AN ANALYSIS OF IMPERFECT BIFURCATION*

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In this article we outline an application of the Thom-Mather singularity theory of C^∞ mappings to a problem in bifurcation theory. A detailed description of the results stated here may be found in Reference 1. Unless otherwise stated all mappings are assumed to be germs of C^∞ functions near the origin.

We view a bifurcation problem as a germ of a C^∞ map G: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ with $G(0, 0) = 0$ where $x \in \mathbb{R}^n$ denotes the state of a system and $\lambda \in \mathbb{R}$ the load or bifurcation parameter. The associated bifurcation diagram is the set $\{G(x, \lambda) = 0\}$.

PROBLEM. Find a classification scheme for all small perturbations of G up to some suitable equivalence.

Note that this is in reality two problems. One must first find a definition for equivalent bifurcation problems and second, perform the classification. Before presenting the technical notion of equivalence, we discuss a simple example.

EXAMPLE. Let $n = m = 1$ and $G(x, \lambda) = x^3 - \lambda x$. The associated bifurcation diagram is the familiar pitchfork. See FIGURE 1. The physical model for G that we consider is a finite element analogue of the Euler beam problem illustrated in FIGURE 2. This system, consisting of two rigid rods of unit length connected by frictionless pins, is subjected to a compressive force λ, which is resisted by a torsional spring of unit strength. The state of the system is the angular derivation x from the horizontal. (We assume that a linear change in the λ coordinate has been made so that $\lambda = 0$ is the bifurcation point.) If we let $V(x, \lambda)$ denote the potential energy then, as is well known,

$$\frac{\partial V}{\partial x}(x, \lambda) = G(x, \lambda) + 0(|x, \lambda|^4).$$

Next we subject this system to two types of imperfections:

(i) An initial curvature b. Here we assume that the spring is at rest at some angular deviation from 0 (FIGURE 3i).

(ii) A central load a (FIGURE 3ii).

*This work was partially supported by the National Science Foundation.

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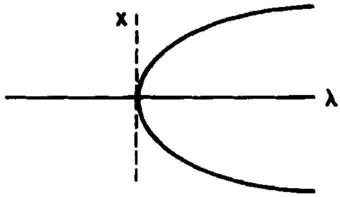

FIGURE 1.

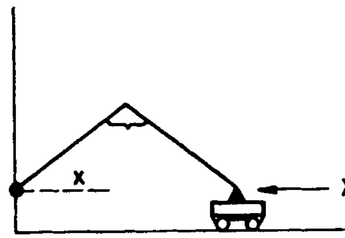

FIGURE 2.

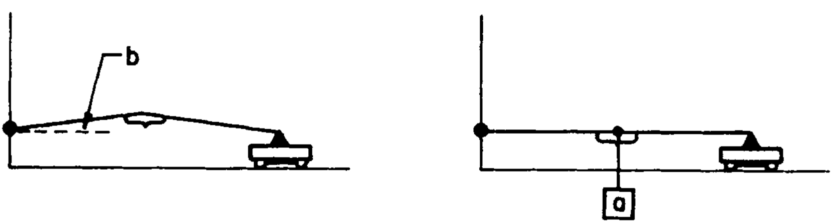

FIGURE 3.

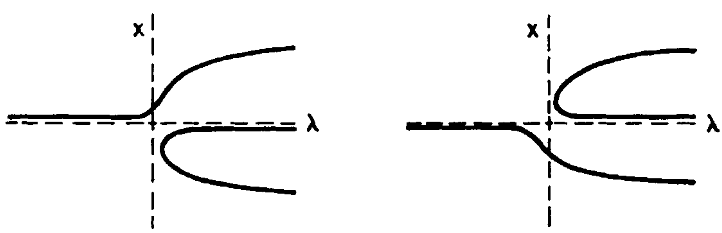

FIGURE 4.

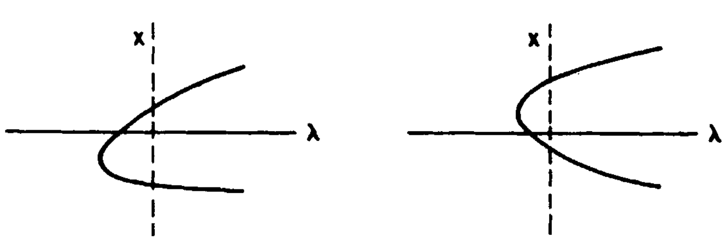

FIGURE 5.

Again it is well known that the presence of either of these imperfections separately will yield bifurcation diagrams similar to those in FIGURE 4. The question that we ask is what happens when both of the above imperfections are present? For example, if the initial curvature b is exactly offset by a central load a, then the bifurcation diagram is as in FIGURE 5. Of course these diagrams assume the exact relation "$a = -b$". Varying these imperfections yields either the diagrams in FIGURE 4 or those in FIGURE 6.

One should observe that the bifurcations diagrams in FIGURE 4 differ from those in FIGURE 6 is a physically discernible way. Moving λ quasistatically in the diagrams of FIGURE 6 can create a hysteresis cycle while in the diagrams of FIGURE 4 no such behavior is possible.

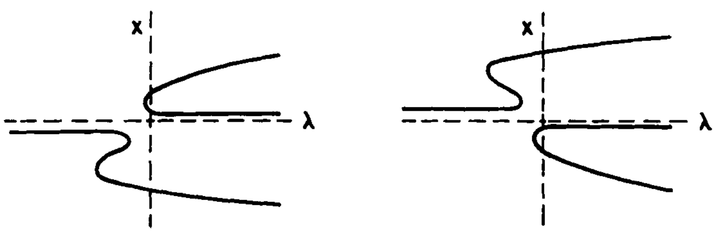

FIGURE 6.

The theorems that we shall describe state that no matter what new imperfections are added to the system no new qualitative behavior will appear.

Our technical notion of equivalence is:

DEFINITION. Two bifurcation problems G, $\overline{G}$: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ are *contact equivalent* if

$$G(x, \lambda) = T_{x,\lambda} \cdot \overline{G}(X(x, \lambda), \Lambda(\lambda)) \tag{1}$$

where for each (x, λ) $T_{x,\lambda}$ is an invertible $m \times m$ matrix and $X(\cdot, 0)$, $\Lambda(\cdot)$ are orientation preserving changes of coordinates with $X(0, 0) = 0$ and $\Lambda(0) = 0$.

COMMENTS. (1) $\partial V/\partial x$ in the example above and $G(x, \lambda) = x^3 - \lambda x$ are contact equivalent. (2) Clearly λ is a distinguished variable in the above change of coordinates. This is demanded on physical grounds; one cannot have the applied force λ depend on the state x. (3) The bifurcation diagrams associated to G and $\overline{G}$ can be obtained from one another by a λ-preserving change of coordinates. (4) One could allow $T_{x,\lambda}$ to be a nonlinear coordinate change for each (x, λ) with $T_{x,\lambda}(0) = 0$ but no generality would be gained as proved by Mather.[2]

A standard approach to classifying small perturbations in singularity theory is given by the so called universal unfolding. We demonstrate its use by the following.

PROPOSITION A. $F(x, \lambda, p, q) = x^3 - \lambda x + px^2 + q$ is a universal unfolding of $x^3 - \lambda x$ (relative to contact equivalence); that is, let $H(x, \lambda, \epsilon) = x^3 - \lambda x + \epsilon_1 h_1(x, \lambda, \epsilon) + \cdots + \epsilon_l h_l(x, \lambda, \epsilon)$. Then for each fixed ϵ near 0, $H(\cdot, \cdot, \epsilon)$ is contact equivalent to $F(\cdot, \cdot, p, q)$ for some p and q. Moreover, the dependence of p and q on ϵ may be chosen to be smooth.

NOTE 1. It is not hard to show that the universal perturbation parameters p and q depend nonsingularly on a and b in the above example. This fact along with Proposition A proves that all qualitative behavior can be obtained by appropriate choices of a and b (up to contact equivalence).

PROPOSITION B. The bifurcation diagrams associated with $F(\cdot,\cdot,p,q)$ are

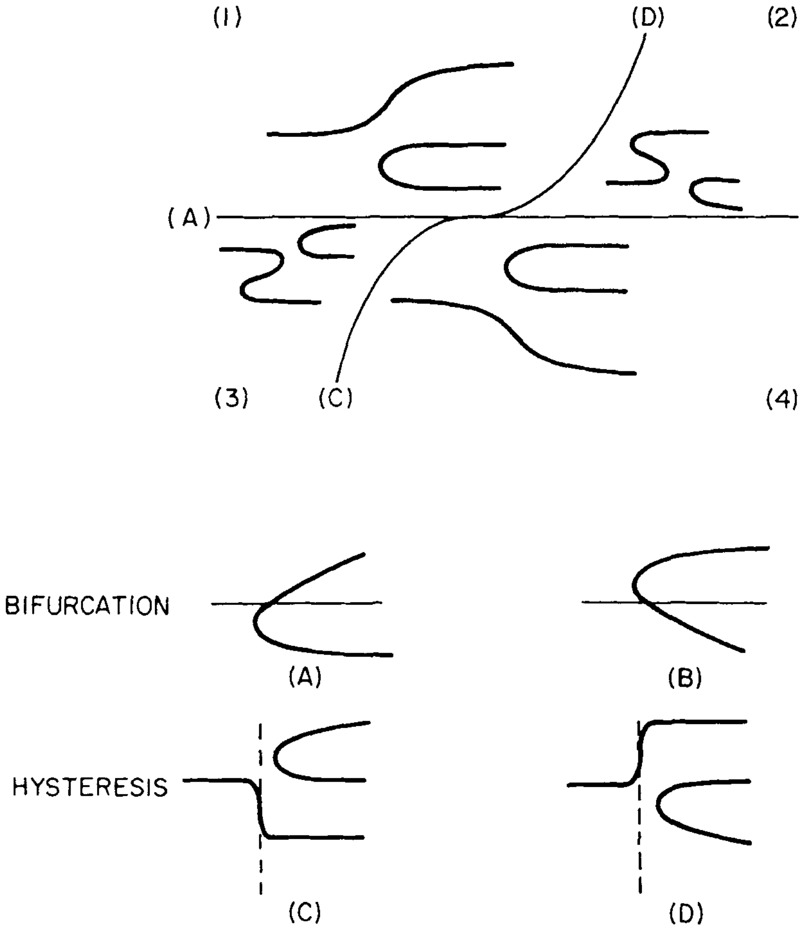

FIGURE 7.

codified (up to contact equivalence) by FIGURE 7. The separating curves are $q = 0$ and $q = p^3/27$.

NOTE 2. FIGURE 7 shows that for (p,q) near $(0,0)$ only the diagrams associated to one of the four connected open regions or one at the four separating branches occur thus giving the desired classification of bifurcation diagrams near $x^3 - \lambda x = 0$.

NOTE 3. The diagrams associated with branches (C) and (D) are not usually considered to have bifurcation. They are not stable under small perturbation as a small perturbation can create a diagram exhibiting hysteresis. We propose the name *hysteresis point* for those points (p, q) on branches (C) and (D). The name *bifurcation point* for the points $(p, 0)$ is clear.

NOTE 4. Proposition A states that a one-parameter family of perturbations of $x^3 - \lambda x$ (ϵ is a single variable) may be represented (up to contact equivalence) by a curve through the origin in pq-space. Since the separating curves of Proposition B are tangent to second order at the origin a typical curve will enter only the regions 1 and 3. In order to observe regions 2 and 4 one must consider two-parameter perturbations. This observation seems consistent with the engineering literature and is substantiated in that both perturbations when considered separately in the model problem yield diagrams from regions 1 and 3.

We now state our main theorems, which draw heavily on standard techniques from singularity theory. Given a bifurcation problem G: $\mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$, let $O_G = \{\overline{G} \mid \overline{G}$ is contact equivalent to $G\}$. Think of O_G as an infinite dimensional "submanifold" of $C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^n)$, the space of all smooth bifurcation problems. Let TG be the tangent space to O_G at G; TG is computed as follows: choose a curve $G_t(x, \lambda)$ in O_G with $G_0 = G$. Compute $(\mathrm{d}G_t/\mathrm{d}t)|_{t=0}$ to obtain a function in TG; in fact, TG is the totality of such functions. For example, if $m = n = 1$, then $G_t(x, \lambda)$ in O_G implies

$$G_t(x, \lambda) = T(x, \lambda, t)\, G(X(x, \lambda, t), \Lambda(\lambda, t)), \tag{2}$$

where $X(x, \lambda, 0) = x$, $\Lambda(\lambda, 0) = \lambda$, and $T(x, \lambda, 0) = 1$ as $G_0 = G$. Differentiating (2) with respect to t (indicated by $\dot{}$) and evaluating at $t = 0$ yields

$$\dot{G}_0(x, \lambda) = \dot{T}(x, \lambda, 0) \cdot G(x, \lambda) + \frac{\partial G}{\partial x}(x, \lambda)\, \dot{X}(x, \lambda, 0) + \frac{\partial G}{\partial \lambda}(x, \lambda)\, \dot{\Lambda}(\lambda, 0).$$

Hence

$$TG = \left\{ a(x, \lambda)\, G + b(x, \lambda) \frac{\partial G}{\partial x} + c(\lambda) \frac{\partial G}{\partial \lambda} \right\},$$

where a, b, and c are arbitrary smooth functions.

DEFINITION. A bifurcation problem G has *codimension* k if there exists a k-dimensional vector subspace K of $C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^m)$ such that $TG \oplus K = C^\infty(\mathbb{R}^n \times \mathbb{R}, \mathbb{R}^m)$.

THEOREM A. Suppose G is a bifurcation problem with codimension k. Let $g_1(x, \lambda), \ldots, g_k(x, \lambda)$ be a basis for K. Then

$$F(x, \lambda, \alpha) = G(x, \lambda) + \alpha_1 g_1(x, \lambda) + \cdots + \alpha_k g_k(x, \lambda)$$

is a universal unfolding for $G(x, \lambda)$; that is, let

$$H(x, \lambda, \epsilon) = G(x, \lambda) + \epsilon_1 h_1(x, \lambda, \epsilon) + \cdots + \epsilon_l h_l(x, \lambda, \epsilon).$$

Then for each ϵ near 0, $H(\cdot, \cdot, \epsilon)$ is contact equivalent to $F(\cdot, \cdot, \alpha)$ for some α. Moreover the dependence of α on ϵ may be chosen to be smooth.

One needs only Taylor's Theorem to show that Theorem A implies Proposition A. For if $G = x^3 - \lambda x$, then

$$TG = \{a \cdot (x^3 - \lambda x) + b \cdot (3x^2 - \lambda) - c \cdot x\}$$
$$= \{\bar{a}(x,\lambda)x^3 + \bar{b}(x,\lambda)(3x^2 - \lambda) + \bar{c}(\lambda)x\}.$$

We claim that $K = \text{span}\{1, x^2\}$ is a complementary subspace to TG. If so, then we are done. Given an arbitrary C^∞ function $f(x,\lambda)$ we have by Taylor's Theorem

$$f(x,\lambda) = f_0(\lambda) + f_1(\lambda)x + f_2(\lambda)x^2 + f_3(x,\lambda)x^3$$
$$\equiv f_0(\lambda) + f_2(\lambda)x^2 \pmod{TG}.$$

Since $\lambda \equiv 3x^2 \pmod{TG}$ we have that $f(x,\lambda) \equiv P_0 + P_2x^2 \pmod{TG}$ where P_0 and P_2 are constants. Thus K is as claimed.

We now give a first step towards the analysis of perturbed bifurcation diagrams. Let $F(x,\lambda,\alpha)$ be a universal unfolding of $G(x,\lambda)$. As described in Note 3 the diagrams associated to points α, which are bifurcation or hysteresis points, are not stable under small perturbation. In general there is a third category of unstable points which we call double limit points (FIGURE 8).

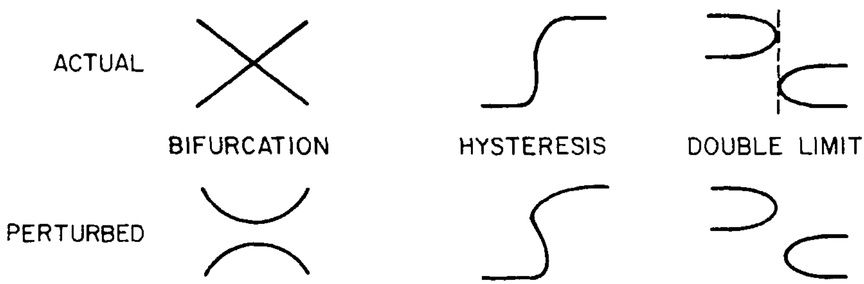

FIGURE 8.

THEOREM B. Let $\Sigma = \{\alpha \mid \alpha$ is one of the three unstable points listed above$\}$. Then any two α terms in the same connected component of the complement of Σ correspond to contact equivalent bifurcation problems.

We analyse the special case $F(x,\lambda,p,q) = x^3 - \lambda x + px^2 + q$. For $F(x_0,\lambda_0,p,q)$ to be a bifurcation point the equations $F = F_x = F_\lambda = 0$ must be satisfied at (x_0,λ_0). The condition $F_\lambda = 0$ implies that $x_0 = 0$ and $F = 0$ implies that $q = 0$. Next observe that for $F(x_0,\lambda_0,p,q)$ to be a hysteresis point the equation $F = F_x = F_{xx} = 0$ must be satisfied at (x_0,λ_0). $F_{xx} = 0$ implies that $x_0 = -p/3$ while $F_x = 0$ implies that $\lambda = -p^2/3$. Finally $F = 0$ implies that $q = p^3/27$. It is not hard to show that double limit points cannot occur in this example. To complete FIGURE 7 we must choose only one (p,q) from each open region and graph the associated bifurcation diagram.

We mention one last result.

THEOREM C. If $G: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^m$ is a bifurcation problem of finite codimension, then G is contact equivalent to a polynomial in x and λ. The universal unfolding $F(x, \lambda, \alpha)$ may be chosen to be a polynomial in x, λ, and α. The set Σ is an algebraic variety of codimension one in α space.

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