, as mentioned above, we apply the Liápunov-Schmidt reduction technique and show how this restricted form of g lets us solve the bifurcation equations by prescribing in advance the required symmetries of solutions. In §4 we state conditions on appropriate coefficients in the general form of g which determine the direction of criticality and the asymptotic stability of these solutions. (We remark that coefficients of high order terms, not just those of degree 3, are involved here, while certain terms of intermediate degree are irrelevant to the stability assignments.) The results for $n = 4$ differ markedly from those for other n. Specifically, if $n \geq 3$ and $n \neq 4$ then (subject to certain nondegeneracy conditions stated in §4) in order for any of the three branches to be stable, all three must be supercritical; further exactly one branch is then stable. When $n = 4$, however, some branches can be stable when others are subcritical; further, two distinct branches may be stable for the same values of the coefficients in f. The detailed situation is summarized in §4. Also in §4 we relate our results for $\mathbf{D}_n$ symmetry, as n becomes large, to the standard results for $\mathbf{O}(2)$ symmetry – the "limiting case" as $n \to \infty$. (See for example Golubitsky and Stewart (1985) §§ 7, 9, 10, and references therein.) In particular we explain how the three distinct types of oscillation occurring for $\mathbf{D}_n$ merge to give only two distinct types for $\mathbf{O}(2)$.

We apply these results in §§5-8 to a system of ODEs representing n nonlinear oscillators coupled together in a ring, with symmetric nearest-neighbour coupling. The precise equations, and their symmetries, are discussed in §5.

In §6 we specialize temporarily to the case of three oscillators. We give conditions on the Jacobian that guarantee the existence of the branches of symmetry-breaking oscillations predicted by our general theory, and describe the corresponding oscillation patterns. When $n = 3$ there are two types of generic Hopf bifurcation, depending on whether the eigenvalues are simple or double. One is that there is a unique branch of periodic (orbits of) solutions, on which all three oscillators have identical waveforms, in phase. The other is that there are three branches of symmetry-breaking oscillations. On one branch the oscillators have the same waveform but are phase-shifted by $2\pi/3$. On the second, two oscillators are identical and in phase, the third behaving differently. On the third branch, two oscillators have identical waveforms but are out of phase by π; and the third has double the frequency. We also support these conclusions with numerical simulations.

In §7 we discuss the case of general n; and in §8 we consider particular examples when $n = 2,4,5$. Again the case $n = 4$ has several peculiarities of its own.

We employ the methods and results of Golubitsky and Stewart (1985), and assume some familiarity with that paper. A brief sketch of the main ideas may be in order, however. Specifically, suppose that $x(t)$ is a periodic solution of (0.1) (with period scaled to 2π) and define an action of $\mathbf{D}_n\times\mathbf{S}^1$ by

$$(\gamma,\theta).x(t) = \gamma x(t-\theta) \qquad (\gamma \in \mathbf{D}_n, \theta \in \mathbf{S}^1)$$

where $\mathbf{S}^1$ is the circle group. Define the *isotropy subgroup* Σ of $x(t)$ to be

$$\Sigma = \{(\sigma,\theta) \mid \sigma x(t-\theta) = x(t)\} \subset \mathbf{D}_n\times\mathbf{S}^1.$$

Then Σ prescribes the spatio-temporal symmetries of the solution $x(t)$. In §4 of Golubitsky and Stewart (1985), or Sattinger (1983), it is shown how the Liapunov-Schmidt procedure may be used to reduce the solution of (0.1), posed on a suitable space of periodic functions, to a finite- dimensional bifurcation problem $g(v,\lambda) = 0$ having related symmetries. In the current context this leads to an action of the group $\mathbf{D}_n\times\mathbf{S}^1$ on $V = \mathbb{R}^4$. For $v \in V$ there is a corresponding notion of isotropy subgroup in the reduced problem, namely

$$\Sigma_v = \{\sigma \in \mathbf{D}_n\times\mathbf{S}^1 \mid \sigma v = v\}.$$

Define the *fixed-point subspace*

$$V^\Sigma = \{w \in V \mid \sigma w = w \text{ for all } \sigma \in \Sigma\}.$$

Then g maps V^Σ to itself. Hence we may find periodic solutions to (0.1) by restricting the reduced bifurcation problem g to the various V^Σ. Theorem 5.1 of Golubitsky and Stewart (1985) guarantees the existence of solutions if $\dim V^\Sigma = 2$. Our strategy is thus to find the isotropy subgroups, compute their fixed-point subspaces, and list those of dimension 2.

In addition, Theorem 8.2 of Golubitsky and Stewart (1985) states that the stability of the solution branch is determined by the eigenvalues of dg. We compute these eigenvalues in §3, taking advantage of the symmetries to simplify the calculations.

1. The Group Action of $\mathbf{D}_n\times\mathbf{S}^1$.

(a) Definition of the Group Action.

We begin by assuming that $\mathbf{D}_n$ $(n \geq 3)$ acts on $\mathbb{C}$ in the standard way as symmetries of the regular n-gon, and on $\mathbb{C}^2$ by the diagonal action

$$\gamma(z_1,z_2) = (\gamma z_1,\gamma z_2).$$

Although $\mathbf{D}_n$ will in general have many distinct irreducible representations (there are $(n+3)/2$ when n is odd, $(n+6)/2$ when n is even) there is no real loss of generality in making this assumption. Essentially it is possible to arrange for a standard action by relabelling the group elements and dividing by the kernel of the action. See §7 for further discussion.

We use the following notation for the elements of $\mathbf{D}_n$. Its cyclic subgroup $\mathbf{Z}_n$ consists of rotations of the plane through $0,\zeta,2\zeta,\ldots,(n-1)\zeta$ where $\zeta = 2\pi/n$. The *flip* κ is reflection in the x-axis. In complex notation $\mathbf{D}_n$ acts on $\mathbf{C}$ as follows:

$$(m\zeta).z = e^{im\zeta}z,$$
$$\kappa.z = \bar{z}.$$

To analyse $\mathbf{D}_n$-equivariant Hopf bifurcation we need to choose a simple form for the action of $\mathbf{D}_n\times\mathbf{S}^1$ on $\mathbf{C}^2$. This we do as follows. The flip κ acts on $\mathbf{C}^2$ by $(z_1,z_2) \mapsto (\bar{z}_1,\bar{z}_2)$. We claim that there exists a 2-dimensional subspace $W_1 \subset \mathbf{C}^2$ such that

(a) $\mathbf{Z}_n$ acts irreducibly on W_1, (1.1)

(b) $W_1 \oplus W_2 = \mathbf{C}^2$ where $W_2 = \kappa W_1$.

For example, we may choose $W_1 = \mathbf{C}\{(1,i)\}$. Since $\mathbf{Z}_n$ acts irreducibly, it follows that $\zeta = 2\pi/n \in \mathbf{Z}_n$ acts by $e^{i\ell\zeta}$ on W_1. However, all of the irreducible subspaces of $\mathbf{Z}_n$ in $\mathbf{C}^2$ have isomorphic representations; thus $\ell=\pm 1$. Replacing ζ by $-\zeta$ if necessary we may assume that $\ell = 1$. Since $\kappa\zeta\kappa = -\zeta$ in $\mathbf{D}_n$ it follows that ζ acts by multiplication by $e^{-i\zeta}$ on W_2.

Next we describe the action of $\mathbf{S}^1$ on $W_1 \oplus W_2$. Since $\mathbf{S}^1$ commutes with $\mathbf{Z}_n$, it follows that $\mathbf{S}^1$ must have the same invariant subspaces as $\mathbf{Z}_n$ (these being 2-dimensional). Thus $\theta \in \mathbf{S}^1$ acts by multiplication by $e^{mi\theta}$ on W_1 for some m. However, since we restrict our original phase shifts to 2π-periodic functions, the action of $\mathbf{S}^1$ must be the standard one, so $m = 1$. Finally, since $\kappa\theta = \theta\kappa$ in $\mathbf{D}_n\times\mathbf{S}^1$, the action of θ on W_2 is identical to the action of θ on W_1.

Thus we have identified the action of $\mathbf{D}_n\times\mathbf{S}^1$ on $(z_1,z_2) \in W_1 \oplus W_2$ with:

(a) $\gamma(z_1,z_2) = (e^{i\gamma}z_1, e^{-i\gamma}z_2)$ $(\gamma \in \mathbf{Z}_n)$

(b) $\kappa(z_1,z_2) = (z_2,z_1)$ (1.2)

(c) $\theta(z_1,z_2) = (e^{i\theta}z_1, e^{i\theta}z_2)$ $(\theta \in \mathbf{S}^1)$.

(b) Isotropy Subgroups of $\mathbf{D}_n\times\mathbf{S}^1$

We wish to compute, up to conjugacy, the isotropy subgroups of $\mathbf{D}_n\times\mathbf{S}^1$. Since points on the same group orbit have conjugate isotropy subgroups our method will be to find a representative point on each orbit and then compute the corresponding isotropy subgroup. There are three distinct cases, depending on whether n is odd, $n \equiv 2 \pmod 4$, or $n \equiv 0 \pmod 4$. The results are given in Tables 1.1 - 1.2.

Table 1.1 Isotropy subgroups of $D_n \times S^1$ acting on $\mathbb{C}^2$, when n is odd.

ORBIT TYPE	ISOTROPY SUBGROUP	FIXED-POINT SPACE	DIMENSION
$(0,0)$	$D_n \times S^1$	$\{(0,0)\}$	0
$(a,0)$	$\tilde{Z}_n = \{(\gamma,-\gamma) \mid \gamma \in Z_n\}$	$\{(z_1,0)\}$	2
(a,a)	$Z_2(\kappa)$	$\{(z_1,z_1)\}$	2
$(a,-a)$	$Z_2(\kappa,\pi)$	$\{(z_1,-z_1)\}$	2
(a,z_2) $z_2 \neq \pm a,0$	$\mathbb{1}$	$\mathbb{C}^2$	4

Table 1.2 Isotropy subgroups of $D_n \times S^1$ acting on $\mathbb{C}^2$, when $n \equiv 2 \pmod 4$. Note that $Z_2{}^c = \{(0,0),(\pi,\pi)\}$.

ORBIT TYPE	ISOTROPY SUBGROUP	FIXED-POINT SPACE	DIMENSION
$(0,0)$	$D^n \times S^1$	$\{(0,0)\}$	0
$(a,0)$	$\tilde{Z}_n = \{(\gamma,-\gamma) \mid \gamma \in Z_n\}$	$\{(z_1,0)\}$	2
(a,a)	$Z_2(\kappa) \oplus Z_2{}^c$	$\{(z_1,z_1)\}$	2
$(a,-a)$	$Z_2(\kappa,\pi) \oplus Z_2{}^c$	$\{(z_1,-z_1)\}$	2
(a,z_2) $z_2 \neq \pm a,0$	$Z_2{}^c$	$\mathbb{C}^2$	4

Table 1.3 Isotropy subgroups of $D_n \times S^1$ acting on $\mathbb{C}^2$, when $n \equiv 0 \pmod 4$. Note that $Z_2{}^c = \{(0,0),(\pi,\pi)\}$.

ORBIT TYPE	ISOTROPY SUBGROUP	FIXED-POINT SPACE	DIMENSION
$(0,0)$	$D^n \times S^1$	$\{(0,0)\}$	0
$(a,0)$	$\tilde{Z}_n = \{(\gamma,-\gamma) \mid \gamma \in Z_n\}$	$\{(z_1,0)\}$	2
(a,a)	$Z_2(\kappa) \oplus Z_2{}^c$	$\{(z_1,z_1)\}$	2
$(a,e^{2\pi i/n}a)$	$Z_2(\kappa\zeta) \oplus Z_2{}^c$	$\{(z_1,e^{2\pi i/n}z_1)\}$	2
(a,z_2) $z_2 \neq \pm a,0$	$Z_2{}^c$	$\mathbb{C}^2$	4

Observe that in Table 1.1 we also list the fixed-point subspaces for the isotropy subgroups. Since three of these fixed-point subspaces are two-dimensional, it follows from Theorem 5.1 of Golubitsky and Stewart (1985) that there are at least three branches of (orbits under $\mathbf{D}_n \times \mathbf{S}^1$ of) periodic solutions occurring generically in Hopf bifurcation with $\mathbf{D}_n$ symmetry. In the remainder of the paper we discuss the generic directions of branching and stabilities of these solutions. We interpret the symmetries in §§ 6–8 in the context of coupled oscillators.

We now verify the entries in Table 1.1. If $(z_1,z_2) = (0,0)$ then trivially the isotropy subgroup is $\mathbf{D}_n \times \mathbf{S}^1$, so we may assume $(z_1,z_2) \neq (0,0)$. By use of θ and κ, if necessary, we may assume that $z_1 = a > 0$, and that $(z_1,z_2) = (a,re^{i\psi})$. We claim that we may assume $0 \le \psi \le \zeta/2 = \pi/n$ [n odd]; $0 \le \psi \le \zeta = 2\pi/n$ [n even]. This is trivial if $r = 0$, so assume $r > 0$. The group elements in $\mathbf{D}_n \times \mathbf{S}^1$ have the form $(m\zeta,\theta)$ and $(\kappa(m\zeta),\theta)$ where $m = 0,1,\ldots,n-1$. These group elements transform $(a,re^{i\psi})$ to:

$$\text{(a)} \quad (ae^{i(m\zeta+\theta)}, re^{i(\psi-m\zeta+\theta)}) \qquad (1.3)$$

$$\text{(b)} \quad (re^{i(\psi-m\zeta+\theta)}, ae^{i(m\zeta+\theta)})$$

respectively. For these group elements to preserve the form (a,z_2) we must assume in (1.3a) that

$$m\zeta+\theta = 2\pi k$$

and in (1.3b) that

$$\psi-m\zeta+\theta = 2\pi k$$

for some integer k. In addition, in (1.3b) it is convenient to interchange the labels r and a. In this way we conjugate $(a,re^{i\psi})$ to

$$\text{(a)} \quad (a,re^{i(\psi-2m\zeta)}) \qquad (1.4)$$

$$\text{(b)} \quad (a,re^{i(2m\zeta-\psi)}).$$

Using (1.4) we can translate ψ by 2ζ and flip ψ to $-\psi$. Now when n is odd, rotations by 2ζ generate the whole group $\mathbf{Z}_n$. Hence we may actually translate ψ by ζ, not just 2ζ. It is now easy to show that every ψ may be assumed in the interval $0 \le \psi \le \zeta/2$ as claimed. On the other hand, when n is even we can only assume ψ is in the interval $0 \le \psi \le \zeta$.

We now consider the three cases: n odd, $n \equiv 2 \pmod 4$, $n \equiv 0 \pmod 4$.

(i) <u>n odd</u>. We first note that if $r \neq a,0$ or $0 < \psi < \zeta/2$, then the isotropy subgroup of $(a,re^{i\psi})$ is $\mathbf{1}$. If $r \neq a$, then (1.3b) shows that no element of the form $(\kappa(m\zeta),\theta)$ can be in the isotropy subgroup. If $r \neq 0$, then (1.3a) shows that the form $(a,re^{i\psi})$ is fixed only by $\theta \equiv -m\zeta \pmod{2\pi}$. Thus (1.4a) implies that $(a,re^{i\psi})$ is fixed only when $\theta = m = 0$, or $\theta = -\pi$, $m = n/2$ when n is even. Since n is odd here, the isotropy subgroup is $\mathbf{1}$.

It also follows from these calculations that $(a,0)$ is fixed precisely by

$(m\zeta,-m\zeta) \in \mathbf{Z}_n \times \mathbf{S}^1$. We have now reduced to the case $r=a$ and $\psi = 0,\zeta/2$. When $\psi = 0$ we have a point (a,a) and its isotropy subgroup is $\mathbf{Z}_2(\kappa) = \{(0,0),\kappa\}$. Similarly, since n is odd, $(a,ae^{i\zeta/2})$ is in the same orbit as $(a,-a)$ and the isotropy subgroup is $\mathbf{Z}_2(\kappa,\pi) = \{(0,0),(\kappa,\pi)\} \subset \mathbf{D}_n \times \mathbf{S}^1$.

Finally, the fixed-point subspaces are easily computed once the isotropy subgroups are known.

(ii) <u>$n \equiv 2 \pmod 4$</u>. This is similar. In fact $\mathbf{Z}_2^c = \{(0,0),(\pi,\pi)\}$ acts trivially on $\mathbf{C}^2$ and hence is contained in every isotropy subgroup. Now $(\mathbf{D}_n \times \mathbf{S}^1)/\mathbf{Z}_2^c \cong \mathbf{D}_{n/2} \times \mathbf{S}^1$ since $\mathbf{D}_{n/2} \times \mathbf{S}^1 \subset \mathbf{D}_n \times \mathbf{S}^1$ and $(\mathbf{D}_{n/2} \times \mathbf{S}^1) \cap \mathbf{Z}_2^c = \mathbf{1}$. There is one subtle point: the induced action of $\mathbf{S}^1$ (when $\mathbf{Z}_2^c$ is thus factored out) is by $e^{2i\theta}$, not $e^{i\theta}$. The same arguments work (with θ replaced by $\theta/2$), but all isotropy subgroups are augmented by $\mathbf{Z}_2^c$.

(iii) <u>$n \equiv 0 \pmod 4$</u>. Again (π,π) fixes $\mathbf{C}^2$, so every isotropy subgroup contains $\mathbf{Z}_2^c$. The previous analysis shows that we may assume

$$(z_1,z_2) = (a,re^{i\psi}) \qquad (0 \le \psi \le \zeta). \tag{1.5}$$

If $r = 0$ then we have $(a,0)$ and the isotropy subgroup is $\tilde{\mathbf{Z}}_n$ as before. Otherwise $r \neq 0$. We claim that the only elements (1.5) with $r \neq 0$ that have isotropy subgroup larger than $\mathbf{Z}_2^c$ are (a,a) and $(a,ae^{2\pi i/n})$.

It follows from (1.4) that for such elements any isotropy subgroup larger than $\mathbf{Z}_2^c$ must contain an element of the form $(\kappa(m\zeta),\theta)$ and hence interchange the coordinates. Therefore $r = a$. From (1.4) we must have

$$2m\zeta-\psi = \psi + 2k\pi$$

so that

$$\psi = m\zeta - k\pi.$$

But $\pi \in \mathbf{D}_n$ when n is even, so $\psi = m\zeta$. Therefore from (1.5) $\psi = 0$ or $\psi = \zeta$. This leads to the two cases (a,a) and $(a,e^{2\pi i/n}a)$. It is easy to check that the isotropy subgroups are as stated in Table 1.3, and that since $n \equiv 0 \pmod 4$ the two isotropy subgroups $\mathbf{Z}_2(\kappa)\oplus\mathbf{Z}_2^c$ and $\mathbf{Z}_2(\kappa\zeta)\oplus\mathbf{Z}_2^c$ are not conjugate.

Remark When $n \equiv 2 \pmod 4$ the element $\kappa\zeta$ is conjugate to $\kappa\pi$. To see this let $q = (n-2)/4$ and compute

$$(-q\zeta)(\kappa\zeta)(q\zeta) = \kappa(q\zeta+\zeta+q\zeta) = \kappa(\tfrac{1}{2}n\zeta) = \kappa\pi.$$

It follows that $\mathbf{Z}(\kappa\zeta)\oplus\mathbf{Z}_2^c$ is conjugate to $\mathbf{Z}(\kappa,\pi)\oplus\mathbf{Z}_2^c$. Therefore the entries in Table 1.3 also apply when $n \equiv 2 \pmod 4$, and provide an alternative description of the orbit structure in that case.

2. Invariant Theory for $D_n \times S^1$

In this section we find a Hilbert basis for the invariant functions $\mathbb{C}^2 \to \mathbb{R}$ and a module basis for the equivariant mappings $\mathbb{C}^2 \to \mathbb{C}^2$. The results here depend only on the parity of n.

Proposition 2.1 Let m = n [n odd], n/2 [n even]; and let $n \geq 3$. Then
(a) Every smooth $D_n \times S^1$-invariant germ $f: W_1 \oplus W_2 \to \mathbb{R}$ has the form

$$f(z_1,z_2) = h(N,P,S,T)$$

where $N = |z_1|^2+|z_2|^2$, $P = |z_1|^2|z_2|^2$, $S = (z_1\bar{z}_2)^m+(\bar{z}_1z_2)^m$, and

$$T = i[z_1^{m+1}\bar{z}_1\bar{z}_2^m + \bar{z}_1^m z_2^{m+1}\bar{z}_2 - z_1\bar{z}_1^{m+1}z_2^m - z_1^m z_2\bar{z}_2^{m+1}].$$

(b) Every smooth $D_n \times S_1$-equivariant map germ $g: W_1 \oplus W_2 \to W_1 \oplus W_2$ has the form

$$g(z_1,z_2) = A\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + B\begin{bmatrix} z_1^2\bar{z}_1 \\ z_2^2\bar{z}_2 \end{bmatrix} + C\begin{bmatrix} \bar{z}_1^{m-1}z_2^m \\ z_1^m\bar{z}_2^{m-1} \end{bmatrix} + D\begin{bmatrix} z_1^{m+1}\bar{z}_2^m \\ \bar{z}_1^m z_2^{m+1} \end{bmatrix}$$

where A,B,C,D are complex-valued $D_n \times S^1$-invariant functions.

Before proving this, we note an easy but useful lemma, which lets us consider the simpler situation of complex-valued invariants.

Lemma 2.2 Let Γ be a group acting on $\mathbb{C}^n$. Let $z = (z_1,\ldots,z_n) \in \mathbb{C}^n$, and define $\bar{z} = (\bar{z}_1,\ldots,\bar{z}_n)$. Suppose that $N_1,\ldots,N_s$ generate (over $\mathbb{C}$) the $\mathbb{C}$-valued Γ-invariant polynomials in $z,\bar{z}$. Then $\mathrm{Re}(N_1),\ldots,\mathrm{Re}(N_s),\mathrm{Im}(N_1),\ldots,\mathrm{Im}(N_s)$ generate (over $\mathbb{R}$) the $\mathbb{R}$-valued Γ-invariant polynomials in $z, \bar{z}$.

Proof The $\mathbb{R}$-valued invariants are those $\mathbb{C}$-valued invariants whose values happen to lie in $\mathbb{R}$. Hence they are generated by the real and imaginary parts of monomials $N_1^{\alpha_1}\ldots N_s^{\alpha_s}$. But if p and q are polynomials in $z,\bar{z}$ over $\mathbb{C}$ then

$$\mathrm{Re}(pq) = \mathrm{Re}(p)\mathrm{Re}(q) - \mathrm{Im}(p)\mathrm{Im}(q),$$
$$\mathrm{Im}(pq) = \mathrm{Re}(p)\mathrm{Im}(q) + \mathrm{Im}(p)\mathrm{Re}(q).$$

An induction argument completes the proof. □

We are now ready for the:

Proof of Proposition 2.1 This is obtained by a direct but not always elegant computation. By Schwarz (1975) and Poénaru (1976) we may assume that the germ is polynomial. We begin with the complex-valued invariants, and later deduce the real-valued ones using Lemma 2.2. A general polynomial map $\varphi: \mathbb{C}^2 \to \mathbb{R}$ is of the form

$$\varphi(z_1,z_2) = \sum A_{\alpha\beta\gamma\delta} z_1{}^{\alpha} \bar{z}_1{}^{\beta} z_2{}^{\gamma} \bar{z}_2{}^{\delta}. \tag{2.1}$$

Invariance under κ implies that

$$A_{\alpha\beta\gamma\delta} = A_{\gamma\delta\alpha\beta}. \tag{2.2}$$

Invariance under $\theta \in \mathbf{S}^1$ implies that

$$\varphi = \sum e^{i\theta(\alpha-\beta+\gamma-\delta)} z_1{}^{\alpha} \bar{z}_1{}^{\beta} z_2{}^{\gamma} \bar{z}_2{}^{\delta},$$

so only terms such that

$$\alpha-\beta+\gamma-\delta = 0 \tag{2.3}$$

can occur. Similarly $\mathbf{Z}_n$-invariance implies that only terms for which

$$\alpha-\beta-\gamma+\delta \equiv 0 \pmod n \tag{2.4}$$

occur.

Define

$$[\alpha\beta\gamma\delta] = z_1{}^{\alpha} \bar{z}_1{}^{\beta} z_2{}^{\gamma} \bar{z}_2{}^{\delta} + z_1{}^{\gamma} \bar{z}_1{}^{\delta} z_2{}^{\alpha} \bar{z}_2{}^{\beta}.$$

Then (2.2) implies that every $\mathbf{C}$-valued invariant is a $\mathbf{C}$-linear combination of terms $[\alpha\beta\gamma\delta]$. We note three obvious invariants

$$N = [1100] = z_1\bar{z}_1 + z_2\bar{z}_2,$$

$$P = \tfrac{1}{2}[1111] = z_1\bar{z}_1 z_2\bar{z}_2, \tag{2.5}$$

$$S = [m00m] = z_1{}^m \bar{z}_2{}^m + \bar{z}_1{}^m z_2{}^m,$$

recalling that $m = n$ [n odd], $n/2$ [n even]. Now (2.3,2.4) imply that

$$\alpha+\gamma = \beta+\delta$$

$$\alpha+\delta \equiv \beta+\gamma \pmod n$$

whence

$$\gamma-\delta \equiv \delta-\gamma \pmod n.$$

Therefore

$$\gamma \equiv \delta \pmod m, \tag{2.6}$$

and

$$\alpha \equiv \beta \pmod m. \tag{2.7}$$

We shall use the following identities:

(a) $[\alpha\beta\gamma\delta] = [\gamma\delta\alpha\beta]$,

(b) $P[\alpha\beta\gamma\delta] = [\alpha+1,\beta+1,\gamma+1,\delta+1]$, (2.8)

(c) $S[\alpha\beta\gamma\delta] = [\alpha+m,\beta,\gamma,\delta+m]+[\alpha,\beta+m,\gamma+m,\delta]$,

(d) $N[\alpha\beta\gamma\delta] = [\alpha+1,\beta+1,\gamma,\delta]+[\alpha,\beta,\gamma+1,\delta+1]$.

We find generators for the invariants by seeking *minimal* terms $[\alpha\beta\gamma\delta]$, not expressible as linear combinations over $\mathbf{C}[P,N,S]$ of terms of smaller total degree $\alpha+\beta+\gamma+\delta$. By (2.8d) minimality implies $\alpha=0$ or $\beta=0$. Using (2.8a) we may eliminate as redundant all terms except $[0\beta\gamma\delta]$ and $[\alpha0\gamma\delta]$. Now

$$[\alpha\beta\gamma\delta] = N[\alpha,\beta,\gamma-1,\delta-1]-P[\alpha,\beta,\gamma-2,\delta-2]$$

so such terms are redundant unless $\gamma < 2$ or $\delta < 2$. Thus there are eight cases to consider:

$$\text{(a)}\ [0\beta 0\delta]\quad \text{(b)}\ [0\beta 1\delta]\quad \text{(c)}\ [0\beta\gamma 0]\quad \text{(d)}\ [0\beta\gamma 1] \qquad (2.9)$$
$$\text{(e)}\ [\alpha 00\delta]\quad \text{(f)}\ [\alpha 01\delta]\quad \text{(g)}\ [\alpha 0\gamma 0]\quad \text{(h)}\ [\alpha 0\gamma 1].$$

In (2.9a) we have $\beta+\delta = 0$, so $\beta = \delta = 0$. But [0000] is just a constant. In (2.9b) $\beta+\delta = 1$, so the term is [0100] (or [0001] which is eliminated by (2.8a)). But (2.7) rules this out. In (2.9c) $\beta = \gamma$ by (2.3), so the term is $[0\beta\beta 0]$. Also $\beta \equiv 0 \pmod m$ by (2.7) so $\beta \geq m$. We have

$$[0\beta\beta 0] = S[0,\beta-m,\beta-m,0]-P[m-1,\beta-m-1,\beta-m-1,m-1]$$

by (2.8b,c). Minimality implies $\beta-m \leq 0$, so $\beta = m$. But $[0mm0] = 2S$ by (2.5,2.8a). In (2.9d) $\gamma = \beta+1$ and the term is $[0,\beta,\beta+1,1]$. Also $\beta \equiv 0 \pmod m$ so $\beta \geq m$. Now

$$[0,\beta,\beta+1,0] = S[0,\beta-m,\beta-m+1,1]-P[m-1,\beta-m-1,\beta-m,m].$$

Minimality implies $\beta \leq m$, so $\beta = m$ and we get $[0,m,m+1,1]$. This is a new generator. The case (2.9e) is the same as (2.9c) by (2.8a). For case (f) we have $\alpha+1 = \delta$, $\alpha \equiv 0 \pmod m$, so we must consider $[\alpha,0,1,\alpha+1]$. But this is equal to $S[\alpha-m,0,1,\alpha+1-m] - P[\alpha-1,m-1,m,\alpha]$. Minimality implies $\alpha = 0$. But $[0011] = [1100]$ by (2.8a), so we get nothing new. In case (2.9g) we have $\alpha+\gamma = 0$ so $\alpha = \gamma = 0$. In case (2.9h) $\alpha+\gamma = 1$, and $\alpha+1 = \delta$, so the only possiblities are [1001] and [0011]. But $\gamma \equiv \delta \pmod m$ rules out the first, and the second is not new.

Thus the $\mathbb{C}$-valued invariants are generated over $\mathbb{C}$ by N,P,S, and $[0,m,m+1,1]$. We now appeal to Lemma 2.2. Each of N,P,S is already real, so their imaginary parts do not contribute. Also $\mathrm{Re}[0,m,m+1,1] = 2NS$ is redundant. So in addition to N, P, S, we require one further generator $T = 2.\mathrm{Im}[0,m,m+1,1]$ to generate (over $\mathbb{R}$) the $\mathbb{R}$-valued invariants. This completes the proof of Proposition 2.1(a).

The equivariants are obtained by a similar argument. If $\Phi:\mathbb{C}^2 \to \mathbb{C}^2$ is $\mathbf{D}_n\times\mathbf{S}^1$-equivariant, write

$$\Phi(z_1,z_2) = (\varphi_1(z_1,z_2),\varphi_2(z_1,z_2)).$$

There is no reality condition. Equivariance under κ implies that

$$\varphi_2(z_1,z_2) = \varphi_1(z_2,z_1).$$

Putting $\varphi_1 = \varphi$ we have

$$\Phi(z_1,z_2) = (\varphi(z_1,z_2),\varphi(z_2,z_1))$$

and it remains to determine the form of φ. We seek generators for the module $\vec{\mathcal{E}}(\mathbf{D}_n\times\mathbf{S}^1)$ of $\mathbf{D}_n\times\mathbf{S}^1$-equivariants over the ring $\mathcal{E}(\mathbf{D}_n\times\mathbf{S}^1)$ of invariants, which by part (1) is $\mathbb{R}[N,P,S,T]$. It suffices to find generators for φ. To see this, suppose that $\varphi = \sum I_jE_j$ where each I_j is invariant, E_j equivariant. Then κ-invariance implies that $I_j(z_1,z_2) = I_j(z_2,z_1)$, so

$$\Phi(z_1,z_2) = \left(\sum I_j(z_1,z_2)E_j(z_1,z_2),\sum I_j(z_2,z_1)E_j(z_2,z_1)\right)$$

$$= \sum I_j(z_1,z_2)(E_j(z_1,z_2),E_j(z_2,z_1)).$$

So the pairs $(E_j(z_1,z_2),E_j(z_2,z_1))$ are generators for $\vec{\mathcal{E}}(D_n \times S^1)$ over $\mathcal{E}(D_n \times S^1)$. Note further that since there is no reality condition on φ, but $\mathcal{E}(D_n \times S^1) = \mathbb{R}[N,P,S,T]$ is real, the generators come in pairs E, iE.

The general φ can be written as a polynomial in $z_1, \bar{z}_1, z_2, \bar{z}_2$ as before. This time define

$$[\alpha\beta\gamma\delta] = z_1^{\alpha}\bar{z}_1^{\beta}z_2^{\gamma}\bar{z}_2^{\delta}.$$

Again we seek to express φ by terms that minimize $\alpha+\beta+\gamma+\delta$. The details are much as in part (a) but there is no analogue of (2.8a). However, (2.8b,c,d) remain true with the new interpretation of $[\alpha\beta\gamma\delta]$. Using (2.8d) we may assume $\alpha = 0$ or $\beta = 0$. In each case $\gamma < 2$ or $\delta < 2$, using (2.8d) again. Thus there are the same eight cases:

$$\text{(a) } [0\beta0\delta] \quad \text{(b) } [0\beta1\delta] \quad \text{(c) } [0\beta\gamma0] \quad \text{(d) } [0\beta\gamma1] \qquad (2.10)$$
$$\text{(e) } [\alpha00\delta] \quad \text{(f) } [\alpha01\delta] \quad \text{(g) } [\alpha0\gamma0] \quad \text{(h) } [\alpha0\gamma1].$$

Now S^1-equivariance implies that only the terms with

$$\alpha-\beta+\gamma-\delta = 1$$

occur; and $\mathbb{Z}_n$-equivariance that

$$\alpha-\beta-\gamma+\delta \equiv 1 \pmod n.$$

Thus

$$\alpha+\gamma = 1+\beta+\delta$$
$$\gamma \equiv \delta \pmod m$$
$$\alpha \equiv \beta+1 \pmod m.$$

Now (2.10a) implies $0 = 1+\beta+\delta$, a contradiction. For (2.10b) we have $\beta = \delta = 0$, and then $1 \equiv 0 \pmod m$, also a contradiction. In case (2.10c) $\gamma = 1+\beta$. But

$$[0,\beta,\beta+1,0] = S[0,\beta-m,\beta+1-m,0]-P[m-1,\beta-m-1,\beta-m,m-1]$$

unless $\beta-m \le 0$. But $\beta \equiv -1 \pmod m$ so $\beta = m-1$ and we get a generator $[0,m-1,m,0]$. Case (2.10d) implies $\gamma = \beta+2$. Now

$$[0,\beta,\beta+2,1] = S[0,\beta-m,\beta+2-m,1]-P[m-1,\beta-m-1,\beta+2-m,m]$$

which similarly yields a generator $[0,m-1,m+1,1]$. In (2.10e) $\alpha = 1+\delta$ and $\delta \equiv 0 \pmod m$. If $\delta = 0$ we get the generator $[1000]$. Otherwise

$$[\delta+1,0,0,\delta] = S[\delta+1-m,0,0,\delta-m]-P[\delta-m,m-1,m-1,\delta-m-1]$$

and is redundant unless $\delta = m$. This yields the term $[m+1,0,0,m]$. But

$$[0,m-1,m+1,1] = N[0,m-1,m,0]-S[1000]+[m+1,0,0,m]$$

and we can remove the generator $[0,m-1,m+1,1]$ and replace it by $[m+1,0,0,m]$. Case (2.10f) similarly leads to a generator $[1011]$ which we replace by $[2100]$. Cases (2.10g,h) lead to nothing new.

Collecting the generators found above we get

[1000]
[2100]
[0,m-1,m,0]
[m+1,0,0,m]

which yield, respectively, the mappings with coefficients A,B,C,D in Proposition 2.1b. This completes the proof. □

Note that when n = 4 there are three cubic equivariants

$$(|z_1|^2+|z_2|^2)\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}, \quad \begin{bmatrix} z_1{}^2\bar{z}_1 \\ z_2{}^2\bar{z}_2 \end{bmatrix}, \quad \begin{bmatrix} \bar{z}_1 z_2{}^2 \\ z_1{}^2\bar{z}_2 \end{bmatrix}.$$

However, when $n \geq 3$, $n \neq 4$, only the first two are cubic. This has an effect on the analysis of branching and stability in $\mathbf{D}_4$-Hopf bifurcation in the next section. This effect also appears in $\mathbf{D}_{4k}$-Hopf bifurcation for representations having kernel $\mathbf{D}_k$.

To tie up one loose end, note that when n = 2 the non-trivial irreducible representations of $\mathbf{D}_2 \cong \mathbf{Z}_2 \oplus \mathbf{Z}_2$ are 1-dimensional and have kernels K such that $\mathbf{D}_2/K \cong \mathbf{Z}_2$, so effectively the problem reduces to $\mathbf{Z}_2 \times \mathbf{S}^1$ acting on $\mathbf{R} \oplus \mathbf{R} \equiv \mathbf{C}$ where $\mathbf{Z}_2$ acts as minus the identity. The invariants and equivariants for this action are the same as those of $\mathbf{S}^1$ on $\mathbf{C}$, namely one invariant generator x^2+y^2 and two equivariants (x,y) and $(-y,x)$. This happens because the $\mathbf{Z}_2$-action is the same as that of the rotation $\pi \in \mathbf{S}^1$.

3. How to Solve the Liapunov-Schmidt Reduced Equations

It now follows from Golubitsky and Stewart (1985) that when $n \geq 3$ the branching equations for $\mathbf{D}_n$-equivariant Hopf bifurcation may be written

$$g = A\begin{bmatrix} z_1 \\ z_2 \end{bmatrix} + B\begin{bmatrix} z_1{}^2\bar{z}_1 \\ z_2{}^2\bar{z}_2 \end{bmatrix} + C\begin{bmatrix} \bar{z}_1{}^{m-1}z_2{}^m \\ z_1{}^m\bar{z}_2{}^{m-1} \end{bmatrix} + D\begin{bmatrix} z_1{}^{m+1}\bar{z}_2{}^m \\ \bar{z}_1{}^m z_2{}^{m+1} \end{bmatrix} = 0$$

where τ is the perturbed period parameter. (If further the original vector field is in Birkhoff normal form, that is, commutes with $\mathbf{D}_n \times \mathbf{S}^1$, then $A = A' - (1+\tau)i$ where $A'(0) = i$.) The way we propose to solve the equations $g = 0$ is by restricting g to each fixed-point subspace V^Σ. This method is possible since $g(V^\Sigma) \subset V^\Sigma$. In Table 3.1 we list the equations for each of the three maximal isotropy subgroups Σ when $n \neq 4$; in Table 3.2 we list them for $n = 4$. Note that each of the branching equations consists of a real and an imaginary part. The imaginary parts of these equations may be solved for τ. The real parts contain the branching information.

Another important aspect of the branching equations is the fact (see

Golubitsky and Stewart (1985) Theorem 8.2 and Taliaferro (1985)) that the asymptotic stability of the solutions is determined by the eigenvalues of dg. The S^1 symmetry forces one eigenvalue of dg to be zero; the signs of the real parts of the remaining three eigenvalues determine the asymptotic stability. We list here, and derive below, the signs of these eigenvalues.

The isotropy subgroup of a solution restricts the form of dg at that solution since for $z = (z_1, z_2)$ we have

$$(dg)_z\gamma = \gamma(dg)_z. \tag{3.1}$$

For each of the three maximal isotropy subgroups Σ, the action of Σ implies that dg has two 2-dimensional invariant subspaces; namely the fixed-point subspace V_0 on which Σ acts trivially, and an invariant complement V_1. See Tables 3.4, 3.5.

Table 3.1 Branching Equations for D_n Hopf bifurcation, $n \geq 3$, n odd or $n \equiv 2 \pmod 4$. Here $m = \begin{cases} n \text{ [n odd]} \\ n/2 \text{ [n even].} \end{cases}$

ORBIT TYPE	BRANCHING EQUATIONS	*SIGNS* OF EIGENVALUES
(0,0)	–	Re $A(0,\lambda)$
(a,0)	$A+Ba^2 = 0$	$\mathrm{Re}(A_N+B) + O(a)$ $-\mathrm{Re}(B)$ [twice]
(a,a)	$A+Ba^2+Ca^{2m-2}+Da^{2m} = 0$	$\mathrm{Re}(2A_N+B) + O(a)$ trace $= \mathrm{Re}(B) + O(a)$ det $= -\mathrm{Re}(B\bar{C}) + O(a)$
(a,–a)	$A+Ba^2-Ca^{2m-2}-Da^{2m} = 0$	$\mathrm{Re}(2A_N+B) + O(a)$ trace $= \mathrm{Re}(B) + O(a)$ det $= \mathrm{Re}(B\bar{C}) + O(a)$

Table 3.2 Branching Equations for $\mathbf{D}_n$ Hopf bifurcation, $n \equiv 0 \pmod 4$, $n \neq 4$. Here $m = n/2$.

ORBIT TYPE	BRANCHING EQUATIONS	*SIGNS* OF EIGENVALUES
$(0,0)$	–	$\mathrm{Re}\, A(0,\lambda)$
$(a,0)$	$A+Ba^2 = 0$	$\mathrm{Re}(A_N+B) + O(a)$ $-\mathrm{Re}(B)$ [twice]
(a,a)	$A+Ba^2+Ca^{2m-2}+Da^{2m} = 0$	$\mathrm{Re}(2A_N+B) + O(a)$ trace $= \mathrm{Re}(B) + O(a)$ det $= -\mathrm{Re}(B\bar{C}) + O(a)$
$(a,e^{2\pi i/n}a)$	$A+Ba^2-Ca^{2m-2}-Da^{2m} = 0$	$\mathrm{Re}(2A_N+B) + O(a)$ trace $= \mathrm{Re}(B) + O(a)$ det $= \mathrm{Re}(B\bar{C}) + O(a)$

Table 3.3 Branching Equations for $\mathbf{D}_4$ Hopf bifurcation.

ORBIT TYPE	BRANCHING EQUATIONS	*SIGNS* OF EIGENVALUES
$(0,0)$	–	$\mathrm{Re}\, A(0,\lambda)$
$(a,0)$	$A+Ba^2 = 0$	$\mathrm{Re}(A_N+B) + O(a)$ $-\mathrm{Re}(B)$ $\lvert B\rvert^2-\lvert C\rvert^2$
(a,a)	$A+(B+C)a^2+Da^4 = 0$	$\mathrm{Re}(2A_N+B+C) + O(a)$ trace $= \mathrm{Re}(B-3C) + O(a)$ det $= \lvert C\rvert^2-\mathrm{Re}(B\bar{C}) + O(a)$
(a,ia)	$A+(B-C)a^2-Da^4 = 0$	$\mathrm{Re}(2A_N+B-C) + O(a)$ trace $= \mathrm{Re}(B+3C) + O(a)$ det $= \lvert C\rvert^2+\mathrm{Re}(B\bar{C}) + O(a)$

Table 3.4 Decomposition of $\mathbb{R}^4$ into invariant subspaces for Σ, when $n \equiv 1,2,3 \pmod 4$.

ISOTROPY	ORBIT REPRESENTATIVE	INVARIANT SUBSPACES
$\tilde{Z}_n = \{(-\gamma,\gamma)\}$	$(a,0)$	$V_0 = \{(w,0)\}$ $V_1 = \{(0,w)\}$; $(-\gamma,\gamma)$ acts by $e^{-2i\gamma}$
$Z_2(\kappa)$ or $Z_2(\kappa)\oplus Z_2^c$	(a,a)	$V_0 = \{(w,w)\}$ $V_1 = \{(w,-w)\}$; κ acts as $-Id$
$Z_2(\kappa,\pi)$ or $Z_2(\kappa,\pi)\oplus Z_2^c$	$(a,-a)$	$V_0 = \{(w,-w)\}$ $V_1 = \{(w,w)\}$; κ acts as $-Id$

Table 3.5 Decomposition of $\mathbb{R}^4$ into invariant subspaces for Σ, when $n \equiv 0 \pmod 4$.

ISOTROPY	ORBIT REPRESENTATIVE	INVARIANT SUBSPACES
$\tilde{Z}_n = \{(-\gamma,\gamma)\}$	$(a,0)$	$V_0 = \{(w,0)\}$ $V_1 = \{(0,w)\}$; $(-\gamma,\gamma)$ acts by $e^{-2i\gamma}$
$Z_2(\kappa)\oplus Z_2^c$	(a,a)	$V_0 = \{(w,w)\}$ $V_1 = \{(w,-w)\}$; κ acts as $-Id$
$Z_2(\kappa\zeta)\oplus Z_2^c$	$(a,e^{2\pi i/n}a)$	$V_0 = \{(w,e^{2\pi i/n}w)\}$ $V_1 = \{(w, e^{-2\pi i/n}w)\}$; κ acts as $-Id$

Of course, the zero eigenvalue has an eigenvector in V_0; hence the other eigenvalue of $dg|V_0$ is given by trace $dg|V_0$. The remaining two eigenvalues of dg are those of $dg|V_1$.

To compute these it is convenient to use the complex coordinates (z_1,z_2). Recall that an $\mathbb{R}$-linear mapping on $\mathbb{C} \equiv \mathbb{R}^2$ has the form

$$w \mapsto \alpha w+\beta\bar{w}$$

where $\alpha,\beta \in \mathbb{C}$. A simple calculation shows that

$$\text{trace} = 2\,\mathrm{Re}(\alpha), \qquad \det = |\alpha|^2 - |\beta|^2. \tag{3.2}$$

We are now in a position to compute the eigenvalues of dg for each maximal isotropy subgroup Σ. For this purpose we write g in coordinates:

$$\text{(a)} \quad Z_1 = Az_1+Bz_1^2\bar{z}_1+C\bar{z}_1^{m-1}z_2^{m}+Dz_1^{m+1}\bar{z}_2^{m}, \tag{3.3}$$

$$\text{(b)} \quad Z_2 = Az_2+Bz_2^2\bar{z}_2+Cz_1^{m}\bar{z}_2^{m-1}+D\bar{z}_1^{m}z_2^{m+1}.$$

In these coordinates dg takes the form

$$dg \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} Z_{1,z_1}w_1+Z_{1,\bar{z}_1}\bar{w}_1+Z_{1,z_2}w_2+Z_{1,\bar{z}_2}\bar{w}_2 \\ Z_{2,z_1}w_1+Z_{2,\bar{z}_1}\bar{w}_1+Z_{2,z_2}w_2+Z_{2,\bar{z}_2}\bar{w}_2 \end{bmatrix}. \tag{3.4}$$

Tables 3.4, 3.5 show that for all n the computations will be very similar for the isotropy subgroups $\tilde{Z}_n$, and for $Z_2(\kappa)[\oplus Z_2^c]$; but the third case, namely $Z_2(\kappa,\pi)[\oplus Z_2^c]$ and $Z_2(\kappa\xi) \oplus Z_2^c$, will differ according to the value of n (mod 4). Further, when n = 4 additional low-order terms occur, so the results will be different in that case.

Case $\tilde{Z}_n$: The first eigenvalue is the trace of $dg|V_0$, which is the mapping

$$w \mapsto (dg)(w,0) = Z_{1,z_1}w + Z_{1,\bar{z}_1}\bar{w}.$$

From (3.3) the trace is 2 Re Z_{1,z_1}. A computation yields

$$Z_{1,z_1}(a,0) = A + A_N a^2 + 2Ba^2 + B_N a^4.$$

Since $A+Ba^2 = 0$ along $\tilde{Z}_n$ solutions we obtain the first entry in Tables 3.1, 3.2, 3.3.

To obtain the remaining eigenvalues in this case, note that when $n \neq 4$, $dg|V_1$ must be a (scalar multiple of a) rotation matrix since it commutes with $e^{-2i\gamma}$. Thus the eigenvalues of $dg|V_1$ are either complex conjugates, or real and equal. In either case the required sign is equal to that of the trace. Now $dg|V_1$ is

$$w \mapsto (dg)(0,w) = Z_{2,z_2}w + Z_{2,\bar{z}_2}\bar{w}.$$

Using (3.3) we see that the trace of this map is 2 Re Z_{2,z_2}. As above we compute $Z_{2,z_2}(a,0) = A = -Ba^2$.

However, if n = 4 then $e^{-2i\gamma} = -1$, so we cannot assume that $dg|V_1$ is a rotation matrix. Thus we must compute both the trace and determinant of $dg|V_1$. We find:

$$\text{trace} = 2\text{Re}(-B)a^2,$$
$$\det = (|B|^2 - |C|^2)a^4.$$

Case $Z_2(\kappa)$ or $Z_2(\kappa)\oplus Z_2^c$: The first eigenvalue in this case is the trace of the mapping $dg|V_0$, which is

$$w \mapsto (dg)(w,w).$$

In coordinates this is

$$w \mapsto (Z_{1,z_1}+Z_{1,z_2})w + (Z_{1,\bar{z}_1}+Z_{1,\bar{z}_2})\bar{w}.$$

Its trace is $2\,\mathrm{Re}\,(Z_{1,z_1}+Z_{1,z_2})$. To evaluate this we compute

$$\begin{aligned}
&\text{(a)} \quad Z_{1,z_1}(a,a) = A + A_{z_1}a+2Ba^2+B_{z_1}a^3+C_{z_1}a^{2m-1}+D_{z_1}a^{2m+1}+(m+1)Da^{2m} \\
&\qquad = A_{z_1}a+Ba^2+B_{z_1}a^3-Ca^{2m-2}+C_{z_1}a^{2m-1}+mDa^{2m}+D_{z_1}a^{2m+1}. \qquad (3.5)\\
&\text{(b)} \quad Z_{1,z_2}(a,a) = A_{z_2}a+B_{z_2}a^3+mCa^{2m-2}+C_{z_2}a^{2m-1}+D_{z_2}a^{2m+1}.
\end{aligned}$$

Observe that $N_{z_1}(a,a) = a = N_{z_2}(a,a)$; $P_{z_1}(a,a) = a^3 = P_{z_2}(a,a)$; $S_{z_1}(a,a) = ma^{2m-1} = S_{z_2}(a,a)$; and $T_{z_1}(a,a) = 0 = T_{z_2}(a,a)$. Thus $f_{z_1} = f_{z_2}$ for any invariant function f, evaluated at (a,a). It follows from (3.6) that trace $dg|V_0$ =

$$\begin{aligned}
&2A_{z_1}a+Ba^2+2B_{z_1}a^3+(m-1)Ca^{2m-2}+2C_{z_1}a^{2m-1}+mDa^{2m}+2D_{z_1}a^{2m+1} \\
&= \begin{cases} (2A_N+B)a^2 + O(a^3) & \text{if } n \neq 4, \\ (2A_N+B+C)a^2 + O(a^3) & \text{if } n = 4. \end{cases}
\end{aligned}$$

This gives the corresponding entry in Tables 3.1, 3.2, 3.3.

To compute the remaining two eigenvalues in this case, we must find det and trace of $dg|V_1$. In coordinates, this mapping is

$$w \mapsto (dg)(w,-w) = (Z_{1,z_1}-Z_{1,z_2})w + (Z_{1,\bar{z}_1}-Z_{1,\bar{z}_2})\bar{w}.$$

Its trace is $\mathrm{Re}(Z_{1,z_1}-Z_{1,z_2})$ which we can compute directly from (3.5) obtaining

$$\begin{aligned}
&Z_{1,z_1}(a,a)-Z_{1,z_2}(a,a) = Ba^2-(m+1)Ca^{2m-2}+mDa^{2m} \\
&= \begin{cases} Ba^2 + O(a^3) & \text{if } n \neq 4, \\ (B-3C)a^2+O(a^3) & \text{if } n = 4. \end{cases} \qquad (3.6)
\end{aligned}$$

To evaluate det $dg|V_1$ we must first compute

$$Z_{1,\bar{z}_1}(a,a)-Z_{1,\bar{z}_2}(a,a) = Ba^2+(m-1)Ca^{2m-2}-mDa^{2m}. \qquad (3.7)$$

From (3.3,3.6,3.7) we have

$$\begin{aligned}
&\det dg|V_1 \\
&= |Ba^2-(m+1)Ca^{2m-2}+mDa^{2m}|^2-|Ba^2+(m-1)Ca^{2m-2}-mDa^{2m}|^2 \\
&= \begin{cases} -4m\,\mathrm{Re}(B\bar{C})a^{2m} + O(a^{2m+1}) & \text{if } n \neq 4, \\ 8(|C|^2 - \mathrm{Re}(B\bar{C}))a^4 + O(a^5) & \text{if } n = 4. \end{cases}
\end{aligned}$$

Case $Z_2(\kappa,\pi)$ or $Z_2(\kappa,\pi)\oplus Z_2^c$: The calculations in this case, which holds when n is odd or $n \equiv 2 \pmod 4$ are almost identical with those in the previous case, hence we shall be brief.

$$dg|V_0 = (Z_{1,z_1}-Z_{1,z_2})w + (Z_{1,\bar z_1}-Z_{1,\bar z_2})\bar w,$$
$$dg|V_1 = (Z_{1,z_1}+Z_{1,z_2})w + (Z_{1,\bar z_1}+Z_{1,\bar z_2})\bar w.$$

The first eigenvalue is trace $dg|V_0 = 2\ \mathrm{Re}(Z_{1,z_1}-Z_{1,z_2})$ evaluated at $(a,-a)$. Calculate

$$Z_{1,z_1}(a,-a) = A+A_{z_1}a+2Ba^2+B_{z_1}a^3-[C_{z_1}a^{2m-1}+D_{z_1}a^{2m+1}+(m+1)Da^{2m}]$$
$$= A_{z_1}a+Ba^2+B_{z_1}a^3+Ca^{2m-2}-[C_{z_1}a^{2m-1}+D_{z_1}a^{2m+1}+mDa^{2m}],$$
$$Z_{1,z_2}(a,-a) = A_{z_1}a+B_{z_2}a^3+mCa^{2m-2}-C_{z_2}a^{2m-1}-D_{z_2}a^{2m+1}.$$

Observe that at $(a,-a)$ we have $N_{z_1} = a = -N_{z_2}$; $P_{z_1} = a^3 = -P_{z_2}$; $S_{z_1} = (-1)^m a^{2m-1} = -S_{z_2}$; and $T_{z_1} = 0 = T_{z_2}$. Therefore $f_{z_1}(a,-a) = -f_{z_2}(a,-a)$ for any invariant function f. It follows that

$$\text{trace } dg|V_0 = 2(A_N+B)a^2 + O(a^3).$$

Also

$$Z_{1,z_1}(a,-a) + Z_{1,z_2}(a,-a) = Ba^2+(m+1)Ca^{2m-2}-mDa^{2m}.$$

Hence

$$\text{trace } dg|V_1 = 2\ \mathrm{Re}(B)a^2 + O(a^3).$$

To compute det $dg|V_1$ we evaluate

$$Z_{1,\bar z_1}(a,-a)+Z_{1,\bar z_2}(a,-a) = Ba^2-(m-1)Ca^{2m-2}+mDa^{2m}.$$

Thus det $dg|V_1$ =

$$|Ba^2+(m+1)Ca^{2m-2}-mDa^{2m}|^2-|Ba^2-(m-1)Ca^{2m-2}-mDa^{2m}|^2$$
$$= 8m\ \mathrm{Re}(B\bar C)a^{2m} + O(a^{2m+1}).$$

Case $\mathbf{Z}_2(\kappa\xi)\oplus\mathbf{Z}_2^c$. This is the final case, and applies for $n \equiv 0 \pmod 4$. Define $\omega = e^{2\pi i/n}$, so that $\omega^m = -1$. Then $V_0 = \{(z,\omega z)\}$ and $V_1 = \{(z,-\omega z)\}$, where the action of $\kappa\xi$ on V_1 is by $-Id$. For this calculation

$$dg|V_0 = (Z_{1,z_1}+\omega Z_{1,z_2})w + (Z_{1,\bar z_1}+\omega^{-1}Z_{1,\bar z_2})\bar w,$$
$$dg|V_1 = (Z_{1,z_1}+\omega^{-1}Z_{1,z_2})w + (Z_{1,\bar z_1}+\omega Z_{1,\bar z_2})\bar w.$$

On the orbit $(a,\omega a)$ we have $N_{z_1}(a,\omega a) = a = \omega N_{z_2}(a,\omega a)$; $P_{z_1}(a,\omega a) = a^3 = \omega P_{z_2}(a,\omega a)$; $S_{z_1}(a,\omega a) = -ma^{2m-1} = \omega S_{z_2}(a,\omega a)$; and $T_{z_1}(a,\omega a) = 0 = \omega T_{z_2}(a,\omega a)$. Thus $f_{z_1} = \omega f_{z_2}$ for any invariant function f, evaluated at $(a,\omega a)$.

When $n > 4$ the calculations are much the same as in the previous case, except for higher order terms. We omit the details.

When $n = 4$ we have $\omega = i$. On V_0 we need only the trace of dg, which is

$$2\ \mathrm{Re}(2A_N+B-C)a^2 + O(a^3).$$

On V_1 we need both trace and determinant. The trace is

$$2\mathrm{Re}(Z_{1,z_1}-iZ_{1,z_2}) = (B+3C)a^2 + O(a^3).$$

The determinant is

$$|Z_{1,z_1}-iZ_{1,z_2}|^2 - |Z_{1,\bar z_1}+iZ_{1,\bar z_2}|^2$$
$$= (|B+3C|^2-|B-C|^2)a^4 + O(a^5)$$
$$= 8(|C|^2 + 2\mathrm{Re}(B\bar C))a^4 + O(a^5).$$

This completes the calculations for Tables 3.1, 3.2, 3.3.

4. The Generic Situation

In this section we use the information contained in Tables 3.1a,b to derive bifurcation diagrams describing the generic $\mathbf{D}_n$-equivariant Hopf bifurcations. When $n \neq 4$ we assume the nondegeneracy conditions

(a) $\mathrm{Re}(A_N+B) \neq 0$,
(b) $\mathrm{Re}(B) \neq 0$,
(c) $\mathrm{Re}(2A_N+B) \neq 0$, (4.1)
(d) $\mathrm{Re}(B\bar C) \neq 0$,
(e) $\mathrm{Re}(A_\lambda) \neq 0$,

where each term is evaluated at the origin. When $n = 4$ we assume the nondegeneracy conditions

(a) $\mathrm{Re}(A_N+B) \neq 0$,
(b) $\mathrm{Re}(B) \neq 0$,
(c) $\mathrm{Re}(2A_N+B+C) \neq 0$,
(d) $\mathrm{Re}(B-3C) \neq 0$,
(e) $\mathrm{Re}(B+3C) \neq 0$, (4.2)
(f) $|B|^2-|C|^2 \neq 0$,
(g) $|C|^2-\mathrm{Re}(B\bar C) \neq 0$,
(h) $|C|^2+\mathrm{Re}(B\bar C) \neq 0$,
(i) $\mathrm{Re}(A_\lambda) \neq 0$.

The main result is:

Theorem 4.1 Assuming the nondegeneracy conditions (4.1) or (4.2), there exists precisely one branch of (an orbit of) small amplitude, near-2π-periodic solutions, for each of the isotropy subgroups $\tilde{\mathbf{Z}}_n$; $\mathbf{Z}_2(\kappa)$ [n odd] or $\mathbf{Z}_2(\kappa)\oplus\mathbf{Z}_2^c$ [n even]; and $\mathbf{Z}_2(\kappa,\pi)$ [n odd], $\mathbf{Z}_2(\kappa,\pi)\oplus\mathbf{Z}_2^c$ [$n \equiv 2 \pmod 4$], $\mathbf{Z}_2(\kappa\zeta)\oplus\mathbf{Z}_2^c$ [$n \equiv 0 \pmod 4$]. Assume that the trivial branch is stable subcritically and loses stability as λ passes through 0. If $n \geq 3$, $n \neq 4$, then:
(a) The $\tilde{\mathbf{Z}}_n$ branch is super- or subcritical according as $\mathrm{Re}(A_N(0)+B(0))$ is

positive or negative. It is stable if $\mathrm{Re}(A_N(0)+B(0)) > 0$, $\mathrm{Re}\, B(0) < 0$.

(b) The $Z_2(\kappa)[\oplus Z_2^c]$ branch is super- or subcritical according as $\mathrm{Re}(2A_N(0)+B(0))$ is positive or negative. It is stable if $\mathrm{Re}(2A_N(0)+B(0)) > 0$, $\mathrm{Re}\, B(0) > 0$, and $\mathrm{Re}\,(B(0)\bar{C}(0)) < 0$.

(c) The $Z_2(\kappa,\pi)[\oplus Z_2^c]$ or $Z_2(\kappa\xi)\oplus Z_2^c$ branch is super- or subcritical according as $\mathrm{Re}(2A_N(0)+B(0))$ is positive or negative. It is stable if $\mathrm{Re}(2A_N(0)+B(0)) > 0$, $\mathrm{Re}\, B(0) > 0$, and $\mathrm{Re}(B(0)\bar{C}(0)) > 0$.

If $n = 4$ then:

(d) The $\tilde{Z}_4$ branch is super- or subcritical according as $\mathrm{Re}(A_N(0)+B(0))$ is postive or negative. It is stable if $\mathrm{Re}(A_N(0)+B(0)) > 0$, $\mathrm{Re}\, B(0) < 0$.

(e) The $Z_2(\kappa)\oplus Z_2^c$ branch is super- or subcritical according as $\mathrm{Re}(2A_N(0)+B(0)+C(0))$ is positive or negative. It is stable if $\mathrm{Re}(2A_N(0)+B(0)+C(0)) > 0$, $\mathrm{Re}(B(0)-3C(0)) > 0$, and $|C|^2-\mathrm{Re}(B(0)\bar{C}(0)) > 0$.

(f) The $Z_2(\kappa\xi)\oplus Z_2^c$ branch is super- or subcritical according as $\mathrm{Re}(2A_N(0)+B(0)-C(0))$ is positive or negative. It is stable if $\mathrm{Re}(2A_N(0)+B(0)-C(0)) > 0$, $\mathrm{Re}(B(0)+3C(0)) > 0$, and $|C|^2+\mathrm{Re}(B(0)\bar{C}(0)) > 0$.

Proof The existence of the stated branches follows from Theorem 5.1 of Golubitsky and Stewart (1985). We consider their direction of criticality and stability, using the results of §3. First suppose $n \geq 3$, $n \neq 4$. To lowest order the three nontrivial branches are given by:

$$\begin{aligned}
&\text{(a)}\quad \lambda = -a^2\,[\mathrm{Re}(A_N(0)+B(0))]/\mathrm{Re}\,A_\lambda(0) + \dots && \tilde{Z}_n\\
&\text{(b)}\quad \lambda = -a^2\,[\mathrm{Re}(2A_N(0)+B(0))]/\mathrm{Re}\,A_\lambda(0) + \dots && Z_2(\kappa)[\oplus Z_2^c] \qquad (4.3)\\
&\text{(c)}\quad \lambda = -a^2\,[\mathrm{Re}(2A_N(0)+B(0))]/\mathrm{Re}\,A_\lambda(0) + \dots && Z_2(\kappa,\pi)[\oplus Z_2^c].
\end{aligned}$$

We assume that our system of ODEs has the form $dz/dt + g(z,\lambda) = 0$, so that eigenvalues of dg with *positive* real part indicate stability. We also assume that the steady state $z = 0$ is stable for $\lambda < 0$ and loses stability when $\lambda > 0$. Hence $\mathrm{Re}\, A_\lambda(0) < 0$. With these assumptions we can draw the bifurcation diagrams in Fig. 4.1. Note that the two Z_2 branches are either both supercritical or both subcritical.

We see that for any periodic solution to be asymptotically stable, all three branches must be supercritical. Then either the $\tilde{Z}_n$ branch is stable (if $\mathrm{Re}\, B(0) < 0$); and precisely one of the Z_2 branches is stable (if $\mathrm{Re}\, B(0) > 0$). Which of these branches is stable depends on the sign of $\mathrm{Re}(B(0)\bar{C}(0))$. Note that $A_N(0)$ and $B(0)$ are cubic coefficients, but $\bar{C}(0)$ is the coefficient of a

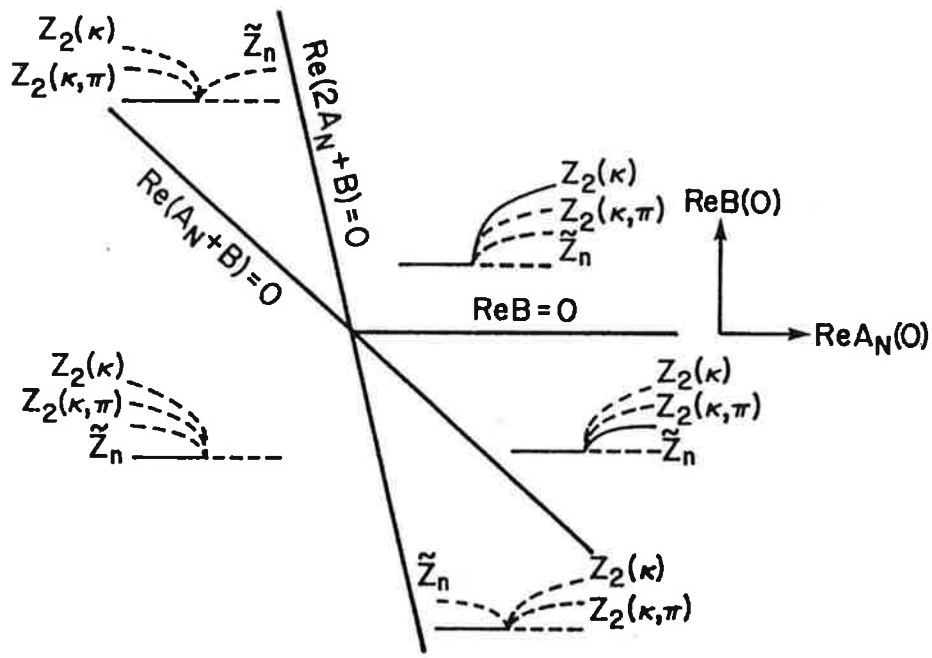


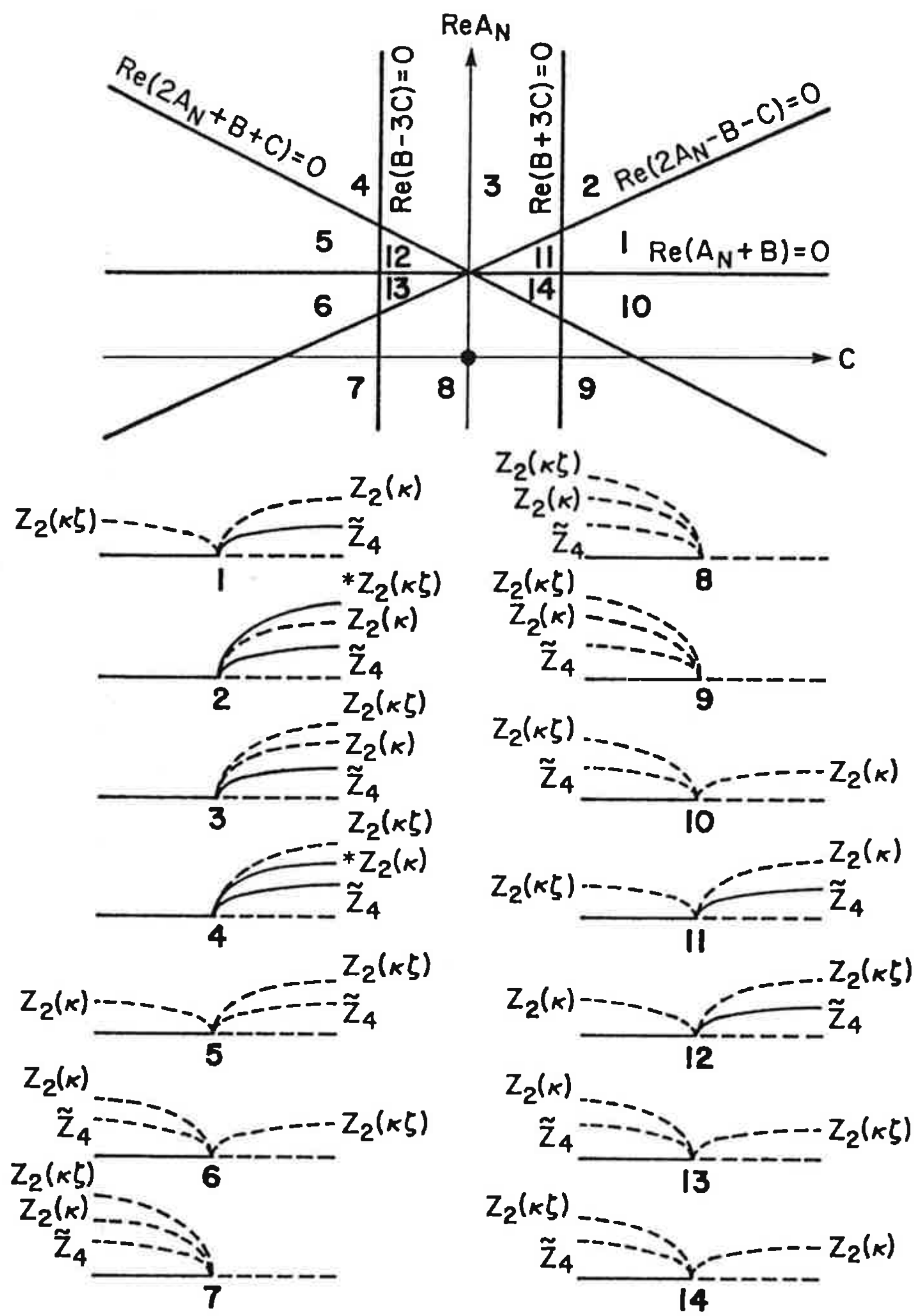
Figure 4.1 Generic branching for $\mathbf{D}_n$ Hopf bifurcation, $n \geq 3$, $n \neq 4$. For any branch to be stable, all three must be supercritical. Exactly one branch, which may correspond to any of the three maximal isotropy subgroups of $\mathbf{D}_n \times \mathbf{S}^1$, is then stable. [Note that the label $\mathbf{Z}_2(k,\pi)$ should be changed to $\mathbf{Z}_2(\kappa\zeta)$ when $n \equiv 0 \pmod 4$, and that the summand $\mathbf{Z}_2{}^c$ is suppressed in the labelling.]

term of degree 2m-1. Moreover, that term is needed to determine which of the $\mathbf{Z}_2$ branches is stable, *independent of the many lower order terms that may exist.*

When n = 4, however, there are *three* linearly independent cubic terms in g. In this case the branching equations take the form

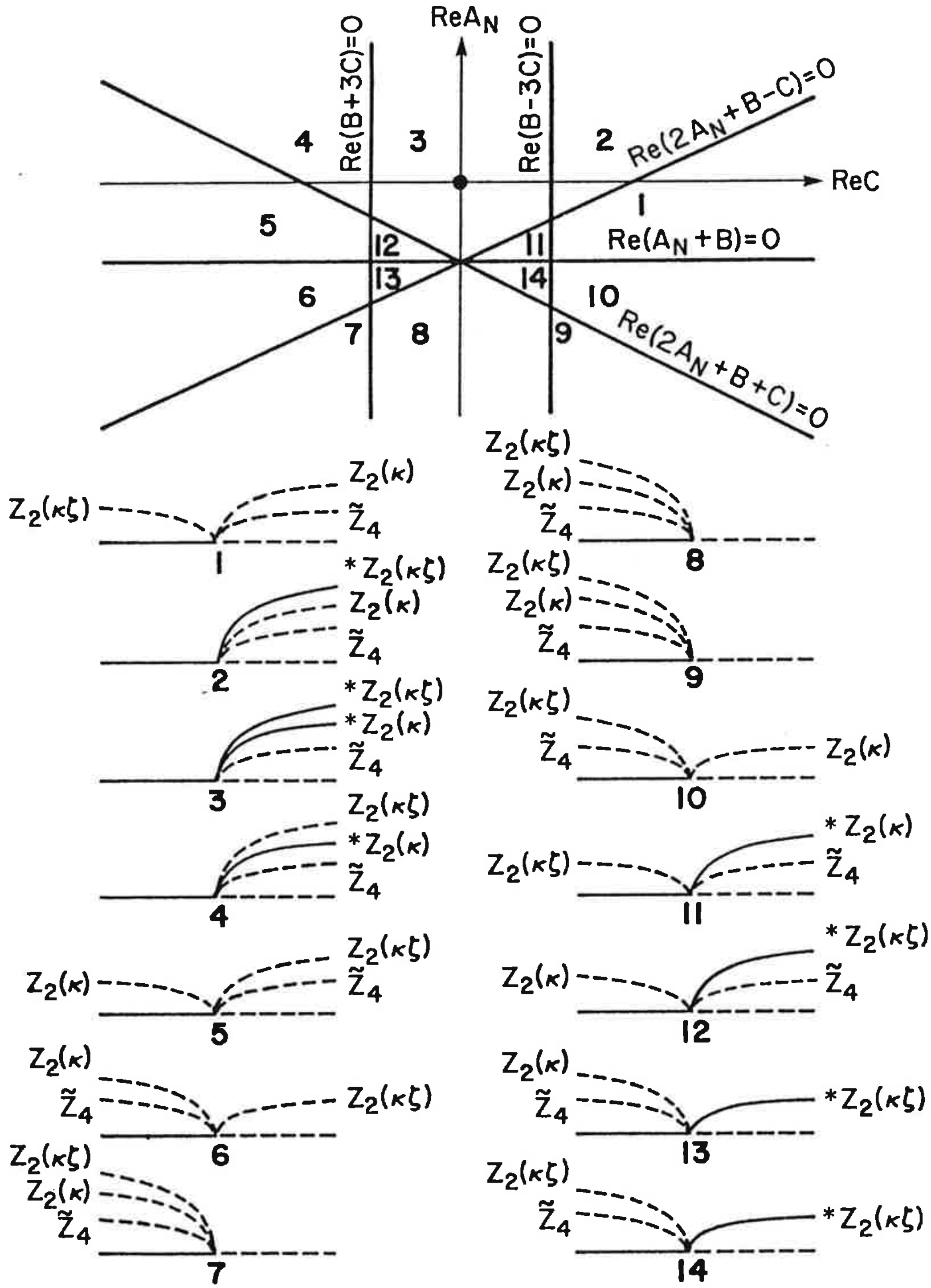
$$\begin{array}{llr} \text{(a)} & \lambda = -a^2\,[\mathrm{Re}(A_N(0)+B(0)]/\mathrm{Re}\,A_\lambda(0) + \ldots & \tilde{\mathbf{Z}}_n \\ \text{(b)} & \lambda = -a^2\,[\mathrm{Re}(2A_N(0)+B(0)+C(0)]/\mathrm{Re}\,A_\lambda(0) + \ldots & \mathbf{Z}_2(\kappa)\oplus\mathbf{Z}_2{}^c \quad (4.4) \\ \text{(c)} & \lambda = -a^2\,[\mathrm{Re}(2A_N(0)+B(0)-C(0)]/\mathrm{Re}\,A_\lambda(0) + \ldots & \mathbf{Z}_2(\kappa\zeta)\oplus\mathbf{Z}_2{}^c. \end{array}$$

The possible configurations of branches that involve *stable* solutions are shown in Fig. 4.2. (Differences in stability assignments of unstable branches



(a) $B < 0$. In regions 1, 3, 5, 11, 12 the $\tilde{Z}_4$ branch is supercritical and stable. In region 2 the $\tilde{Z}_4$ branch is supercritical and stable; the $Z_2(\kappa\zeta)$ branch is supercritical and can also be stable depending on a combination of higher order coefficients. In region 4 a similar statement holds for the $Z_2(\kappa)$ branch.

Figure 4.2 Generic branching for D_4 Hopf bifurcation. An asterisk denotes a branch whose stability depends on high order terms in Table 3.3, not included in the coordinates of the diagram. Solid lines indicate stable branches, dotted lines unstable.



(b) $B > 0$. The $\tilde{Z}_4$ branch is never stable. In regions 2 and 13 the $Z_2(\kappa\zeta)$ branch can be stable for suitable high order coefficients. In regions 4 and 11 the $Z_2(\kappa)$ branch can be stable for suitable high order coefficients. In region 3 either the $Z_2(\kappa)$ branch, or the $Z_2(\kappa\zeta)$ branch, or both, can be stable for suitable high order coefficients; however, they cannot both be unstable.

are not indicated, for simplicity, but may be derived from Table 3.1b.) We note two interesting features where the case $n = 4$ differs from all others (assuming the standard action of $\mathbf{D}_n$):

(a) The two $\mathbf{Z}_2$ branches need not have the same direction of criticality.

(b) More than one branch may be stable for the same values of the coefficients of g. That is, the system may have non-unique (orbits of) stable states. Indeed for any choice of two out of the three isotropy subgroups we are considering, there are parameter values that make both of these two branches stable, but the third unstable. It is *not* possible to have all three branches stable simultaneously.

Remark In Figure 4.2 those branches marked with an asterisk are stable or unstable depending on the high-order coefficients noted in Table 3.3. Both possibilities can occur with suitable choices of coefficients.

In one case, namely case 3 of Figure 4.2b, there are two branches whose stabilities depend upon higher order coefficients. Either one of these, or both, may be stable; but it is easy to see that they cannot both be unstable, since this would require $|C|^2$ to be negative.

Relation with O(2)-symmetric systems

Dihedral group symmetry $\mathbf{D}_n$ often arises when a continuous system, having circular $\mathbf{O}(2)$ symmetry, is approximated by a discrete one. It may seem curious that there are *three* classes of isotropy subgroups with 2-dimensional fixed-point spaces for $\mathbf{D}_n$, however large n is, but only *two* classes for the "limit" $\mathbf{O}(2)$. An analysis of this sheds some light on the approximation of a continuous system by a discrete one.

In algebraic terms what happens is that the solutions with the two $\mathbf{Z}_2$ isotropy subgroups "merge" as $n \to \infty$. Solutions of these types differ by a vanishingly small amount for large n. For example, consider for definiteness a tower

$$\mathbf{D}_4 \subset \mathbf{D}_8 \subset \mathbf{D}_{16} \subset \dots .$$

The $\mathbf{Z}_2$ solutions are those with isotropy subgroups $\mathbf{Z}_2(\kappa)\oplus\mathbf{Z}_2^c$ and $\mathbf{Z}_2(\kappa\zeta)\oplus\mathbf{Z}_2^c$. Now κ and $\kappa\zeta$ are both reflections: the first in the real axis, the second in a line making an angle π/n with the real axis. These lines approach each other for large n.

In addition, the coefficient $\bar{C}$ that determines which of the two branches is stable is attached to a term of increasingly high order as $n \to \infty$. Thus the distinction between the stabilities becomes more delicate.

Geometrically, we can picture the relevant periodic solutions in the following way. For D_n think of a torus, on which n stable periodic trajectories are separated by n unstable ones, as in Figure 4.3. These correspond to the pair of Z_2 branches. As $n \to \infty$ we get increasingly many closed trajectories, and the degree of instability weakens. We approach a torus foliated by a continuous family of periodic orbits, *neutrally* stable to displacements around the torus.

Figure 4.3 A torus on which n stable periodic trajectories are separated by n unstable ones. Each family of n trajectories corresponds to an orbit under D_n of one of the two Z_2 solutions.

5. Oscillators Coupled in a Ring

As an example of a problem having D_n symmetry we consider a system of n identical oscillators coupled in a ring, as in Fig. 5.1.

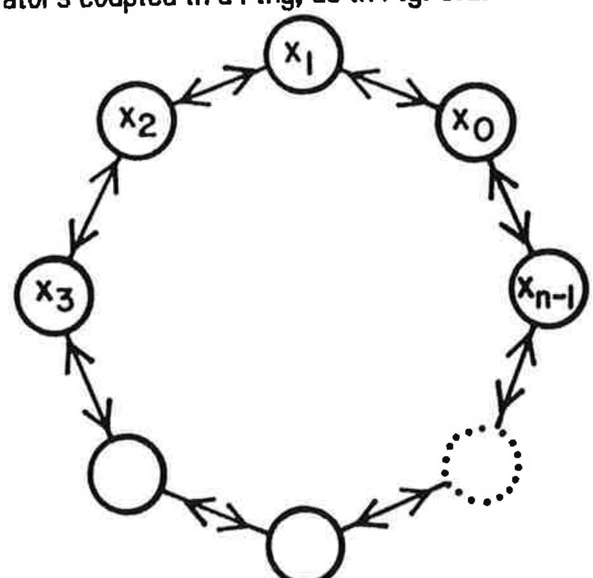


Figure 5.1 A ring of coupled identical nonlinear oscillators.

Such problems have been studied by a number of authors, including Alexander and Auchmuty (1986), and Van Gils and Valkering (1985). Applications include in particular chemical oscillators and biological oscillators (cells coupled via membrane transport of ions). The situation is considered in the seminal paper by Turing (1952) on morphogenesis. Tsotsis (1981) discusses the problem in the context of chemical reactors. Surveys in the literature include Winfree (1980), De Kleine, Kennedy and MacDonald (1982), and Kopell (1983). The case of two oscillators has been discussed by Othmer and Scriven (1971), Smale (1974), and Howard (1979). We shall describe the generic behaviour, paying particular attention to the cases $n \le 5$. The results should prove applicable in a number of physical contexts. In this section we set up the equations and discuss their symmetries. In §6 we analyse the three-oscillator case, finding conditions under which purely imaginary eigenvalues can occur, deriving the corresponding representations of $\mathbf{D}_3$, and listing the patterns of oscillation that generically can occur. In §7 we generalize the methods to the general case of n oscillators. In §8 we discuss as examples the cases $n = 2,4,5$ and compare our results with those of Alexander and Auchmuty (1986).

For simplicity we assume that the coupling is "nearest neighbour" and that it is symmetrical in the sense that the interaction between any neighbouring pair of oscillators takes the same form. [The "nearest neighbour" assumption may be relaxed, with minor changes in our conclusions, provided the symmetry is retained; and the method may be applied to systems having other kinds of symmetry - for example four oscillators with any two interacting identically, a system with tetrahedral symmetry.]

For purposes of illustration we shall use a fairly concrete system of equations with linear interaction. Oscillator p will be described by two state variables (x_p,y_p) and p will be taken mod n. Then we have a system of n equations

$$^{d}/_{dt}(x_p,y_p) = F(x_p,y_p,\lambda)+K(\lambda).(2x_p-x_{p-1}-x_{p+1},2y_p-y_{p-1}-y_{p+1}). \qquad (5.1)$$

Here $F:\mathbb{R}^2 \to \mathbb{R}^2$ is an arbitrary smooth function and λ is a bifurcation parameter. For each λ, the matrix K is constant:

$$K = \begin{bmatrix} k_{11} & k_{12} \\ k_{21} & k_{22} \end{bmatrix}.$$

The k_{ij} (which depend only on λ, and might be constant) represent "coupling strengths". The form of the equations is chosen so that the coupling term in K vanishes if all oscillators behave identically. (The bifurcation parameter λ may be present in F, or K, or both, depending on interpretation.)

For theoretical discussion and proofs we abstract from this system its symmetries, and consider the more general system

$$dx_p/dt = g(x_{p-1}, x_p, x_{p+1}; \lambda). \tag{5.2}$$

Here p is taken mod n and runs from 0 to n-1, and $x_p \in \mathbf{R}^k$ for some k, so the system (5.2) is on $\mathbf{R}^{nk}$. (We shall see below that $k \geq 2$ is required for Hopf bifurcation, and henceforth assume this.) We require $g: \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{R}^k \times \mathbf{R} \to \mathbf{R}^k$ to be a smooth function with the symmetry property

$$g(u,v,w;\lambda) = g(w,v,u;\lambda). \tag{5.3}$$

It is easy to check that (5.1) has this symmetry when written in the form (5.2). Note that in (5.3) we do *not* assume linearity of the coupling.

The system (5.2) has $\mathbf{D}_n$ symmetry. The action of $\mathbf{Z}_n \subset \mathbf{D}_n$ is to permute the x_p cyclically; the action of the flip κ sends x_p to x_{-p}. This symmetry is reflected in the types of generic Hopf bifurcation that may occur, as we shall now see.

6. Three Oscillators

In this section we find conditions under which equation (5.2) can possess purely imaginary eigenvalues, and use group theory to analyse the resulting generic patterns of oscillation. To avoid complicating the argument with group-theoretic generalities we first consider the case of three oscillators, which illustrates the main principles. The general case is similar, and will be easier to describe once the method has been illustrated for n = 3.

Equation (5.2) becomes

$$\begin{aligned} dx_0/dt &= g(x_2,x_0,x_1;\lambda) \\ dx_1/dt &= g(x_0,x_1,x_2;\lambda) \\ dx_2/dt &= g(x_1,x_2,x_0;\lambda) \end{aligned} \tag{6.1}$$

where $x_p \in \mathbf{R}^k$ and λ is a bifurcation parameter. Suppress the λ-dependence in the notation and write (6.1) as

$$dx/dt = G(x)$$

where $x = (x_0,x_1,x_2) \in \mathbf{R}^{3k}$. Then we summarize the results of this section as:

Theorem 6.1 Suppose that $L = (dG)_{(0,\lambda)}$ has a pair of purely imaginary eigenvalues $\pm i$ (without loss of generality at $\lambda = 0$) which cross the imaginary axis with nonzero speed as λ passes through 0. Let A, B be the matrices of partial derivatives $d_{x_1}G$ and $d_{x_0}G$, restricted to the spaces of

x_1, x_0-variables respectively. Assume that the corresponding Liapunov-Schmidt reduced system is nondegenerate in the sense of (4.1). Then either:

(a) The purely imaginary eigenvalues are those of A+2B, and are simple. There is a Hopf bifurcation at $\lambda = 0$ and all three oscillators have the same waveform and the same phase.

(b) The purely imaginary eigenvalues are those of A-B and have multiplicity 2. There are three branches of symmetry-breaking oscillations, with the following patterns:

Isotropy subgroup $\tilde{\mathbf{Z}}_3$: The oscillators have the same waveforms but with phase shifts of $2\pi/3$ from one to the next.

Isotropy subgroup $\mathbf{Z}_2(\kappa)$: Two oscillators have the same waveform and same phase; the third oscillates with the same period but a different waveform.

Isotropy subgroup $\mathbf{Z}_2(\kappa,\pi)$: Two oscillators have the same waveform but are π out of phase; the third oscillates with half the period.

Remarks 6.2

(a) We can rescale time so that any given pair of purely imaginary eigenvalues is $\pm i$, and the corresponding periodic solutions will then have periods near 2π.

(b) The assumption of nondegeneracy in this theorem is *generic* for the full oscillator equations (6.1). We sketch a proof. First observe that the form of G in (6.1) is the most general $\mathbf{D}_3$-equivariant mapping on $\mathbf{R}^{3k}$. Let W be the eigenspace for the appropriate purely imaginary eigenvalues of G. Then the Liapunov-Schmidt reduced function g is defined on W, and determines the types of Hopf bifurcation that may occur. The direction of criticality of any branch predicted by our theory, and its stability, are determined by a finite number of Taylor coefficients of g. If τ is the perturbed period parameter, these coefficients do not involve τ-derivatives. Let k be the highest degree involved. Then the Liapunov-Schmidt reduction defines a polynomial mapping ρ from the coefficients of degree $\leq k$ in the original mapping G to those of g. We claim that ρ is a surjection to the space of nonzero coefficients (of terms not involving τ) of the $\mathbf{D}_3 \times \mathbf{S}^1$-equivariant mappings on W. To see this, note that any $\mathbf{D}_3 \times \mathbf{S}^1$-equivariant mapping h on W extends to a $\mathbf{D}_3 \times \mathbf{S}^1$-equivariant mapping H on $\mathbf{R}^{3k}$, for an appropriate $\mathbf{S}^1$-action. Since such an H is in normal form, it follows from Golubitsky and Stewart (1985) Proposition 4.3 that the Liapunov-Schmidt reduced mapping for H agrees with h for coefficients not involving τ-derivatives. Hence ρ is surjective for $\mathbf{D}_3 \times \mathbf{S}^1$-equivariant mappings, hence also for D_3-equivariant mappings. But G is an *arbitrary* D_3-equivariant mapping. Therefore the nondegeneracy conditions for Hopf bifurcations in the reduced problem are satisfied for an open subset of coefficients of degree $\leq k$ in the original problem; and any combination of signs of the relevant coefficients can occur in the reduced problem.

Proof The first step is to find conditions under which the linearization $L = (dG)_{(0,\lambda)}$ can possess purely imaginary eigenvalues. Now $g(x_0,x_1,x_2) = g(x_2,x_1,x_0)$ and we have

$$d_{x_1}G = A$$
$$d_{x_0}G = B$$
$$d_{x_2}G = B$$

for certain k×k matrices A,B. Hence in k×k block form,

$$L = \begin{bmatrix} A & B & B \\ B & A & B \\ B & B & A \end{bmatrix}.$$

There is a k-dimensional space V_0 of vectors $[v,v,v]$ ($v \in \mathbb{R}^k$) that is invariant under L. Indeed

$$L\,[v,v,v] = L\begin{bmatrix} v \\ v \\ v \end{bmatrix} = \begin{bmatrix} A & B & B \\ B & A & B \\ B & B & A \end{bmatrix}\begin{bmatrix} v \\ v \\ v \end{bmatrix} = \begin{bmatrix} (A+2B)v \\ (A+2B)v \\ (A+2B)v \end{bmatrix}. \qquad (6.2)$$

Hence the eigenvalues of $L|V_0$ are those of A+2B.

Let $\omega = e^{2\pi i/3}$ be a cube root of unity in $\mathbb{C}$. Then, complexifying from $\mathbb{R}^{nk}$ to $\mathbb{C}^{nk}$, we can find two other subspaces invariant under L:

$$V_1 = \{[v,\omega v,\omega^2 v] | v \in \mathbb{R}^k\},$$
$$V_2 = \{[v,\omega^2 v,\omega v] | v \in \mathbb{R}^k\}.$$

A similar calculation to (6.2) shows that the eigenvalues of $L|V_1$ are those of $A+\omega B+\omega^2 B$, which is A−B since $1+\omega+\omega^2 = 0$. Similarly $L|V_2$ has the eigenvalues of $A+\omega^2 B+\omega B = A-B$. We conclude that the eigenvalues of L are:

The eigenvalues of A+2B, (6.3)

The eigenvalues of A−B, repeated twice.

Hence L has purely imaginary eigenvalues, giving the possibility of Hopf bifurcation in (5.2), if and only if either A+2B or A−B has purely imaginary eigenvalues. For this to occur, we must have $k \geq 2$. We shall see that the two cases correspond to different representations of D_3, and hence lead to different patterns of oscillation.

In our standard notation, we have $\mathbf{D}_3 \supset \mathbf{Z}_3 = \{1,\zeta,\zeta^2\}$, and the flip is $\kappa \in \mathbf{D}_3$. The standard action of ζ on the plane $\mathbf{C}$ is rotation through $2\pi/3$, that is, multiplication by $\omega = e^{2\pi i/3}$. The actions on $\mathbf{R}^{3k}$ are as follows:

$$\zeta(x_0,x_1,x_2) = (x_1,x_2,x_0) \quad (6.4)$$
$$\kappa(x_0,x_1,x_2) = (x_0,x_2,x_1).$$

We seek to decompose $\mathbf{R}^{3k}$ into irreducible subspaces for the action of $\mathbf{D}_3$, noting that every eigenspace for L is $\mathbf{D}_3$-invariant. Clearly both ζ and κ act trivially on vectors [v,v,v] $\in V_0$, so $\mathbf{D}_3$ acts on V_0 by k copies of the trivial action on $\mathbf{R}$. We assert that $\mathbf{R}^{3k} = V_0 \oplus W_0$ where W_0 is the sum of k copies of the nontrivial action of $\mathbf{D}_3$ on $\mathbf{R}^2 \equiv \mathbf{C}$. This can easily be seen directly, because $\mathbf{D}_3$ has only two distinct irreducible representations, and the trivial one occurs only on V_0. That is, if [u,v,w] is fixed by $\mathbf{D}_3$ then u = v = w, which is obvious from (6.2). (For $\mathbf{D}_n$ with n > 3 a little more care must be taken at this stage of the analysis since there are several nontrivial representations: see §7.)

To summarize: $\mathbf{R}^{3k}$ breaks up as $\mathbf{R}^k \oplus \mathbf{R}^{2k}$ where $\mathbf{D}_3$ acts on $\mathbf{R}^k$ by k copies of the trivial representation, and on $\mathbf{R}^{2k}$ by k copies of the nontrivial representation. Explicitly, $V_0 = \mathbf{R}^k$ is spanned by all [v,v,v] and $W_0 = \mathbf{R}^{2k}$ by all [v,w,-v-w] (since this is obviously an invariant complement to V_0).

Thus generically there are two cases. The first is when the purely imaginary eigenvalue of L (which of course must be part of a complex conjugate pair) comes from A+2B. It is then (generically) simple, and there is a standard Hopf bifurcation. Since $\mathbf{D}_3$ acts trivially on V_0, all three oscillators behave identically (same waveform, same phase). To put it another way, $\mathbf{D}_3$ lies in the isotropy subgroup of such a solution.

The second case is when the purely imaginary eigenvalue of L comes from A-B. By (6.1) this will have multiplicity at least 2, and generically (in the world of $\mathbf{D}_3$ symmetry) exactly 2. We are then in the situation of Theorem 5.1 of Golubitsky and Stewart (1985), with $\Gamma = \mathbf{D}_3$ in its standard representation. We conclude that there will be (at least) three branches of oscillations, with isotropy subgroups $\tilde{\mathbf{Z}}_3$, $\mathbf{Z}_2(\kappa)$, and $\mathbf{Z}_2(\kappa,\pi)$. We describe the interpretation of each in turn.

The first, $\tilde{\mathbf{Z}}_3$, is the discrete analogue of a rotating wave. The waveform is fixed under cyclic permutation of the oscillators, provided phase is shifted by $2\pi/3$. The oscillations thus have the identical waveform in all three oscillators, but a phase lag of $2\pi/3$ from each oscillator to the next. (The phase lag may also be $-2\pi/3$, which is a physically distinct solution only when the the numbering of the oscillators has been chosen: it is in the same orbit under $D_3 \times S^1$. By "waveform" we mean the trajectory of x_p in phase space $\mathbb{R}^k$.

On the $Z_2(\kappa)$ branch, the waveform is identical when x_1 and x_2 are interchanged. In other words, oscillators 1 and 2 behave identically, while oscillator 0 has a different waveform (not prescribed by the symmetry).

On the $Z_2(\kappa,\pi)$ branch the waveform is identical if x_1 and x_2 are interchanged *and* the phase is shifted by π. In other words, oscillators 1 and 2 have the same waveform but are exactly π out of phase, whereas oscillator 0 is "π out of phase with itself". That is, a phase shift of π produces the same waveform in oscillator 0, which is therefore oscillating with *half* the period of oscillators 1 and 2. The exact shape of the waveform is not prescribed by the symmetry approach, but will be nearly sinusoidal close to the bifurcation point. □

At least with our current techniques, it is a more complicated matter to determine whether the bifurcation is super- or subcritical, and which branch (if any) is stable. Before the results of §4 can be used, G must be Liapunov-Schmidt reduced to the appropriate eigenspace. Certain cubic terms will distinguish stability of $\tilde{Z}_n$ or some Z_2 branch; suitable fifth order terms will determine whether $Z_2(\kappa)$ or $Z_2(\kappa,\pi)$ is stable. We shall not pursue this problem here.

Remark In the notation of (5.1) we have

$$A = (dF)_{(0,\lambda)} + 2K$$

$$B = -K$$

so the relevant eigenvalues are those of

$$(dF)_{(0,\lambda)} \text{ (once)}, (dF)_{(0,\lambda)} + 3K \text{ (twice)}.$$

In the second case, where the bifurcation breaks symmetry to $\tilde{Z}_3$, $Z_2(\kappa)$, or $Z_2(\kappa,\pi)$, it is possible for $(dF)_{(0,\lambda)}$ to have eigenvalues with negative real part. That is, if the coupling is removed ($K = 0$) the individual "oscillators" need not be capable of oscillating on their own. This effect was noticed for two coupled oscillators by Smale (1974). Loosely speaking, the coupling, rather than instabilities of individual components, can be the source of the oscillation.

Numerical Simulations. We have performed some numerical experiments on a system of the form (5.1), taking

$$F(x,y) = \begin{bmatrix} -4 & 1 \\ -1 & -4 \end{bmatrix}\begin{bmatrix} x \\ y \end{bmatrix} + p(x^2+y^2)\begin{bmatrix} x \\ y \end{bmatrix} + q(x^2+y^2)\begin{bmatrix} -y \\ x \end{bmatrix} - 2K\begin{bmatrix} x \\ y \end{bmatrix} \qquad (6.5)$$

where

$$K = -\alpha\begin{bmatrix} -4 & 2 \\ -2 & -4 \end{bmatrix}.$$

When $\alpha = 1.05$, p=5, q=30 we observe $\tilde{Z}_3$ symmetry. The three waveforms (plotted for the x-variable) are shown in Figure 6.1, and the fact they they are identical but phase-shifted by $2\pi/3$ is clear. Plotting trajectories in the three phase planes (x_p,y_p) we observe identical limit cycles, with the three phase points traversing them at three roughly equally spaced positions.

When $\alpha = 1.2$, p = 5, q = -50 we observe $Z_2(\kappa,\pi)$ symmetry. Now two oscillators traverse the same limit cycle π out of phase, while the third remains *steady* at the origin. By varying additional fifth order terms the $Z_2(\kappa)$ solution should be obtainable, although we have not attempted this. (We remark that other kinds of behaviour appear to be observed in the numerical solutions, notably quasiperiodic oscillation. This might be expected as a secondary bifurcation linking distinct primary branches. Figure 6.3 shows an interesting subharmonic oscillation with a great deal of structure.)

The fact that one oscillator is steady in this simulation deserves comment. Suppose that F is odd, so that $F(-x,-y) = -F(x,y)$. Then $D_3 \times Z_2 \subset D_3 \times S^1$ commutes with the vector field, where $Z_2 = \{0,\pi\}$. It follows that the fixed-point suspace for $\Sigma = Z_2(\kappa,\pi) \subset D_3 \times Z_2$ is invariant under the flow. But this subspace is the set of vectors $[0,v,-v]$. In other words, oscillator 0 stays at the origin, hence is steady.

Even if F is not odd, we can put the vector field into Birkhoff normal form up to arbitrarily high order (for a discussion in this context, see Golubitsky and Stewart (1985) Proposition 8.6) so that in particular the normal form commutes with Z_2. It is tempting to conclude that for the $Z_2(\kappa,\pi)$ branch one oscillator is *always* steady. That this is *not* the case is shown by Fig. 6.2, obtained from (6.5) by adding terms

$$r\begin{bmatrix} (x^2+y^2)^2 \\ 0 \end{bmatrix} + s\begin{bmatrix} 0 \\ (x^2+y^2)y \end{bmatrix}$$

to F, again at $\alpha = 1.1$, p = 5, q = -50, r = 10, s = 0. Instead of the third oscillator being steady, it exhibits *a small-amplitude oscillation at double the frequency* of the other two, and these latter are out of phase with each other. This is precisely what is predicted by the symmetry analysis. Thus in this example the dynamic behaviour can definitely be changed by a symmetry-breaking "tail" of arbitrarily high order occuring in the reduction to Birkhoff normal form.

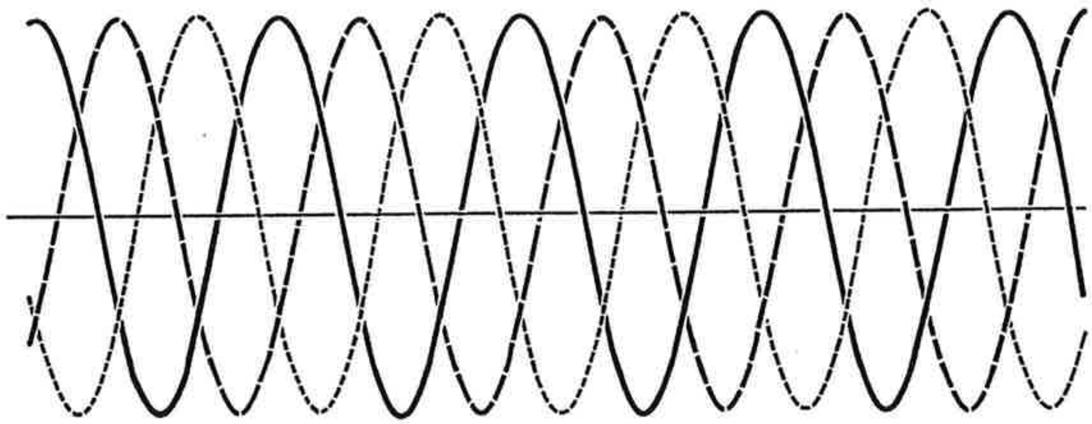


Figure 6.1 Numerical solution showing three identical waveforms, $2\pi/3$ out of phase. Here $\alpha = 1.05$, $p = 5$, $q = 30$, $r = 0$, $s = 0$.

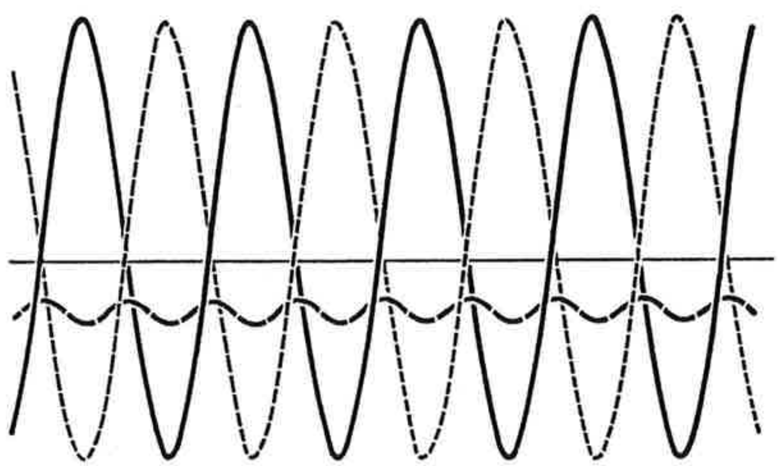


Figure 6.2 Numerical solution showing two identical waveforms, π out of phase, and a third at double the frequency. Here $\alpha = 1.1$, $p = 5$, $q = -50$, $r = 10$, $s = 0$.

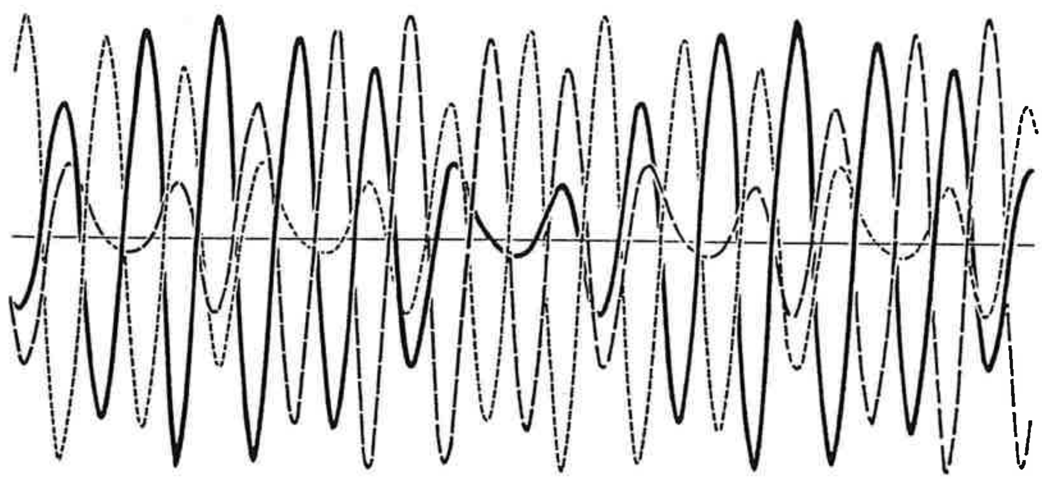


Figure 6.3. A subharmonic oscillation with striking symmetry features observed numerically when $\alpha = 1.1$, $p = 5$, $q = 50$, $r = 10$, $s = 0$.

7. The General Case

We now turn to the general case of n oscillators, described by (5.1). We will prove:

Theorem 7.1 Suppose that $L = (dG)_{(0,\lambda)}$ has a pair of purely imaginary eigenvalues $\pm i$ (without loss of generality at $\lambda = 0$) which cross the imaginary axis with nonzero speed as λ passes through 0. Let A, B be the matrices of partial derivatives $d_{x_1}G$ and $d_{x_0}G$, restricted to the spaces of x_1, x_0-variables respectively. Then generically the purely imaginary eigenvalues of L will be those of $A+\ell_j B$ where $\ell_j = 2 \cos 2\pi j/n$, $0 \leq j \leq n-1$. They are simple if $j = 0$, or if $j = n/2$ [n even]; double otherwise. Assume that the corresponding Liapunov-Schmidt reduced system is nondegenerate in the sense of (4.1,4.2). Then the resulting oscillation patterns may be described using the results of §4, for a particular representation ρ_j of $\mathbf{D}_n$.

Remarks

(a) To avoid group-theoretic generalities we describe the use of ρ_j below and give examples in §8.

(b) The nondegeneracy assumptions are generic for the equations (5.1), for the reasons noted in Remark 6.2b.

Proof Let $\varsigma = e^{2\pi i/n}$ be a primitive n^{th} root of unity in $\mathbb{C}$. (Note that we also use ς to denote a generator of $\mathbf{D}_n$, but since this acts on $\mathbb{C}$ as multiplication by $e^{2\pi i/n}$ no confusion should arise.) Recall that

$$\varsigma^{-1} = \bar{\varsigma} = \varsigma^{n-1}, \qquad \varsigma^{-j} = (\bar{\varsigma}^j) = \varsigma^{n-j}.$$

To find the eigenvalues of L we may complexify. Define the space of vectors

$$V_j = \{[v, \varsigma^j v, \varsigma^{2j} v, \ldots, \varsigma^{(n-1)j} v] \mid v \in \mathbb{R}^k\}.$$

Then

$$\mathbb{C}^{nk} = V_0 \oplus V_1 \oplus \ldots \oplus V_{n-1}.$$

The linearization $L = (dG)_{(0,\lambda)}$ now takes the form

$$n \quad \begin{bmatrix} A & B & & & & & & B \\ B & A & B & & & & & \\ & B & A & B & & & & \\ & & B & A & B & & & \\ & & & & \cdots & & & \\ & & & & & \cdots & & \\ & & & & & B & A & B \\ B & & & & & & B & A \end{bmatrix} \tag{7.1}$$

where A and B are k×k matrices. This banded ("circulant") structure of course comes from the nearest-neighbour coupling. Now

$$L[v, \varsigma^j v, \ldots, \varsigma^{(n-1)j} v] = (A + (\varsigma^j + \varsigma^{(n-1)j})B)[v, \varsigma^j v, \ldots, \varsigma^{(n-1)j} v],$$

whence the eigenvalues of $L|V_j$ are those of $A+(\varsigma^j+\varsigma^{(n-1)j})B$, that is, of $A+(\varsigma^j+\bar{\varsigma}^j)B = A+\ell_j B$ where $\ell_j = 2\cos 2\pi j/n$.

For example, when $n = 3$ we have

$j=0$: $A + 2\cos 0 . B = A+2B$

$j=1$: $A + 2\cos(2\pi/3)B = A-B$

$j=2$: $A + 2\cos(4\pi/3)B = A-B$

as before.

Note that since $\cos(-\theta) = \cos\theta$ these occur in pairs, $\ell_j = \ell_{n-j}$, except for $j = 0$ [n odd] and $j = 0, n/2$ [n even].

Provided $k \geq 2$ there can be purely imaginary eigenvalues of $A+\ell_j B$ for any j. Generically these will be simple on V_j. Because of the way the ℓ_j pair up, purely imaginary eigenvalues of L will generically be simple for $j = 0$ [n odd], and $j = 0, n/2$ [n even]; and of multiplicity 2 for all other $j = 1,2,\ldots[n/2]$ [n odd], $1,2,\ldots,\frac{1}{2}n-1$ [n even]. The simple eigenvalue case corresponds to ordinary Hopf bifurcation when $j = 0$ and to a type of $\mathbf{Z}_2$ Hopf bifurcation when $j = n/2$ (see below); the double eigenvalue case is symmetric Hopf bifurcation of the type studied in Golubitsky and Stewart (1985). □

The relevant action of $\mathbf{D}_n$ depends on the value of j, and we describe the results below. They may be proved by a complexification argument, by direct calculation, or by abstract representation theory. They are sufficiently natural for a proof to be superfluous. Let ρ_j be the representation of $\mathbf{D}_n$ on $\mathbb{C}$ given by

$$z \mapsto \zeta^j z \qquad \text{(rotation due to } \zeta \in \mathbf{Z}_n\text{)}$$
$$z \mapsto \bar{z} \qquad \text{(flip } \kappa\text{)}$$

when $j \neq 0, n/2$. Let ρ_0 be the trivial representation on $\mathbb{R}$. If n is even let $\rho_{n/2}$ be the representation on $\mathbb{R}$ given by

$$z \mapsto -z \qquad \text{(rotation)}$$
$$z \mapsto z \qquad \text{(flip).}$$

Suppose that a Hopf-type bifurcation occurs when eigenvalues of $A+\ell_j B$ cross the imaginary axis with nonzero speed, with generic assumptions on multiplicity (simple for j = 0,n/2; double for all other j). Then the action of $\mathbf{D}_n$ on the corresponding imaginary eigenspace is (isomorphic to) ρ_j. From this we can predict the possible patterns of oscillation. To do this, observe that ρ_j can be obtained by composing the standard representation of a suitable dihedral group $\mathbf{D}$ with a homomorphism

$$\varphi_j : \mathbf{D}_n \to \mathbf{D}_n$$

sending the rotation generator $\zeta \in \mathbf{D}_n$ to ζ^j and sending κ to itself. The group $\mathbf{D}$ is the image of $\mathbf{D}_n$ under φ_j and is a dihedral group $\mathbf{D}_q$ where

$$q = n/h, \quad h = \gcd(n,j).$$

Thus we may apply the general theory for $\mathbf{D}_q$ and re-interpret the results for the original system of n oscillators. Essentially the effect is to bunch them into sets of size h (corresponding to cosets of $\mathbf{Z}_h$ in $\mathbf{Z}_n$), all oscillators in one such set behaving identically; in addition the waveforms on the q sets of oscillators so obtained behave like the waveforms for a system of q oscillators with standard $\mathbf{D}_q$-action. Rather than prove these assertions in general (they are combinatorially somewhat complicated exercises in elementary group theory) we describe in the next section examples, for low n, which exhibit the typical features of the analysis.

8. Examples

We have already studied the case n = 3; here we look at n = 2,4, and 5.

(a) n = 2

There are exceptional features to $\mathbf{D}_2$ - for example its irreducible representations are all 1-dimensional - but the general theory above still applies provided we take care with the group actions. We have $\ell_0 = 2$, $\ell_1 = -2$, so the eigenvalues of L are those of A±2B. The corresponding representations of $\mathbf{D}_2$ on $\mathbb{R}^2 \equiv \mathbb{C}$ are trivial ($\zeta.z = z$) or "standard" ($\zeta.z = -z$, $\kappa.z = \bar{z}$). In the first case the isotropy group is $\mathbf{Z}_2$ (both oscillators have identical waveforms and are in phase). In the second case it is $\mathbf{Z}_2{}^c = \{(0,0),(\pi,\pi)\}$. So a spatial rotation by π (interchange oscillators) is compensated for by a phase shift of π. In other words, they oscillate with identical waveforms but π out of phase. We thus recover a result of Othmer and Scriven (1971).

(b) n = 4

Now

$$\ell_0 = 2\cos 0 = 2 \qquad \ell_1 = 2\cos \pi/2 = 0$$

$$\ell_2 = 2\cos \pi = -2 \qquad \ell_3 = 2\cos 3\pi/2 = 0.$$

There are three cases: purely imaginary eigenvalues of A+2B (simple, yielding the representation ρ_0 of $\mathbf{D}_4$); A (double, ρ_1); and A-2B (simple,ρ_2). The typical oscillation patterns in these cases are shown schematically in Fig. 8.1.

The case ρ_2 deserves further comment as it is a prototype for ρ_ℓ on $\mathbf{D}_n$ when n and ℓ have a common factor. The invariant subspace on which ρ_2 occurs consists of all vectors [v,-v,v,-v] for $v \in \mathbb{R}^k$. On this, ζ acts as -Id and κ acts trivially. So we get k copies of the 1-dimensional representation

$$\zeta.x = -x, \qquad \kappa.x = x$$

on $\mathbb{R}$. There is a unique isotropy subgroup generated by ζ^2, κ, and (ζ,π). This yields the pattern of Fig. 8.1 for ρ_2.

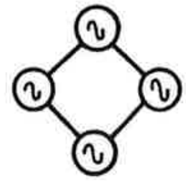
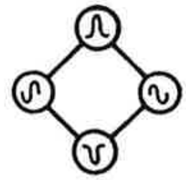
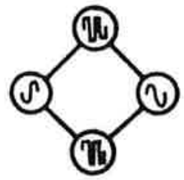
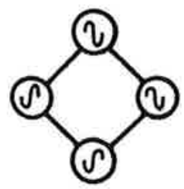
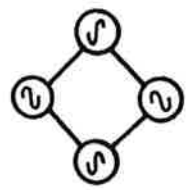
REPRESENTATION	ISOTROPY	PATTERN	COMMENTS
ρ_0	$\mathbf{D}_4$		Identical waveforms in phase
ρ_1	$\tilde{\mathbf{Z}}_4$		Travelling Wave Phase lags $\pm \pi/2$
ρ_1	$\mathbf{Z}_2(\kappa)\oplus\mathbf{Z}_2^c$		Oscillators 13 identical in phase with twice frequency;0,2 identical and π out of phase
ρ_1	$\mathbf{Z}_2(\kappa\zeta)\oplus\mathbf{Z}_2^c$		0,1 identical; 2,3 identical and π out of phase with 0,1
ρ_2	$\langle 2\zeta,\kappa,(\zeta,\pi)\rangle$		0,2 identical; 1,3 identical and π out of phase with 0,2

Figure 8.1 Generic oscillation patterns for a four-oscillator ring. Here $\zeta = \pi/2$.

(c) n = 5

Here

$$\ell_0 = 2 \qquad \ell_1 = \ell_4 = 2\cos 2\pi/5 = \tau - 1$$
$$\ell_2 = \ell_3 = 2\cos 4\pi/5 = -\tau$$

where τ is the golden number $\frac{1}{2}(\sqrt{5}+1)$. Again there are three cases: purely imaginary eigenvalues of $A+2B$ (simple, ρ_0); $A+\tau^{-1}B$ (double, ρ_1); and $A-\tau B$ (double, ρ_2). The corresponding typical oscillation patterns are shown schematically in Figure 8.2.

The case ρ_2 is a prototype for ρ_ℓ on $\mathbf{D}_n$ when n and ℓ are coprime but $\ell \neq 1$. We can obtain ρ_2 by composing the standard ρ_1 with the map $\zeta \to \zeta^2$. This has the effect of re-ordering the oscillators from 01234 to 02413

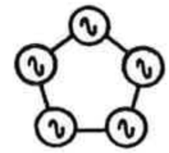
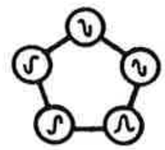
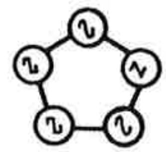
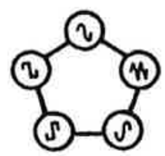
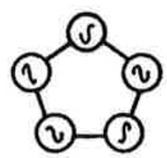
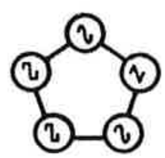
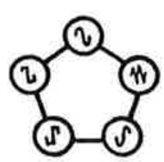
REPRESENTATION	ISOTROPY	PATTERN	COMMENTS
ρ_0	D_5		Identical waveforms in phase
ρ_1	$\tilde{Z}_5$		Travelling Wave Phase lags $\pm 2\pi/5$
ρ_1	$Z_2(\kappa)$		Oscillators 1,4 identical in phase; 2,3 identical and in phase
ρ_1	$Z_2(\kappa,\pi)$		1,4 π out of phase; 2,3 π out of phase; 0 of twice frequency
ρ_2	$\tilde{Z}_5$		Phase lags $\pm 4\pi/5$
ρ_2	$Z_2(\kappa)$		Oscillators 1,4 identical in phase; 2,3 identical and in phase
ρ_2	$Z_2(\kappa,\pi)$		1,4 π out of phase; 2,3 π out of phase; 0 of twice frequency

Figure 8.2 Generic oscillation patterns for a five-oscillator ring.

(replacing the pentagon with the pentacle). If the patterns for ρ_1 are relabelled in this order, the patterns for ρ_2 result. Note that the $\mathbf{Z}_2(\kappa)$ solutions for ρ_1 and ρ_2 yield the same pattern. This happens because the map $\gamma \mapsto \gamma^2$ defines an automorphism of $\mathbf{D}_n$ which preserves the orbits of $\mathbf{Z}_2(\kappa)$ and interchanges the two representations ρ_1 and ρ_2. Similar phenomena occur for representations ρ_k of $\mathbf{D}_n$ when n and k are coprime.

We end by comparing our results and methods with those of Alexander and Auchmuty (1986). They work in a slightly different context, but there is an area of overlap to which we address our comments. They seek solutions in which all oscillators have *identical* waveforms $x_p(t)$, with possible phase lags. This ansatz rules out solutions other than those with isotropy subgroup $\tilde{\mathbf{Z}}_n$ (and possibly $\mathbf{Z}_2(\kappa)$ from those found above. In their Corollary to Theorem 2 they prove (in a "global" context) the existence of this $\tilde{\mathbf{Z}}_n$ branch. Further, when n = 4 (or more generally n = 4k) they find a torus of solutions of the form

$$\begin{aligned} x_0(t) &= p(t) \\ x_1(t) &= p(t+\chi) \\ x_2(t) &= p(t+\pi) \\ x_3(t) &= p(t+\pi+\chi) \end{aligned} \tag{8.1}$$

where χ is an arbitrary phase shift. When $\chi = \pi$ this is our $\mathbf{Z}_2(\kappa)$ solution. But if $\chi \neq 0,\pi$ then their solution is not found by our method, and this deserves explanation. The answer is that their proof requires the equations for each individual oscillator on $\mathbf{R}^p$ to be $\mathbf{Z}_2$-symmetric (odd), and the coupling to be linear.

Acknowledgements

We are grateful to Giles Auchmuty, Brian Gray, Dan Luss, Lenny Pismen, and John Guckenheimer for helpful discussions. We are especially grateful to Bernold Fiedler who read an earlier draft in great detail, corrected several errors, and resolved a number of difficulties. The research of MG was supported in part by NASA Grant NAG-279 and by NSF Grant DMS-8402604. The research of INS was supported in part by NASA Grant NAG-279, SERC Grant GR/D/33359, and a travel grant from the Royal Society.

References

J.C.Alexander and G.Auchmuty (1986), Global Bifurcations of Phase-locked Oscillators, *Arch. Rational Mech. Anal.*, to appear.

H.A.De Kleine, E.Kennedy, and N.MacDonald (1982), *A study of coupled chemical oscillators*, preprint.

S.A. van Gils and T.Valkering (1985), *Hopf bifurcation and Symmetry: Standing and travelling waves in a circular chain*, preprint #279, Vrije Universiteit Amsterdam.

S.A. van Gils and J. Mallet-Paret (1985), *Hopf bifurcation and Symmetry: travelling and standing waves on the circle*, in preparation.

M.Golubitsky and I. Stewart (1985), Hopf Bifurcation in the presence of symmetry, *Arch. Rational Mech. Anal.* **87** 107-165.

L.N.Howard (1979), Nonlinear oscillations, in *Oscillations in Biology*, F.R. Hoppensteadt ed., AMS Lectures in Applied Mathematics **17** 1-69.

N.Kopell (1983), Forced and coupled oscillators in biological applications, *Proc. Int. Congr. Math. Warsaw*.

H.G.Othmer and L.E.Scriven (1971), Instability and dynamic pattern in cellular networks, *J. Theoret. Biol.* **32** 507.

V.Poénaru (1976), *Singularités C^∞ en Présence de Symétrie*, Lecture Notes in Math. **510**, Springer, New York.

D.H.Sattinger (1983), *Branching in the Presence of Symmetry*, CBMS-NSF Regional Conf. series in Appl. Math. **40**, SIAM, Philadelphia.

D.H.Sattinger (1984), *Petit cours dans les méthodes des groupes dans la bifurcation*, Cours de Troisième Cycle, École Polytechnique Fédérale de Lausanne, to appear.

G.Schwarz (1975), Smooth functions invariant under the action of a compact Lie group, *Topology* **14** 63-68.

S.Smale (1974), A mathematical model of two cells via Turing's equation, in *Some Mathematical Questions in Biology V*, J.D.Cowan ed.,AMS Lecture Notes on Mathematics in the Life Sciences **6** 15-26.

S.D.Taliaferro (1985) *Stability of bifurcating solutions in the presence of symmetry*, preprint, Texas A&M.

T.T.Tsotsis (1981), Nonuniform steady states in systems of interacting catalyst particles: the case of negligible interparticle mass transfer coefficient. *Chem. Eng. Commun.* **11**, 27-58.

A.Turing (1952), The chemical basis of morphogenesis, *Phil. Trans. Roy. Soc. London* **B237** 37-72.

A.T.Winfree (1980), *The Geometry of Biological Time*, Springer, New York.

Martin Golubitsky
Mathematics Department
University of Houston
University Park
HOUSTON TX 77004
USA

Ian Stewart
Mathematics Institute
University of Warwick
COVENTRY CV4 7AL
England