Offprint from "Archive for Rational Mechanics and Analysis",
Volume 87, Number 2, 1985, pp. 107–165

Hopf Bifurcation in the Presence of Symmetry

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Abstract

Using group theoretic techniques, we obtain a generalization of the HOPF Bifurcation Theorem to differential equations with symmetry, analogous to a static bifurcation theorem of CICOGNA. We discuss the stability of the bifurcating branches, and show how group theory can often simplify stability calculations. The general theory is illustrated by three detailed examples: $O(2)$ acting on R^2, $O(n)$ on R^n, and $O(3)$ in any irreducible representation on spherical harmonics.

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§ 1. Introduction

The aim of this paper is to generalize the Hopf bifurcation theorem (on branching to periodic solutions) to systems of differential equations with symmetry.

We state our results for ordinary differential equations, though we expect to be able to apply them to suitable partial differential equations by standard reductions. Specifically, suppose we have a system of ordinary differential equations

$$\frac{dv}{dt} + f(v, \lambda) = 0 \tag{1.1}$$

where $v(t) \in R^m$, $\lambda \in R$ is a bifurcation parameter, and $f: R^m \times R \to R^m$ is a smooth (C^∞) mapping commuting with the action of a compact Lie group Γ on R^m. That is,

$$f(\gamma v, \lambda) = \gamma f(v, \lambda), \gamma \in \Gamma. \tag{1.2}$$

Further assume $f(0, \lambda) = 0$, so that there is a trivial solution.

Let $(df)_{v,\lambda}$ be the $n \times n$ Jacobian matrix of derivatives of f with respect to the variables v, evaluated at (v, λ). The most important hypothesis of the standard Hopf theorem is that $(df)_{0,0}$ should have a pair of *simple* purely imaginary eigenvalues. In the presence of a symmetry group Γ, it is possible to arrange for eigenvalues of df to be purely imaginary by placing suitable restrictions on the action of Γ. However, in these cases Γ often forces these eigenvalues to be multiple. Hence, the standard Hopf theorem cannot be applied.

While the symmetries complicate the analysis by forcing multiple eigenvalues, they also potentially simplify it by placing restrictions on the form of the mapping f; in particular, the terms that can appear in the Taylor expansion. The same is true in static bifurcation theory (see SATTINGER [1979, 1983]). Often the simplification of the Taylor series is sufficient to counterbalance the complication of the eigenvalues. We exploit such a phenomenon here.

There are two ways in which a group Γ admits multiple imaginary eigenvalues for $(df)_{0,0}$ that can occur generically in a 1-parameter family.

(a) $R^m = V \oplus V$ where Γ acts absolutely irreducibly on V and by the diagonal action in $V \oplus V$.

(b) Γ acts irreducibly but not absolutely irreducibly on R^m.

Recall that Γ acts *absolutely irreducibly* on V if the only linear maps on V commuting with Γ are multiples of the identity. In this paper we consider only case (a). In § 2 we show that the equivariance condition (1.2) implies that $(df)_{0,\lambda}$ has two eigenvalues each with multiplicity $m/2$. It is possible for these eigenvalues to be complex conjugate pairs $\sigma(\lambda) \pm i\phi(\lambda)$. We consider the case where $\sigma(0) = 0$, $\phi(0) \neq 0$. In this case the system (1.1) has an m-dimensional center manifold. We prove our existence theorem assuming the usual transversality condition on the eigenvalues that cross the imaginary axis,

$$\sigma'(0) \neq 0. \tag{1.3}$$

One particular example of case (a), the group $O(2)$ acting on $R^2 \oplus R^2$, has been studied in detail by SCHECTER [1976] using methods of RUELLE [1973], and by BAJAJ [1982] using perturbation theory. These authors all emphasize the role of symmetry. The occurrence of both standing and rotating waves for $O(2)$ is demonstrated in ERNEUX & HERSCHKOWITZ-KAUFMANN [1977]; the stability properties of these waves were known to SATTINGER [1980]. Very recently this example has been analyzed by SATTINGER [1983], VAN GILS [1984], CHOSSAT & IOOSS [1984], as well as ourselves: see §§ 7, 10. Each of these papers proceeds from a slightly different point of view and the precise statements of the results therefore differ slightly.

The $O(2)$ example has been considered in the context of discrete dynamical systems by RUELLE [1973], who indicates that the case of ordinary differential equations can also be analyzed by his methods (see SCHECTER [1976]). RAND [1982] and RENARDY [1982] consider the related question of $SO(2)$ acting on $\mathbf{R}^2$, which falls into category (b) above.

Here we stress a general approach to such problems that is independent of the symmetry group Γ. Our main contribution is the idea of relating the existence of certain types of periodic solution to that of isotropy subgroups satisfying certain algebraic criteria.

Hopf bifurcation from a multiple eigenvalue has been studied, without symmetry assumption, by CHOW, MALLET-PARET, & YORKE [1978]. Under mild hypotheses they establish the existence of a branch of periodic solutions to (1.1). When symmetry is present, our methods yield the existence of many solution branches, each having prescribed symmetries.

To explain the general idea in its simplest setting, we now state and prove a simple Hopf theorem and indicate how this theorem is generalized to our main result, Theorem 5.1. In order to state the simple Hopf theorem, we need to introduce several group-theoretic ideas.

An important feature of bifurcations with symmetry is the occurrence of *spontaneous symmetry breaking*. The symmetry of a bifurcating branch of solutions decreases to a proper subgroup Σ of Γ. This subgroup Σ is called the isotropy subgroup of the solution. More precisely, if Γ acts on V and $v \in V$, then

$$\Sigma_v = \{\sigma \in \Gamma \mid \sigma v = v\} \tag{1.4}$$

is the *isotropy subgroup* of v. The *fixed-point subspace* of a subgroup $\Sigma \subset \Gamma$ is

$$V^\Sigma = \{v \in V \mid \sigma v = v \text{ for all } \sigma \in \Sigma\}, \tag{1.5}$$

it consists of all points in V whose symmetries include Σ.

Fixed-point subspaces have one important feature, which is the basis of our analysis here: Suppose $f\colon V \to V$ commutes with Γ, *i.e.*, $f(\gamma v) = \gamma f(v)$ for $v \in V$, $\gamma \in \Gamma$; then

$$f\colon V^\Sigma \to V^\Sigma. \tag{1.6}$$

To prove (1.6) observe that if $v \in V^\Sigma$ and $\sigma \in \Sigma$, then $f(v) = f(\sigma v) = \sigma f(v)$. Hence, σ fixes $f(v)$ and $f(v) \in V^\Sigma$ as claimed.

We can now state a very simple generalization of HOPF's theorem having an almost trivial proof. Let Γ act on $V \oplus V = \mathbf{R}^m$ by the diagonal action.

Theorem 1.1. (*The simple Hopf theorem*). *Suppose Σ is an isotropy subgroup of Γ such that* $\dim (V \oplus V)^\Sigma = 2$. *If* (1.3) *holds, then there is a branch of small amplitude periodic solutions to* (1.1) *whose group of symmetries is Σ.*

Proof. The commutativity of f in (1.1) implies that f maps $(V \oplus V)^{\Sigma} \times \boldsymbol{R}$ to $(V \oplus V)^{\Sigma}$. Restricting the system (1.1) to the plane $(V \oplus V)^{\Sigma}$ yields a 2×2 system of differential equations that satisfy the hypotheses of Hopf's standard theorem. *Cf.* MARSDEN & MCCRACKEN [1976] or HASSARD, KAZARINOFF & WAN [1981]. Solutions in this plane have the appropriate symmetry. □

Remark. If the trivial solution is stable subcritically, then subcritical branches are unstable. This follows by exchange of stability in the standard Hopf theorem. However, supercritical branches may or may not be stable, since directions not in $(V \oplus V)^{\Sigma}$ will be involved.

The basic observation in this paper is that a further generalization of HOPF's theorem is possible. One of the standard proofs of Hopf bifurcation is obtained via a Liapunov-Schmidt reduction. This reduction leads to an additional $\boldsymbol{S}^1$ (circle group) symmetry, arising from the dynamic phase-shift symmetry of periodic solutions. (This observation appears also in SATTINGER [1983].) More precisely, we define the isotropy subgroup of a periodic solution $v(t)$ of (1.1) to be

$$\Sigma_{v(t)} = \{(\gamma, \theta) \in \Gamma \times \boldsymbol{S}^1 \mid v(t) = \gamma v(t + \theta)\}. \tag{1.7}$$

Thus we now envisage a combination of both spatial and temporal symmetries.

As we show in § 4, the Liapunov-Schmidt reduction leads to a reduced bifurcation equation on $V \otimes \boldsymbol{C}$ where $\Gamma \times \boldsymbol{S}^1$ acts on $V \otimes \boldsymbol{C}$ by $(\gamma, \theta)\, v \otimes z = \gamma v \otimes e^{i\theta} z$. See § 3 for a discussion of this action. The isotropy subgroups (1.7) can be identified with subgroups of $\Gamma \times \boldsymbol{S}^1$ acting on $V \otimes \boldsymbol{C}$. See Theorem 4.1. Our main theorem states that if Σ is an isotropy subgroup of $\Gamma \times \boldsymbol{S}^1$ for which

$$\dim (V \otimes \boldsymbol{C})^{\Sigma} = 2, \tag{1.8}$$

then (locally) there exists a branch of small-amplitude periodic solutions to (1.1) having spatio-temporal symmetry Σ. See Theorem 5.1.

This result may be viewed as an analogue, for periodic bifurcations, of a static bifurcation theorem due to VAN DER BAUWHEDE [1980] and CICOGNA [1982], where the corresponding assumption is that the fixed-point subspace of Σ is 1-dimensional. (A special case follows from MICHEL [1972].) The point of this assumption on the fixed-point subspace is that maximal isotropy subgroups, with minimal-dimensional fixed-point subspaces, lead to solutions. This remark is discussed in detail in § 12.

The remainder of the paper splits into three parts.

(a) A discussion of which subgroups of $\Gamma \times \boldsymbol{S}^1$ can be isotropy subgroups, and how to compute $\dim (V \otimes \boldsymbol{C})^{\Sigma}$ for subgroups of $\Gamma \times \boldsymbol{S}^1$. See § 6 and § 13.

(b) How to compute the (orbital asymptotic) stability of these periodic solutions. See § 8.

(c) The computation of specific examples. Here we discuss $\Gamma = \boldsymbol{O}(2)$ acting on $V = \boldsymbol{R}^2$, §§ 7, 9, 10; $\Gamma = \boldsymbol{O}(n)$ acting on $\boldsymbol{R}^n$, § 11, and every irreducible representation of $\Gamma = \boldsymbol{O}(3)$, §§ 14, 15.

Our discussion of stability exploits a suggestion of John Guckenheimer to assume that the vector field f is in Γ-normal form, and hence commutes with $\Gamma \times \boldsymbol{S}^1$, not just Γ. In this case, the Liapunov-Schmidt reduction may be computed explicitly. See Proposition 4.3. We use the extra S^1-symmetry to solve explicitly the Floquet equations (Lemma 8.3) and show that the eigenvalues of the linearized reduced equations give the Floquet multipliers. See Theorem 8.2.

As mentioned above, the specific example of $O(2)$ acting on $\boldsymbol{R}^2$ was considered by SCHECTER [1976] and others. There are two types of periodic solutions: a spatial branch with $\boldsymbol{Z}_2$ symmetry and a branch of *rotating waves* with symmetry

$$\widetilde{SO}(2) = \{(\psi, -\psi) \mid \psi \in SO(2) \subset O(2)\}.$$

Both branches occur together provided (1.3) holds. Rotating waves were noted by AUCHMUTY [1978] in reaction-diffusion equations. To obtain the rotating wave solutions by our methods, the simple Hopf theorem 1.1 is insufficient, and one needs the more general existence theorem 5.1. Indeed solutions with nonspatial symmetry *are* a generalized form of rotating wave. We recover the existence results by our group-theoretic techniques in § 7.

Schecter and others also analyzed the stabilities of branches. We recover these results in § 10, using general group-theoretic ideas to complete the stability analysis, at least when f is in Γ-normal form. We find that a bifurcating branch (of either type) can be stable only when both branches are supercritical. When both branches are supercritical, exactly one is orbitally asymptotically stable. Which branch is stable depends on third-order terms in the reduced equation. See § 10 and the bifurcation diagrams in Figure 10.1. Interestingly, these results carry over, with little additional computation, to $O(n)$ acting in its standard representation on $\boldsymbol{R}^n$. In particular, rotating waves may be found for $O(n)$ acting diagonally on $\boldsymbol{R}^n \oplus \boldsymbol{R}^n$, and can in some cases be stable. See § 11.

The final sections, §§ 12–14, concern $O(3)$ acting in any of its irreducible representations on spherical harmonics V_l. (For the analogous static bifurcation problem see IHRIG & GOLUBITSKY [1984].) In particular, we compute here all isotropy subgroups of $O(3) \times S^1$ with 2-dimensional fixed-point subspaces. A surprise here is the appearance of a branch with *tetrahedral* symmetry (twisted into a spatial and temporal part), when $l = 2, 4, 5, 6, 7, 9$. *Cf.* § 14. The existence of such periodic solutions seems to be a new phenomenon. In addition, there are a variety of spatial and rotating wave solutions. See Tables 14.1 and 14.2.

Throughout our work we have placed emphasis on general considerations related to symmetry, resorting to detailed calculations only after making the effects of symmetry explicit. The examples show how efficient this approach can be when the group action is sufficiently well behaved.

For simplicity we have assumed throughout that f in (1.1) is defined on $V \oplus V$, with Γ acting absolutely irreducibly on V. However, standard methods can be used to extend the results to a more general situation. In particular we have:

Theorem 1.2. *Consider the differential equation*

$$\frac{dv}{dt} + F(v, \lambda) = 0 \tag{1.9}$$

where $F: \boldsymbol{R}^M \times \boldsymbol{R} \to \boldsymbol{R}^M$ *commutes with an action of* Γ *on* $\boldsymbol{R}^M$. *Suppose* $(dF)_{0,0}$ *has imaginary eigenspace* $V \oplus V \subset \boldsymbol{R}^M$ *where* Γ *acts absolutely irreducibly on* V.

Identify $V \oplus V$ *with* $V \otimes \boldsymbol{C}$, *and let* $\Sigma \subset \Gamma \times S^1$ *be an isotropy group with 2-dimensional fixed-point subspace. Then there exists a branch of small amplitude periodic solutions to* (1.9) *whose group of symmetries is* Γ.

Theorem 1.2 may be proved in the same way as Theorem 5.1, because the Liapunov-Schmidt reduction leads to the same situation. Theorem 1.2 is a setting for our results that extends naturally to partial differential equations. □

Remark 1.3. Theorem 1.2, as stated, yields a periodic solution branch for each suitable isotropy subgroup. However, one effect of symmetry is to relate many of these solutions to each other. To see this, let $x(t)$ be a solution of (1.1) with isotropy subgroup Σ, and let (γ, θ) belong to $\Gamma \times S^1$. Then $\gamma x(t+\theta)$ is a periodic solution to (1.1) with isotropy subgroup $(\gamma, \theta)\Sigma(\gamma, \theta)^{-1} \subset \Gamma \times S^1$. We wish to enumerate the distinct trajectories of these symmetry-related solutions. There are two sources of redundancy. First, change of phase yields solutions with identical trajectories. Second, the action of elements of the isotropy subgroup Σ yields exactly the same solution. Letting $\pi: \Gamma \times S^1 \to \Gamma$ be projection, we see that the manifold $\Gamma/\pi(\Sigma)$, of dimension $\dim \Gamma - \dim \Sigma$, parametrizes the distinct trajectories. The union of these trajectories is an invariant submanifold for (1.1) diffeomorphic to $(\Gamma \times S^1)/\Sigma$, of dimension $\dim \Gamma - \dim \Sigma + 1$, and foliated by periodic trajectories. For example, if $\Gamma/\pi(\Sigma)$ is diffeomorphic to a circle, then this invariant submanifold is a 2-torus.

In general, this invariant submanifold is the union of a finite number of mutually diffeomorphic connected components. Typically, in applications, these components correspond to physically observable distinctions between solutions, such as direction of rotation.

Remark 1.4. One consequence of Remark 1.3 is that we require only one representative from each conjugacy class of isotropy subgroups, when applying Theorem 1.2. These representatives may be found by computing the isotropy subgroup of a representation of each orbit of $\Gamma \times S^1$ on $V \otimes \boldsymbol{C}$. We follow this procedure in the examples below.

§ 2. Conditions for Imaginary Eigenvalues

Let $f: \boldsymbol{R}^m \times \boldsymbol{R} \to \boldsymbol{R}^m$ commute with the action of a compact Lie group Γ. The requirement that the Jacobian $(df)_{0,0}$ have purely imaginary eigenvalues imposes restrictions on the representation of Γ. To see this, decompose $\boldsymbol{R}^m$ into a direct sum of irreducible Γ-invariant subspaces

$$\boldsymbol{R}^m = V_1 \oplus \ldots \oplus V_k. \tag{2.1}$$

We assert (and prove below) that if $(df)_{0,0}$ has purely imaginary eigenvalues, then either:

(a) some irreducible representation of Γ occurs at least twice in the decomposition (2,1) or (2.2)
(b) the action of Γ on some V_j is not absolutely irreducible.

Recall that a representation of Γ is *absolutely irreducible* if the only linear maps commuting with Γ are multiples of the identity.

We shall not discuss (2.2b) further in this paper. The simplest way for (2.2a) to occur is for $\mathbf{R}^m$ to be the direct sum of two isomorphic absolutely irreducible subspaces. By identifying these subspaces suitably we may assume that

$$\mathbf{R}^m = V \oplus V \tag{2.3}$$

where Γ acts diagonally:

$$\gamma(v, w) = (\gamma v, \gamma w), \qquad \gamma \in \Gamma. \tag{2.4}$$

In this paper, we focus attention on the representation of Γ defined by (2.3) and (2.4) even though some of our results could be stated more generally.

To prove our assertion, we must show that $(df)_{0,0}$ has real eigenvalues if conditions (2.2) fail (that is, if each V_i in (2.1) is a distinct absolutely irreducible subspace). Now f commutes with Γ, that is, $f(\gamma u, \lambda) = \gamma f(u, \lambda)$ for $\gamma \in \Gamma$. Differentiate this relation using the chain rule to obtain

$$(df)_{\gamma u,\lambda}\gamma = \gamma(df)_{u,\lambda}, \qquad \gamma \in \Gamma. \tag{2.5}$$

Since $\gamma 0 = 0$, $(df)_{0,0}$ commutes with Γ. Each V_j is a distinct irreducible representation of Γ, so that $(df)_{0,0}(V_j) \subset V_j$ (see DORNHOFF [1971]). Since Γ acts absolutely irreducibly on V_j, (2.5) implies that on V_j, $(df)_{0,0}$ is a multiple of the identity. Thus all eigenvalues of $(df)_{0,0}$ are real.

We now consider the case (2.3) where $f: (V \oplus V) \times \mathbf{R} \to V \oplus V$ commutes with Γ. Let $n = \dim V$.

Lemma 2.1. *Suppose $(df)_{0,0}$ has i as an eigenvalue. Then*

(a) *The eigenvalues of $(df)_{0,\lambda}$ consist of a complex conjugate pair $\sigma(\lambda) \pm i\phi(\lambda)$, each of multiplicity n. Moreover, σ and ϕ are smooth functions of λ.*

(b) *There is an invertible linear map $S: V \oplus V \to V \oplus V$, commuting with Γ, such that*

$$(df)_{0,0} = SJS^{-1}$$

where

$$J \equiv \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}. \tag{2.6}$$

Remark. In fact, in the proof of Lemma 2.1(a) we show that each eigenvalue of $(df)_{0,0}$ has multiplicity at least n.

Proof. (a) Let $T: V \oplus V \to V \oplus V$ be a linear map commuting with Γ and write T in the block form

$$T = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where A, B, C, D are $n \times n$ matrices. T commutes with the diagonal action of Γ if and only if each of A, B, C, D commutes with the action of Γ on V. The absolute irreducibility of Γ acting on V implies that

$$T = \begin{pmatrix} aI & BI \\ cI & dI \end{pmatrix}. \tag{2.7}$$

It is well known that if A, B, C and D commute, then

$$\det \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \det (AD - BC).$$

(*Cf.* HALMOS [1974] p. 102, ex. 9.) It follows that the characteristic polynomial of T is

$$\det (T - \mu I) = [(a - \mu)(d - \mu) - bc]^n. \tag{2.8}$$

Thus each of the eigenvalues of T occurs with multiplicity at least n.

Since $(df)_{0,\lambda}$ commutes with Γ and has a pair of complex conjugate eigenvalues it follows that each eigenvalue occurs with multiplicity n. The smoothness of $\sigma(\lambda)$ and $\phi(\lambda)$ also follows from (2.8).

(b) As above, $(df)_{0,0}$ has the form $\begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}$. From (2.8) we see that for $(df)_{0,0}$ to have i as an eigenvalue we must have $a + d = 0$, $ad - bc = 1$. We now conjugate $(df)_{0,0}$ so that $a = d = 0$. Assuming $a \neq 0$, define

$$R_\theta = \begin{pmatrix} \cos\theta I & -\sin\theta I \\ \sin\theta I & \cos\theta I \end{pmatrix},$$

which commutes with Γ, and choose θ so that

$$\cot(2\theta) = \frac{b + c}{2a}.$$

Then

$$R_\theta (df)_{0,0} R_\theta^{-1} = \begin{pmatrix} 0 & hI \\ -h^{-1}I & 0 \end{pmatrix}$$

for $h \in \mathbf{R}$. Finally, note that

$$J = \begin{pmatrix} I & 0 \\ 0 & -hI \end{pmatrix} \begin{pmatrix} 0 & hI \\ -h^{-1}I & 0 \end{pmatrix} \begin{pmatrix} I & 0 \\ 0 & -h^{-1}I \end{pmatrix}.$$

□

§ 3. The Action of $\Gamma \times S^1$

In the next section we show that our search for periodic solutions of (1.1) leads to a mapping

$$\phi : (V \oplus V) \times \mathbf{R} \times \mathbf{R} \to V \oplus V \tag{3.1}$$

which commutes not only with the diagonal action of Γ, but also with an action of the circle group S^1. In this section we give two presentations of the representation of $\Gamma \times S^1$ on $V \oplus V$ and we show that they are equivalent. Then, we describe, in general, the linear maps that commute with $\Gamma \times S^1$.

(*a*) *The representation of* $\Gamma \times S^1$ *on* $V \oplus V$. Fix an orthonormal basis of V and let the vector $(x_1, \dots, x_n)$ denote the coordinates of the vector $x \in V$ in this basis. The action of Γ on V allows us to identify each $\gamma \in \Gamma$ with an $n \times n$ matrix $\varrho(\gamma)$ acting on (the coordinates of) V.

We may identify $(x, y) \in V \oplus V$ with the $n \times 2$ matrix

$$Z = \begin{pmatrix} x_1 & y_1 \\ \vdots & \vdots \\ x_n & y_n \end{pmatrix}. \tag{3.2}$$

Define an action of $\Gamma \times S^1$ on $V \oplus V$ by

$$(\gamma, \theta) \cdot Z = \varrho(\gamma)\, Z R_\theta \tag{3.3}$$

where

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \tag{3.4}$$

is the 2×2 rotation through the angle $\theta \in S^1$.

This action of $\Gamma \times S^1$ may also be understood using tensor product notation. Identify $V \oplus V$ with $V \otimes \mathbf{C}$ by

$$(x, y) \to x \otimes 1 + y \otimes i. \tag{3.5}$$

Now define an action of $\Gamma \times S^1$ on $V \otimes \mathbf{C}$ by

$$(\gamma, \theta) \cdot (v \otimes z) = (\gamma v) \otimes e^{i\theta} z. \tag{3.6}$$

Alternatively, we may calculate (γ, θ) acting on $v \otimes z = x \otimes 1 + y \otimes i$ and obtain

$$(\cos\theta\, x - \sin\theta\, y) \otimes 1 + (\sin\theta\, x + \cos\theta\, y) \otimes i.$$

Using the identifications (3.5) and (3.2) we see that the action $\Gamma \times S^1$ defined by (3.6) and the one defined by (3.3) are identical.

Remark 3.1. In later sections we sometimes consider the diagonal action of Γ on $V \oplus V$, and sometimes the action of $\Gamma \times S^1$ on $V \oplus V$ just defined. To reduce confusion we will use $V \oplus V$ in the former case and $V \otimes \mathbf{C}$ in the latter.

(*b*) *The Linear Equivariants of* $\Gamma \times S^1$.

Lemma 3.2. *Let* $T: V \otimes \mathbf{C} \to V \otimes \mathbf{C}$ *be a linear map that commutes with* $\Gamma \times S^1$. *Then*

$$T = aI + cJ \tag{3.7}$$

where $a, c \in \mathbf{R}$ *and*

$$I = \begin{pmatrix} I_n & 0 \\ 0 & I_n \end{pmatrix} \quad \text{and} \quad J = \begin{pmatrix} 0 & -I_n \\ I_n & 0 \end{pmatrix}.$$

Remarks 3.3. (a) The matrix J may be defined in a coordinate-free manner using the action of S^1 and differentiation. Using (3.3) we compute

$$J(1,\theta)\begin{pmatrix} x \\ y \end{pmatrix} = -\frac{d}{d\theta}(1,\theta)\cdot\begin{pmatrix} x \\ y \end{pmatrix} \tag{3.8}$$

where the $2n$-vector $\begin{pmatrix} x \\ y \end{pmatrix}$ is identified with the $n \times 2$ matrix Z in the obvious way.

(b) J is the matrix form of the action of $(1, -\frac{1}{2}\pi) \in \Gamma \times S^1$. In this way we see that J commutes with the action of $\Gamma \times S^1$ on $V \otimes \mathbf{C}$. In particular, every matrix T of the form (3.7) commutes with $\Gamma \times S^1$.

(c) Note that Lemma 3.2 implies that the action of $\Gamma \times S^1$ on $V \otimes \mathbf{C}$ is *not* absolutely irreducible, even though it is irreducible.

Proof. As in the proof of Lemma 2.1(a) we may write T in the block matrix form $\begin{pmatrix} A & B \\ C & D \end{pmatrix}$. Since T commutes with the diagonal action of Γ, T must have the form (2.7), that is, $T = \begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}$. We now note that T commutes with S^1 only if the 2×2 matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ commutes with R_θ, $\theta \in S^1$. This happens precisely when $d = a$ and $b = -c$; that is, when (3.7) is satisfied. □

§ 4. The Liapunov-Schmidt Reduction

It is well known that the Hopf theorem can be proved using a Liapunov-Schmidt reduction (see HALE [1969]). Here we adapt this approach to the symmetric case. We follow the exposition of GOLUBITSKY & LANGFORD [1981] where the role of the S^1-symmetry arising from phase shifts is emphasized. See GOLUBITSKY & SCHAEFFER [1985] for a similar treatment.

Consider the differential equation

$$\left.\frac{dv}{dt}\right| + f(v, \lambda) = 0 \tag{4.1}$$

where $f: (V \oplus V) \times \mathbf{R} \to V \oplus V$ is infinitely differentiable and commutes with the diagonal action of Γ. (As above, we assume that Γ is a compact Lie group acting absolutely irreducibly on V.) Here $\lambda \in \mathbf{R}$ is a distinguished parameter, assumed to be near 0, that will act as a bifurcation parameter.

Note that (provided $\dim V > 1$) the commutativity of f with Γ guarantees the existence of a trivial steady state solution to (4.1); that is, $f(0, \lambda) \equiv 0$. (To prove this fact, observe that if $\dim V > 1$ and V is irreducible, then $\gamma \cdot x = x$ for all $\gamma \in \Gamma$ only if $x = 0$. It follows that $\gamma \cdot (x, y) = (x, y)$ for all γ only if $(x, y) = (0, 0)$. Now use the commutativity of f to conclude that $\gamma \cdot f(0, \lambda) = f(0, \lambda)$ for all $\gamma \in \Gamma$ and that $f(0, \lambda)$ is zero.)

The implementation of the Liapunov-Schmidt reduction requires one assumption:

$$\text{(H1)}\quad (df)_{0,0} = J.$$

Remarks. (a) If $(df)_{0,0}$ has non-zero purely imaginary eigenvalues, then we may rescale time t in (4.1) so that the eigenvalues of $(df)_{0,0}$ are precisely $\pm i$.

(b) We may now apply Lemma 2.1(b) to observe that $(df)_{0,0}$ is similar to the matrix J, where the similarity commutes with Γ. Therefore, we may make a linear change of coordinates on V, and hence on $V \oplus V$, so that the system (4.1) satisfies (H1). We now see that (H1) is equivalent, up to a linear change of coordinates, to the assumption that $(df)_{0,0}$ has a non-zero purely imaginary eigenvalue.

(c) We do not require the imaginary eigenvalues of $(df)_{0,0}$ to be simple. Indeed, they will have multiplicity $n = \dim V$. So the usual hypothesis of the Hopf theorem does not apply.

We now state three results. Proofs are given later in this section.

Theorem 4.1. *Assume* (H1). *Then there exists a* reduced bifurcation equation

$$\phi(x, y, \lambda, \tau) = 0 \tag{4.2}$$

where $\phi: (V \otimes \mathbf{C}) \times \mathbf{R} \times \mathbf{R} \to V \otimes \mathbf{C}$ *is infinitely differentiable and commutes with* $\Gamma \times \mathbf{S}^1$, *such that the small amplitude periodic solutions of* (4.1) *with period* $2\pi/(1 + \tau)$ *are in one-to-one correspondence with those solutions* (x, y, λ, τ) *to* (4.2) *which are near* (0, 0, 0, 0).

The equivariance of ϕ with the action of $\Gamma \times S^1$ allows us to compute explicitly several terms in the Taylor expansion of ϕ. Since ϕ commutes with S^1 it commutes with R_π, so ϕ is an odd function. Using Lemma 3.2 (which characterizes the linear maps commuting with $\Gamma \times S^1$) we see that

$$\phi(x, y, \lambda, \tau) = P(\lambda, \tau)\begin{pmatrix} x \\ y \end{pmatrix} + Q(\lambda, \tau)\begin{pmatrix} -y \\ x \end{pmatrix} + \dots \tag{4.3}$$

where $+ \dots$ indicates term of degree at least three in x and y.

Using Lemma 2.1(a) we know that the eigenvalues of $(df)_{0,\lambda}$ are $\sigma(\lambda) \pm i\phi(\lambda)$ where $\sigma(0) = 0$, $\phi(0) = 1$ and $\sigma'(0)$ exists. Using the notation of (4.3), we have

Proposition 4.2.

$$\begin{aligned} &\text{(a)}\quad P(0, 0) = 0, \quad Q(0, 0) = 0,\\ &\text{(b)}\quad P_\tau(0, \tau) = 0, \quad Q_\tau(0, \tau) = -1,\\ &\text{(c)}\quad P_\lambda(0, 0) = \sigma'(0). \end{aligned} \tag{4.4}$$

There is a special case for which the reduced bifurcation equation ϕ may be computed explicitly in terms of f.

Proposition 4.3. Suppose that $f: (V \oplus V) \times \boldsymbol{R} \to V \oplus V$ commutes with the action of $\boldsymbol{S}^1$, as well as with the action of Γ. Then

$$\phi(x, y, \lambda, \tau) = f(x, y, \lambda) - (1 + \tau)\, J \begin{pmatrix} x \\ y \end{pmatrix}. \tag{4.5}$$

Note that (4.4) may be verified directly from (4.5) in this special case.

In the next section, we use Theorem 4.1 and Proposition 4.2 to prove the existence of certain types of periodic solution to (4.1). In § 8 we use Proposition 4.3 to discuss the (orbital) stability of these solutions. The remainder of this section is devoted to proofs of the above results. Since most of the ideas for these proofs may be found in the proof of the standard Hopf theorem, we shall be brief. Notation is chosen to conform with that of GOLUBITSKY & SCHAEFFER [1985].

Proof of Theorem 4.1. Let $C_{2\pi}$ and $C^1_{2\pi}$ respectively denote the Banach spaces of continuous and continuously differentiable 2π-periodic mappings $u: \boldsymbol{R} \to V \oplus V$, with norms $|u| = \max_{0 \leqq s \leqq 2\pi} |u(s)|$ and $|u|_1 = |u| + \left|\frac{du}{ds}\right|$.

In this proof we exploit an action of $\Gamma \times \boldsymbol{S}^1$ on $C_{2\pi}$. The action is given by

$$(\gamma, \theta) \cdot u(s) = \gamma u(s + \theta); \tag{4.6}$$

so Γ acts spatially and θ acts by a phase shift.

In order to look only for 2π-periodic solutions we rescale time t in (4.11) by

$$s = (1 + \tau)\, t, \qquad u(s) = v(t). \tag{4.7}$$

Then we may rewrite (4.1) as a nonlinear operator equation:

$$\Phi(u, \lambda, \tau) \equiv (1 + \tau) \frac{du}{ds} + f(u, \lambda) = 0 \tag{4.8}$$

where $\Phi: C^1_{2\pi} \times \boldsymbol{R} \times \boldsymbol{R} \to C_{2\pi}$. Note that if $\Phi(u, \lambda, \tau) = 0$ then $v(t) = u(s)$, as in (4.7), is a $2\pi/(1 + \tau)$-periodic solution to (4.1). Also observe that Φ commutes with the action of $\Gamma \times \boldsymbol{S}^1$ defined in (4.6).

To solve the equation $\Phi = 0$ we apply the Liapunov-Schmidt procedure. Let L be the linearization of Φ about the trivial solution $u = 0$, that is,

$$L \equiv (d\Phi)_{0,0,0} = \frac{d}{ds} + J \tag{4.9}$$

using (H1). In particular, (H1) implies that there are $2n$ linearly independent 2π-periodic solutions to the linear differential equation $Lu = 0$. These solutions are

$$\mathrm{Re}\,(e^{is}E_j), \qquad \mathrm{Im}\,(e^{is}E_j), \qquad j = 1, \ldots, n \tag{4.10}$$

where E_j is the $2n$-vector with i in the j^{th} position and 1 in the $(n + j)^{\text{th}}$ position.

We digress to observe that we may identify $\ker L$ with $V \otimes \boldsymbol{C}$ in such a way that the action of $\Gamma \times \boldsymbol{S}^1$ on $\ker L$ given by (4.6) becomes the standard action of $\Gamma \times \boldsymbol{S}^1$ on $V \otimes \boldsymbol{C}$ described in § 3. Since Φ commutes with $\Gamma \times \boldsymbol{S}^1$, so does the linearization L. It follows that both $\ker L$ and range L are invariant subspaces under the action of $\Gamma\times S^1$. To determine this action explicitly, we identify $(x, y)\in V\oplus V$ with

$$u(s) = \sum_{j=1}^{n} x_j \operatorname{Re}(e^{is}E_j) + y_j \operatorname{Im}(e^{is}E_j). \tag{4.11}$$

A computation using (4.11) shows that the action of $\Gamma\times S^1$ on $u(s)$, defined by (4.6), leads to the standard action of $\Gamma\times S^1$ on the $n\times 2$ matrix (x, y) defined by (3.3).

We now proceed with the Liapunov-Schmidt reduction. Since the formal adjoint of L is

$$L^* = -\frac{d}{ds} + J^t = -\frac{d}{ds} - J = -L,$$

we may use the Fredholm alternative to write

$$C_{2\pi} = \ker L^* \oplus \operatorname{range} L = \ker L \oplus \operatorname{range} L. \tag{4.12}$$

Using the splitting (4.12), let

$$E: C_{2\pi} \to \operatorname{range} L$$

be the projection whose kernel is $\ker L$. Now split the equation $\Phi = 0$ into

$$\begin{aligned} &\text{(a)}\quad E\Phi(u, \lambda, \tau) = 0,\\ &\text{(b)}\quad (I-E)\,\Phi(u, \lambda, \tau) = 0. \end{aligned} \tag{4.13}$$

If we write $u = v + w$ where $v\in\ker L$, $w\in\operatorname{range} L$, then by the implicit function theorem we may solve (4.13a) for w, obtaining $w = W(v, \lambda, \tau)$, so that

$$E\Phi(v + W(v, \lambda, \tau), \lambda, \tau) \equiv 0. \tag{4.14}$$

Substituting for w in (4.13b) leads to the reduced bifurcation equation

$$\phi(v, \lambda, \tau) = (I-E)\,\Phi(v + W(v, \lambda, \tau), \lambda, \tau) \tag{4.15}$$

where $\phi: \ker L\times \mathbf{R}\times \mathbf{R} \to \ker L$. Now the generalities of the Liapunov-Schmidt procedure in the presence of symmetries imply that ϕ commutes with the action of $\Gamma\times S^1$ on $\ker L$ and that the linear terms vanish identically, that is,

$$(d\phi)_{0,0,0} \equiv 0. \tag{4.16}$$

See Sattinger [1979, 1983] or Golubitsky & Schaeffer [1985].

Moreover, the Liapunov-Schmidt procedure shows that locally the zeros of ϕ are in one-to-one correspondence with the zeros of Φ. □

Proof of Proposition 4.2. Observe that $P(0, 0)$ and $Q(0, 0)$ are linear terms in ϕ and must vanish by (4.16). Next note that the subspaces

$$V_k = \mathbf{R}\{\operatorname{Re}(e^{is}E_k), \quad \operatorname{Im}(e^{is}E_k)\} \subset \ker L,$$

$k = 1, \ldots, n$, are S^1-invariant. One may now perform the same calculations as in GOLUBITSKY & LANGFORD [1981], for ϕ restricted to V_k, to obtain (4.4b, c). See also GOLUBITSKY & SCHAEFFER [1985]. These calculations require only the linear terms in v. □

Proof of Proposition 4.3. We claim that under the assumption that f commutes with S^1 as well as Γ, the choice $W(v, \lambda, \tau) \equiv 0$ provides a solution to (4.14). The main observation needed to prove this is the following:

$$\text{If } v(s) \in \ker L, \quad \text{then } f(v(s), \lambda) \in \ker L. \tag{4.17}$$

Suppose that $W(v, \lambda, \tau) \equiv 0$ is a solution of (4.14). By uniqueness of solutions given by the implicit function theorem, we know that $W = 0$ is the only solution. A computation using (4.15) and (4.8) shows that

$$\phi(v, \lambda, \tau) = (I - E)\ [(1 + \tau)\frac{dv}{ds} + f(v, \lambda)].$$

We assert:

$$\text{If } v \in \ker L, \quad \text{then } \frac{dv}{ds} = -Jv \in \ker L. \tag{4.18}$$

Since $I - E$ equals the identity on $\ker L$, it follows from (4.17) and (4.18) that ϕ has the form (4.5). Here we use the coordinates on $\ker L$ given by (4.11).

To complete the proof we must verify (4.18), (4.17), and show that $W = 0$ satisfies (4.14.) We begin by observing from the definition of L in (4.9) that $Lv = 0$ implies $dv/ds = -Jv$. Moreover, since f and hence Φ commute with the spatial symmetry S^1, it follows that L commutes with S^1. Since J is just rotation by $-\frac{1}{2}\pi$ (see Remark 3.3(i)), it follows that L commutes with J and that $Jv \in \ker L$. This verifies (4.18).

To verify (4.17) we observe that every $v(s) \in \ker L$ has the form

$$v(s) = (1, s) \cdot v_0 \tag{4.19}$$

where v_0 is a fixed vector in $V \oplus V$ and s acts on v_0 via the spatial action of S^1 given in (3.3). To see this, first note, using (3.8), that every v of the form (4.19) is a solution to $Lv = 0$. Now use a dimension count to show that every periodic function in $\ker L$ has the form (4.19). The observation (4.17) follows from a short calculation:

$$\begin{aligned} Lf(v) &= \frac{d}{ds} f(v(s), \lambda) + Jf(v(s), \lambda), \\ &= \frac{d}{ds}[(1, s) f(v_0, \lambda)] + J \cdot (1, s) f(v_0, \lambda), \\ &= -J(1, s) \cdot f(v_0, \lambda) + J \cdot (1, s) f(v_0, \lambda), \\ &= 0. \end{aligned}$$

Here the second equality uses the commutativity of f with S^1 and the third equality uses (4.18).

Finally, we show that $W=0$ satisfies (4.14). Use (4.17) and (4.18) to observe that $\Phi(v, \lambda, \tau) \in \ker L$ and note that $\ker E = \ker L$. □

Remarks 4.4. (a) We have shown that if f commutes with $\Gamma \times S^1$, then the periodic solutions to (4.8) lie in $\ker L$ and hence are circles in $V \oplus V$. *Cf.* (4.19).

(b) Proposition 4.3 shows that there are in general no additional restrictions on the form of the reduced mapping ϕ, other than those required by $\Gamma \times S^1$-symmetry and the occurrence of purely imaginary eigenvalues in $(df)_{0,0}$. This implies, for example, that the various combinations of stabilities allowed by Theorem 10.1 below can all occur for suitable choices of f.

§ 5. A Hopf Theorem with Symmetry

The main result of this section is an analogue, for periodic solutions, of a static equivariant bifurcation theorem of VAN DER BAUWHEDE [1980] and CICOGNA [1981]. In order to state this theorem we need to describe precisely the types of periodic solutions we seek.

Let $u(s)$ be in $C_{2\pi}$. Recall from (4.6) that $\Gamma \times S^1$ acts on $C_{2\pi}$ by

$$(\gamma, \theta)\, u(s) = \gamma u(s+\theta). \tag{5.1}$$

The *isotropy subgroup* of $\Gamma \times S^1$ corresponding to $u(s)$ is

$$\Sigma_u = \{(\gamma, \theta) \in \Gamma \times S^1 : (\gamma, \theta) \cdot u = u\}. \tag{5.2}$$

We speak of Σ_u as the *symmetries of the periodic mapping u*. Observe that these symmetries are a combination of spatial (Γ) and temporal (S^1) symmetries.

We need two hypotheses in addition to

$$\text{(H1)} \quad (df)_{0,0} = J$$

in order to state our theorem. First, we need the standard transversality condition of HOPF:

$$\text{(H2)} \quad \sigma'(0) \neq 0$$

where $\sigma(\lambda)$ is the real part of the eigenvalues of $(df)_{0,\lambda}$ as in Lemma 2.1(a) and Proposition 4.2(c). Second, we need to restrict attention to periodic solutions of (4.1) with certain kinds of isotropy subgroup. Let Σ be an isotropy subgroup of the action of $\Gamma \times S^1$ on $V \otimes C$. (We may identify Σ with the symmetries of a periodic mapping in $\ker L$ since the action (5.1) of $\Gamma \times S^1$ on $\ker L \subset C_{2\pi}$ may be identified with the action of $\Gamma \times S^1$ on $V \otimes C$ defined in § 3. See (3.6).)

Every isotropy subgroup Σ has a fixed-point subspace, defined as follows. Let G be a group acting on a vector space W. Choose $w \in W$ and let Σ be the isotropy subgroup of G corresponding to w. Then the *fixed-point subspace* of Σ is

$$W^\Sigma = \{v \in W \mid \sigma v = v \text{ for every } \sigma \in \Sigma\}. \tag{5.3}$$

W^Σ consists in all points in W which have at least Σ as their group of symmetries.

We can now state our third hypothesis. As above, let Σ be an isotropy subgroup of $\Gamma \times S^1$ acting on $V \otimes C$. Assume

$$\text{(H3)} \quad \dim (V \otimes C)^\Sigma = 2.$$

Proposition 6.2. Let Σ be an isotropy subgroup of $\Gamma \times S^1$ acting on $V \otimes C$, as above, with $\Sigma \neq \Gamma \times S^1$. Then

(a) $H \equiv P(\Sigma)$ is isomorphic to Σ.

(b) There is a homomorphism $\theta : H \to S^1$ such that $\Sigma = H^\theta$.

Proof. (a) Since the action of S^1 on $V \otimes C$ is fixed-point-free (*cf.* (3.6)), $\Sigma \cap S^1 = 1$. Here we use the fact that Σ is an isotropy subgroup. Thus $\ker P \cap \Sigma = 1$ and $H \cong \Sigma$.

(b) Every $\sigma \in \Sigma$ may be written uniquely as $\sigma = (h, \theta(h))$ for some $\theta(h) \in S^1$. We must show that $\theta : H \to S^1$ is a homomorphism. However, the fact that Σ is a subgroup of $\Gamma \times S^1$ leads to

$$(h, \theta(h)) \cdot (k, \theta(k)) = (hk, \theta(h)\,\theta(k)).$$

Hence $\theta(hk) = \theta(h) \cdot \theta(k)$ and θ is a homomorphism. $\square$

Remarks. (a) If θ is trivial, then $H^\theta = H \subset \Gamma$. Otherwise, we think of H as being twisted by θ and refer to θ as the *twist*.

(b) A homomorphism $H \to S^1$ is often called a *character* of H. To each character there corresponds an orthogonal representation of H on $R^2 = C$ defined by $h \to e^{i\theta(h)}$, and conversely. We shall use the term "character" in a slightly different sense in § 13: we prefer the term "twist" here.

We think of elements of Γ as *spatial* symmetries (acting on V and $V \oplus V$) and we think of elements of S^1 (acting on periodic solutions by phase shift) as *temporal* symmetries. In this sense, an element $\sigma = (h, \theta(h)) \in \Gamma \times S^1$ is a spatial symmetry if $\theta(h) = 0$ and a combined *spatio-temporal* symmetry if $h \neq 1$ and $\theta(h) \neq 0$.

For a given isotropy subgroup H^θ of $\Gamma \times S^1$, the spatial symmetries form a subgroup

$$K = \ker \theta.$$

We would like to think of a periodic solution having only spatial symmetries in its isotropy subgroup ($H = H^\theta = K$) as being spatially symmetric. However, this leads to certain technical difficulties regarding the implementation of Theorem 5.1.

We prefer:

Definition 6.3. An isotropy subgroup $\Sigma \subset \Gamma \times S^1$ is *spatial* if

$$\dim (V \otimes C)^\Sigma = \dim (V \otimes C)^K \tag{6.1}$$

and *temporal* otherwise.

Thus a spatial isotropy subgroup Σ may include some spatio-temporal symmetries, but these symmetries do not impose any additional restriction on the fixed-point subspace of Σ. For an example, see § 7.

We can now characterize those spatial isotropy subgroups having 2-dimensional fixed-point subspaces.

Proposition 6.4. Let H^θ be a spatial isotropy subgroup of $\Gamma \times S^1$. Then

(a) $\dim (V \otimes C)^{H^\theta} = 2$ if and only if $\dim V^K = 1$.

(b) Let K be an isotropy subgroup of Γ acting on V with $\dim V^K = 1$. Then there exists a unique spatial isotropy subgroup H^θ of $\Gamma \times S^1$ with $\dim (V \otimes C)^{H^\theta} = 2$ and $K = \ker \theta$.

Remark. It follows from Proposition 6.4 that the simple Hopf Theorem of § 1 corresponds precisely to the spatial isotropy subgroups H^θ for which $\dim V^K = 1$.

Proof. (a) Using the definition of spatial isotropy subgroup H^θ we see that (a) will follow from:

$$\dim (V \otimes C)^K = 1 \quad \text{if and only if} \quad \dim V^K = 1,$$

where $K = \ker \theta \subset \Gamma$. Since the action of Γ on $V \otimes C$ is just the diagonal action of Γ on $V \oplus V$ we conclude that

$$\gamma(v, w) = (v, w) \quad \text{if and only if} \quad \gamma v = v, \ \gamma w = w.$$

Thus $(v, w) \in (V \otimes C)^K$ if and only if $v, w \in V^K$. It follows that

$$\dim (V \otimes C)^K = 2 \dim V^K \tag{6.2}$$

and (a) is proved.

(b) Let K be an isotropy subgroup of Γ acting on V and let $v \in V$ be a vector whose isotropy subgroup is K. Since $\dim V^K = 1$, we see from (a) that K fixes every element $v \otimes z \in V \otimes C$ and no other.

Let H^θ be the isotropy subgroup in $\Gamma \times S^1$ fixing $v \otimes z$, $z \neq 0$. We assert that $\ker \theta = K$. On the one hand, K consists of spatial symmetries and fixes $v \otimes z$; thus $K \subset \ker \theta$. On the other hand, $\ker \theta$ fixes $v \otimes z$ and hence v. Thus, $\ker \theta \subset K$ since K is an isotropy subgroup.

Next we note that $v \otimes z = v' \otimes z'$ when v, v', z, z' are all nonzero only if $v' = \varepsilon v$, $z' = \varepsilon z$ where $\varepsilon = \pm 1$. It follows that $(h, \theta(h)) (v \otimes z) = v \otimes z$ if and only if $hv = \varepsilon v$ and $\theta(h) z = \varepsilon z$. Thus

$$H^\theta = \ker \theta \cup \{(h, \theta(h)) \mid \theta(h) = \pi, hv = -v\}. \tag{6.3}$$

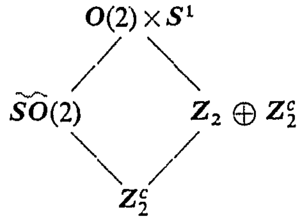
It follows from (6.3) that H^θ fixes every vector in $V \otimes C$ and $\dim (V \otimes C)^{H^\theta} = 2$. Moreover, (6.3) uniquely characterizes the isotropy subgroup of $\Gamma \times S^1$ containing K. □

§ 7. $O(2)$-Symmetric Hopf Bifurcation

We now apply our results to the case where $\Gamma = O(2)$ acts on R^2 by its standard representation. This situation has been studied by (among others) SCHECTER [1976] and BAJAJ [1982]: we recover their main results by exploiting the symmetry directly. We show that the lattice of (conjugacy classes of) isotropy subgroups is the one given in Table 7.1 where Z_2^c acts trivially on $R^2 \otimes C$ and

$\widetilde{SO}(2)$ is a twisted copy of $SO(2)$ inside $O(2)\times S^1$. We also show that the fixed-point spaces of $\widetilde{SO}(2)$ and $Z_2 \oplus Z_2^c$ are each 2-dimensional, so Theorem 5.3 gives the existence of two distinct periodic solution branches (when the usual transversality condition (H3) $\sigma'(0) \neq 0$ holds).

Table 7.1. The lattice of isotropy subgroups of $O(2)\times S^1$



In addition, $Z_2 \oplus Z_2^c$ is an example of a spatial isotropy subgroup and $\widetilde{SO}(2)$, which occurs in twisted form, is an example of a temporal isotropy subgroup. The subgroup $\widetilde{SO}(2)$ leads to rotating waves. The stability of these solutions is discussed in § 10.

The submaximal isotropy subgroup Z_2^c acts trivially and has a 4-dimensional fixed-point subspace. Possible solutions with this symmetry are discussed briefly in § 10.

Note that although $O(2)$ acts faithfully on R^2 (that is, the representation has a trivial kernel) the action of $O(2)\times S^1$ is not faithful: the kernel is Z_2^c. This is one of the reasons for the choice of Definition 6.3 as the appropriate one for a spatial isotropy subgroup.

We write the action of $O(2)\times S^1$ in the matrix form (3.2); that is,

$$(\gamma, \theta)\cdot Z = \varrho(\gamma)\, ZR_\theta \tag{7.1}$$

where the 2×2 matrix

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

and the column vectors (a, c) and (b, d) are vectors in $V = R^2$.

Lemma 7.1. *Every orbit of* $O(2)\times S^1$ *contains a diagonal matrix* $\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$ *where* $a \geqq d \geqq 0$. *Moreover, there are four orbit types. Orbit representatives, isotropy subgroups and fixed-point subspaces are given in Table 7.2.*

We define the action in (7.1) more explicitly. Recall from (3.4) that

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}. \tag{7.2}$$

The action of the group $O(2)$ on R^2 is generated by rotations R_ψ and a flip (or complex conjugation)

$$\varkappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{7.3}$$

Thus the elements of $O(2)\times S^1$ acting on Z are either (R_ψ, R_θ) or $(\varkappa R_\psi, R_\theta)$.

Table 7.2. Isotropy subgroups and fixed-point subspaces for $O(2)\times S^1$

Orbit representative	Isotropy group	Fixed-point subspace
(a) $\begin{pmatrix}0 & 0\\ 0 & 0\end{pmatrix}$	$O(2)\times S^1$	0
(b) $\begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}$ $a>0$	$\widetilde{SO}(2)=\{\psi, -\psi\}$	$\begin{pmatrix}a & b\\ -b & a\end{pmatrix}$ $a, b \in \mathbf{R}$
(c) $\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}$ $a>0$	$\mathbf{Z}_2\oplus\mathbf{Z}_2^c=\{(0,0),(\pi,\pi),(\varkappa,0),(\varkappa\pi,\pi)\}$ where $\varkappa$ is defined in (7.3)	$\begin{pmatrix}a & b\\ 0 & 0\end{pmatrix}$ $a, b \in \mathbf{R}$
(d) $\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}$ $a>d>0$	$\mathbf{Z}_2^c=\{(0,0),(\pi,\pi)\}$	$\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ $a, b, c, d \in \mathbf{R}$

Proof of Lemma 7.1. If Z is symmetric, then $R_{-\theta}ZR_\theta$ is diagonal for suitable θ (diagonalization by an orthogonal transformation). If Z is diagonal, then multiplication (on the left) by a suitable choice of $\begin{pmatrix}\pm 1 & 0\\ 0 & \pm 1\end{pmatrix}$ allows us to assume that $a, d \geqq 0$. Finally, multiplication on the left by $\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}$, if necessary, allows us to assume $a\geqq d\geqq 0$. Thus it suffices to show that each orbit contains a symmetric matrix.

Let $Z=\begin{pmatrix}a & b\\ c & d\end{pmatrix}$ and assume $b \neq c$. Then $R_\psi B$ is symmetric if $\psi = \cot^{-1}\left(\dfrac{a+d}{b-c}\right)$.

Next we compute the data in Table 7.1 assuming that Z is diagonal. We distinguish four cases:

$$\begin{aligned}&\text{(a)}\quad a=d=0,\\ &\text{(b)}\quad a=d>0,\\ &\text{(c)}\quad a>d=0,\\ &\text{(d)}\quad a>d>0.\end{aligned}\tag{7.4}$$

It is easy to verify that each of the isotropy subgroups Σ fixes the listed fixed-point subspaces. Moreover, it is clear that $O(2)\times S^1$ fixes only $Z=0$. We need to check that the isotropy subgroups in cases (b), (c) and (d) are no larger than those listed. Note that $\det Z>0$ in cases (b) and (d). Therefore, the isotropy subgroups in these cases are contained in $SO(2)\times S^1$. In case (b) we compute

$$R_\psi aIR_\theta = aR_{\psi+\theta}.$$

Now $R_{\theta+\psi}=I$ only if $\theta=-\psi$, so $\Sigma=\widetilde{SO}(2)$.

To complete cases (c) and (d) we observe that $R_\psi Z R_\theta$ is diagonal if and only if

$$\begin{aligned} &\text{(a)}\quad (a+d)\sin(\psi+\theta)=0,\\ &\text{(b)}\quad (a-d)\sin(\psi-\theta)=0. \end{aligned} \tag{7.5}$$

In cases (7.4c, d), equation (7.5) is satisfied only if $\psi=\theta=0$ or $\psi=\theta=\pi$. Thus case (7.4d) has the isotropy subgroup Z_2^c. Note that $\varkappa$ is in the isotropy subgroup of (7.4c). It then follows that the isotropy subgroup is $Z_2 \oplus Z_2^c$ in this case.

Remarks. (a) The $\widetilde{SO}(2)$-symmetric solutions are the *rotating waves* of AUCHMUTY [1979]. To see this, recall how $\Sigma = \widetilde{SO}(2)$ acts on $C_{2\pi}$. In particular, if $u(s)$ is a periodic solution with isotropy subgroup $\widetilde{SO}(2)$, then $u(s) = R_\psi u(s-\psi)$ or

$$u(s+\psi) = R_\psi u(s), \tag{7.6}$$

which is the standard definition of a rotating wave.

(b) The existence of rotating waves in $O(2)$-symmetric systems of partial differential equations was proved by AUCHMUTY [1979]. The Liapunov-Schmidt reduction of § 4 may be adapted to partial differential equations and should provide another method for obtaining AUCHMUTY'S results.

(c) Pictures of rotating waves found numerically by ERNEAUX & HERSCHKOWITZ-KAUFMAN [1977] in the Brusselator on a disc are in AUCHMUTY [1978].

(d) Recall Remark 1.3, in which we show how to enumerate distinct trajectories of symmetry-related solutions. We stated there that a given isotropy subgroup Σ gives rise to a family of trajectories parametrized by $\Gamma/\pi(\Sigma)$, where $\pi : \Gamma \times S^1 \to \Gamma$ is projection. In this example, $\Gamma = O(2)$, with two choices for Σ. If $\Sigma = Z_2 \oplus Z_2^c$, then $\Gamma/\pi(\Sigma)$ is diffeomorphic to a circle, and this family of periodic trajectories forms an invariant 2-torus. If $\Sigma = \widetilde{SO}(2)$, then $\Gamma/\pi(\Sigma)$ consists in two points, yielding two isolated periodic trajectories. These differ in their direction of rotation, in the sense that one satisfies (7.6) and the other satisfies

$$u(s+\psi) = R_{-\psi} u(s). \tag{7.7}$$

Compare SCHECTER [1976] § 4, example 2.

(e) To analyze the branching direction and stabilities for $O(2)$-symmetric Hopf bifurcation we shall need more information on the invariant theory of $O(2) \times S^1$. This is obtained in § 9.

§ 8. Asymptotic Stability

We now investigate the stability of periodic solutions constructed using Theorem 5.1. Our approach is to use a combination of Floquet Theory, normal form theory and group theory. We wish to thank JOHN GUCKENHEIMER for pointing out the utility of assuming that the vector field is in normal form when making stability calculations.

Let $s = (1 + \tau)\, t$ be rescaled time as in § 4, and let $u(s)$ be a 2π-periodic solution of

$$(1 + \tau)\frac{du}{ds} + f(u, \lambda) = 0 \tag{8.1}$$

where $f\colon (V \oplus V) \times \boldsymbol{R} \to V \oplus V$ commutes with Γ. Consider the linearization of (8.1) about the solution $u(s)$; this is the *Floquet equation*

$$\frac{dz}{ds} + \frac{1}{1 + \tau}(df)_{u(s),\lambda}\, z = 0. \tag{8.2}$$

Let $z(s)$ be a solution to (8.2) and define

$$M_u z(0) = z(2\pi). \tag{8.3}$$

M_u is linear, since (8.2) is linear. M_u is the *Floquet matrix* and its eigenvalues are the *Floquet multipliers*.

We show below that one of the Floquet multipliers is forced to be 1; this follows from the S^1-symmetry of phase shift. The standard Floquet Theorem states that if the remaining eigenvalues of M_u lie inside the unit circle, then $u(s)$ is an asymptotically stable periodic solution to (8.1). Cf. CODDINGTON & LEVINSON [1955] or HALE [1969].

For symmetric periodic solutions, this assumption on Floquet multipliers is rarely satisfied. The reason is that many eigenvalues of M_u may be forced to be 1. More precisely, we have:

Proposition 8.1. Let $u(s)$ be a 2π-periodic solution of (8.1) with isotropy subgroup Σ. Then M_u has at least d_Σ eigenvalues equal to 1 where

$$d_\Sigma = \dim \Gamma + 1 - \dim \Sigma. \tag{8.4}$$

Proof. Let γ_t be a smooth curve in $\Gamma \times S^1$ where, say, $t \in (-1, 1)$ with $\gamma_0 = 1$. Let $w(s) = \frac{d}{dt}\gamma_t\, u(s)|_{t=0}$. We assert and prove below that $w(s)$ is a 2π-periodic solution to the Floquet equation (8.2). It follows from (8.3) that

$$M_u w_0 = w_0. \tag{8.5}$$

Whenever $w_0 \neq 0$, M_u has an eigenvector with eigenvalue 1.

Observe that if $\gamma_t \in \Sigma$ for every t then $\gamma_t u(s) = u(s)$ and $w(s) \equiv 0$. Conversely, if γ_t is transverse to Σ in $\Gamma \times S^1$ then $w_0 \neq 0$. In particular, one can construct $d_\Sigma = \dim \Gamma \times S^1 - \dim \Sigma$ such independent eigenvectors with eigenvalue 1.

Note: If $\gamma_t = t \in S^1$ then $\gamma_t u(s) = u(s + t)$. Thus, $w_0 = \frac{du}{ds}(0)$, which in standard Floquet theory is the eigenvector with eigenvalue 1.

To prove the assertion, we must show that $w(t)$ satisfies the Floquet equation. Compute

$$\begin{aligned}\frac{dw}{ds}(s) &= \frac{d}{ds}\frac{d}{dt}\gamma_t u(s)\big|_{t=0},\\ &= \frac{d}{dt}\gamma_t\frac{d}{ds}u(s)\big|_{t=0},\\ &= \frac{d}{dt}\gamma_t\left(-\frac{1}{1+\tau}f(u(s),\lambda)\right),\\ &= -\frac{1}{1+\tau}\frac{d}{dt}\gamma_t f(u(s),\lambda)\big|_{t=0}.\end{aligned} \tag{8.6}$$

Since γ_t consists in spatial symmetries that commute with f, and phase shifts that also commute with f, we have $\gamma_t f(u(s), \lambda) = f(\gamma_t u(s), \lambda)$. It follows from (8.6) that

$$\begin{aligned}\frac{dw}{ds}(s) &= -\frac{1}{1+\tau}\frac{d}{dt}f(\gamma_t u(s),\lambda)\big|_{t=0}\\ &= -\frac{1}{1+\tau}(df)_{u(s),\lambda}\, w(s),\end{aligned}$$

the last equality following from the chain rule and $\gamma_0 = 1$. □

Let $n = \dim V$. The standard Floquet Theorem may be modified to show that if the $2n - d_\Sigma$ eigenvalues of M_u, which are not forced to equal 1 by the group action, lie strictly inside the unit circle, then $u(s)$ is *orbitally asymptotically stable*. That is, any solution starting sufficiently close to u tends towards a periodic solution γu for some $\gamma \in \Gamma \times S^1$. *Cf.* HALE & STOKES [1960].

The purpose of this section is to provide a theoretical basis for determining whether or not the remaining $2n - d_\Sigma$ eigenvalues lie inside the unit circle. Although our results are far from complete, they are sufficiently detailed to enable us to compute stabilities in the $O(2) \times S^1$ case. However, we must assume that (8.1) is in normal form.

The idea in normal form theory for vector fields is to perform (polynomial) changes of coordinates to simplify the form of the k-th order terms in f. More precisely, one simplifies the second order term, then the third, etc. This simplification procedure is described by TAKENS [1973], HASSARD *et al.* [1981], and GUCKENHEIMER & HOLMES [1983], for example. The end result is to find, to any desired *finite order* k, a set of coordinates in which the terms of f up to order k commute with S^1 acting as a group of spatial rather than temporal symmetries. See Proposition 8.6.

There is an implication of this normal form procedure in our case. By performing changes of coordinates that commute with Γ acting on $V \oplus V$ we can think of the terms in f up to order k as a mapping of $(V \otimes C) \times R$ into $V \otimes C$ that commutes with $\Gamma \times S^1$, not just Γ. (In this way S^1 now acts as spatial symmetries.) We have already seen a trivial instance of this process when we showed under the assumption of purely imaginary eigenvalues that we could assume (H1): $(df)_{0,0} = J$.

For the stability calculations below we augment (H1) by

$$\text{(H4)} \quad f:(V \otimes C) \times R \to V \otimes C \quad \text{commutes with} \quad \Gamma \times S^1.$$

Hypothesis (H4) is unlike (H1) in that we cannot, in general, prove that our initial f can be put *exactly* into a form satisfying (H4) by an appropriate change of coordinates. However, to any finite order k, this can be done. Since we are trying to estimate eigenvalues (do they lie inside the unit circle?), the assumption (H4) should not lead to any difficulties in the general case. We have not attempted to verify this point. Nevertheless our arguments *are* valid for f's satisfying (H4) and this is a reasonable class to consider.

The remainder of this section is divided into three parts:

(a) Stability from the reduced bifurcation equation.
(b) Ways that isotropy subgroups help in the analysis of stability.
(c) Normal forms for vector fields with symmetry.

(*a*) *Stability and Liapunov-Schmidt.* The main result in this subsection relates the eigenvalues of the (linearization of the) reduced bifurcation equation to the (orbital asymptotic) stability of the corresponding periodic solution of (8.1).

We begin by letting $\phi:(V \otimes C) \times R \times R \to V \otimes C$ be the reduced bifurcation equation obtained from (8.1) by the Liapunov-Schmidt reduction (Theorem 4.1). Let (u_0, λ_0, τ_0) be a solution to $\phi = 0$. Assuming that f is in normal form (hypothesis (H4)) we recall two facts from § 4:

$$\text{(a)} \quad \phi = f - (1 + \tau) J \quad (\textit{Cf.}\ \text{Proposition 4.3}) \tag{8.7}$$

and

$$\text{(b)} \quad u(s) = (1, s) \cdot u_0,\ (1, s) \in \Gamma \times S^1 \quad \text{is the periodic solution}$$

to (8.1) corresponding to u_0 (*Cf.* Remark 4.4).

Since ϕ commutes with $\Gamma \times S^1$ we know that $2n - d_\Sigma$ of the eigenvalues of $(d\phi)_{u_0,\lambda_0,\tau_0}$ equal zero. The proof is analogous to Proposition 8.1 since $\phi(\cdot, \lambda_0, \tau_0)$ vanishes on the orbit of u_0 under the action of $\Gamma \times S^1$. Thus $(d\phi)_{u_0,\lambda_0,\tau_0}$ vanishes on the tangent space of that orbit, which has dimension d_Σ. We claim that the remaining $2n - d_\Sigma$ eigenvalues of $d\phi$ control the orbital asymptotic stability of $u(s)$.

Theorem 8.2. *Assume that f satisfies* (H4). *Then the periodic solution $u(s)$ to* (8.1) *is orbitally asymptotically stable if the $2n - d_\Sigma$ eigenvalues of $(d\phi)_{u_0,\lambda_0,\tau_0}$ that are not forced by the group action to be zero, have positive real parts. The solution is unstable if one of these eigenvalues has negative real part.*

The first step in the proof of Theorem 8.2 is showing that the form of u in (8.7b) allows us to convert the Floquet equation (8.2) to a linear equation with constant coefficients. This is reminiscent of rotating wave solutions; *cf.* RENARDY [1982] and IOOSS & JOSEPH [1981]. (In fact, one might say that the imposition of normal form (H4) on f serves to convert all periodic solutions to (8.1) into rotating waves. This point can be made rigorous; we do not pursue it here.)

Lemma 8.3. *For a periodic solution u of the form* (8.7b) *the Floquet matrix is given by*

$$M_u = \exp\left(-2\pi\left[\frac{1}{1+\tau_0}(df)_{u_0,\lambda_0} - J\right]\right). \tag{8.8}$$

Proof. Let

$$z(s) = (1, s)\cdot w(s) \tag{8.9}$$

and compute (8.2) as follows:

$$\frac{dz}{ds} + \frac{1}{1+\tau_0}(df)_{u(s),\lambda_0}\, z(s) = \frac{d}{ds}(1, s)\cdot w + (1, s)\cdot\frac{dw}{ds} + \frac{1}{1+\tau_0}(df)_{(1,s)u_0,\lambda_0}(1, s)\, w,$$

$$= (1, s)\cdot\left[-Jw + \frac{dw}{ds} + \frac{1}{1+\tau_0}(df)_{u_0,\lambda_0} w\right]$$

where the last equality uses (2.5) and (3.8). Thus the Floquet equation is satisfied when

$$\frac{dw}{ds} + \left[\frac{1}{1+\tau_0}(df)_{u_0} - J\right] w = 0.$$

This linear system may be solved explicitly to yield

$$w(s) = \exp\left(-\left[\frac{1}{1+\tau_0}(df)_{u_0,\lambda_0} - J\right] s\right). \tag{8.10}$$

From (8.10) observe that $w(0) = z(0)$ and $w(2\pi) = z(2\pi)$. Now use the definition of M_u in (8.3) to verify (8.8).

Proof of Theorem 8.2. It follows from (8.7a) that

$$(d\phi)_{u_0,\lambda_0,\tau_0} = (df)_{u_0,\lambda_0} - (1+\tau_0)\, J.$$

Thus (8.8) implies

$$M_u = \exp\left(-\frac{2\pi}{1+\tau_0}(d\phi)_{u_0,\lambda_0,\tau_0}\right).$$

Thus, if the $2n - d_\Sigma$ eigenvalues of $d\phi$ have positive real parts, then the corresponding Floquet multipliers lie inside the unit circle and $u(s)$ is orbitally asymptotically stable. □

(*b*) *Isotropy subgroups and eigenvalues of $d\phi$.* Let $u(s)$ be a periodic solution to (8.1) with isotropy subgroup Σ. Assume (H4) is valid so that $\Sigma \subset \Gamma \times S^1$ may be viewed as spatial symmetries in $V \otimes C$. As above, let $\phi(u, \lambda, \tau)$ be the reduced bifurcation equation. We claim that the existence of Σ restricts the form of $d\phi$ sufficiently to allow, in certain cases, explicit computation of the eigenvalues of $d\phi$.

To begin, we write

$$V \otimes C = W_1 \oplus \dots \oplus W_l \tag{8.11}$$

where each W_j is invariant under Σ and each W_j is the direct sum of irreducible subspaces of Σ of a given isomorphism type. That is, irreducible subspaces of W_j and W_k, $j \neq k$, do not have isomorphic actions of Σ.

Note. The fixed-point subspace $(V \otimes C)^{\Sigma}$ consists of all subspaces of $V \otimes C$ on which Σ acts as the identity representation. Thus $(V \otimes C)^{\Sigma} = W_j$ for some j. For definiteness we set

$$W_1 = (V \otimes C)^{\Sigma}.$$

Lemma 8.4. *Using the notation above, we have*

$$(d\phi)_{u_0,\lambda_0,\tau_0}(W_j) \subset W_j, \quad j = 1, \dots, l.$$

Proof. It follows from (2.5) that $(d\phi)_{u_0,\lambda_0,\tau_0}$ commutes with Σ. Hence, $d\phi$ maps any irreducible subspace $\tilde{W}$ of W_j to an irreducible subspace of W with an isomorphic representation. But W_j contains all such subspaces; *cf.* DORNHOFF [1971], Lemma 21.1.

Observe that if (H3) is valid, that is, if $\dim (V \otimes C)^{\Sigma} = 2$, then f maps $(V \otimes C)^{\Sigma} \times \mathbf{R}$ to $(V \otimes C)^{\Sigma}$ and one can apply the simple Hopf Theorem of § 1. It follows from the standard Hopf theorem that one of the eigenvalues of $d\phi \mid (V \otimes C)^{\Sigma}$ is zero and the sign of the other determines the stability of periodic solutions to the restricted system $f \mid (V \otimes C)^{\Sigma} \times \mathbf{R}$. In this case exchange of stability holds. In particular, we have proved:

Lemma 8.5. *Assume* (H4). *If the steady-state solution is stable subcritically, then any subcritical branch of periodic solutions, with isotropy subgroup Σ satisfying* (H3), *is unstable.*

(c) *Normal forms and Symmetry*. The method of normal forms provides an answer to the question: Given the linear part L of a vector field X, which higher-order terms in X cannot be transformed away by smooth changes of coordinates? To state the answer we introduce some notation. Let H_k be the linear space of vector fields X whose coordinate function are homogeneous polynomials of degree k. Let

$$\operatorname{ad} L(Y) = [L, Y]$$

denote the Lie bracket of vector fields. Since L is linear, $\operatorname{ad} L$ maps H_k to itself. Let G_k be any vector space complement to $\operatorname{ad} L(H_k)$ in H_k. Then the answer to the above question may be phrased as follows (see GUCKENHEIMER & HOLMES [1983] Theorem 3.3.1): For any integer N there exists a polynomial change of coordinates h such that

$$h_* X = L + Y_2 + \dots + Y_N + R_{N+1} \tag{8.12}$$

where

$$h_* X(x) = (dh)^{-1} X(hx)$$

is the vector field X in the new coordinates, $Y_k \in G_k$ $(k = 2, \dots, n)$, and R_{N+1} vanishes through order N.

Suppose that L is the linear vector field on $V \oplus V$ associated with (degenerate) Hopf bifurcations, that is.

$$L = \begin{pmatrix} 0 & -I \\ I & 0 \end{pmatrix} x.$$

Then there is a natural choice for the complements G_k in H_k. Let S^1 act as usual on $V \otimes \mathbf{C}$ (identified with $V \oplus V$). Then we may take

$$G_k = \{X \in H_k \mid (R_\theta)_* X = X, \theta \in S^1\}. \tag{8.13}$$

This fact seems to be well known (see GUCKENHEIMER & HOLMES [1983] p. 144, GUCKENHEIMER [1984], RUELLE [1973]) but we give a proof below for completeness. Moreover, this proof extends to symmetric vector fields, which are our main concern here. Let Γ be a (compact Lie) group of symmetries and define $H_k(\Gamma)$ to be the set of Γ-equivariant vector fields in H_k. Then we shall give a short Lie-theoretic proof of the following result, suggested to us by JOHN GUCKENHEIMER:

Proposition 8.6. Let X be a Γ-equivariant vector field on $V \oplus V$ with $(dX)_0 = L$. Then for every integer N there exists a polynomial change of coordinates h such that

$$h_* X = L + Y_2 + \dots + Y_N + R_{N+1}$$

where $Y_k \in H_k(\Gamma \times S^1)$ and R_{N+1} vanishes through order N.

Proof. First, assume $\Gamma = 1$. W need to prove that

$$\operatorname{Im} \operatorname{ad} L \oplus G_k = H_k \tag{8.14}$$

when $G_k = H_k(S^1)$ as in (8.13). Let $\varrho : H_k \to G_k$ be defined by averaging over S^1, that is,

$$\varrho(X) = \int_{S^1} (R_\theta)_* X \, d\theta .$$

It is well known (*cf.* ADAMS [1969], Proposition 3.15) that ϱ is a projection onto G_k. Therefore

$$\ker \varrho \oplus G_k = H_k . \tag{8.15}$$

Since ad L is linear we have

$$\dim \ker \operatorname{ad} L + \dim \operatorname{Im} \operatorname{ad} L = \dim H_k . \tag{8.16}$$

Now (8.14) follows from (8.15) and (8.16) provided we can show that

$$\ker \operatorname{ad} L = G_k, \tag{8.17}$$

$$\operatorname{Im} \operatorname{ad} L \subset \ker \varrho. \tag{8.18}$$

First we prove (8.17). The definition of the Lie derivative of a vector field (SPIVAK [1979]), and the fact that $\frac{d}{d\theta} R_\theta \cdot x = Lx$, imply that

$$[L, Y] = \frac{d}{d\theta} (R_\theta)_* Y.$$

If $Y \in G_k$ then $(R_\theta)_* Y = Y$, so $[L, Y] = 0$ and $Y \in \ker \operatorname{ad} L$. Conversely if $y \in \ker \operatorname{ad} L$ then $[L, Y] = 0$ implies that $(R_\theta)_* Y = Y$ by SPIVAK [1979] pp. 217–218.

To prove (8.18) observe that

$$\varrho[L, Y] = [L, \varrho Y] \subset [L, G_k] = 0$$

by (8.17).

We have now proved that (8.13) provides a suitable choice of complement to $\operatorname{Im} \operatorname{ad} L$ for the standard normal form procedure, described in GUCKENHEIMER & HOLMES [1983]. For the general symmetric version, let Γ act on $V \oplus V$, define $H_k(\Gamma)$ as above, and set

$$G_k(\Gamma) = G_k \cap H_k(\Gamma),$$

$$\operatorname{ad}_\Gamma L = \operatorname{ad} L \mid H_k(\Gamma).$$

Since the action of S^1 on $V \otimes C$ commutes with that of Γ, it follows that $\operatorname{ad}_\Gamma L$ maps $H_k(\Gamma)$ to itself. We assert that

$$\operatorname{Im} \operatorname{ad}_\Gamma L \oplus G_k(\Gamma) = H_k(\Gamma). \tag{8.19}$$

This follows from the proof of (8.14) because all constructions used commute with Γ. But now the usual normal form procedure applies in this Γ-symmetric setting. □

§ 9. Invariant Theory for $O(2) \times S^1$

In order to determine the precise direction of branching (sub- or supercritical) and the stability of given branches of periodic solutions, we must explicitly compute the $\Gamma \times S^1$-*equivariants*, that is, the mappings that commute with $\Gamma \times S^1$. We carry out this calculation here when $\Gamma = O(2)$ acting on R^2 by its standard representation.

Recall the matrix notation for $R^2 \otimes C$ in (3.2). Specifically, let

$$Z = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be in $R^2 \otimes C$. The action of $O(2) \times S^1$ on $R^2 \otimes C$ is defined by (3.3) which we restate here as:

$$(\gamma, \theta) \cdot Z = \gamma Z R_\theta \tag{9.1}$$

where γ is a matrix in $O(2)$ and

$$R_\theta = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$$

as in (3.4). There are two functions which are obviously invariant under the action (9.1), namely

$$\begin{aligned} &\text{(a)} \quad N = a^2 + b^2 + c^2 + d^2, \\ &\text{(b)} \quad \delta^2 \quad \text{where} \quad \delta = ad - bc. \end{aligned} \tag{9.2}$$

The function N is just the norm of Z; since the action of $O(2) \times S^1$ is orthogonal with respect to this norm, N is invariant. The function δ is just det Z; δ itself is not invariant since $\det \gamma$ can be -1. However, δ^2 is invariant. In fact, these two functions generate all invariants, as part (a) of the next theorem shows.

Theorem 9.1. (a) *Let* $h: R^2 \otimes C \to R$ *be infinitely differentiable and invariant under* $O(2) \times S^1$. *Then there is a smooth function* $k: R^2 \to R$ *such that*

$$h(Z) = k(N, \delta^2). \tag{9.3}$$

(b) *Let* $\phi: R^2 \otimes C \to R^2 \otimes C$ *be infinitely differentiable and commute with* $O(2) \times S^1$. *Then there exist invariant functions* p, q, r, s *such that*

$$\phi = p\begin{pmatrix} a & b \\ c & d \end{pmatrix} + q\begin{pmatrix} -b & a \\ -d & c \end{pmatrix} + r\delta\begin{pmatrix} -c & -d \\ a & b \end{pmatrix} + s\delta\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}. \tag{9.4}$$

Remark. If h in (9.3) or ϕ in (9.4) depend smoothly on parameters $\alpha \in R^l$, then one can prove that (9.3) and (9.4) still hold, where k, p, q, r, s also depend smoothly on α.

First we establish that the function ϕ in (9.4) does commute with $O(2) \times S^1$. This is immediate from:

Lemma 9.2. *The generators*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \begin{pmatrix} -b & a \\ -d & c \end{pmatrix}, \quad \delta\begin{pmatrix} -c & -d \\ a & b \end{pmatrix}, \quad \delta\begin{pmatrix} d & -c \\ -b & a \end{pmatrix}$$

commute with $O(2) \times S^1$.

Proof. The first two are I and $-J$, which commute with $O(2) \times S^1$ (Lemma 3.2). Let $E = \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$. The other two generators are δE and $-\delta JE$. Now δ is invariant under $SO(2) \times S^1$ and is sent to $-\delta$ by $\varkappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \in O(2)$. It suffices to show that E commutes with $SO(2)\times S^1$ and is sent to $-E$ by $\varkappa$. Note that $E = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Since $\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SO(2)$ it follows that E commutes since $SO(2)\times S^1$ is abelian. The $\varkappa$-action is easily computed. □

The proof of Theorem 9.1 makes use of an extension argument involving the diagonal and antidiagonal matrices. (A more direct computational proof is possible but lengthy.) Let

$$D = \left\{ \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} \right\}, \quad A = \left\{ \begin{pmatrix} 0 & b \\ c & 0 \end{pmatrix} \right\}. \tag{9.5}$$

Observe that

$$V \otimes C = D \oplus A.$$

Proof of Theorem 9.1. First note that ϕ may be written uniquely in coordinates as

$$\phi = \phi_D + \phi_A$$

where $\phi_D : R^2 \otimes C \to D$ and $\phi_A : R^2 \otimes C \to A$. Moreover, the matrix J acts on Z by

$$J \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} b & -a \\ d & -c \end{pmatrix}.$$

In particular, $J : D \to A$ and $J : A \to D$ are isomorphisms with $J^2 = -Id$. Thus, we can write

$$\phi = \phi_D - J(J\phi_A), \tag{9.6}$$

where ϕ_D and $J\phi_A$ both map $R^2 \otimes C \to D$.

A second remark is that ϕ is determined by its values on D. This follows since every orbit of the action of $O(2)\times S^1$ intersects D (see Lemma 7.1) and ϕ commutes with $O(2)\times S^1$. In symbols, if $Z = \gamma \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}$, then

$$\phi(Z) = \gamma\phi\left(\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right). \tag{9.7}$$

Now even though it is true that $\phi \mid D$ determines ϕ, it is not true that $\phi \mid D$ is arbitrary. That is, not every smooth mapping of $D \to R^2 \otimes C$ extends to a smooth mapping of $R^2 \otimes C \to R^2 \otimes C$ commuting with $O(2)\times S^1$. We can see this in two different ways. First we assume that (9.4) is valid and write

$$\begin{aligned} \phi \mid D &= p \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + q \begin{pmatrix} 0 & a \\ -d & 0 \end{pmatrix} + r\,\delta \begin{pmatrix} 0 & -d \\ a & 0 \end{pmatrix} + s\,\delta \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}, \\ &= p \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + s\,\delta \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix} - J\left(q \begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix} + r\,\delta \begin{pmatrix} d & 0 \\ 0 & a \end{pmatrix}\right), \end{aligned} \tag{9.8}$$

where $\delta = a\,d$ and p, q, r, s are functions of $N = a^2 + d^2$ and $\delta^2 = a^2\,d^2$.

Note that

$$\begin{aligned}&\text{(a)}\quad \phi_D \mid D = p\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix} + s\,\delta\begin{pmatrix} d & 0\\ 0 & a\end{pmatrix},\\ &\text{(b)}\quad (J\phi_A)\mid D = q\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix} + r\,\delta\begin{pmatrix} d & 0\\ 0 & a\end{pmatrix},\end{aligned} \tag{9.9}$$

so that although $\phi_D \mid D$ and $(J\phi_A)\mid D$ have the same structure, they are *not* arbitrary smooth functions.

The second way see we that $\phi \mid D$ is not arbitrary is to observe that the subgroup Δ of $O(2)\times S^1$ which leaves D invariant is nontrivial. In fact, from (7.5) we see that Δ is generated by the flip $\varkappa = \begin{pmatrix} 1 & 0\\ 0 & -1\end{pmatrix}$ and all (R_ψ, R_0) in $SO(2)\times S^1$ for which

$$\sin(\psi+\theta) = 0 = \sin(\psi-\theta).$$

Therefore, Δ consists in the eight elements $(\psi, \theta) =$

$$\begin{gathered}(0,0),\ (0,\pi),\ (\pi,0,),\ (\pi,\pi),\\ \left(\frac{\pi}{2},\frac{\pi}{2}\right),\ \left(\frac{\pi}{2},\frac{3\pi}{2}\right),\ \left(\frac{3\pi}{2},\frac{\pi}{2}\right),\ \left(\frac{3\pi}{2},\frac{3\pi}{2}\right)\end{gathered} \tag{9.10}$$

along with $\varkappa$ times each of these elements. Since $\phi_D : D \to D$ and $(J\phi_A): D \to D$, each of these functions must commute with Δ.

We assert that given any two smooth mappings g_D and Jg_A which map $D \to D$ and which commute with Δ there exists a smooth extension of

$$g \mid D = g_D - J(Jg_A) \tag{9.11}$$

mapping $R^2 \otimes C \to R^2 \otimes C$, and commuting with $O(2)\times S^1$. The extension of (9.11) is obtained using (9.7). Moreover, in proving this fact we derive the forms (9.3) and (9.4).

To prove the assertion, we first need to identify Δ and its action on D more precisely. Since $Z_2^c = \{(0,0), (\pi,\pi)\}$ acts as the identity on $R^2 \otimes C$, the effective action of Δ on D is given by the eight-element group Δ/Z_2^c. One may check that this quotient group is just the dihedral group D_4. Moreover, the dihedral group D_4 acts naturally on R^2 as symmetries of a square. One may also check that the action of Δ on R^2 is this natural action. This is clear, since

$$\varkappa(a,d) = (a,-d) \tag{9.12}$$

flips the square, and

$$\varkappa\left(\frac{\pi}{2},\frac{\pi}{2}\right)\cdot(a,d) = (-d,a) \tag{9.13}$$

rotates R^2 by a right angle.

Now the commuting maps and the invariant functions for the dihedral group $\boldsymbol{D}_n$ acting on $\boldsymbol{R}^2$ as symmetries of an n-gon are well known. To describe them, we identify $\boldsymbol{R}^2$ with $\boldsymbol{C}$. Then the action of $\boldsymbol{D}_4$ is generated by

$$z \to \bar{z} \quad \text{and} \quad z \to e^{\pi i/2} z.$$

By (9.12), (9.13) this identification may be achieved by letting

$$z = a + i\,d.$$

The invariant functions are generated by the quadratic $z\bar{z}$ and the quartic $\operatorname{Re} z^4$. The mappings of $\boldsymbol{C} \to \boldsymbol{C}$ commuting with $\boldsymbol{D}_4$ have the form

$$A(z\bar{z}, \operatorname{Re} z^4)\, z + B(z\bar{z}, \operatorname{Re} z^4)\, \bar{z}^3.$$

This is known for polynomials, and extends to smooth functions as usual by SCHWARZ [1975], POÉNARU [1976]. *Cf.* SATTINGER [1983].

In particular, the vector space of invariant quadratics is 1-dimensional ($\boldsymbol{R}\{z\bar{z}\}$) and the vector space of invariant quartics is 2-dimensional ($\boldsymbol{R}\{(z\bar{z})^2, \operatorname{Re} z^4\}$). By a dimension count, each of these invariants extends to $\boldsymbol{R}^2 \otimes \boldsymbol{C}$, since N is quadratic and N^2 and δ^2 are quartic invariants. This verifies (9.3).

To verify (9.4) observe that both $\phi_D \mid D$ and $-(J\phi_A) \mid D$ each have one linear generator $(z \to z)$ and one cubic generator $z \to \bar{z}^3$. These correspond to the two linear and two cubic generators in (9.4).

In fact, we can exhibit the invariants and equivariants (commuting maps) for $\boldsymbol{D}_4$ on $\boldsymbol{C}$ explicitly as restrictions of those for $O(2) \times S^1$ on $\boldsymbol{R}^2 \otimes \boldsymbol{C}$, as follows:

$$z\bar{z} = a^2 + b^2 = N \mid D,$$

$$\operatorname{Re}(z^4) = a^4 - 6a^2 d^2 + d^4 = (N^2 - 8\delta^2) \mid D,$$

$$z = a + i\,d = I \mid D,$$

$$\bar{z}^3 = (a^3 - 3a\,d^2) + i(d^3 - 3a^2\,d) = (NI + 4\delta E) \mid D,$$

where E is as in the proof of Lemma 9.2. □

§ 10. Branching and Stability for $O(2)$

Having obtained the explicit description of $O(2) \times S^1$-equivariant mappings in (9.4), we are now in a position to apply our theory to derive specific results about the direction of branching and stability of the two types of periodic solutions guaranteed by Theorem 5.1. See § 7. This completes the recovery, from the group-theoretic viewpoint, of the results of SCHECTER [1976] and BAJAJ [1982].

In order to compute stability we assume that our system of differential equations is in normal form; that is, that (H4) is valid. We also assume the transversality condition (H2). Since the vector field f is assumed $O(2) \times S^1$-equivariant we have

$$f(z, \lambda) = p \begin{pmatrix} a & b \\ c & d \end{pmatrix} + q \begin{pmatrix} -b & a \\ -d & c \end{pmatrix} + r\,\delta \begin{pmatrix} -c & -d \\ a & b \end{pmatrix} + s\,\delta \begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \tag{10.1}$$

where p, q, r, s are functions of N, δ^2, λ and $p(0) = 0$, $q(0) = 1$. The last restriction corresponds to $(df)_{0,0} = J$. The reduced bifurcation equation has the form

$$\phi = P\begin{pmatrix} a & b \\ c & d \end{pmatrix} + Q\begin{pmatrix} -b & a \\ -d & c \end{pmatrix} + R\,\delta\begin{pmatrix} -c & -d \\ a & b \end{pmatrix} + S\,\delta\begin{pmatrix} d & -c \\ -b & a \end{pmatrix} \qquad (10.2)$$

where $P = p$, $Q = q - 1 - \tau$, $R = r$, $S = s$ using Proposition 4.3.

Remark. If we had not assumed that f is in normal form (H4) then ϕ would still be $O(2)\times S^1$-equivariant (Theorem 4.1); however, we would not be able to make the above simple correspondence between f and ϕ in that case. In principle, we could determine the first few coefficients of ϕ, say up to order three, for a general f by explicitly putting f in normal form up to order three. We have not attempted this calculation.

Recall that $P_\lambda(0, 0) = \sigma'(0) \neq 0$ by (4.4c), and (H2) states that $\sigma'(0) \neq 0$.

Theorem 10.1. *Assume f is in $O(2)\times S^1$ normal form and satisfies* (H2). *Let* $\varepsilon = \operatorname{sgn}(\sigma'(0))$. *Assume in addition that*

$$P_N(0) \neq 0, \qquad 2P_N(0) + S(0) \neq 0, \qquad S(0) \neq 0. \qquad (10.3)$$

Then (*a*) *the branch of spatial periodic solutions with isotropy subgroup $Z_2 \oplus Z_2^c$ is supercritical if $\varepsilon P_N(0) < 0$ and subcritical if $\varepsilon P_N(0) > 0$. This solution is orbitally stable if $P_N(0) > 0$ and $S(0) > 0$.*

(*b*) *The branch of rotating wave solutions with isotropy subgroup $SO(2)$ is supercritical if $\varepsilon(2P_N(0) + S(0)) < 0$ and subcritical if $\varepsilon(2P_N(0) + S(0)) > 0$. This solution is orbitally stable if $2P_N(0) + S(0) > 0$ and $S(0) < 0$.*

(*c*) *There are* (*locally*) *no periodic solutions with period near 2π, other than the ones listed in* (*a*) *and* (*b*).

Remarks. (a) Spatial solutions (a) are unstable if $P_N(0) < 0$ or $S(0) < 0$; rotating waves (b) are unstable if $2P_N(0) + S(0) < 0$ or $S(0) > 0$. If any of these inequalities become equalities, the stabilities depend on higher order terms in ways we have not attempted to determine.

(b) We summarize the results of Theorem 10.1 in the schematic bifurcation diagrams of Figure 10.1. The branches represent $O(2)\times S^1$*-orbits* of periodic solutions. We consider the case $\varepsilon = \operatorname{sgn}\sigma'(0) < 0$, where the steady-state solution is stable subcritically and unstable supercritically.

We see that one of the bifurcating periodic solutions can be stable only if *both* bifurcate supercritically. The other is then *unstable*, assuming the nondegeneracy condition (10.3). Which one is stable depends on the sign of $S(0)$.

Proof of Theorem 10.1. (a) First consider the solutions with isotropy subgroup $Z_2 \oplus Z_2^c$. When looking for $Z_2 \oplus Z_2^c$-solutions to $\phi = 0$, we need look only at ϕ evaluated at points $Z = \begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$ where $a > 0$, since ϕ vanishes on $O(2)\times S^1$-orbits. See Table 7.2. For such Z, $\phi = 0$ reduces to the equation

$$P(a^2, 0, \lambda) = 0, \qquad Q(a^2, 0, \lambda, \tau) = 0.$$

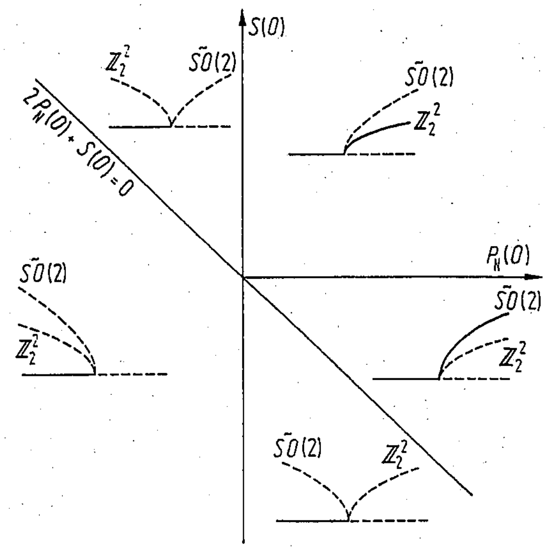


Fig. 10.1. Branching and stability for the two types of periodic solutions in $O(2)$-symmetric Hopf bifurcation. The $(P_N(0), S(0))$ plane divides into 5 regions: for values interior to these the schematic bifurcation diagrams are as shown. (Solid lines correspond to stable branches, dotted lines to unstable branches.) Here we assume $\varepsilon = \operatorname{sgn}(\sigma'(0)) < 0$

Note that $\tau = q(a^2, 0, \lambda) - 1$ satisfies $Q = 0$. Since $P_\lambda(0) \neq 0$ we can solve $P(a^2, 0, \lambda) = 0$ for

$$\lambda = -\frac{P_N(0)}{P_\lambda(0)} a^2 + \dots, \qquad a > 0, \tag{10.4}$$

which yields the super/subcritical result.

To find the stability of these solutions we must compute the eigenvalues of $d\phi$ along the solution branch. Recall that d_Σ of the four eigenvalues are zero, where

$$d_\Sigma = \dim \Gamma + 1 - \dim \Sigma = 1 + 1 - 0 = 2.$$

See Proposition 8.1. Moreover, we can decompose $R^2 \otimes C$ into subspaces invariant under the action of Σ as follows:

$$R^2 \otimes C = W_1 \oplus W_2 \tag{10.5}$$

where $W_1 = (R^2 \otimes C)^\Sigma = \left\{\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix}\right\}$ and $W_2 = \left\{\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix}\right\}$. To prove that W_2 is invariant under Σ, note that $\varkappa = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is the only element in $Z_2 \oplus Z_2^c$ which acts nontrivially, and W_2 is the -1 eigenspace of $\varkappa$. Because $\varkappa$ acts differently on W_1 and W_2, we know from Lemma 8.4 that $(d\phi)(W_j) \subset W_j$, $j = 1, 2$, Moreover, $d\phi$ restricted to the fixed-point subspace has only one zero eigenvalue. Since $d_\Sigma = 2$, it follows that $d\phi \mid W_1$ and $d\phi \mid W_2$ each have one possibly nonzero eigenvalue.

These can be computed as $\operatorname{tr}(d\phi \mid W_1)$ and $\operatorname{tr}(d\phi \mid W_2)$. We write ϕ in coordinates as

$$\phi = \begin{pmatrix} A & B \\ C & D \end{pmatrix} = \begin{pmatrix} Pa - Qb - R\,\delta c + S\,\delta d & Pb + Qa - R\,\delta d - S\,\delta c \\ Pc - Q\,d + R\,\delta a - S\,\delta b & Pd + Qc + R\,\delta b + S\,\delta a \end{pmatrix}. \tag{10.6}$$

Then

$$\begin{aligned} &\text{(a)} \quad \operatorname{tr}(d\phi \mid W_1) = \frac{\partial A}{\partial a} + \frac{\partial B}{\partial b}, \\ &\text{(b)} \quad \operatorname{tr}(d\phi \mid W_2) = \frac{\partial C}{\partial c} + \frac{\partial D}{\partial d}, \end{aligned} \tag{10.7}$$

where each of these derivatives is evaluated at $\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}$, $P = 0$ and $Q = 0$. It is straightforward to compute the necessary derivatives; the results are

$$\begin{aligned} &\text{(a)} \quad \frac{\partial A}{\partial a} + \frac{\partial B}{\partial b} = 2a^2 P_N + O(a^3), \\ &\text{(b)} \quad \frac{\partial C}{\partial c} + \frac{\partial D}{\partial d} = Sa^2. \end{aligned} \tag{10.8}$$

The nondegeneracy conditions (10.3) allow us to conclude that the signs of the two nonzero (real) eigenvalues of $d\phi$ are given by $\operatorname{sgn}(P_N(0))$ and $\operatorname{sgn}(S(0))$. Now apply Theorem 8.2.

(b) Next we consider the solutions with isotropy subgroup $\widetilde{SO}(2)$. Referring to Table 7.2, we see that to solve $\phi = 0$ we need only consider points $Z = \begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}$, $a > 0$. For such Z, the equation $\phi = 0$ reduces to

$$P + a^2 S = 0, \qquad Q - a^2 R = 0. \tag{10.9}$$

The second equation can be solved for τ. Observe that $N = 2a^2$, $\delta = a^2$ for these solutions so that the first equation may be solved for λ, yielding

$$\lambda = -\frac{2P_N(0) + S(0)}{P_\lambda(0)} a^2 + \dots .$$

To compute the stability of these solutions, we first observe that $d_\Sigma = 1$ and that

$$\boldsymbol{R}^2 \otimes \boldsymbol{C} = W_1 \oplus W_2$$

where $W_1 = (\boldsymbol{R}^2 \otimes \boldsymbol{C})^\Sigma = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right\}$ and $W_2 = \left\{\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\right\}$. To see that W_2 is invariant under $\dot{S}O(2) = \{(R_\theta, R_{-\theta})\}$ observe that W_2 consists of symmetric trace-zero matrices, which are preserved by similarity transformations $Z \to R_\theta R_\theta^{-1}$. A calculation shows that $(R_\theta, R_\theta^{-1})$ acts as a rotation through angle 2θ on W_2, which is irredusible for Σ and distinct from W_1.

The next step in the proof is to compute the eigenvalues of $d\phi \mid W_1$ and $d\phi \mid W_2$. Here we use Lemma 8.4 again. Observe that one of the eigenvalues of $d\phi \mid W_1$ is zero, since W_1 is the fixed-point subspace of $\Sigma = \widetilde{SO}(2)$. The other eigenvalue is $\operatorname{tr}(d\phi \mid W_1)$. Since Σ acts as the group of rotations on W_2 and $d\phi$ commutes with Σ (cf. (2.5)) we see that the 2×2 matrix $d\phi \mid W_2$ commutes with rotations. The only linear maps which commute with rotations are themselves multiples of rotations. Thus $d\phi \mid W_2$ either has two equal real eigenvalues, or a complex conjugate pair. In either case the real parts of these eigenvalues are $\frac{1}{2}\operatorname{tr}(d\phi \mid W_2)$. It now follows from Theorem 8.2 that a rotating wave solution is orbitally stable when

$$\operatorname{tr}(d\phi \mid W_1) > 0 \quad \text{and} \quad \operatorname{tr}(d\phi \mid W_2) > 0. \tag{10.10}$$

(Condition (10.10) could also have been obtained by a lengthy direct calculation of $d\phi$ on rotating wave solutions, and indeed this can even be done *without* assuming that f is in Γ-normal form.)

Use the obvious bases $\left\{\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix}, \begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix}\right\}$ for W_1 and $\left\{\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}, \begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\right\}$ for W_2, and compute the trace of $d\phi$ on each of these subspaces. The answer is:

$$\begin{aligned} &\text{(a)} \quad \operatorname{tr}(d\phi \mid W_1) = \frac{\partial A}{\partial a} + \frac{\partial A}{\partial b} + \frac{\partial B}{\partial b} - \frac{\partial B}{\partial c} \\ &\text{(b)} \quad \operatorname{tr}(d\phi \mid W_2) = \frac{\partial A}{\partial a} - \frac{\partial A}{\partial d} + \frac{\partial B}{\partial b} + \frac{\partial B}{\partial c}, \end{aligned} \tag{10.11}$$

in the notation of (10.6). We now compute the right-hand side of (10.11) explicitly at points $Z = \begin{pmatrix}a & 0\\ 0 & a\end{pmatrix}$, $a > 0$ satisfying (10.9). First calculate:

$$\begin{aligned} &\text{(a)} \quad \frac{\partial A}{\partial a} = \frac{\partial P}{\partial a} a + \frac{\partial S}{\partial a} a^3, \\ &\text{(b)} \quad \frac{\partial A}{\partial d} = \frac{\partial P}{\partial d} a + 2Sa^2 + \frac{\partial S}{\partial d} a^3, \\ &\text{(c)} \quad \frac{\partial B}{\partial b} = P + \frac{\partial Q}{\partial b} a - \frac{\partial R}{\partial b} a^3, \\ &\text{(d)} \quad \frac{\partial B}{\partial c} = \frac{\partial Q}{\partial c} a - \frac{\partial R}{\partial c} a^3 - Sa^2. \end{aligned} \tag{10.12}$$

Using (10.11) and (10.12), we compute

$$\begin{aligned} &\text{(a)} \quad \operatorname{tr}(d\phi \mid W_1) = 2(2P_N(0) + S(0))\, a^2 + O(a^3), \\ &\text{(b)} \quad \operatorname{tr}(d\phi \mid W_2) = -4S(0)\, a^2 + O(a^3). \end{aligned} \tag{10.13}$$

In deriving (10.13), we use the fact that $\lambda = O(a^2)$ along this solution branch. The stability conditions for rotating waves now follow from (10.10), (10.13) and the nondegeneracy condition (10.3).

(c) In theory there are other possible solutions with isotropy subgroup Z_2^c. From Table 7.2 we see that orbits of such possible solutions intersect $Z = \begin{pmatrix} a & 0 \\ a & d \end{pmatrix}$ with $a > d > 0$. For such A, the equation $\phi = 0$ reduces to the equation

$$P = Q = R = S = 0. \tag{10.14}$$

Use (10.6), However, the nondegeneracy condition $S(0) \neq 0$ in (10.3) will be violated by any such solution. □

To round off the dicussion of $O(2)$ acting on R^2 we briefly discuss solutions to a degenerate system, *not* satisfying (10.3), wich have isotropy subgroup Z_2^c. As we saw, such solutions must satisfy the system (10.14) of four equations in four unknowns N, δ^2, λ, τ. For such a solution to exist we need to assume two extra conditions

$$R(0) = S(0) = 0. \tag{10.15}$$

We know that we can always solve $Q = 0$ for τ, and under the hypothesis (H2) we can solve $P = 0$ for $\lambda = \Lambda(N, \delta^2)$. Thus (10.14) reduces to the two equations

$$R(N, \delta^2, \Lambda(N, \delta^2)) = 0, \quad S(N, \delta^2, \Lambda(N, \delta^2)) = 0. \tag{10.16}$$

Typically, solutions of (10.16) have to be isolated orbits of solutions in (Z, λ)-space. (The possible existence of such solutions is noted in passing by BAJAJ [1982].)

Now if we arrange for (10.15) to be satisfied by specifying the values of two auxiliary parameters α_1, α_2, then we can ask: "What happens to the solution to (10.15) as these parameters are varied?". The answer is that for a typical point in the (α_1, α_2)-plane there will be an *isolated* value of λ for which a periodic solution with isotropy subgroup Z_2^c occurs. We emphasize that this periodic solution does *not* lie on a branch of periodic solutions obtained by varying λ, while keeping (α_1, α_2) constant. At first glance, this fact seems bizarre. However, JOHN GUCKENHEIMER provided us with a plausible explanation for such a solution—if one considers the dynamics *in toto*.

Any solution with isotropy subgroup $\Sigma = Z_2^c$ lies on an invariant 2-torus in $R^2 \otimes C$ which is foliated by 2π-periodic solutions conjugate under $O(2) \times S^1$. This fact follows from $\dim Z_2^c = 0$. Suppose the invariant torus persists as λ is varied. It could happen that 2π-periodic flows on the torus would occur only for isolated values of λ. At other values of λ the flow could either be quasiperiodic or periodic with period incommensurate with 2π. In either case, such flows would not appear in the Banach space $C_{2\pi}$. This conjecture points out one weakness in the Liapunov-Schmidt approach to dynamics.

§ 11. Remarks on $O(n)$ acting on R^n

We now prove that the analysis for $\Gamma = O(2)$ acting on R^2 generalizes easily to $O(n)$ acting on R^n its by standard representation, with essentially identical results. The argument uses group theory and requires little additional computation.

Let $V = R^n$ be equipped with the usual inner product, denoted by $\langle v, w\rangle$ for $v, w \in R^n$,. For $\gamma \in O(n)$ we have

$$\langle \gamma v, \gamma w\rangle = \langle v, w\rangle.$$

Let $O(n)\times S^1$ act on $R^n \otimes C$, thought of as $n\times 2$ matrices as in (3.3).

We begin by showing that the lattices of isotropy subgroups of $O(n)\times S^1$ acting on $R^n \otimes C$ has the same form for general n as it does for $n = 2$. See Table 11.1. Moreover, the computations for general n depend on the results for $n = 2$, as follows. Since $O(n)$ acts transitively on planes in R^n it follows that for every

$$Z = \begin{pmatrix} v_1 & w_1 \\ \vdots & \vdots \\ v_n & w_n \end{pmatrix} = (v \mid w)$$

there exists $\gamma \in O(n)$ such that γv and γw both lie in the space

$$W = \{(x, y, 0, \ldots, 0)\}.$$

Therefore

$$\gamma z = \begin{pmatrix} a & b \\ c & d \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \equiv \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

Now use Lemma 7.1 and the action of $O(2)\times S^1 \subset O(n)\times S^1$ to see that every orbit of $O(n)\times S^1$ contains an element $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ where $a \geqq d \geqq 0$. It is easy to see that the isotropy subgroups of the elements listed in Table 11.1 contain the listed subgroups; we assert that these subgroups are precisely the isotropy subgroups. For $\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$ and $\begin{bmatrix} a & 0 \\ 0 & 0 \end{bmatrix}$ this assertion may be verified directly.

Next we consider elements $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$, $a \geqq d > 0$. Let B be the 4-dimensional space of matrices $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$. Note that the action of S^1 on $R^n \otimes C$ has B as an invariant subspace. Moreover, an argument using determinants shows that if the columns in $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ are independent then so are the columns in $\begin{bmatrix} a & b \\ c & d \end{bmatrix} R_\theta$. It follows that if (γ, R_θ) fixes $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix}$ then $\gamma : R^2 \to R^2$, since the columns in $\begin{bmatrix} a & 0 \\ 0 & d \end{bmatrix} R_\theta$ are independent in R^2 and the image of these columns under

Table 11.1. The lattice of isotropy subgroups of $O(n)\times S^1$

$$O(n)\times S^1 \quad \left(\begin{bmatrix}0 & 0\\ 0 & 0\end{bmatrix}\right)$$

$$Z_2^c\oplus O(n-1) \quad \left(\begin{bmatrix}a & 0\\ 0 & 0\end{bmatrix}\right) \qquad \widetilde{SO}(2)\times O(n-2) \quad \left(\begin{bmatrix}a & 0\\ 0 & a\end{bmatrix}\right)$$

$$Z_2^c\times O(n-2) \quad \left(\begin{bmatrix}a & 0\\ 0 & d\end{bmatrix},\ a>d>0\right)$$

γ also lies in R^2. Now since $\gamma\in O(n)$, the $(n-2)$-dimensional space $(R^2)^\perp$ is also invariant under γ. It now follows that for $\begin{bmatrix}a & 0\\ 0 & d\end{bmatrix}$ the isotropy subgroup is the isotropy subgroup of $O(2)\times S^1$ acting on $\begin{bmatrix}a & b\\ c & d\end{bmatrix}$ augmented with $O(n-2)$. This completes Table 11.1.

Observe that the fixed-point spaces for the two maximal isotropy subgroups are 2-dimensional, so again we can use Theorem 5.1 directly to obtain an existence theorem for two branches of periodic solutions for $O(n)$ acting on R^n. The analogy between $O(n)$ and $O(2)$ continues.

Next consider the invariant theory of $O(n)\times S^1$. We assert that this is determined from $O(2)\times S^1$ by extension. Observe that the subspace B is the fixed-point subspace of $O(n-2)$, that is, $B=(R^n\otimes C)^{O(n-2)}$. Thus if $f\colon R^n\otimes C \to R^n\otimes C$ is $O(n)\times S^1$-equivariant, then it follows that

$$f\colon B\to B$$

and $f\,|\,B$ commutes with $O(2)\times S^1$. Moreover, we showed above that every orbit of $O(n)\times S^1$ on $R^n\otimes C$ intersects B. Thus f is uniquely determined by $f\,|\,B$. We now show that every smooth $O(2)\times S^1$-equivariant map $g\colon B\to B$ extends to a smooth $O(n)\times S^1$-equivariant map $f\colon R^n\otimes C\to R^n\otimes C$. We do this by explicitly extending the generators of the $O(2)\times S^1$-invariants and equivariants listed in Theorem 9.1.

Let $Z=(v\,|\,w)$, $v, w\in R^n$. Then N and δ^2 extend to

$$\text{(a)}\quad \bar{N}=\langle v,v\rangle+\langle w,w\rangle,$$
$$\text{(b)}\quad \bar{\Delta}=\langle v,v\rangle\langle w,w\rangle-\langle v,w\rangle^2, \tag{11.1}$$

which are clearly $O(n)$-invariant since $\langle\gamma v,\gamma w\rangle=\langle v,w\rangle$. They are also S^1-invariant since without loss of generality $(v\,|\,w)\in B$ where $O(2)\times S^1$-invariance and hence S^1-invariance holds by Theorem 9.1.

Similarly, the equivariant generators I, J, $K = \delta \begin{pmatrix} -c & -d \\ a & b \end{pmatrix}$, $L = -JK$ extend to

$$\begin{aligned} &\text{(a)} \quad \bar{I} = (v \mid w), \\ &\text{(b)} \quad \bar{J} = (-w \mid v), \\ &\text{(c)} \quad \bar{K} = (\langle v, v\rangle w - \langle v, w\rangle v \mid \langle w, w\rangle v - \langle v, w\rangle w), \\ &\text{(d)} \quad \bar{L} = -JK. \end{aligned} \tag{11.2}$$

Again $O(n)$-equivariance is clear, and S^1-equivariance follows from S^1-equivariance on B (Lemma 9.2). This proves:

Proposition 11.1. *The generators for $O(2)\times S^1$-invariants and -equivariants extend uniquely to provide generators for $O(n)\times S^1$-invariants and -equivariants on $R^n \otimes C$.*

The branching directions of the two families of periodic solutions for $O(n)\times S^1$ now follow directly from Theorem 10.1, the corresponding results for $O(2)\times S^1$.

Finally, we look at the stabilities. We consider the two isotropy subgroups $Z_2^c \times O(n-1)$ and $\widetilde{SO}(2)\times O(n-2)$ separately. In the case $\Sigma = Z_2^c\times O(n-1)$ we find that

$$W_1 = (R^n \otimes C)^{\Sigma}, \qquad W_2 = (W_1)^{\perp} \tag{11.3}$$

where $\dim W_1 = 2$. Moreover, the action of Σ is the action of $O(n-1)$ on $R^{n-1} \oplus R^{n-1}$; here we use the notation of (8.11). Since $\dim O(n) = (n-1)\,n/2$ we calculate $d_\Sigma = \dim O(n) + 1 - \dim O(n-1) = n$. One of the eigenvalues of $d\phi$ forced to be zero occurs in W_1, the other $n-1$ in W_2. Since $d\phi \mid W_2$ commutes with Σ we see that

$$d\phi \mid W_2 = \begin{pmatrix} \alpha I_{n-1} & \beta I_{n-1} \\ \gamma I_{n-1} & \varepsilon I_{n-1} \end{pmatrix}.$$

There are two eigenvalues of this matrix, each with multiplicity $n-1$. One of these eigenvalues is zero. To determine orbital stability for these solutions we need only check whether

$$\operatorname{tr}(d\phi \mid W_1) > 0, \qquad \operatorname{tr}(d\phi \mid W_2) > 0.$$

We have now reduced to the case of $O(2)\times S^1$. See Theorem 10.1 and (10.7), in particular. The result is identical.

Finally, we consider orbital stability for the rotating waves, $\Sigma = \widetilde{SO}(2)\times O(n-2)$. Note that $d_\Sigma = 2n-3$. Here all but three of the eigenvalues of $d\phi$ are forced by the group action to be zero. Observe that Σ decomposes $R^n \otimes C$ into

$$R^n \otimes C = W_1 \oplus W_2 \oplus W_3$$

where

$$W_1 = (R^n \otimes C)^{\Sigma} = \left\{\begin{bmatrix} a & b \\ -b & a \end{bmatrix}\right\},$$

$$W_2 = \left\{\begin{bmatrix} a & b \\ b & -a \end{bmatrix}\right\},$$

$$W_3 = B^{\perp}.$$

Note that $W_1 \oplus W_2 = B$. One may verify that $d\phi \mid W_3 \equiv 0$, which contributes $2n-4$ zero eigenvalues, the other zero eigenvalue occurring in $d\phi \mid W_1$. In any case, orbital stability is determined by

$$\operatorname{tr} d\phi \mid W_1 \quad \text{and} \quad \operatorname{tr} d\phi \mid W_2$$

as in Theorem 10.1 and (10.10). These results and those for $O(2)\times S^1$ are identical.

§ 12. Remarks on Maximal Isotropy Subgroups

A key ingredient in the proof of Theorem 5.1 is the assumption (H3) that the fixed-point subspace of the isotropy subgroup Σ is 2-dimensional. In this section we present a context in which this hypothesis seems more natural. We show that Σ is a maximal isotropy subgroup, and make some observations about these maximal isotropy subgroups.

Definition 12.1. A proper isotropy subgroup Σ is *maximal* if no other proper isotropy subgroup contains Σ.

Let Σ be an isotropy subgroup of a group G acting linearly on a vector space W. Let $N_G(\Sigma)$ be the normalizer of Σ in G. Since Σ is normal in $N_G(\Sigma)$ we can form the quotient group $D(\Sigma) = N_G(\Sigma)/\Sigma$. We assert that $D(\Sigma)$ acts naturally on the fixed-point subspace W^{Σ}. Let $w \in W^{\Sigma}$, $\delta \in N_G(\Sigma)$. For every $\sigma \in \Sigma$ there exists $\sigma' \in \Sigma$ such that $\sigma\delta = \delta\sigma'$. It follows that $\sigma\delta w = \delta\sigma' w = \delta w$. Hence Σ fixes δw and $\delta w \in W^{\Sigma}$. Hence $N^G(\Sigma)$ maps W^{Σ} to W^{Σ}. Since Σ acts trivially on W^{Σ}, by definition, there is an induced (linear) action of $D(\Sigma)$ on W^{Σ}.

Lemma 12.2. *Let Σ be a maximal isotropy subgroup of G acting on W. Assume $W^G = \{0\}$. Then the induced action of $D(\Sigma)$ on W^{Σ} is free of fixed-points.*

Proof. Suppose the action has a fixed point, that is, there exists $\delta \in N_G(\Sigma) \sim \Sigma$ and $0 \neq w \in W^{\Sigma}$ such that $\delta w = w$. Then the isotropy subgroup of w contains Σ, and $\delta \notin \Sigma$. Since $w \neq 0$, this isotropy subgroup is proper, and Σ is not a maximal isotropy subgroup.

The importance of Lemma 12.2 is that actions of compact Lie groups which are free of fixed points are relatively rare, and can be classified. This observation leads to:

Theorem 12.3. *Let G be a compact Lie group acting on W. Assume $W^G = \{0\}$. Let Σ be a maximal isotropy subgroup of G and let D^0 be the connected component of the identity in $D(\Sigma)$. Then either*

(a) $D^0 = 1$,

(b) $D^0 \cong S^1$ *and W^Σ is the direct sum of irreducible subspaces under S^1, each isomorphic to C, with S^1 acting on C by complex multiplication.*

(c) $D^0 \cong S^3 = SU(2)$ *and W^Σ is the direct sum of irreducible subspaces under S^3, each isomorphic to the quaternions H, with S^3 acting by quaternionic multiplication (via the identification of S^3 with the unit quaternions).*

Proof. The classification of fixed-point-free actions of compact connected Lie groups is given in BREDON [1972]. A sketch is also given in GOLUBITSKY [1983].

In fact, the possibilities for $D(\Sigma)$ can be classified, not just D^0. There is a lengthy list for case (a), see WOLF [1967]. For case (b) $D \cong SO(2)$ or $O(2)$, and for case (c) $D \cong SU(2)$; see BREDON [1972].

Definition 12.4. A maximal isotropy subgroup Σ of G is said to be *real* if $D^0 = 1$, *complex* if $D^0 = S^1$, and *quaternionic* if $D^0 = S^3$.

Note that for the complex case, $\dim W^\Sigma \equiv 0 \pmod 2$, and in the quaternionic case, $\dim W^\Sigma \equiv 0 \pmod 4$.

Proposition 12.5. (a) The action of $\Gamma \times S^1$ on $V \otimes C$ cannot have real maximal isotropy subgroups.

(b) If $\dim (V \otimes C)^\Sigma = 2$, then Σ is a complex maximal isotropy subgroup.

Proof. (a) In Proposition 6.1 we showed that $S^1 \cap \Sigma = 1$ for any proper isotropy subgroup Σ. Since $S^1 \subset N_{\Gamma\times S^1}(\Sigma)$ it follows that S^1 embeds in D^0.

(b) If $\dim (V \otimes C)^\Sigma = 2$, then Σ must be a maximal isotropy subgroup. (Any larger isotropy subgroup would have a fixed-point subspace of smaller dimension, but this is impossible since there are no real maximal isotropy subgroups.) Σ cannot be quaternionic as then $\dim (V \otimes C)^\Sigma \geqq 4$. □

Note. Proposition 12.5(a) provides another proof that $\dim (V \otimes C)^\Sigma \geqq 2$ for any isotropy subgroup Σ in $\Gamma \times S^1$.

In static bifurcation theory there is a theorem of VAN DER BAUWHEDE [1980] and CICOGNA [1981] that (in this language) states that a branch of steady-state solutions exists for every real maximal isotropy subgroup whose fixed-point subspace is 1-dimensional, as long as an eigenvalue-crossing condition analogous to (H2) is valid. Stated loosely: real maximal isotropy subgroups with minimal dimensional fixed-point subspaces yield steady-state solutions.

Our Theorem 5.1 provides an analogous result for complex maximal isotropy subgroups. Stated loosely: complex maximal isotropy subgroups with minimal dimensional fixed-point subspaces yield periodic solutions.

This analogy leads directly to the question: What natural class of solutions corresponds to quaternionic maximal isotropy subgroups with minimal (4-)dimensional fixed-point subspaces?

As a side note, we know that in order to find solutions for complex maximal isotropy subgroups Σ with $\dim (V \otimes C)^{\Sigma} = 2$ it is necessary (mathematically) to introduce an auxiliary parameter, in addition to the bifurcation parameter λ. In Hopf bifurcation this auxiliary parameter is τ, the perturbed period.

For quaternionic maximal isotropy subgroups Σ with $\dim W^{\Sigma} = 4$ we can show that it will be necessary to introduce *three* auxiliary parameters in order to find solutions. We are, however, unable to find a naturally occurring context for such an analysis.

§ 13. Dimensions of Fixed-Point Subspaces

In this section we derive criteria for determining when a fixed-point subspace of $V \otimes C$ is 2-dimensional, in terms of the dimension of fixed-point subspaces of V. These criteria will be required in § 15 to classify isotropy subgroups of $O(3) \times S^1$ having 2-dimensional fixed-point subspaces. The reader willing to take the results of § 15 on trust may skip this section.

Recall from Proposition 6.2 that every isotropy subgroup Σ of $\Gamma \times S^1$ may be written in twisted form as H^{θ}, where $H \subset \Gamma$ is a subgroup and $\theta : H \to S^1$ is a homomorphism. We have two main results in this section.

Theorem 13.1. *Let* $K = \ker \theta \subset H$. *Then* $\dim (V \otimes C)^{H^{\theta}} = 2$ *if*

(a) $\theta(H) = 1$ *and* $\dim V^H = 1$,

(b) $\theta(H) = Z_2$ *and* $\dim V^K - \dim V^H = 1$,

(c) $\theta(H) = Z_3$ *and* $\dim V^K - \dim V^H = 2$.

Remarks. (i) Theorem 13.1(a) was already proved in Proposition 6.4(a) and is included here for completeness.

(ii) Similar formulas exist for $\theta(H) = Z_4$ or Z_6 (for example) but not for Z_5, Z_7, *etc.*

For each twist $\theta : H \to S^1$ there is an irreducible representation of H defined as follows.

First, suppose $\theta(H) = 1$ or $\theta(H) = Z_2$. Then H acts on R by

$$h \cdot x = \cos \theta(h)\, x, \qquad x \in R.$$

Otherwise, H acts on $R^2 \cong C$ by

$$h \cdot z = e^{i\theta(h)} z, \qquad z \in C.$$

Denote these representations of H by ϱ_{θ}.

Theorem 13.2. $\dim (V \otimes C)^{H^\theta}$ *is equal to twice the multiplicity with which* ϱ_θ *occurs in the action of H on V. Thus* $\dim (V \otimes C)^{H^\theta} = 2$ *precisely when* ϱ_θ *occurs once in the action of H on V.*

Remark. In case $\theta = 0$, ϱ_θ is the identity representation and the multiplicity with which ϱ_θ occurs in the action of H is just $\dim V^H$. This special case is proved by (6.2).

We begin with a discussion of Theorem 13.1, whose proof follows directly from:

Proposition 13.3. Let H^θ be a subgroup of $\Gamma \times S^1$ and let $K = \ker \theta$.

(a) If $\theta(H) = Z_2$, then

$$\dim (V \otimes C)^{H^\theta} = 2 (\dim V^K - \dim V^H).$$

(b) If $\theta(H) = Z_3$, then

$$\dim (V \otimes C)^{H^\theta} = \dim V^K - \dim V^H.$$

The proof of Proposition 13.3 requires the trace formula, which we digress to explain. Let ϱ be a representation of a compact Lie group G on a vector space W. For each $\gamma \in G$, the map $\varrho(\gamma): W \to W$ is linear; denote its trace by $\operatorname{Tr} \varrho(\gamma)$. We state the well known trace formula. (*Cf.* Ihrig & Golubitsky [1984], Sattinger [1983], Michel [1980], Adams [1969] Prop. 3.3.)

Proposition 13.4. Let H be a closed subgroup of G. Then

$$\dim W^H = \int_H \operatorname{Tr} \varrho(h). \tag{13.1}$$

Remark. The integral in (13.1) is with respect to the invariant (Haar) measure on H, which we assume normalized so that the total volume vol (H) is 1. Integration with respect to this measure defines the *normalized Haar integral.* (*Cf.* Adams [1969] or Kirillov [1976].)

Next we apply the trace formula to an isotropy subgroup H^θ of $\Gamma \times S^1$ acting on $V \otimes C$.

Lemma 13.5. Let ϱ be the representation of Γ on V. Then

$$\dim (V \otimes C)^{H^\theta} = 2 \int_H \operatorname{Tr} \varrho(h) \cos \theta(h).$$

Proof. Let ϱ' be the representation of $\Gamma \times S^1$ on $V \otimes C$ and let $h \varepsilon H$. In block matrix form we have

$$\begin{aligned} \varrho'(h, \theta(h)) &= \varrho'(h, 0) \cdot \varrho'(1, \theta), \\ &= \begin{pmatrix} \varrho(h) & 0 \\ 0 & \varrho(h) \end{pmatrix} \begin{pmatrix} \cos \theta I & -\sin \theta I \\ \sin \theta I & \cos \theta I \end{pmatrix}, \\ &= \begin{pmatrix} \varrho(h) \cos \theta & -\varrho(h) \sin \theta \\ \varrho(h) \sin \theta & \varrho(h) \cos \theta \end{pmatrix}. \end{aligned}$$

Therefore

$$\dim (V \otimes C)^{H^\theta} = \int_{H^\theta} \operatorname{Tr} \varrho'(h, \theta(h)),$$
$$= 2 \int_H \varrho(h) \cos \theta,$$

as desired. (This fact can also be proved from the $V \otimes C$ structure via the formula $\operatorname{Tr}(\alpha \times \beta) = \operatorname{Tr}(\alpha) \operatorname{Ttr}(\beta)$. □

Remark. Suppose H^θ is spatial, *i.e.*, $\theta \equiv 0$. Then Lemma 13.5 implies that $\dim (V \otimes C)^H = 2 \dim H$, which yields another proof of Theorem 13.1(a).

Proof of Proposition 13.3. (a) Since $\theta(H) = Z_2$ we must have $\theta(h) = 0$ or π and $\cos(\theta(h)) = \pm 1$. In fact, $\cos(\theta(h)) \equiv 1$ on K and $\cos(\theta(h)) \equiv -1$ on $H \sim K$. Now use Lemma 13.5 to compute

$$\dim (V \otimes C)^{H^\theta} = \frac{2}{\operatorname{vol} H}\left[\int_K \operatorname{Tr} \varrho(h) - \int_{H \sim K} \operatorname{Tr} \varrho(h)\right], \tag{13.2}$$

to avoid confusion with the normalizations, we use a fixed normalization of Γ.

Next observe that

$$2 \dim V^H = \frac{2}{\operatorname{vol} H}\left[\int_K \operatorname{Tr} \varrho(h) + \int_{H \sim K} \operatorname{Tr} \varrho(h)\right]. \tag{13.3}$$

Adding (13.2) and (13.3) yields

$$\dim (V \otimes C)^{H^\theta} + 2 \dim V^H$$
$$= \frac{4}{\operatorname{vol} H} \int_K \operatorname{Tr} \varrho(h)$$
$$= \frac{2}{\operatorname{vol} K} \int_K \operatorname{Tr} \varrho(h)$$
$$= 2 \dim V^K$$

since $\operatorname{vol} H = 2 \operatorname{vol} K$. This proves part (a).

(b) Similarly, when $\theta(H) = Z_3$ we have $\cos \theta(h) \equiv 1$ on K and $\cos(\theta(h)) \equiv -\frac{1}{2}$ on $H \sim K$. Hence Lemma 13.5 implies that

$$\dim (V \otimes C)^{H^\theta} = \frac{2}{\operatorname{vol} H}\left[\int_K \operatorname{Tr} \varrho(h) - \tfrac{1}{2} \int_{H \sim K} \operatorname{Tr} \varrho(h)\right]. \tag{13.4}$$

Further, (13.3) is still valid. Adding $\frac{1}{2}$ (13.3) to (13.4) yields

$$\dim (V \otimes C)^{H^\theta} + \dim V^H = \frac{3}{\operatorname{vol} H} \int_K \operatorname{Tr} \varrho(h),$$
$$= \frac{1}{\operatorname{vol} K} \int_K \operatorname{Tr} \varrho(h),$$
$$= \dim V^K,$$

since $\operatorname{vol} H = 3 \operatorname{vol} K$ in this case. □

The proof of Theorem 13.2 uses orthogonality of characters. We summarize the required results here.

In representation theory the function

$$\chi(\gamma) = \operatorname{Tr} \varrho(\gamma)$$

is called the *character* of ϱ, and it determines ϱ uniquely up to equivalence. Let χ_1 and χ_2 be the characters of irreducible representations ϱ_1, ϱ_2 of G. Then we have the *orthogonality relations* (ADAMS [1969]) which for real representations take the form:

$$\int_G \chi_1(\gamma)\,\chi_2(\gamma) = \begin{cases} 0 & (\varrho_1, \varrho_2 \text{ inequivalent}) \\ 1 & (\varrho_1 \sim \varrho_2 \text{ absolutely irreducible}) \\ 2 & (\varrho_1 \sim \varrho_2 \text{ not absolutely irreducible}). \end{cases} \tag{13.5}$$

Here $\sim$ denotes equivalence of representations, that is, isomorphism of actions.

Decompose W into irreducible subspaces for G:

$$W = W_1 \oplus \ldots \oplus W_k.$$

Let ϱ be an irreducible representation of G. Then its *multiplicity* $\mu(\varrho)$ is the number of W_j for which the representation of G on W_j is equivalent to ϱ.

Proof of Theorem 13.2. Let χ_θ be the character of ϱ_θ, and χ the character of $\varrho \mid H$. Let $\mu(\theta)$ be the multiplicity of ϱ_θ in $\varrho \mid H$. We need to show that $\dim (V \otimes C)^{H^\theta} = 2\mu(\theta)$.

Note that

$$\chi_\theta = \operatorname{Tr} \varrho_\theta = \begin{cases} \cos\theta & \text{if } |\theta(H)| \leqq 2 \\ 2\cos\theta & \text{otherwise}. \end{cases}$$

By Lemma 13.5,

$$\dim (V \otimes C)^{H^\theta} = 2 \int_H \chi(h) \cos\theta. \tag{13.6}$$

If $|\theta(H)| \leqq 2$, then the right-hand side of (13.6) equals

$$2 \int_H \chi(h)\,\chi_\theta(h).$$

By the orthogonality relations (13.5) this is $2\mu(\theta)$, since ϱ_θ is absolutely irreducible. If $|\theta(H)| > 2$, then (13.6) is equal to

$$\int_H \chi(h)\,\chi_\theta(h).$$

Since ϱ_θ is irreducible but not absolutely irreducible, this is also $2\mu(\theta)$. □

§ 14. $O(3)$-Symmetric Hopf Bifurcation

In this section and the next we consider the orthogonal group $O(3)$, which occurs in many problems with spherical symmetry. This group has two distinct irreducible representations in each odd dimension $2l+1$, expressed in terms of the spherical harmonics V_l of degree l. If $-I \in O(3)$ acts as the identity on V_l we have the *plus* representation; if $-I$ acts as minus the identity then we have the *minus* representation. The "natural" representation of $O(3)$, arising in most applications, is plus when l is even, minus when l is odd. (These are the natural representations induced on spherical harmonics by the standard actions of $O(3)$ on the 2-sphere in $\mathbf{R}^3$.)

Our aim is to classify, for both plus and minus representations and for *all* l, the isotropy groups of $O(3) \times S^1$ on $V_l \otimes \mathbf{C}$ having 2-dimensional fixed-point spaces. For each such group Theorem 5.1 yields the existence of a branch of periodic solutions.

The subgroups of $O(3)$, and the dimensions of their fixed-point spaces on V_l, are described in IHRIG & GOLUBITSKY [1984], whose notation we shall follow. Similar results are stated without proof by MICHEL [1980]. In order to state the main results of this section, we briefly describe the subgroups.

We have

$$O(3) = \mathbf{Z}_2^c \oplus SO(3)$$

where $\mathbf{Z}_2^c = \{\pm 1\}$. There are three types of subgroups:

I: Subgroups of $SO(3)$.

II: Subgroups containing $\mathbf{Z}_2^c$, of the form $\mathbf{Z}_2^c \oplus H$ where $H \subset SO(3)$.

III: Subgroups intersecting $\mathbf{Z}_2^c$ trivially but not contained in $SO(3)$.

These are classified up to conjugacy as follows:

Type I $SO(3)$, $O(2)$, $SO(2)$, $\mathbf{I}$ (icosahedral group), $\mathbf{O}$ (octahedral), $\mathbf{T}$ (tetrahedral), $\mathbf{D}_n$ (dihedral), $\mathbf{Z}_n$ (cyclic).

Type II $\mathbf{Z}_2^c \oplus$ Type I.

Type III $O(2)^-$, $\mathbf{O}^-$, $\mathbf{D}_n^z$ $(n \geqq 2)$, $\mathbf{D}_{2n}^d$ $(n \geqq 2)$, $\mathbf{Z}_{2n}^-$ $(n > 1)$.

The superscripts in type III indicate subgroups isomorphic, but not conjugate in $O(3)$, to their unsubscripted counterparts of type I. The groups of type III are constructed as follows. If H is of type III then it is uniquely determined by H_1, its projection to $SO(3)$, and by $H_0 = H \cap SO(3)$. We have $|H_1 : H_0| = 2$ and any pair of subgroups (H_0, G_1) of $SO(3)$ with $|H_1 : H_0| = 2$ can occur: see IHRIG & GOLUBITSKY [1984] Lemma 1.7 (or argue as in Proposition 6.2 above). There is a homomorphism $\eta : H_1 \to \mathbf{Z}_2^c$ such that

$$\eta(h) = \begin{cases} 1 & h \in H_0 \\ -1 & h \in H_1 \sim H_0 \end{cases}.$$

Define $H^\eta = \{(h, \eta(h)) \mid h \in H_1\}$. This is the subgroup of type III. (The construction is essentially a $\mathbf{Z}_2$-twist on H_0, but now the twist map is into $\mathbf{Z}_2^c$, not S^1.) For $H_1 = O(2)$, $\mathbf{O}$, or $\mathbf{Z}_{2n}$ there is a unique subgroup of index 2, $H_0 = SO(2)$,

T, or Z_n, respectively. For these groups we denote H_1^η by H_1^-. For $(H_1, H_0) = (D_n, Z_n)$ we denote H_1^η by D_n^z; for $(H_1, H_0) = (D_{2n}, D_n)$ we denote it D_{2n}^d.

Let H^θ be an isotropy group of $O(3)$, with 2-dimensional fixed-point space. It turns out that H must be of type II, so that $H = Z_2^c \oplus J$ where $J \subset SO(3)$. In almost all cases, H^θ is determined up to conjugacy by the type (I, II, or III) of $K = \ker\theta$. We summarize the results of the computation in Table 14.1. For details

Table 14.1. Isotropy subgroups of $O(3)\times S^1$ on $V_l \otimes C$, having 2-dimensional fixed-point subspaces

J (see note [3])	Type of K	Twist $\theta(H)$	Value of l: Plus representation	Value of l: Minus representation
$O(2)$	II	1	even l	
$O(2)$	I	Z_2		even l
$O(2)$	II	Z_2	odd l	
$O(2)$	III	Z_2		odd l
$SO(2)$	II	S^1 $[k = 1, \ldots, l]$	all l	
$SO(2)$	III	S^1 $[k = 1, \ldots, l]$ (see note [1])		all l
I	II	1	6, 10, 12, 16, 18, 20, 22, 24, 26, 28, 32, 34, 38, 44; 21, 25, 27, 31, 33, 35, 37, 39, 41, 43, 47, 49, 53, 59	
I	I	Z_2		6, 10, 12, 16, 18, 20, 22, 24, 26, 28, 32, 34, 38, 44; 21, 25, 27, 31, 33, 35, 37, 39, 41, 43, 47, 49, 53, 59
O	II	1	4, 6, 8, 10, 14; 9, 13, 15, 17, 19, 23	
O	I	Z_2		4, 6, 8, 10, 14; 9, 13, 15, 17, 19, 23
O	II	Z_2	6, 10, 12, 14, 16, 20; 3, 7, 9, 11, 13, 17	
O	III	Z_2		6, 10, 12, 14, 16, 20; 3, 7, 9, 11, 13, 17
T	II	Z_3	2, 4, 6; 5, 7, 9	
T	I	Z_6		2, 4, 6; 5, 7, 9
D_n	I	Z_2		$l/2 < n \leqq l$
D_n	IIA	Z_2	$l/2 < n \leqq l$	
D_n	IIB (see note [2])	Z_2	$l < n \leqq 2l$	

Notes: [1] For S^1-twists, $\theta: SO(2) \to S^1$ is given by $\theta(\phi) = k\phi$ and $k = 1, \ldots, l$ occur.
[2] For D_n, IIA is $Z_2^c \oplus Z_n$; and IIB is $Z_2^c \oplus D_{n/2}$ when n is even.
[3] Here $H^\theta = Z_2^c \oplus J$ is the isotropy subgroup, $K = \ker\theta$.

and further explanation of the terms used, see below. For concreteness, we summarize the classification for the "natural" representation of $O(3)$, of sign $(-1)^l$, for $l \leqq 6$, in Table 14.2. This is just a reformulation of parts of Table 14.1 and may easily be extended to all l by combining the even l entries in the "plus" column of Table 14.1 with the odd l entries in the "minus" column. Note in particular the Z_3- or Z_6-twisted tetrahedral symmetry occurring when $l = 2, 4, 5, 6, 7, 9$. For static $O(3)$-equivariant bifurcation problems the tetrahedral group does not appear as a maximal isotropy group, see IHRIG & GOLUBITSKY [1984]. Also note that the case $l = 1$ was obtained in § 11.

The verification of Tables 14.1 and 14.2 is a case-by-case computation, making use of the results of § 13. It is given in the next section.

Table 14.2. Low dimensional isotropy subgroups of $O(3) \times S^1$ on $V_l \otimes C$, with the natural $(-1)^l$ representation, having 2-dimensional fixed-point subspaces

l	Twist $\theta(H)$	J	Number of branches of periodic solutions given by Theorem 5.1
1	Z_2	$O(2)$	2
	S^1 $[k = 1]$	$SO(2)$	
2	1	$O(2)$	6
	Z_2	D_2, D_4	
	Z_3	T	
	S^1 $[k = 1, 2]$	$SO(2)$	
3	Z_2	$O(2), O, D_2, D_3$	7
	S^1 $[k = 1, 2, 3]$	$SO(2)$	
4	1	$O(2), O$	11
	Z_2	D_3, D_4, D_6, D_8	
	Z_3	T	
	S^1 $[1 \leqq k \leqq 4]$	$SO(2)$	
5	Z_2	$O(2), D_3, D_4, D_5$	10
	Z_6	T	
	S^1 $[1 \leqq k \leqq 5]$	$SO(2)$	
6	1	$O(2), I, O$	17
	Z_2	$O, D_4, D_5, D_6, D_8, D_{10}, D_{12}$	
	Z_3	T	
	S^1 $[1 \leqq k \leqq 6]$	$SO(2)$	

Note: $H = Z_2^c \oplus J$ twisted by θ.

§ 15. Calculations for $O(3)$

We end by detailing the somewhat technical calculations required to verify Tables 14.1 and 14.2 above. We split the computation into subsections according to the twist type $\theta(H)$, which may be $SO(2)$, trivial, Z_3 (and Z_6), or Z_2. The Z_2-twisted case is the most complicated.

Note that although $\theta(H)$ does not determine the twisted group H^θ uniquely (because there may be several twist maps θ with the same image) it *will* determine the conjugacy class uniquely when $\theta(H) = \mathbf{Z}_2$, $\mathbf{Z}_3$, or $\mathbf{Z}_6$, because the generators of these groups are unique up to sign, and H^θ is conjugate to $H^{-\theta}$ in $O(3) \times S^1$.

A: Possible twists

Lemma 15.1. *For the plus representation, $(-I, 0)$ lies in every isotropy subgroup and $(-I, \pi)$ in none. For the minus representation, $(-I, \pi)$ lies in every isotropy subgroup and $(-I, 0)$, in none.*

Proof. $(0, \pi)$ acts as $-Id$; $(-I, 0)$ acts as $\pm Id$. □

Every isotropy group is of the form H^θ where $\theta : H \to S^1$, and H is a closed subgroup of $O(3)$. Let $K = \ker \theta$, and say that the *twist type* of H is $\theta(H)$. That is, H has a $\mathbf{Z}_3$-twist if $\theta(H) = \mathbf{Z}_3 \subset S^1$, and so on. The only closed subgroups of S^1 are S^1 and $\mathbf{Z}_k$, $k = 1, 2, 3, \ldots$; so these are the only twist types.

Now H is the projection of H^θ on $O(3)$. By Lemma 15.1 this must contain $\mathbf{Z}_2^c$, so H is always type II, with $H = \mathbf{Z}_2^c \oplus J$ for $J \subset SO(3)$. From now on we use J to specify H in this sense. Let

$$H' = \langle g^{-1} h^{-1} gh \mid g, h \in H \rangle$$

be the *commutator subgroup*. Then $K \supset H'$ since $\theta(H) \subset S^1$ is abelian. Hence, we may read off the possible twist types for H from the *abelianization* $H^{ab} = H/H'$. Table 15.1 lists all possibilities: it is easily obtained form IHRIG & GOLUBITSKY [1984].

Table 15.1. Twist types for closed subgroups of $O(3)$

J	H	H'	H^{ab}	Twist Types
$SO(3)$	$O(3)$	$SO(3)$	$\mathbf{Z}_2$	$1, \mathbf{Z}_2$
$O(2)$	$\mathbf{Z}_2^c \oplus O(2)$	$SO(2)$	$\mathbf{Z}_2^2$	$1, \mathbf{Z}_2$
$SO(2)$	$\mathbf{Z}_2^c \oplus SO(2)$	1	$SO(2) \oplus \mathbf{Z}_2$	$1, S^1, \mathbf{Z}_2$
$\mathbf{I}$	$\mathbf{Z}_2^c \oplus \mathbf{I}$	$\mathbf{I}$	$\mathbf{Z}_2$	$1, \mathbf{Z}_2$
$\mathbf{O}$	$\mathbf{Z}_2^c \oplus \mathbf{O}$	$\mathbf{T}$	$\mathbf{Z}_2^2$	$1, \mathbf{Z}_2$
$\mathbf{T}$	$\mathbf{Z}_2^c \oplus \mathbf{D}_n$	$\mathbf{D}_2$	$\mathbf{Z}_3 \oplus \mathbf{Z}_2 = \mathbf{Z}_6$	$1, \mathbf{Z}_2, \mathbf{Z}_3, \mathbf{Z}_6$
$\mathbf{D}_n$	$\mathbf{Z}_c^2 \oplus \mathbf{D}_n$	$\mathbf{Z}_n$ or $\mathbf{Z}_{n/2}$ [1]	$\mathbf{Z}_2^2$ or $\mathbf{Z}_2^3$ [2]	$1, \mathbf{Z}_2$
$\mathbf{Z}_n$	$\mathbf{Z}_c^2 \oplus \mathbf{Z}_n$	1	$\mathbf{Z}_n \oplus \mathbf{Z}_2$	$1, \mathbf{Z}_d$ ($d \mid 2n$ or $d \mid n$) [3]

Notes:

[1] $\mathbf{Z}_n$ if n is odd, $\mathbf{Z}_{n/2}$ is n is even.
[2] $\mathbf{Z}_2^2$ if n is odd, $\mathbf{Z}_2^3$ if n is even.
[3] $d \mid 2n$ if n is odd, $d \mid n$ if n is even.

B: Recognizing Isotropy Subgroups

Our computations will show that a given H^θ has a 2-dimensional fixed-point space, but this does not of itself imply that H^θ is an isotropy subgroup. To decide this, we use:

Lemma 15.2. *Let* $H^\theta \subset O(3)\times S^1$ *have a 2-dimensional fixed-point space. Then the following are equivalent.*

(a) H^θ *is an isotropy subgroup.*
(b) H^θ *is a maximal isotropy subgroup.*
(c) *Whenever* $H^\theta \subsetneq L^\phi$, *the fixed-point space of* L^ϕ *has dimension* <2 (*hence* 0).

Proof. All fixed-point spaces of isotropy groups of $O(3)\times S^1$ have dimension $\geqq 2$ by Proposition 12.5 (indeed the dimension is even, by Theorem 12.3). So (a) implies (b). Clearly (b) implies (c).

To prove (c) implies (a), suppose H^θ is not an isotropy subgroup. Let $0 \neq x \in (V_l \otimes C)^{H^\theta}$. Then $H^\theta \subsetneq \Sigma_x = L^\phi$ for suitable L, ϕ. Hence $x \in (V_l \otimes C)^{L^\phi}$. But $(V_l \otimes C)^{L^\phi} \subset (V_l \times C)^{H^\theta}$ and dimensions are $\geqq 2$, so $(V_l \otimes C)^{L^\phi} = (V_l \otimes C)^{H^\theta}$. This proves (c) implies (a) so all statements are equivalent.

It is usually easy to decide when $H^\theta \subset L^\phi$. This occurs if and only if

$$H \subset L \quad \text{and} \quad \phi \text{ extends } \theta. \tag{15.1}$$

It follows that $\ker\theta \subset \ker\phi$.

Our strategy for finding isotropy subgroups with 2-dimensional fixed-point spaces is to classify first by twist type, and second by the type (I, II, or III) of $K = \ker\theta$. Both of these are conjugacy-invariants for $O(3)\times S^1$.

For the plus and minus representation, Z_2^c must be respectively untwisted and twisted by Lemma 15.1. (That is, Z_2^c is, or is not, in $\ker\theta$). Hence

$$\text{For the plus representation, } K \text{ is of type II.} \tag{15.2}$$

$$\text{For the minus representation, } K \text{ is of type I or III.} \tag{15.3}$$

C: S^1-Twists

$SO(2)$ is a maximal torus of $O(3)$, and the root-space decomposition of V_l

$$V_l = W_0 \oplus W_1 \oplus \dots \oplus W_l \tag{15.4}$$

where $\dim W_0 = 1$, $\dim W_k = 2$ for $k > 0$. If $\phi \in SO(2)$, then ϕ acts on W_k as rotation by $k\phi$. *Cf.* ADAMS [1969].

By Theorem 13.2, if $\theta : SO(2) \to S^1$ is a twist with $\theta(\phi) = k\phi$, then $\dim (V \otimes C)^{SO(2)^\theta}$ is a 2 precisely when the representation of $SO(2)$ corresponding to θ occurs exactly once in the restriction of the $O(3)$-action on V_l to $SO(2)$. By (15.4) this happens for $k = 1, 2, \dots, l$ only. (Note: a twist by $-k\phi$ gives a group conjugate in $O(3)\times S^1$ to that given by $+k\phi$, so we may assume $k \geqq 1$.) Hence $(Z_2^c \oplus SO(2))^\theta$ has a 2-dimensional fixed-point space when

$$\theta(\phi) = k\phi \qquad (k = 1, \ldots, l),$$

$$\theta(-I) = \begin{cases} 0 & \text{(plus representation)} \\ \pi & \text{(minus representation).} \end{cases}$$

This leads to the entries with S^1-twists in Table 14.1.

D: Cyclic Subgroups

We can now avoid what would otherwise be a complicated calculation: the case $J = Z_n$. There are many twists of $Z_n \oplus Z_2^c$. However, as in subsection C we may consider the effect of the twist separately on Z_n and on Z_2^c. The effect of the twist on Z_2^c is determined by the parity (plus or minus) of the representation; and the fixed-point space is the same as for the twisted Z_n.

Let x generate Z_n. The possible twists are $x \to \dfrac{2k\pi}{n}$, $k = 0, \ldots, n$. By Theorem 13.2 the fixed-point space has dimension 2 if and only if this representation occurs with multiplicity 1 in the restriction of the $O(3)$-action to Z_n. But $Z_n \subset SO(2)$, so the root-space decomposition implies that $k = 1, 2, \ldots, l$. The twist on Z_n therefore extends to the corresponding twist on $SO(2)$, and by Lemma 15.2, Z_n^θ cannot be an isotropy group with 2-dimensional fixed-point space. Hence $J = Z_n$ does not occur in Table 14.1.

E: Dimension Formulas for $O(3)$

We now state formulas for the dimensions $d(H)$ of fixed-point spaces V_l^H of $H \subset O(3)$ acting on V_l. These are reproduced from Ihrig & Golubitsky [1984].

Plus Representation. $J \subset SO(3)$ and $Z_2^c \oplus J$ have the same fixed-point space, so $d(Z_2^c \oplus J) = d(J)$. Their values are given by:

$$d(Z_n) = 2\left[\frac{l}{n}\right] + 1 \qquad (n \geqq 1),$$

$$d(D_n) = \begin{cases} \left[\dfrac{l}{n}\right] & l \text{ odd} \\ \left[\dfrac{l}{n}\right] + 1, & l \text{ even} \end{cases} \qquad (n \geqq 2),$$

$$d(SO(2)) = 1,$$

$$d(O(2)) = \begin{cases} 0 & l \text{ odd} \\ 1 & l \text{ even} \end{cases},$$

$$d(T) = 2\left[\frac{l}{3}\right] + \left[\frac{l}{2}\right] - l + 1,$$

$$d(O) = \left[\frac{l}{4}\right] + \left[\frac{l}{3}\right] + \left[\frac{l}{2}\right] - l + 1,$$

$$d(I) = \left[\frac{l}{5}\right] + \left[\frac{l}{3}\right] + \left[\frac{l}{2}\right] - l + 1.$$

We shall not require the value of $d(H)$ of a type III group H in the plus representation.

Minus Representation. If $J \subset SO(3)$ then $d(J)$ is as above. Clearly $d(Z_2^c \oplus J) = 0$. For type III groups the results are:

$$d(Z_n^-) = 2\left[\frac{l+n}{2n}\right],$$

$$(dD_n^z) = \begin{cases} \left[\frac{l}{n}\right] & l \text{ even} \\ \left[\frac{l}{n}\right] + 1 & l \text{ odd} \end{cases},$$

$$d(D_{2n}^d) = \left[\frac{l+n}{2n}\right],$$

$$d(O^-) = \left[\frac{l+2}{4}\right] + \left[\frac{l}{3}\right] + \left[\frac{l+1}{2}\right] - l,$$

$$d(O(2)^-) = \begin{cases} 0 & l \text{ even} \\ 1 & l \text{ odd.} \end{cases}$$

The values for the exceptional groups I, O, T, O^- are "periodic" in the sense that

$$d(I)(l+30) = d(I)(l) + 1$$

$$d(O)(l+12) = d(O)(l) + 1$$

$$d(T)(l+6) = d(T)(l) + 1$$

$$d(O^-)(l+12) = d(O^-)(l) + 1.$$

Their values for $l \leqq 30, 12, 6, 12$ respectively are given in IHRIG & GOLUBITSKY [1984]. This periodicity is useful in the calculations sketched below.

F: Untwisted Groups

Consider an untwisted group $H \subset O(3)$, with $\theta = 0$. By (15.2) only the plus representation can apply, and $H = K = Z_2^c \oplus J$. We have $\dim (V \otimes C)^H = 2 \dim V^J$ as in (6.2). Hence we seek J such that $\dim J^J = 1$. By § 15.E we find those subgroups whose fixed-point subspace are 2-dimensional. By inspection (using Lemma 15.2(c)) we determine which of these are isotropy subgroups. The resulting list is:

$Z_2^c \oplus O(2)$: all even l.

$Z_2^c \oplus I$: $l = 6, 10, 12, 16, 18, 20, 22, 24, 26, 28, 32, 34, 38, 44;$
$21, 25, 27, 31, 33, 35, 37, 39, 41, 43, 47, 49, 53, 59.$

$Z_2^c \oplus O$: $l = 4, 6, 8, 10, 14; 9, 13, 15, 17, 19, 23.$

(We have separated the even and odd l for later convenience, when considering the $(-1)^l$-representation.) This gives the entries in Table 14.1 with $\theta(H) = 1$.

G: Z_3- and Z_6-twists

These arise only when $H = Z_2^c \oplus T$. In the plus representation, $K = Z_2^c \oplus D_2$ and the twist is Z_3; in the minus representation $K = D_2$ and the twist is Z_6. (The subgroup D_2 is defined geometrically as the symmetries of the tetrahedron formed by rotating it about the lines joining mid-points of opposite edges. Alternatively, think of T as the even permutation on the vertices 1, 2, 3, 4 of the tetrahedron, and D_2 as the Klein Four-Group $\{Id, (12)(34), (13)(24), (14)(23)\}$.) In either case, in order for T^0 to have a 2-dimensional fixed-point subspace we must have by Theorem 13.1(b)

$$\dim V_l^{D_2} - \dim V_l^T = 2.$$

By inspection from § 15.E we find that $l = 2, 4, 5, 6, 7, 9$.

H: Z_2-twists

This is the only remaining case. Thanks to the dihedral groups, it requires detailed analysis, depending on the type of $K = \ker \theta$. Using Theorem 13.1 and Lemma 15.1 we can tabulate the conditions for a 2-dimensional fixed-point space (Table 15.2).

Table 15.2. Conditions for a 2-dimensional fixed-point space with Z_2 twist

$H = Z_2^c \oplus J$	Plus Representation	Minus Representation
K type I $K = J$	None	$\dim V^J = 1$
K type II $K = Z_2^c \oplus L$	$\dim V^L = \dim V^J + 1$	None
K type III $K = M^-$	None	$\dim V^{M^-} = 1$

Note that a type III kernel can occur only for the minus representation, as asserted in § 15.E. From Table 15.1, the possible J are $O(2)$, $SO(2)$, I, O, T, and D_n. (Z_n is eliminiated by § 15.D). Assuming $d(H^0) = 2$ we may use § 15.E) to obtain Table 15.3, where we abbreviate the lists of values of l for exceptional groups as follows.

6–44: 6, 10, 12, 16, 18, 20, 22, 24, 26, 28, 32, 36, 38, 44.
21–59: 21, 25, 27, 31, 33, 35, 37, 39, 41, 43, 47, 49, 53, 59.
4–14: 4, 6, 8, 10, 14.
9–23: 9, 13, 15, 17, 19, 23.
6–20: 6, 10, 12, 14, 16, 20.
3–17: 3, 7, 9, 11, 13, 17.

Note that for $J = SO(2)$, I, and T, only a kernel K of type I is possible since J has no subgroup of index 2.

Table 15.3. Z_2-twisted groups with $d(H^0) = 2$

J	K	Plus Representation	Minus Representation
$O(2)$	I: $O(2)$		l even
	II: $Z_2^c \oplus SO(2)$	l odd	
	III: $O(2)^-$		l odd
$SO(2)$	I: $SO(2)$		all l
I	I: I		6–44; 21–59
O	I: O		4–14; 9–23
	II: $Z_2^c \oplus T$	6–20; 3–17	
	III: O^-		6–20; 3–17
T	I: T		4, 8; 3, 7, 11
D_n	I: D_n		$l < n$ (l even) $n \leqq l < 2n$ (l odd)
	IIA: $Z_2^c \oplus Z_n$	$n \leqq l < 2n$ (l even) $l < n$ (l odd)	
($n = 2m$)	IIB: $Z_2^c \oplus D_m$	$m \leqq l < 3m$	
	IIIA: D_n^z		l even
($n = 2m$)	IIIB: D_n^d		$(2k+1)\,m \leqq l < (2k+2)\,m$ (any k, l odd)

I: Conjugacy Problems

In obtaining Table 15.3 we must consider conjugacy classes under $O(3)$ of *pairs* (H, K) where $|H/K| = 2$, two such pairs (H_i, K_i) being conjugate if $\gamma H_1 \gamma^{-1} = H_2$, $\gamma K_1 \gamma^{-1} = K_2$ for the same γ. This is not necessarily the same as considering pairs of conjugacy classes (where γ may differ between H_i and K_i). If H has a unique subgroup of index 2 (or more generally if K is a characteristic subgroup, invariant under automorphisms of H), then any γ that conjugates H_1 to H_2 automatically conjugates K_1 to K_2. This justifies the entries for $J = SO(2)$, I, and T. If H has a characteristic subgroup $L \subset SO(3)$ such that $H/L = Z_2 \oplus Z_2$ then there are three possible groups K, of types I, II, III respectively. This justifies the entries for $J = O(2)$, O, and D_n where n is odd. When $n = 2m$ is even, $Z_2^c \oplus D_{2m}/Z_m \oplus Z_2^3$ and there are seven possible K. Three of these, types I, IIA, IIIA, contain Z_{2m} which is characteristic; types IIB and IIIB occur as $O(3)$-conjugate pairs, since they are conjugate in D_{4n} which normalizes D_{2n}. This justifies the remaining entries.

J: *Elimination of redundancy*

To complete the proof that Table 14.1 is correct we must eliminate from Table 15.3 all groups H^θ contained in a larger group L^ϕ also in Table 15.3. The $SO(2)$ entries for l even extend to $O(2)$ with K of type I; for l odd, to $O(2)$ with K of type III. Similarly the T entries extend to O, For D_n, the cases I (l even), IIA (l odd), IIA, IIIB extend to $O(2)$. In case IIB we have $\left[\frac{l}{3}\right] < m \leqq l$, and we must also eliminate $m \leqq \left[\frac{l}{2}\right]$ since $2m$ is then in the same range. Once this is done, Table 14.1 results. It is easy to check that there are no repetitions, up to conjugacy.

Acknowledgement. We are grateful to Stuart Antman, John Guckenheimer and Chris Jones for many helpful suggestions and comments. This research was supported in part by grants from NASA-Ames (NAG 2-279) and from NSF (MCS-8101580).

The work of the second author was also supported by a visiting position in the Department of Mathematics, University of Houston.

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Note added in proof: Independently a result similar to our Theorem 5.1 has been obtained by DAVID SATTINGER [1984].

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(Received April 3, 1984)