BIFURCATION PROBLEMS WITH HIDDEN SYMMETRIES

M. Golubitsky, J. Marsden
and D. Schaeffer

Published in:

Partial Differential Equations and Dynamical Systems

W.E. Fitzgibbon III (Ed)

Research Notes in Mathematics, No. **101**
Pitman, London (1984) pp. 181-210

M GOLUBITSKY, J MARSDEN & D SCHAEFFER

Bifurcation problems with hidden symmetries

§1. INTRODUCTION

Basic theory for bifurcation problems with symmetry was developed by Sattinger [1979] and Golubitsky and Schaeffer [1979b]. A symmetry group usually forces the bifurcation to be rather degenerate but simultaneously, one can take advantage of the symmetry to render some interesting problems computable.

Recent papers of Hunt [1981], [1982] make use of symmetries in what, at first sight, appears to be a nonstandard fashion. This enables him to arrive at a parabolic umbilic description for the buckling of a right circular cylinder under end loading (see also Hui and Hansen [1981]). The purpose of this paper is to establish the following points:

1. The scheme of Hunt is consistent with the general theory of Golubitsky and Schaeffer [1979].

2. There is a simple abstract procedure involving "hidden symmetries" which enables one to simplify calculations and to arrive at Hunt's procedure as a special case in a natural way.

3. The scheme proposed by Hunt for the buckling of shells can be derived by starting with, for example, the partial differential equations of Kirchhoff shell theory, and

4. The stability assignments can be computed for the bifurcation problem considered by Hunt.

A crucial $\mathbb{Z}_2$ symmetry on a subspace is used by Hunt to obtain a description of the bifurcation in terms of the parabolic umbilic. This symmetry is derived by him in a heuristic way. We show that it arises by a natural abstract construction that is verifiable for a Kirchhoff shell model.

The name "hidden" symmetry arises from two facts. First, it is a symmetry defined only on a subspace of state space. Second, this symmetry is revealed by working in a larger space that does not fix the phases of the relevant modes. This larger space is where the framework of Golubitsky and Schaeffer [1979b] holds. We shall explain these statements in more detail shortly in §2.

As Hunt notes, there are other bifurcation problems that can be dealt with by the 'hidden symmetry method', such as the buckling of stiffened structures. Another example is Schaeffer's [1980] analysis of the Taylor problem. In particular, the use of hidden symmetries enables one to see directly that certain terms in the bifurcation equation vanish. This was done by direct calculation in Schaeffer [1980]. As will be noted later, hidden symmetries also appear to play an important role in the analysis of other bifurcation problems as well, such as the Bénard problem. This is briefly discussed in Golubitsky, Swift and Knobloch [1984] and Ihrig and Golubitsky [1984].

In some physical problems, solutions of a partial differential equation on a bounded domain satisfying appropriate boundary conditions are in one-to-one correspondence with periodic solutions on an infinite domain which have additional reflection symmetries. The periodic problem is a mathematical device which helps in the understanding of the given problem in the finite domain. In particular, this device enables one to understand how hidden symmetries in the problem can be understood in the abstract formulation as symmetries on a subspace. This procedure shows why our abstract formulation includes more cases than one might at first expect.

In Section 2 we explain in more detail how the periodic extensions give rise to hidden symmetries by means of a simple example. Section 3 gives the abstract infinite dimensional formulation of bifurcation problems with hidden symmetries and Section 4 applies the methods developed to Hunt's problem of buckling cylinders. Finally in Section 5 we discuss the stability and bifurcation diagrams for the cylinder problem.

In this paper we have had to make a choice between the variational approach (based on an energy function) and the direct approach (based on the equations). In the variational approach, one is given an energy functional which is invariant under the action of the symmetry group. One then applies the splitting lemma of Gromoll and Meyer to obtain a reduced function $f: \mathbb{R}^n \to \mathbb{R}$ whose critical points are (locally) in one to one correspondence with the critical points of the energy functional. (See Golubitsky and Marsden [1983] and Buchner, Marsden, and Schecter [1983] for a general view of this approach.) Moreover, the reduced function f inherits the symmetries of the original energy function and is, itself, invariant under the group action.

The second way to obtain symmetries is to start with a differential equation whose associated differential operator has a linearization which is Fredholm of index 0. Then one may use the Liapunov-Schmidt procedure to obtain a (reduced) mapping $g: \mathbb{R}^n \to \mathbb{R}^n$ whose zeros are (locally) in one to one correspondence with the solutions to the original differential equations. Moreover, if the original differential operator is equivariant with respect to a group of symmetries, then (under suitable hypotheses), so is g.

The difference between the two approaches is significant when the unfolding (or imperfection sensitivity) problem is studied. This latter topic is discussed here only briefly. To be consistent with the spirit of

Hunts paper and with elastic buckling in general, we shall adopt the variational (or catastrophe theory) point of view.

Acknowledgements. We thank Giles Hunt for stimulating conversations which inspired this work. We also thank Stuart Antman, David Chillingworth, Ed Ihrig, and Steve Wan for several useful comments.

§2. HIDDEN SYMMETRIES AND PERIODIC BOUNDARY CONDITIONS

In this investigation of the buckling of cylindrical shells, Hunt noted that the parabolic umbilic, $\pm x^4 \pm xy^2$, appeared in a context where some less degenerate singularity (such as the elliptic or hyperbolic umbilic) seemingly should have been expected. Taking the point of view that one should attempt to explain unexpected degeneracies, Hunt looked for a context in which the parabolic umbilic would occur naturally. He found one, which he calls symmetries on a subspace. In this paper we give a context, namely that of hidden symmetries, which reveals Hunt's symmetries on a subspace in a natural way.

Let us first give a prototype example (due to Hunt) which shows how the parabolic umbilic arises from the imposition of a symmetry on a subspace. In the second half of this section we show how this situation can arise by means of a simple example.

Let $g(x,v)$ be a real-valued function satisfying

$$\begin{aligned} &\text{a)}\quad g(-x,v) = g(x,v) \quad\text{and}\quad g(0,-v) = g(0,v)\ , \text{ and} \\ &\text{(b)}\quad g(0) = 0\ ,\quad (dg)(0) = 0,\quad (d^2g)(0) = 0 \end{aligned} \tag{2.1}$$

Conditions (2.1a) state that f has a reflectional symmetry in the x-variable and a reflectional symmetry on the subspace consisting of the v-axis. Conditions (2.1b) state that g has a degenerate singularity at the origin.

Note. The function g could arise via the splitting lemma from an infinite dimensional variational problem. Conditions (2.1b) state that the kernel of the Hessian of the original variational problem is two-dimensional, while conditions (2.1a) reflect certain symmetry properties of this variational problem.

Writing the first few terms of the Taylor expansion of f consistent with (2.1) one finds

$$g(x,v) = Ax^2v + Bv^4 + Cx^2v^2 + Dx^4 + \ldots\ldots$$

The important point to note here is that x^2v is the only cubic term which can be non-zero. The symmetry on the subspace forces the coefficient of v^3 to be 0. Now if $A \cdot B \neq 0$ then f is right equivalent to the parabolic umbilic (cf. Zeeman and Trotman [1975]). More precisely, there exists a diffeomorphism $\phi: \mathbb{R}^2 \to \mathbb{R}^2$ such that

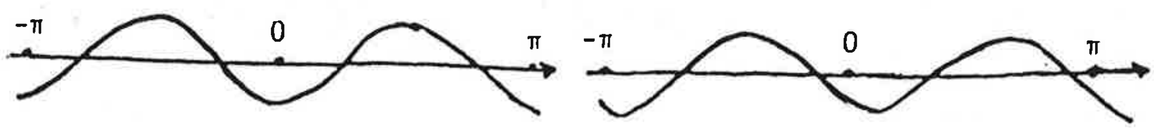
$$g(\phi(x,v)) = \varepsilon v^4 + x^2v$$

where $\varepsilon = \operatorname{sgn} B$. If the coefficient of v^3 were nonzero one would obtain either the elliptic or hyperbolic umbilic.

To motivate the abstract set up in the following section, consider the following simple example. Suppose that one has a bifurcation problem in variational form which is posed on the interval $[-\pi,\pi]$ with periodic and possibly other boundary conditions assumed. Often such problems have $O(2)$ as a symmetry group; the rotations $SO(2)$ act by translation $\zeta \mapsto \zeta + \theta$ where θ is an element of $SO(2) \approx S^1$ and the orientation reversing element of $O(2)$ acts by flipping the interval $\zeta \mapsto -\zeta$. Typically, the kernel of the linearization of the bifurcation problem at an eigenvalue of this linearization will be the 2-dimensional space V_k generated by $\{\cos k\zeta, \sin k\zeta\}$. Solutions to the original bifurcation problem which correspond to V_k by a splitting lemma argument are said to have <u>wave number</u> k. However there is often an extra parameter in the original bifurcation problem, such as an aspect ratio, which alters the eigenvalue structure. For special values of this parameter it is possible to have an eigenvalue of multiplicity 4. Typically, in such instances, the wave numbers are consecutive. For definiteness, suppose the corresponding eigenspace is $V_k \oplus V_{k+1}$. In the example below we study the case $\mathbb{R}^4 = V_1 \oplus V_2$ with explicit coordinates given by

$$(x,y,v,w) \to x \cos \zeta + y \sin \zeta + v \cos 2\zeta + w \sin 2\zeta. \qquad (2.2)$$

We now discuss why one studies bifurcation problems on an interval $[-\pi,\pi]$ with periodic boundary conditions. Often one has a physical problem posed on the finite interval which one tries to solve by solving a corresponding problem on the infinite interval and looking for periodic solutions of period 2π. This reformulation introduces $O(2)$ as a group of symmetries. However, in the original problem on the finite interval there may be additional boundary conditions besides periodicity which limit the periodic solutions allowable. For example, the only solutions to the transformed problem on the infinite interval which may be relevant are the ones which start and end symmetrically; i.e., those solutions which are invariant under the flip $\gamma_0(\zeta) = -\zeta$. See Figure 2-1 which illustrates this for waves with $k = 2$.



flip invariant | periodic but not flip invariant

Figure 2.1

A similar situation occurs in the analysis by Schaeffer [1981] of the Taylor problem and, as we shall see, can be used as the basis for the analysis by Hunt [1982] of the buckling of a cylindrical shell though the situation in the latter case is yet more complicated.

The context hypothesized above allows hidden symmetries. Let $\mathbb{R}^4 = V_1 \oplus V_2$. The translation $\zeta \to \zeta + \theta$ of $SO(2)$ acts on V_k by rotation through the angle $k\theta$. The flip γ_0 acts on $\mathbb{R}^4$ with the coordinates (2.2) by

$$\gamma_0(x,y,v,w) = (x,-y,v,-w).$$

Let Δ be the 2-element group generated by γ_0. Let F_Δ, the fixed point set for Δ, be defined by

$$F_\Delta = \{(x,0,v,0)\} .$$

Note that F_Δ corresponds to the periodic solutions $x \cos \zeta + v \cos 2\zeta$ which are exactly the periodic solutions which begin and end symmetrically.

We are thus interested in the elements of F_Δ. Consider the "naive symmetry group" $N(\Delta)$ consisting of those elements of $O(2)$, which leave F_Δ setwise invariant; that is, $N(\Delta)$ is generated by the flip γ_0 and the translation by half a period, $h(\zeta) = \zeta + \pi$. The action of h on F_Δ is given in the coordinates (x,v) by

$$h(x,v) = (-x,v).$$

We now ask, "Is there a hidden symmetry in this problem?", that is, is there a symmetry on a subspace other than the elements of $N(\Delta)$? The answer is yes. Let $q(\zeta) = \zeta + \frac{\pi}{2}$ be translation by a quarter period. Then q acts on $\mathbb{R}^4$ by $q(x,y,v,w) = (y,-x,-v,-w)$. In particular, on the fixed point subspace of $N(\Delta)$, namely $F_{N(\Delta)} = \{(0,v)\} \subset F_\Delta$, q acts by $q(0,v) = (0,-v)$.

Now one sees that if one were to solve the hypothetical problem above by a splitting lemma argument one would be looking for the critical point structure of a function $f\colon \mathbb{R}^4 \to \mathbb{R}$ which is invariant under the action of $O(2)$. By looking for "physically reasonable" solutions one tries to find the critical point structure of $g\colon \mathbb{R}^2 \to \mathbb{R}$ where $g = f \mid F_\Delta$ and g satisfies the symmetry conditions (2.1a); in particular, g satisfies the hypotheses of symmetry on a subspace studied by Hunt.

As noted above, the analysis of the buckling of the cylinder proposed by Hunt [1982] is somewhat more complicated though the end result is similar. The reason for this complication is that in the buckling problem two copies of $O(2)$ act as symmetries. More precisely, one copy of $O(2)$ occurs because the finite cylinder is replaced by the infinite cylinder with periodic boundary conditions imposed. The second copy of $O(2)$ acts on the problem since the cylinder itself is invariant under rotation about its axis.

§3. ABSTRACT FORMULATION

We begin with the following situation. Let Γ be a subgroup of $O(n)$ and assume $g\colon \mathbb{R}^n \to \mathbb{R}$ is invariant under Γ: $g(\gamma x) = g(x)$ for $x \in \mathbb{R}^n$ and $\gamma \in \Gamma$. Suppose we are interested in the critical points of g that possess a given symmetry. That is, let $\Delta \subset \Gamma$ be a given subgroup and let F_Δ be its fixed point set:

$$F_\Delta = \{y \in \mathbb{R}^n \mid \delta y = y \text{ for all } \delta \in \Delta\};$$

we are interested in critical points of g that lie in F_Δ. Let $h = g|F_\Delta$.

Lemma 1. Let $y \in F_\Delta$. Then g has a critical point at y if and only if h does.

Remark. This lemma is a special case of the "principle of symmetric criticality" due to Palais [1979]. We give a direct proof for the case at hand.

Proof. If g has a critical point at $y \in F_\Delta$, then obviously y is also a critical point for h. Conversely, let $y \in F_\Delta$ be a critical point for h. To show y is a critical point of g, we use the following remark. Let $G\colon \mathbb{R}^n \to \mathbb{R}^n$ be equivariant with respect to Γ: i.e. $g(\gamma x) = G(x)$ for all $\gamma \in \Gamma$. Now $G(F_\Delta) \subset F_\Delta$ since for $x \in F_\Delta$ and $\delta \in \Delta$, $G(x) = G(\delta x) = \delta G(x)$ and so $G(x) \in F_\Delta$. Now let $H = G|F_\Delta\colon F_\Delta \to F_\Delta$. Then for $x \in F_\Delta$, it is clear that

$$G(x) = 0 \quad \text{if and only if} \quad H(x) = 0 \tag{3.1}$$

To complete the proof of the lemma, let $G(x) = \nabla g(x)$. Then since $\Gamma \subset O(n)$, we have $G(\gamma x) = G(x)$ and $H(x) = \nabla h(x)$ for $x \in F_\Delta$. The result therefore follows by (3.1). ∎

We shall call a subgroup $\Lambda \subset \Gamma$ <u>fixed point complete</u> if

$$\Lambda = \{\gamma \in \Gamma | \gamma y = y \text{ for all } y \subset F_\Lambda\} \tag{3.2}$$

If our symmetry subgroup Δ is not fixed point complete, we can always enlarge Δ to symmetry subgroup $\bar{\Delta}$ that is fixed point complete and for which $F_\Delta = F_{\bar{\Delta}}$. This is reasonable since we are looking for critical points in F_Δ and augmenting Δ by group elements that pointwise fix F_Δ does not change the fixed point set.

To study the critical points of h on F_Δ, it is useful to find the symmetries of h. To locate these, we first consider the subgroup $N(\Delta) \subset \Gamma$ defined by

$$N(\Delta) = \{\gamma \in \Gamma | \gamma(F_\Delta) \subset F_\Delta\} \tag{3.3}$$

It is clear that h is invariant under $N(\Delta)$, so $N(\Delta)$ is a symmetry group for h. The notation $N(\Delta)$ is used because of the next lemma.

<u>Lemma 2</u>. $N(\Delta)$ <u>is the normalizer of</u> $\bar{\Delta}$ <u>in</u> Γ.

<u>Proof</u>. Recall that if H is a subgroup of a group G, its normalizer is defined by $N_H = \{g \in G | g^{-1}Hg \subset H\}$.

First, suppose that $\gamma \in N(\Delta)$. To show that $\gamma \in N_{\bar{\Delta}}$, let $\delta \in \bar{\Delta}$; we must prove that $\gamma^{-1}\delta\gamma \in \bar{\Delta}$. But if $y \in F_\Delta$ then $\gamma y \in F_\Delta$, so $\delta\gamma y = \gamma y$ and $\gamma^{-1}\delta\gamma y = \gamma^{-1}\gamma y = y$. Thus $\gamma^{-1}\delta\gamma \in \bar{\Delta}$ since $\bar{\Delta}$ is fixed-point complete.

Conversely, suppose $\gamma \in N_{\bar{\Delta}}$. If $y \in F_\Delta$ and $\delta \in \bar{\Delta}$, then $\gamma^{-1}\delta\gamma \in \bar{\Delta}$, so $\gamma^{-1}\delta\gamma y = y$ or $\delta\gamma y = \gamma y$ for all $\delta \in \bar{\Delta}$. Thus y is fixed by all $\delta \in \bar{\Delta}$, so by definition of $F_\Delta = F_{\bar{\Delta}}$, $\gamma y \in F_\Delta$. Thus $\gamma \in N(\Delta)$ by (3.3). ∎

Since $\bar{\Delta}$ acts trivially on F_Δ, we can "discard" it from our symmetry group of h. In view of lemma 2 one can do this by letting the group $D(\Delta)$

be defined by

$$D(\Delta) = N(\Delta)/\overline{\Delta}$$

and calling it (or $N(\Delta)$) the naive group of symmetries of h. Notice that there is a well defined action of $D(\Delta)$ on F_Δ and that h is invariant under this action.

There is a second way h can inherit symmetries from Γ. Let Σ be a proper fixed point complete subgroup of Γ and assume $\Delta \subset \Sigma$. Thus, $F_\Sigma \subset F_\Delta$. As above, $N(\Sigma)$ leaves F_Σ invariant and $h|F_\Sigma$ is invariant under $D(\Sigma)$. The new symmetries obtained this way are the hidden symmetries. Here is the formal definition.

Definition. A hidden symmetry of h is a nontrivial element of $N(\Sigma)$ for some proper subgroup Σ of Γ containing Δ, which is not in $N(\Delta)$.

Remarks 1. One could take the view that one is searching for F_Σ as much as for Σ; given F_Σ, Σ can be defined as the isotropy group of typical points in F_Σ. In the example we shall see that Σ and F_Σ are found simultaneously.

2. If we are looking for zeros of a map $H:F_\Delta \to F_\Delta$ commuting with $N(\Delta)$ (rather than critical points of an invariant function h), then the mere existence of $F_\Sigma \subsetneq F_\Delta$ can put restrictions on H. Indeed, H must map F_Σ to itself, a fact that does not in general follow just from the equivariance of H under $N(\Delta)$.

3. In Hunt's example we shall see that one can choose $\Sigma = N(\Delta)$. The group theoretic reason for this is given in remark 4 below. In other examples, one probably will need to understand the lattice of fixed point complete subgroups of Γ and the lattice of isotropy subgroups. For

examples involving convection, either the planar Bénard problem or spherical convection, these lattices can be computed and in certain cases hidden symmetries are important in understanding solutions with a given symmetry group Δ. For example, in Buzano and Golubitsky [1983] this occurs with the rectangular solutions (see Golubitsky, Swift and Knobloch [1984], §IV for a discussion). For the spherical Bénard problem, the lattice of isotropy subgroups for representations of $O(3)$ is worked out in Ihrig and Golubitsky [1984]. They find that bifurcating solutions corresponding to fixed point sets of dimension one (with maximal isotropy subgroups--these are found using a theorem of Cicogns) are often unstable. In this case one needs to go to the next level in the lattice and there hidden symmetries may be important. See Golubitsky [1983] for a general introduction to ideas and examples involving the lattice of isotropy subgroups.

4. We now ask whether there can be symmetries more subtle than hidden symmetries. In general, the answer seems to be yes. However, in Hunt's example and in the case where Δ is a maximal isotropy subgroup these subtle symmetries do not exist. This last fact was pointed out to us by Ed Ihrig, cf. Remark 5 below.

Before discussing subtle symmetries we review our description of hidden symmetries. In our discussion we have assumed the existence of a function $g: \mathbb{R}^n \to \mathbb{R}$ invariant under Γ and a fixed point complete subgroup Δ of Γ. Our interest lies in understanding the restrictions placed on $h = g|F_\Delta$. So far we have observed two types of restrictions on h. The first observation states that h is invariant under the group $N(\Delta)$ which acts naturally on F_Δ. We have called elements in $N(\Delta)$ "naive symmetries". In the second observation we have shown, by iteration, that if $\Sigma \supsetneq \Delta$ is a proper, fixed point complete subgroup of Γ then $h|F_\Sigma$ is invariant under the subgroup $N(\Sigma)$. We have called elements in $N(\Sigma)$ "hidden symmetries". Moreover, Hunt's symmetry on a subspace is just a specific instance of a hidden symmetry.

We now question whether there are any additional group theoretic restrictions placed on h by the existence of the large group Γ. Such additional symmetries we call subtle symmetries. The only way that subtle symmetries may arise is as follows. Suppose there is an element γ in $\Gamma \sim N(\Delta)$ for which

$$\gamma^{-1}(F_\Delta) \cap F_\Delta \supsetneq F_\Gamma \, . \tag{3.4}$$

(Aside: F_Γ is contained in each fixed point space and all symmetries fix vectors in F_Γ. Thus (3.4) states that the intersection in the LHS of (3.4) contains a nontrivial vector. Moreover, since $\gamma \notin N(\Delta)$ the intersection on the LHS of (3.4) is a proper subspace of F_Δ.) Observe that when (3.4) is valid, we obtain a further restriction on h. For if $w \in \gamma^{-1}(F_\Delta) \cap F_\Delta$ then both w and γw are in F_Δ; thus $h(\gamma w) = h(w)$. We call such elements γ which are not hidden symmetries, subtle symmetries.

We may clarify the issue of subtle symmetries as follows. We claim that $F_\Delta \cap \gamma^{-1}(F_\Delta)$ is itself a fixed point subspace. Observe that

$$\begin{aligned} &\text{(a)} \quad F_{\gamma^{-1}\Delta\gamma} = \gamma^{-1}(F_\Delta), \text{ and} \\ &\text{(b)} \quad F_G \cap F_H = F_{<G,H>} \end{aligned} \tag{3.5}$$

where G, H are subgroups of Γ and $<G,H>$ is the subgroup they generate. To prove the claim, let $T = <\Delta, \gamma^{-1}\Delta\gamma>$. It follows from (3.5) that

$$F_T = F_\Delta \cap \gamma^{-1}(F_\Delta)$$

Moreover,

$$\Delta \subsetneqq T \subsetneqq \Gamma$$

since (3.4) is assumed valid and $\gamma \notin N(\Delta)$.

Thus, when searching for subtle symmetries we look for fixed point subspaces F_T satisfying

$$\begin{aligned} &(a) \quad F_\Gamma \subsetneqq F_T \subsetneqq F_\Delta\,, \\ &(b) \quad \gamma(F_T) \subset F_\Delta, \text{ and} \\ &(c) \quad \gamma(F_T) \neq F_T\,. \end{aligned} \tag{3.6}$$

Note that (3.6c) follows from the fact that γ is not a hidden symmetry; hence $\gamma \notin N(T)$.

Thus we see that naive symmetries correspond to elements of Γ which leave F_Δ invariant, hidden symmetries correspond to elements of Γ which leave some subspace F_Σ of F_Δ invariant, and subtle symmetries correspond to elements γ of Γ which map a subspace F_T of F_Δ onto a different subspace $\gamma(F_T)$ of F_Δ.

5. There are no hidden symmetries and there are no subtle symmetries when Δ is a maximal isotropy subgroup.(Proof due to Ed Ihrig.) In order to find a hidden symmetry or a subtle symmetry we would need to find a fixed point subspace F_T satisfying (3.6a). Now suppose that F_T is any fixed point subspace, then we claim that F_T is the union of fixed point subspaces of isotropy subgroups. First observe that if $v \in F_T$ then $F_{\Sigma_v} \subset F_T$ where Σ_v is the isotropy subgroup of v. (This follows from the fact that T fixes v and thus is contained, by definition, in Σ_v.) Since $v \in \Sigma_v$ it follows that $F_T = \bigcup_{v \in T} F_{\Sigma_v}$. Second, use (3.6a) to observe that $F_{\Sigma_v} \subset F_T \subset F_\Delta$ and hence that $\Delta \subset \Sigma_v$. By (3.6a) we can choose a vector $v \in F_T \sim F_\Gamma$ and for such a v, $\Sigma_v \neq \Gamma$. It now follows that if Δ is a maximal isotropy

subgroup we must have $\Delta = \Sigma_v$. Hence $F_\Delta = F_{\Sigma_v} = F_T$ which contradicts the second equality in (3.6a).

We now discuss how one reduces an infinite dimensional problem to the situation of looking for critical points of a function $g: \mathbb{R}^n \to \mathbb{R}$ invariant under a group $\Gamma \subset O(n)$.

Let X be a Banach space and $\langle\ ,\ \rangle$ a (not necessarily complete) inner product on X. (In many examples X is a $W^{s,p}$ Sobolev function space, and $\langle\ ,\ \rangle$ is an L^2, H^1 or H^2 inner product).* Let $f:X \to \mathbb{R}$ be a C^∞ function defined on a neighborhood of $0 \in X$ with $f(0) = 0$ and $Df(0) = 0$. Eventually f will depend on parameters but this is suppressed for the moment.

Let K be the kernel of $D^2f(0)$; i.e.

$$K = \{v \in X | D^2f(0)\cdot(v,w) = 0 \text{ for all } w \in X\}$$

Assume that f admits a smooth $\langle\ ,\ \rangle$ gradient $\nabla f:X \to X$ so $\nabla f(0) = 0$ and let $T = D\nabla f(0):X \to X$. It is easy to check that $\langle D(\nabla f(0))\cdot v,w\rangle = D^2f(0)\cdot(v,w)$, so $K = \operatorname{Ker} T$ and T is $\langle\ ,\ \rangle$ - symmetric.

Assume T is Fredholm of index zero; in particular, X admits the following $\langle\ ,\ \rangle$ -orthogonal decomposition into closed subspaces.

$$X = K \oplus \text{Range } T\ .$$

Under these hypotheses, the critical points of f are in one to one correspondence for those of a reduced function

$$g:K \to \mathbb{R}$$

---

* The generalization to the case in which X is a manifold and $\langle\ ,\ \rangle$ is a weak Riemannian metric on X is routine.

defined in a neighborhood $0 \in K$. This correspondence is by means of the graph of an implicitly defined function $\phi: K \to \text{Range } T$ defined by solving the equation $P \circ \nabla f = 0$ where P is the projection to Range T. One can also show that the problem of finding normal forms for f can be reduced to that for g, which satisfies $g(0) = 0$, $Dg(0) = 0$ and $D^2 g(0) = 0$. This is the splitting lemma, which is the variational analogue of the Liapunov-Schmidt procedure (see Golubitsky and Marsden [1983] for details and references).

Suppose that Γ is a group of isometries which act continuously on X by linear transformations and leave f invariant; that is

$$f(\gamma x) = f(x) \quad \text{for all} \quad x \in X, \quad \gamma \in \Gamma .$$

By differentiating this relation, it readily follows that Γ leaves K invariant and g is invariant as well; that is, $\gamma x \in K$ if $\gamma \in \Gamma$ and $x \in K$ and

$$g(\gamma x) = g(x) \quad \text{for} \quad \gamma \in \Gamma, \quad x \in K.$$

<u>Aside</u>. In the buckling problem of Hunt [1982], X corresponds to a Sobolev space of 2L-periodic displacements of a right circular cylinder, where periodic means with respect to movement along the z-axis, the axis of the cylinder, and $\Gamma = O(2) \times O(2)$ is the natural symmetry group of the problem. We take $[-L,L]$ as the fundamental interval along the cylinder's axis. Hunt is interested in solutions which are symmetric with respect to reflection about the midpoint of this interval. These solutions comprise the fixed point set for a subgroup Δ of Γ. This and the discussion in §2 are motivations for the construction of F_Δ given above. Another motivation is provided by Schaeffer [1980], in which functions satisfying a desired set

of boundary conditions for the Navier-Stokes equation on a fixed domain $\Omega \times [-L,L]$, where $\Omega \subset \mathbb{R}^2$, can be characterized as periodic functions on $\Omega \times \mathbb{R}$ which are invariant under reflection with respect to the planes $z = \pm L$. Again the states satisfying these boundary conditions can be characterized as the fixed point set for a subgroup Δ.

In many examples, including Hunt's, there is a further reduction in dimension that can be done by finding a cross section for the action. The method is similar to that of Golubitsky and Schaeffer [1981] in which reduction from a five dimensional kernel to a two dimensional subspace was important (interestingly, in this example, the subspace was a fixed point set for the group D_2).

Suppose $V \subset F_\Delta$ is a subspace satisfying:

$$\begin{aligned}&\text{(a)}\quad V \text{ saturates } F_\Delta; \text{ that is, } \bigcup_{\gamma \in N(\Delta)} \gamma V = F_\Delta, \text{ and}\\ &\text{(b)}\quad \text{for each } x \in V,\ T\mathcal{O}_x + \bigcup_{\gamma \in N(\Delta)_x} \gamma V = F_\Delta\end{aligned} \tag{3.7}$$

where $\mathcal{O}_x$ is the $N(\Delta)$ orbit of x in F_Δ and $N(\Delta)_x$ is the isotropy subgroup of x in $N(\Delta)$. By Lemma 1, we seek the critical points of $h = g|F_\Delta$ on F_Δ. We now show that the critical points of h are determined by $k = h|V$.

Lemma 3. Let V satisfy (3.7) and let $k = h|V$. Then the critical points of h are the $N(\Delta)$ orbits of the critical points of k.

Proof. By invariance of h under $N(\Delta)$, it is enough to show that for $x \in V$, x is a critical point of h if and only if it is one for k. From $h(\gamma x) = h(x)$ we get, in terms of differentials,

$$dh(\gamma x) \circ \gamma = dh(x).$$

[In terms of gradients, since the action is orthogonal we have $\nabla h(\gamma x) = \gamma\nabla h(x)$.] Obviously if h has a critical point at $x \in V$ so does k and the orbit of x consists of critical points. Suppose $x \in V$ is a critical point of k. Then $dh(x)|V = 0$ and so $dh(x)|\gamma V = 0$ for $\gamma \in N(\Delta)_x$ by $dh(\gamma x)\circ\gamma = dh(x)$. Since h is constant on $\mathcal{O}_x$, $dh(x)|T_x\mathcal{O}_x = 0$. Thus by (3.7b), $dh(x) = 0$. By (3.7a) we have not missed any critical points. ∎

Thus, no information is lost by restricting attention to the cross-section V. As in Golubitsky and Schaeffer [1981], we expect one can prove that no information is lost in the unfolding theory as well.

§4. BUCKLING OF A CYLINDRICAL SHELL

We now describe a context in which one can in principle rigorously arrive at Hunt's model for cylindrical shell buckling. We do not provide a complete exposition, but only indicate a framework with enough details so the symmetry groups become apparent and the hidden symmetry is revealed. We use a shell model for simplicity, but one could in principle also use a three dimensional model.

First we outline a framework for nonlinear Kirchhoff shell theory (cf. Naghdi [1972], Marsden and Hughes [1983] and references therein). Let M be a reference two manifold and $\mathcal{C}$ a space of deformations $\phi: M \to \mathbb{R}^3$. Each $\phi \in \mathcal{C}$ is required to be an embedding of M into $\mathbb{R}^3$ and is to satisfy any relevant displacement boundary conditions. For each ϕ, let $F = D\phi$ be the deformation gradient, F^T its transpose, and $C = F^TF$, a positive definite symmetric two tensor on M, the Cauchy Green tensor. (Apart from the positioning of tensor indices, C is the pull-back of the Eucliden metric on $\mathbb{R}^3$ to M). Let k denote the (referential) second fundamental form of the embedding ϕ; i.e. k is the second fundamental form of the deformed surface $\phi(M)$, regarded as a symmetric two tensor on

M.

Kirchhoff shell theory deals with elastic stored energy functions of the form $W(C,k)$. We shall assume that the shell is homogeneous, i.e. W is independent of the reference point $X \in M$, is isotropic and is invariant under isometries of M. The real shell being modelled has a finite thickness which is incorporated into $W(C,k)$.

If $\underset{\sim}{\lambda}$ denotes a prescribed dead load on ∂M, then the energy function is

$$V_{\underset{\sim}{\lambda}} : C \to \mathbb{R}$$

$$V_{\underset{\sim}{\lambda}} = \int_M W(C,k)\, dA - \int_{\partial M} \underset{\sim}{\lambda} \cdot \phi \, ds$$

Equilibrium configurations of the shell are the critical points of $V_{\underset{\sim}{\lambda}}$.

We choose the length scale so that the radius of the cylinder is unity. For a cylindrical shell with periodic boundary conditions, we choose

$$M = S^1 \times [0,L]$$

where S^1 is the unit circle in the plane. We consider deformations $\phi: M \to \mathbb{R}^3$ of Sobolev class H^s, $s \geq 4$ which are immersions that map $z =$ constant planes to $z =$ constant planes and which remain H^s when extended periodically in z; thus letting ϕ^x, ϕ^y and ϕ^z be the components of ϕ, the extension satisfies

$$\phi^x(\theta,z+L) = \phi^x(\theta,z), \; \phi^y(\theta,z+L) = \phi^y(\theta,z)$$

and

$$\phi^z(\theta,z+L) = \phi^z(\theta,z) + \phi^z(\theta,L)$$

where $\theta \in S^1$.

Note that the deformed cylinder has a well-defined length, say aL.

We choose X to be the space of H^s displacements u defined by $u(\theta,z) = \phi(\theta,z) - (\cos\theta, \sin\theta, z)$, so $V_{\underset{\sim}{\lambda}}$ becomes defined on a neighborhood of $u = 0$.

We assume that the linearized problem at (and hence near) $u = 0$ is strongly elliptic (so the Fredholm alternative is available -- see for example Marsden and Hughes [1983, Ch. 6]). The linearized equations are fourth order and the linearized energy is quadratic in second derivatives of u. Motivated by analogous examples for the Morse lemma (see Golubitsky and Marsden [1983] and Buchner, Marsden and Schecter [1983]), we find that $\langle\ ,\ \rangle$ may be chosen to be the H^2 inner product.

Let us assume that the linearized elastic moduli, the length L and a parallel end load of magnitude λ are chosen so that the modes described in Hunt [1982] (and references therein) comprise the kernel K of the linearized equations in X. This kernel has dimension six. It contains the two basic displacements shown in Figure 4-1 (adapted from Hunt [1982]).

The general element of the six dimensional space K has the form

$$u = \left[\alpha_1 \cos\frac{\pi z}{L}\cos 2\theta + \alpha_2 \cos\frac{\pi z}{L}\sin 2\theta,\ \alpha_3 \sin\frac{\pi z}{L}\cos 2\theta + \alpha_4 \sin\frac{\pi z}{L}\sin 2\theta,\ \beta_1 \cos\frac{2\pi z}{L} + \beta_2 \sin\frac{2\pi z}{L}\right]. \tag{4.1}$$

This function may be characterized by two amplitudes namely $q_1 = \{\Sigma\alpha_i^2\}^{1/2}$ and $q_2 = (\beta_1^2 + \beta_2^2)^{1/2}$, and various phases. The phases associated with the four dimensional subspace corresponding to the α's are a little subtle and fortunately do not matter for what follows. The case illustrated in Figure 4-1 has $\alpha_2 = \alpha_3 = \alpha_4 = 0$. In this case all displacements vanish for $z = L/2$, at which points the cross section is circular. In general there is a plane where the displacements vanish if and only if $\alpha_1\alpha_4 = \alpha_2\alpha_3$.

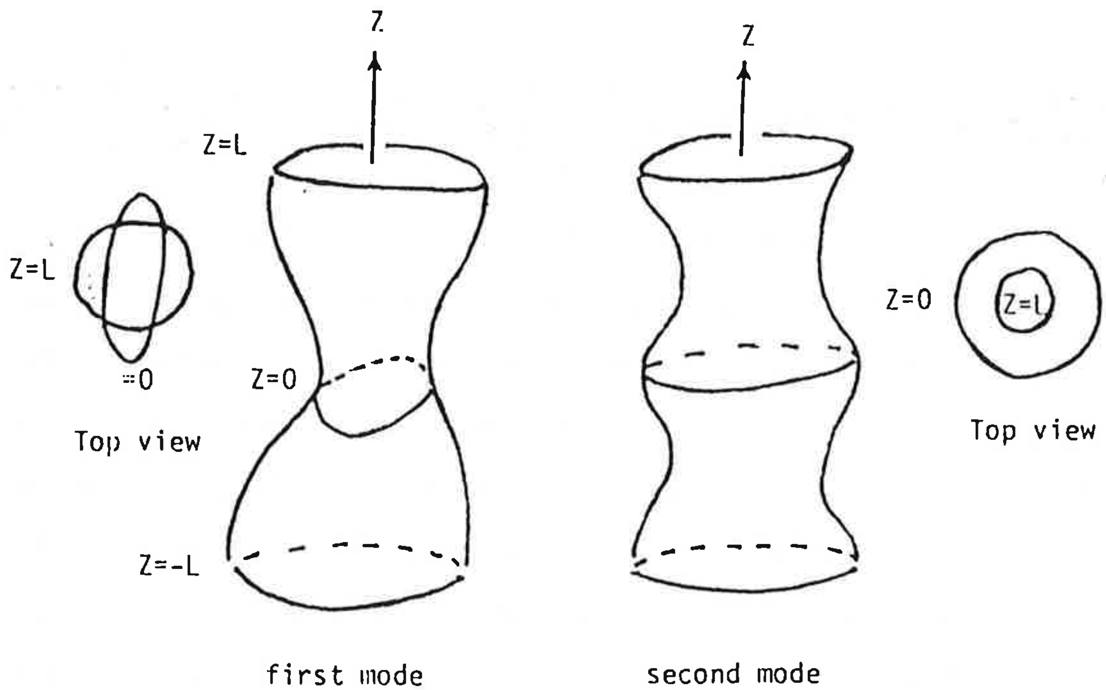


Figure 4-1

This fact will not be needed below, since we will shortly be restricting to the subspace with $\alpha_3 = \alpha_4 = 0$.

As we have hinted at in the preceding paragraph, a full bifurcation analysis directly on K is somewhat complicated. However, our interest is in F_Δ and the analysis on F_Δ is greatly simplified by symmetries and hidden symmetries.

The group Γ is taken to be $O(2) \times O(2)$; the first $O(2)$ consists of rotations about the z-axis and reflections in vertical planes. The second $O(2)$ consists of (2L-periodic) translations along the z-axis and reflections in z = constant planes. These $O(2)$ actions induce an action of Γ on M and $\mathbb{R}^3$ and hence on X. Elements $\gamma \in \Gamma$ act on configurations ϕ and displacements u by $\gamma\phi = \gamma \circ \phi \circ \gamma^{-1}$ and $\gamma u = \gamma \circ u \circ \gamma^{-1}$). It is clear that the potential function V_λ is invariant under Γ. Note that Γ also acts on K using the given representation.

We let Δ be the $\mathbb{Z}_2$ subgroup of $O(2) \times O(2)$ consisting of the identity element and the vertical reflection in $z = 0$. This vertical reflection acts on $\alpha\beta$-space by $(\alpha_1,\alpha_2,\alpha_3,\alpha_4,\beta_1,\beta_2) \to (\alpha_1,\alpha_2,-\alpha_3,-\alpha_4,\beta_1,-\beta_2)$. The fixed point set of Δ in K consists of displacements u in (4.1) whose third component vanishes on $z = 0$ and is odd in z and whose first two components are even in z.

Note that F_Δ consists of modes whose vertical phase is fixed. Thus F_Δ is three dimensional and may be parametrized by a pair (ξ,q_2) where $\xi = \alpha_1 + i\alpha_2 \in \mathbb{C}$ and $q_2 = \beta_1 \in \mathbb{R}$.

The normalizer $N(\Delta)$ is $O(2)$ corresponding to rotations about the z-axis and reflections in the vertical planes $\theta = \theta_0$. It acts on F_Δ by

$$(\xi,q_2) \mapsto (e^{i\theta}\xi,q_2) \qquad \text{(rotation by an angle } \theta)$$

and

$$(\xi,q_2) \mapsto (e^{2i\theta}\,\bar{\xi},q_2) \quad \text{(reflection in the plane } \theta = \theta_0)$$

Since $g|F_\Delta$ is invariant under $N(\Delta)$ it must be a function of $|\xi|^2$ and q_2.

To look for hidden symmetries we look at the fixed point set $F_{N(\Delta)}$ of $N(\Delta)$. This is the set of axisymmetric displacement; that is, $F_{N(\Delta)} = \{(\xi,q_2) \in F_\Delta | \xi = 0\}$. Observe using (4.1) that the quarter-period vertical translation maps elements of $F_{N(\Delta)}$ to their negatives. This is the hidden symmetry in Hunt's problem.

The set V in Lemma 3 can be chosen to be the set in F_Δ on which ξ is real, i.e. $\alpha_1 = 0$. By Lemma 3, we can restrict to V with no loss of information. Since $F_{N(\Delta)} \subset V$ the hidden symmetry still restricts $k = h|V$. Letting $q_1 = \alpha_1$ we see that k is a function of q_1^2, q_2^2 and $q_1^2 q_2$. (we note in passing that V is the fixed point set of the $\mathbb{Z}_2$ action $\xi \to \bar{\xi}$).

§5. REMARKS ON BIFURCATION AND STABILITY

Expanding k in a Taylor series, we get the form

$$k = \frac{1}{2}(aq_1^2 + bq_2^2) + cq_1^2 q_2 + dq_1^4 + eq_2^4 + fq_1^2 q_1^2 + \text{h.o.t.}$$

where "h.o.t." means "higher order terms". At a bifurcation point such as (2.1) k satisfies $a = b = 0$. Set

$$f = cq_1^2 q_2 + dq_1^4 + eq_2^4 + fq_1^2 q_2^2 + \text{h.o.t.}$$

As noted in §2, one sees that f is right equivalent to the parabolic umbilic, assuming $c, e \neq 0$. More precisely, f is right equivalent to

$$g = \frac{1}{4}\delta q_2^4 + \varepsilon q_1^2 q_2 ,$$

where $\delta = \text{sgn } e$ and $\varepsilon = \text{sgn } c$. The universal unfolding of the parabolic umbilic generally requires four parameters.

However, we consider here only those terms in the universal unfolding which are consistent with the present symmetry. This leads to an unfolding with just two free parameters, of the form

$$\tilde{G} = \frac{1}{2}(\alpha q_1^2 + \beta q_2^2) + \varepsilon q_1^2 q_2 + \frac{1}{4} q_2^4 \tag{5.1}$$

where we have chosen the + sign for the q_2^4 term. We expect one can show that $\tilde{G}$ is a universal unfolding for the parabolic umbilic in the context of hidden symmetries. Formal calculations indicate this is in fact correct; a rigorous argument may be possible directly or using the work of Damon [1983].

The bifurcation diagram for (5.1) is shown in Figure 5-1 for $\varepsilon = +1$. In this figure the $\alpha\beta$-plane is divided into 6 regions by the curves $\alpha = 0$, $\beta = 0$ and $\beta = -\alpha^2$. The unfolding $\tilde{G}$ has the same number of critical points for any two pairs of (α,β) which lie in the same region. The number of these critical points, along with the signs of the eigenvalues of $d^2\tilde{G}$ at those critical points, is given in Figure 5-1. We use s to indicate a negative eigenvalue and u to indicate a positive eigenvalue.

If we now consider a bifurcation problem $H(q_1,q_2,\lambda)$ depending on the distinguished parameter λ for which $H(q_1,q_2,0) = g(q_1,q_2)$. Then the unfolding theorem, which we assume valid in the context of hidden symmetries, allows us to identify $H(q_1,q_2,\lambda)$ for fixed λ with $G(q_1,q_2,\alpha(\lambda),\beta(\lambda))$ where $\alpha(0) = \beta(0) = 0$. Therefore, the bifurcation problem $H(q_1,q_2,\lambda)$ may be identified with a smooth path $(\alpha(\lambda),\beta(\lambda))$ in the unfolding space $\tilde{G}$. Cf. Golubitsky and Schaeffer [1979a].

For the particular problem at hand we identify λ with the load as in Hunt [1982] and related bifurcation problems. Our reasoning is that the load λ should enter the potential $V_{\underset{\sim}{\lambda}}$ of §3 as the coefficient of a positive definite quadratic term plus, perhaps, higher order terms. Under this assumption we identify $V_{\underset{\sim}{\lambda}}$ with the path $\alpha(\lambda) = \beta(\lambda) = -\lambda$; that is, we consider the bifurcation problem

$$H(q_1,q_2,\lambda) = -\frac{1}{2}\lambda(q_1^2 + q_2^2) + q_1^2q_2 + \frac{1}{4}q^4 \qquad (5.2)$$

Note we have taken $\varepsilon = +1$.

We now ask in which ways can the bifurcation problem H in (5.2) be perturbed. Since the curve $\alpha(\lambda) = \beta(\lambda) = -\lambda$ intersects each of the dividing curves in Figure 5-1 transversely, it is likely that the universal unfolding of H will depend on just one additional parameter σ.

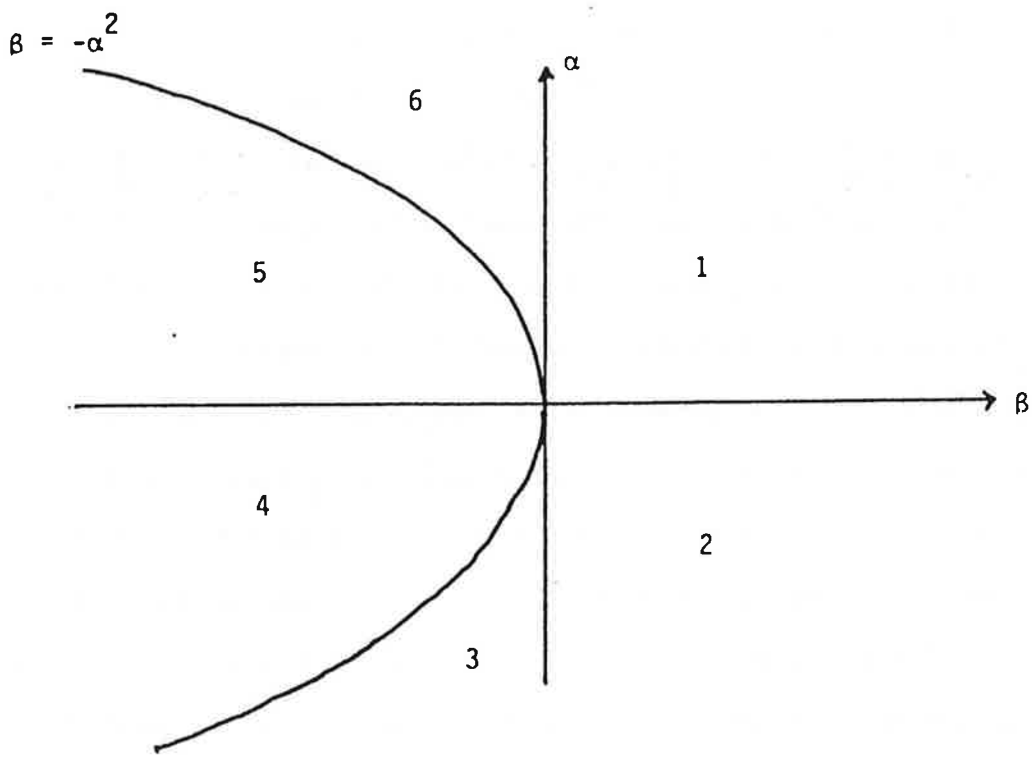


Figure 5-1.

We rewrite the unfolding (5.1) as

$$G(q_1,q_2,\lambda,\sigma) = -\frac{1}{2}\lambda(q_1^2 + q_2^2) + \frac{1}{2}\sigma(q_1^2 - q_2^2) + q_1^2 q_2 + \frac{1}{4}q^4 \tag{5.3}$$

where we have taken $\varepsilon = +1$. We wish to think of (5.3) as an unfolding with λ as the distinguished parameter and σ as an imperfection parameter.

We now study the bifurcation diagrams associated with G for fixed values of σ. For the reasons stated above, we feel that it is likely that G is a universal unfolding of H; however, we cannot prove this fact. In any case, G, itself, will give us sample kinds of behavior which may be found in Hunt's context.

By a bifurcation diagram we mean the set defined by $G_{q_1} = G_{q_2} = 0$ for fixed σ. These bifurcation diagrams are shown in Fig. 5-2 for $\sigma < 0$, $\sigma = 0$ and $\sigma > 0$. The stabilities may be computed directly (or recovered from Figure 5-1) and are shown schematically. Since G is even in q_1, we show only the orbits of the branches. In this figure the labels s and u refer to eigenvalues of the Hessian; s for a negative eigenvalue and u for a positive one.

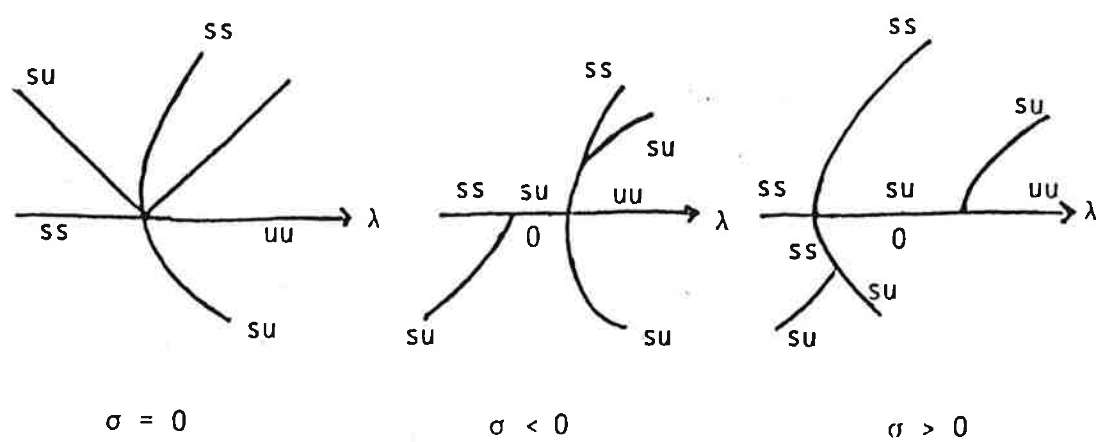


Figure 5-2

In the unperturbed problem ($\sigma = 0$) notice that as λ increases, the trivial (unbuckled) solution becomes unstable, the stability being picked up by the q_2 mode alone with $q_1 = 0$. Note that of the two axisymmetric states which are possible, concave or convex buckling, only one is stable. For the sake of argument we assume convex as in the first mode of Fig. 4-1. An interesting perturbation occurs for $\sigma > 0$. Here the shell buckles into one of the two asixymmetric states either convex or concave. Should the buckling be concave then the bifurcation diagram predicts that this solution will loose stability with a snap-through bifurcation to a convex axisymmetric state. For $\sigma < 0$ the existence of an interval in λ with no stable solutions suggests that the shell buckles to a state involving other modes, a situation not covered by this analysis.

As should be clear from the discussion above, we have not attempted to develop the theory of universal unfoldings for potentials with hidden symmetries either with or without a distinguished parameter. What we have tried to indicate is that such a theory might generate new and intersting behavior.

REFERENCES

1. M. Buchner, J. Marsden and S. Schecter [1983]. Examples for the infinite dimensional Morse lemma, SIAM J. Math. Anal. 14 (1983) 1045-1055.
2. E. Buzano and M. Golubitsky [1983]. Bifurcation on the hexagonal lattice and the planar Bénard problem, Phil. Trans. Roy. Soc. Lon. A308, 617-667.
3. J. Damon [1983]. The unfolding and deferminary theorems for subgroups of A and K insingularities, Proc. Symp. Pure Math. 40 pp. 233-254.
4. M. Golubitsky [1983]. The Bénard problem, symmetry and the lattice of isotropy subgroups. Bifurcation Theory, Mechanics and Physics. C.P. Bruter etal (eds). D. Reidel Publishing Co., 1983. 225-256.
5. M. Golubitsky and J. Marsden [1983]. The Morse lemma in infinite dimensions via singularity theory, SIAM J. Math. Anal. 14 (1983) 1037-1044.
6. M. Golubitsky and D. Schaeffer [1979a]. A theory for imperfect bifurcation via singularity theory. Comm. Pure. Appl. Math. 32, 21-98.

a situation not covered by this analysis.

As should be clear from the discussion above, we have not attempted to develop the theory of universal unfoldings for potentials with hidden symmetries either with or without a distinguished parameter. What we have tried to indicate is that such a theory might generate new and interesting behavior.

Acknowledgement Research supported in part by ARO contract DAAG-29-79-C-0086 (J.M., M.G.), DOE contract DE-AT03-82ER12097 (J.M.), NSF grant MCS-8101580 (MG) and NSF grant MCS-79-02010 (D.S.).

REFERENCES

1. M. Buchner, J. Marsden and S. Schecter [1983]. Examples for the infinite dimensional Morse lemma, SIAM J. Math. Anal. 14, 1045-1055.

2. E. Buzano and M. Golubitsky [1983]. Bifurcation on the hexagonal lattice and the planer Bénard problem, Phil. Trans. Roy. Soc. Lon. A308, 617-667.

3. J. Damon [1983]. The unfolding and deferminary theorems for subgroups of A and K. Singularities, Proc. Symp. Pure Math. 40, 233-254.

4. M. Golubitsky [1983]. The Bénard problem, symmetry and the lattice of isotropy subgroups. Bifurcation Theory, Mechanics and Physics. C.P. Bruter et al (eds). D. Reidel Publishing Co., 225-256.

5. M. Golubitsky and J. Marsden [1983]. The Morse lemma in infinite dimensions via singularity theory, SIAM J. Math. Anal. 14, 1037-1044.

6. M. Golubitsky and D. Schaeffer [1979a]. A theory for imperfect bifurcation via singularity theory. Commun. Pure. Appl. Math. 32, 21-98.

7. M. Golubitsky and D. Schaeffer [1979b]. Imperfect bifurcation in the presence of symmetries. Commun. Math. Phys. 69, 205-232.

8. M. Golubitsky and D. Schaeffer [1981]. Bifurcation with O(3) symmetry including applications to the Bénard problem. Commun. Pure Appl. Math. 35, 81-109.

9. M. Golubitsky, J. Swift and E. Knobloch [1984]. Symmetries and pattern selection in Rayleigh-Bénard convection. Physica D. To appear.

10. D. Hui and J.S. Hansen [1981]. The parabolic umbilic catastrophe and its application in the theory of elastic stability. Quart. Appl. Math. 39, 201-220.

11. G.W. Hunt [1981]. An algorithm for the nonlinear analysis of compound bifurcation. Phil. Trans. Roy. Soc. Lon. 300, 443-471.

12. G.W. Hunt [1982]. Symmetries of elastic buckling, Eng. Struct. 4, 21-28.

13. E. Ihrig and M. Golubitsky [1984]. Pattern selection with O(3) symmetry. Physica D. Nonlinear Phenomena. To appear.

14. J. Marsden and T.J.R. Hughes [1983]. Mathematical foundations of elasticity, Prentice-Hall.

15. P. Naghdi [1972]. The theory of Shells, in Handbuch der Physik (C. Truesdell, ed.) VIa/2, Springer.

16. R.S. Palais [1979]. The principle of symmetric criticality, Commun. Math. Phys. 69, 19-30.

17. D. Sattinger [1979]. Group theoretic methods in bifurcation theory, Springer Lecture Notes in Math. #762.

18. D. Schaeffer [1980]. Qualitative analysis of a model for boundary effects in the Taylor problem, Math. Proc. Camb. Phil. Soc. 87, 307-337.

19. E.C. Zeeman and D.J.A. Trotman [1975]. The classification of elementary catastrophes of codimension ≤ 5, Springer Lect. Notes Math. 525, 263-327.

Martin Golubitsky, Jerrold Marsden and David Schaeffer

University of Houston, University of California - Berkeley and Duke University