Bifurcations with $O(3)$ Symmetry Including Applications to the Bénard Problem

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0. Introduction

This paper has two purposes: to study from the singularity theory point of view certain bifurcation problems which commute with the five-dimensional irreducible representation of the orthogonal group $O(3)$ and to give some implications of this study for the Bénard problem in spherical geometry. We shall now describe in general terms our results and compare them with the work of Chossat [3] whose paper motivated our own interest in the problem.

The Bénard problem is concerned with convection in a viscous fluid when it is heated from below. The fluid is assumed to be confined in a spherical shell of outer radius R_0 and inner radius ηR_0, where η is near 0.3. This choice of η is partially motivated by considering convection within the molten layer of the core of the earth (see Busse [1] for further discussion).

We consider the Bénard problem in the Boussinesq approximation. In this model there is a trivial solution representing pure heat conduction radially outward. As the temperature on the inner sphere (that is, the Rayleigh number R) is increased, this trivial solution loses stability, say at $R = R^*$. The Bénard problem is the study of the resulting bifurcation. Chossat studies this problem using the Lyapunov-Schmidt reduction. In the Boussinesq approximation the fluid is driven from the pure conduction state by a term in the momentum equation involving the gravity vector $\mathbf{g}(r)$ and a term in the temperature equation involving the gradient of the conduction temperature $\nabla \mathbf{T}_0$. Assuming that the production of heat and the density are uniform throughout the shell, these vectors have the form

$$\mathbf{g}(r) = \left(\frac{\gamma_1}{r^3} + \gamma_2\right)\mathbf{r}, \tag{0.1}$$

$$\nabla \mathbf{T}_0 = \left(\frac{\beta_1}{r^3} + \beta_2\right)\mathbf{r},$$

Communications on Pure and Applied Mathematics, Vol. XXXV, 81–111 (1982)
 CCC 0010-3640/82/010081-31$03.10

where γ_i, β_i are constants. If

$$\gamma_1/\gamma_2 = \beta_1/\beta_2, \tag{0.2}$$

the equations governing the motion linearized about the pure conduction state are selfadjoint. We shall refer to the *selfadjoint* case when (0.2) is satisfied.

For the selfadjoint case the kernel of the linearized Boussinesq equations at $R = R^*$ is five-dimensional, having the angular dependence of the spherical harmonics of order 2. We denote this five-dimensional space by Y. The Lyapunov-Schmidt reduction shows that solving the full Boussinesq equations for steady state solutions near the pure conduction solution is equivalent to solving reduced bifurcation equations of the form

$$H(x,\lambda) = 0, \tag{0.3}$$

where $H : Y \times \mathbb{R} \to Y$ is C^∞ and $\lambda = R - R^*$. Moreover this reduction implies that the linear terms $(d_x H)$ are zero and that H commutes with the action of $O(3)$ on spherical harmonics, i.e.,

$$H(\gamma \cdot x,\lambda) = \gamma \cdot H(x,\lambda) \tag{0.4}$$

for every $\gamma \in O(3)$. We discuss the above issues in more detail in Section 6. We remark that Chossat shows in the selfadjoint case that the quadratic terms in (0.3) are also zero.

In this paper we study the form the reduced bifurcation equations (0.3) can assume consistent with the symmetry (0.4). This analysis provides a specific example of the general theory developed in [4], [5]. Our goals are two-fold:

(A) To give conditions on H and its derivatives at the bifurcation point which ensure that (0.3) may be put into a simple normal form by an appropriate change of coordinates.

(B) To enumerate *all* qualitatively distinct, small perturbations of the equations (i.e., enumerate these perturbations up to changes of coordinates). We consider two cases in detail:

(i) The quadratic term in (0.3) is non-zero.

(ii) The least degenerate situation where the quadratic terms in (0.3) are zero.

Our results concerning these two problems are summarized in Theorem 4.7. In Section 5 we discuss the solution sets of (0.3) in the two cases.

The solution of problem (B) for case (ii) will allow us to comment on non-selfadjoint perturbations of the selfadjoint problem. This is the major point in which our analysis differs from Chossat's. For example Chossat shows that the bifurcating solutions are always axisymmetric in the selfadjoint case. We show that nonaxisymmetric solutions can result from arbitrarily small non-selfadjoint perturbations.

We also correct a statement of [3] that the stability of the bifurcating solutions depends "on the sign of a certain coefficient." It turns out that a stable solution is *always* present; the coefficient in question merely determines whether the preferred motion is upwards at the equator and downwards at the two poles or the reverse. (See Sections 5 and 6 for details.)

Our presentation relies more heavily on group theory than [3]. An attribute of the five-dimensional irreducible representation of $O(3)$ will permit us to reduce studying the 5×5 system of equations (0.3) depending on a parameter to a 2×2 system depending on a parameter. The reduced system will have a solution set consisting of curves in $\mathbb{R}^3$ while the original system has a solution set consisting of three- and four-dimensional varieties in $\mathbb{R}^6$. This is a significant simplification allowing one explicitly to calculate and picture the bifurcation diagrams associated with this problem. Moreover, the singularity theory analysis is technically simpler in the 2×2 case, a point on which we will elaborate later.

It should be noted here that the reduced problem

$$G(x, \lambda) = 0, \tag{0.5}$$

where $G : \mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ commutes with the standard representation of the permutation group S_3 (thought of as symmetries of a triangle) on $\mathbb{R}^2$. This problem has independent interest of its own; see, for example, the work of Buzano, Geymonat, and Poston [2] on the buckling of a triangular beam. Our results, of course, apply equally to this case.

Finally, we note that our analysis relies on the explicit calculations of Chossat regarding the mapping H in (0.3) obtained from the Lyapunov-Schmidt reduction. In his analysis Chossat was forced to make several non-degeneracy assumptions about higher-order terms in H; we are forced to make the same assumptions. Aside from elegance, the advantage of the singularity theory approach is to allow the partial study of the non-selfadjoint case while using *no new* additional information other than that needed to study the selfadjoint case.

This paper has the following organization. In the first section we describe explicitly the needed group theory. In the second section we calculate the general information required to do singularity theory on (0.5) and apply these results in Section 3 to study the two singularities of interest mentioned above. Section 4 contains the reduction from the 5×5 system to the 2×2 system along with the interpretation of the results of Section 3 for the 5×5 case. In Section 5, we present explicitly the bifurcation diagrams associated with the problems of Section 3 along with the linearized stability analysis of the corresponding solutions. The last section contains the interpretation of the results for the Bénard problem.

1. Group Theoretical Preliminaries

The irreducible five-dimensional representation of $O(3)$, the one we consider in this paper, has a very special form which distinguishes it from higher-dimensional irreducible representations of $O(3)$. This special form permits the reduction of (0.3) from a 5×5 system to the 2×2 system (0.5). In the language of Lie groups, this representation is the representation of k on p in the $k \oplus p$ decomposition of $Sl(3, \mathbb{R})$. Kostant and Rallis [6] study such representations and some of our results below are a special case of their theory. More pragmatically, let

$$V = \{3 \times 3 \text{ symmetric matrices } A \text{ with } \operatorname{tr} A = 0\}. \tag{1.1}$$

Then $O(3)$ acts on V by similarity, i.e.,

$$\gamma \cdot A = \gamma A \gamma' \quad \text{for} \quad \gamma \in O(3). \tag{1.2}$$

This is the presentation of the five-dimensional irreducible representation of $O(3)$ with which we work. We note that this presentation is considered in [8]. Let $D \subset V$ be the two-dimensional subspace of diagonal matrices.

LEMMA 1.3. *Let $H : V \to V$ be equivariant with respect to the action of $O(3)$ given in* (1.2). *Then D is invariant under H and H is determined by its restriction to D.*

Proof: The assumption that H is equivariant means that $H(\gamma \cdot A) = \gamma \cdot H(A)$. Note that if γ fixes A, then γ also fixes $H(A)$ as

$$H(A) = H(\gamma \cdot A) = \gamma \cdot H(A). \tag{1.4}$$

Let

$$\gamma_1 = \begin{bmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \quad \text{and} \quad \gamma_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{bmatrix}.$$

It is easily seen that $D = \{A \in V \mid \gamma_1 \cdot A = \gamma_2 \cdot A = A\}$. Hence, if A is in D, then A and $H(A)$ are fixed by γ_1 and γ_2. So $H(A) \in D$. Finally, for A in V, choose γ so that $\gamma \cdot A$ is in D. (This is possible since every symmetric matrix may be diagonalized by an orthogonal matrix.) Then $H(A) = \gamma' \cdot H(\gamma \cdot A)$.

Note that the subgroup of $O(3)$ which both preserves D and acts faithfully on D is the group S_3 of permutations on three letters (the diagonal entries). We shall show in Section 4—though Lemma 1.3 indicates why—that studying bifurcation problems H commuting with the five-dimensional representation of $O(3)$ is equivalent to studying bifurcation problems

$$G : D \times \mathbb{R} \to D \tag{1.5}$$

commuting with this action of S_3 on D, where G is the restriction of H to $D \times R$. Note that Lemma 1.3 implies that the image of G is D. In Sections 1–3 we concentrate on problems of the form (1.5).

We shall use the notation Δ for the permutation group S_3 which acts on D. This group also acts naturally on $\mathbb{C}$, being generated (as a group) by the operations

$$(1.6) \qquad z \to \bar{z}, \qquad z \to e^{i\alpha} z,$$

where $\alpha = \frac{2}{3}\pi$. Here $\bar{z}$ denotes the complex conjugate of z. There is a (real) linear isomorphism of the (real) two-dimensional space D with $\mathbb{C}$ such that the action of Δ on D assumes the form (1.6). Indeed one such isomorphism is given by

$$(1.7) \qquad d_1 = x, \quad d_2 = \tfrac{1}{2}(-x + \sqrt{3}\, y), \quad d_3 = \tfrac{1}{2}(-x - \sqrt{3}\, y),$$

where d_i are the entries of a diagonal matrix and (x, y) are Cartesian coordinates in the complex plane. Below we use $z = x + iy$ for the complex coordinate on D associated with this isomorphism. We find it preferable to use the respresentation of Δ on $\mathbb{C}$ since there one can compute the zero set of G more easily than in (1.5).

A point which we should emphasize is that the techniques of singularity theory are local in nature. Although we write the domain of our functions as $\mathbb{R}^5$, V, D, $\mathbb{C}$, etc., we are implicitly thinking of these domains as some unspecified neighborhood of the origin in these spaces. No confusion should result from this convention except perhaps when we explicitly graph the zero set of G. Then we shall disregard branches of $G = 0$ which occur far away from the origin. The technical device which permits us to work with an unspecified neighborhood of 0 is the notion of a germ. A *germ* is an equivalence class of mappings, two mappings being equivalent if they are identical on some neighborhood of the origin. Pragmatically, one can work with germs as if they were functions.

We now set our notation. Let g be a compact group acting orthogonally on $\mathbb{R}^n$; denote the action of $\gamma \in g$ on $x \in \mathbb{R}^n$ by $\gamma \cdot x$. A function $f: \mathbb{R}^n \to \mathbb{R}$ is *invariant* if $f(\gamma \cdot x) = f(x)$. The set of germs of invariant C^∞ functions forms a ring denoted by $\mathcal{E}_n^g$. A mapping $G: \mathbb{R}^n \to \mathbb{R}^n$ is *equivariant* if $G(\gamma \cdot x) = \gamma \cdot G(x)$. In this case one can also say that G commutes with the action of g. The set of germs of equivariant C^∞ mappings, denoted by $\mathcal{E}_{n,n}^g$, forms a module over the ring $\mathcal{E}_n^g$. The singularity theory calculations require the use of a second module. Consider those mappings $T: \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n$ which are linear in the second variable and satisfy $T(\gamma \cdot x, \gamma \cdot w) = \gamma \cdot T(x, w)$. The space of such T's is also a module and is denoted by $\mathcal{M}_{n,n}^g$. We study here bifurcation problems $G: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. The equivariance condition in this case is $G(\gamma \cdot x, \lambda) = \gamma \cdot G(x, \lambda)$; that is, the group acts trivially on the bifurcation parameter λ. Thus, we shall use the notation $\mathcal{E}_{n+1}^g$, $\mathcal{E}_{n+1,n}^g$, and $\mathcal{M}_{n+1,n}^g$ to indicate that the bifurcation parameter λ is implicitly included in the discussion.

We note that we shall consider only two groups in this paper $\Gamma = O(3)$ and $\Delta = S_3$. We shall also consider only one representation of each of these groups though in each case we look at two presentations of these representations. As we have discussed, we shall use the representation of Γ on spherical harmonies of order 2 and the representation of Γ on V. We shall use the representation of Δ on D and the representation of Δ on $\mathbb{C}$. One of our main points is that the singularity theory analysis of bifurcation problems H commuting with Γ is identical to the singularity theory analysis of bifurcation problems G commuting with Δ. Moreover, for the remainder of this paper we use H when referring to equivariant mappings relative to Γ, and G when referring to equivariant mappings relative to Δ.

Recall that a finitely generated module A over a ring R is said to be *free* if there is a set $\{f_1, \cdots, f_k\}$ in A such that every $f \in A$ is represented uniquely as $f = a_1 f_1 + \cdots + a_k f_k$ where each a_i is in R. The set $\{f_1, \cdots, f_k\}$ is called a *basis* for A and we use the notation $A = R\{f_1, \cdots, f_k\}$.

We now analyze the two-dimensional representation of Δ on $\mathbb{C}$.

PROPOSITION 1.8. (a) *Let f be in $\mathcal{E}_2^\Delta$. Then there is a C^∞ germ $g(u,v)$ such that*

$$f(z) = g(|z|^2, \mathfrak{Re}\, z^3). \tag{$*$}$$

Moreover, if f is a polynomial, then there is a unique polynomial $g(u,v)$ satisfying equation $(*)$

(b) $\mathcal{E}_{2,2}^\Delta = \mathcal{E}_2^\Delta\{z, \bar{z}^2\}$,

(c) $\mathfrak{M}_{2,2}^\Delta = \mathcal{E}_2^\Delta\{w, \bar{z}\bar{w}, z^2\bar{w}, z^3 w\}$.

Note. We shall refer to the four generators of $\mathfrak{M}_{2,2}^\Delta$ by T_0, T_1, T_2, T_3, where the subscript i refers to the degree of homogeneity of T_i in the z-variable.

Proof: (a) A theorem of G. Schwarz [9] states that ϕ is onto if $|z|^2$ and $\mathfrak{Re}\,(z^3)$ form a Hilbert basis for the ring of invariant polynomials. Let

$$f(z) = \Sigma a_{jk} z^j \bar{z}^k \tag{1.9}$$

be invariant and real-valued. Thus

$$a_{jk} = a_{kj}, \quad a_{jk} = 0 \quad \text{unless} \quad j = k \pmod 3, \quad \text{and} \quad a_{jk} = \overline{a_{kj}}\,. \tag{1.10}$$

It follows that a basis for the vector space of invariant polynomials is given by

$$z^j \bar{z}^k + \bar{z}^j z^k = z^k \bar{z}^k (z^{3l} + \bar{z}^{3l}), \quad \text{where} \quad j - k = 3l \geqq 0. \tag{1.11}$$

Observe that

$$\begin{aligned} z^{3l} + \bar{z}^{3l} &= (z^3 + \bar{z}^3)(z^{3(l-1)} + \bar{z}^{3(l-1)}) \\ &\quad - z^3\bar{z}^3(\bar{z}^{3(l-2)} + z^{3(l-2)}) \quad \text{for} \quad l \geqq 2. \end{aligned} \tag{1.12}$$

We see by induction, using (1.12), that $z^{3l} + \bar{z}^{3l}$ can be written as a polynomial in $z\bar{z}$ and $\mathfrak{Re}\, z^3$. Finally, if g is a polynomial satisfying $g(z\bar{z}, \mathfrak{Re}\, z^3) \equiv 0$, then $g \equiv 0$ since the image of $z \to (z\bar{z}, \mathfrak{Re}\, z^3)$ contains an open set in $\mathbb{R}^2$.

(b) Again, we work first on the level of polynomials. Let $G: \mathbb{C} \to \mathbb{C}$ be an equivariant polynomial map where

$$G(z) = \Sigma a_{jk} z^j \bar{z}^k. \tag{1.13}$$

The conditions on the a_{jk} induced by equivariance under (1.6) are

$$a_{jk} = \bar{a}_{jk} \quad \text{and} \quad a_{jk} = 0 \quad \text{unless} \quad j = k + 1 \pmod 3). \tag{1.14}$$

Thus a basis for the equivariant polynomials in $\mathcal{E}^{\Delta}_{2,2}$ is

$$(z\bar{z})^k z^{3l+1}, \quad j > k, \quad \text{and} \quad (z\bar{z})^j \bar{z}^{3l+2}, \quad j < k, \tag{1.15}$$

where $l \geqq 0$. Observe that

$$\begin{aligned} z^{3l} z &= (z^{3l} + \bar{z}^{3l}) z - (z\bar{z}) \bar{z}^{3(l-1)+2}, \\ \bar{z}^{3l} \bar{z}^2 &= (z^{3l} + \bar{z}^{3l}) \bar{z}^2 - (z\bar{z})^2 z^{3(l-1)+1}. \end{aligned} \qquad l \geqq 1, \tag{1.16}$$

By induction, we see that z and $\bar{z}^2$ generate the module of equivariant polynomials over the ring of invariant polynomials. A generalization of Schwarz's theorem by Poenaru [7], page 65, states in this case that every equivariant function may be written as

$$f(|z|^2, \mathfrak{Re}\, z^3) z + g(|z|^2, \mathfrak{Re}\, z^3) \bar{z}^2. \tag{1.17}$$

To show that z and $\bar{z}^2$ are independent assume that (1.17) is equal to 0 and that $z = x + iy$. Then, assuming $y \neq 0$,

$$f = 2xg \quad \text{and} \quad xf + (x^2 - y^2) g = 0. \tag{1.18}$$

Thus

$$(3x^2 - y^2) g = 0 \tag{1.19}$$

and $g = 0$ by continuity. Hence $f = 0$.

(c) Let $T(z, w)$ be a polynomial in $\mathfrak{M}^{\Delta}_{2,2}$. Then

$$T(z, w) = \Sigma \left(a_{jk} z^j \bar{z}^k w + b_{jk} z^j \bar{z}^k \bar{w} \right). \tag{1.20}$$

The equivariance conditions generated by (1.6) imply

$$
\begin{aligned}
&a_{jk} = \bar{a}_{jk} \quad \text{and} \quad a_{jk} = 0 \quad \text{unless} \quad j = k \pmod 3, \\
&b_{jk} = \bar{b}_{jk} \quad \text{and} \quad b_{jk} = 0 \quad \text{unless} \quad j = k + 2 \pmod 3.
\end{aligned}
\tag{1.21}
$$

The proof that the four generators of $\mathfrak{M}^{\Delta}_{2,2}$ are as listed in Proposition 1.8(c) and that those generators are independent proceeds in a fashion similar to the proof of (b). The details are left to the reader.

One has, of course, a corresponding statement for the representation of Δ on D; namely,

PROPOSITION 1.22. (a) *Let f be in $\mathcal{E}^{\Delta}_2$. Then*

$$(*) \quad f(A) = g\big(\operatorname{tr}(A^2), \det A\big), \quad \textit{where} \quad A \in D.$$

(b) $\mathcal{E}^{\Delta}_{2,2} = \mathcal{E}^{\Delta}_2\{A, A^2 - \frac{1}{3}\operatorname{tr}(A^2)I\}$, *where I is the 2×2 identity matrix.*

(c) $\mathfrak{M}^{\Delta}_{2,2} = \mathcal{E}^{\Delta}_2\{B, AB - \frac{1}{3}\operatorname{tr}(AB)I, A^2B - \frac{1}{3}\operatorname{tr}(A^2B)I, \operatorname{tr}(A^2B)A\}$, *where* $B \in D$.

Proposition 1.22 may be proved in a fashion analogous to the proof of Proposition 1.8, or proved directly using Proposition 1.8 as follows. From (a) of Proposition 1.8 one sees that there is precisely one invariant polynomial homogeneous of degree 2 ($|z|^2$) and one homogeneous of degree 3 ($\mathfrak{Re}\, z^3$). Since $\operatorname{tr} A^2$ and $\det A$ are homogeneous with the same degrees, (a) of Proposition 1.22 is proved. Similarly, by Proposition 1.8 (b) there is only one equivariant map homogeneous of degree 1 and one homogeneous of degree 2, so (b) of Proposition 1.22 follows. Note that there is one term in $\mathfrak{M}^{\Delta}_{2,2}$ homogeneous of degree 0 (T_0), one homogeneous of degree 1 (T_1), two homogeneous of degree 2 (T_2 and $|z|^2T_0$) and three homogeneous of degree 3 ($T_3, |z|^2T_1, \mathfrak{Re}\, z^3T_0$). If we label the generators of $\mathfrak{M}^{\Delta}_{2,2}$ in (c) of Proposition 1.22 by S_0, S_1, S_2, S_3, respectively, one can show that the same structure (in degrees of homogeneity) holds. In particular, S_2 and $\operatorname{tr}(A_2)S_0$ are independent as are $S_3, \operatorname{tr}(A^2)S_1$ and $\det(A)S_0$. Hence (c) is proved.

There are several observations which follow directly from Proposition 1.22. The first involves an issue left unresolved in Lemma 1.3.

LEMMA 1.23. *Let $G: D \to D$ be in $\mathcal{E}^{\Delta}_{2,2}$. Then G extends uniquely to an $H: V \to V$ in $\mathcal{E}^{\Delta}_{5,5}$.*

Proof: In Lemma 1.3 we showed that G extends uniquely to an H; what we could not show at that point was that H is a smooth germ. However in view of Proposition 1.22, the map G has the form

$$G(A) = a(\operatorname{tr} A^2, \det A)A + b(\operatorname{tr} A^2, \det A)\big(A^2 - \tfrac{1}{3}\operatorname{tr}(A^2)I\big).$$

Since $\operatorname{tr}(A^2)$, $\det A$, and $A^2 - \frac{1}{3}\operatorname{tr}(A^2)I$ clearly extend to smooth invariant and equivariant functions on V, the result follows trivially.

In exactly the same fashion one has:

Lemma 1.24. *Let S be in $\mathfrak{M}^{\Delta}_{2,2}$. Then S extends to a mapping in $\mathfrak{M}^{\Gamma}_{5,5}$.*

The proof is identical to that of Lemma 1.23 since the generators for the module $\mathfrak{M}^{\Delta}_{2,2}$ (S_0, S_1, S_2, and S_3) extend smoothly. Note that we do not claim that this extension is unique; it is not. Consider, for example, $\operatorname{tr}(AB)A$ which has the same restriction to D as $-\frac{1}{2}\operatorname{tr}(A^2)B + 3[A^2B - \frac{1}{3}\operatorname{tr}(A^2B)I]$.

We mentioned in the introduction that it is technically simpler to work with the two-dimensional representation of Δ than the five-dimensional representation of Γ. The reason is that it is much simpler to find generators for $\mathfrak{M}^{\Delta}_{2,2}$ than for $\mathfrak{M}^{\Gamma}_{5,5}$.

2. Computations of $\tilde{\Delta}G$

Let $G : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ be a bifurcation problem with symmetry group g, i.e., $G(\gamma x, \lambda) = \gamma G(x, \lambda)$ for all $\gamma \in g$. Let G and H be two such bifurcation problems. As defined in [5], G and H are *g-equivalent* if

$$H(x,\lambda) = T(x,\lambda) \cdot G(X(x,\lambda), \Lambda(\lambda)), \tag{2.1}$$

where $X(0,0) = 0$, $\Lambda(0) = 0$, $\det(d_xX)(0,0) > 0$, $\Lambda'(0) > 0$, and $T(x,\lambda)$ is an invertible $n \times n$ matrix. Moreover, T and X satisfy $T(\gamma x, \lambda)\gamma = \gamma T(x,\lambda)$ and $X(\gamma x, \lambda) = \gamma X(x,\lambda)$.

Observe that for the representations which we study the linear part of X, $(d_xX)(0,0)$, is cI for some real number c. So $\det(d_xX)(0,0) > 0$ if either n is even or n is odd and $c > 0$. In this paper we use two irreducible representations, the five-dimensional representation of $O(3)$ and the two-dimensional representation of S_3, and we relate these representations by a restriction mapping. Thus if we allow S_3-equivalences which have $c < 0$, then such equivalences will correspond to $O(3)$-equivalences which are *orientation reversing*. Therefore, we assume in this paper that S_3-equivalences always satisfy $c > 0$.

We showed in [5] that the computation of a universal unfolding relative to Γ depends on computing a basis for the vector space

$$\mathcal{E}^g_{n+1,n} / \tilde{g}G, \tag{2.2}$$

where $\tilde{g}G$ is the submodule of $\mathcal{E}^g_{n+1,n}$ over the ring $\mathcal{E}^g_{n+1}$ generated by

$$\tilde{g}G = \{(d_xG)(X) + T \cdot G\}, \tag{2.3}$$

X and T satisfying the equivariance conditions in (2.1) (though not the invertibility conditions). The way that $\tilde{g}G$ is obtained is to consider a one-parameter family $G_t(x,\lambda)$ of equivariant bifurcation problems such that $G_0 = G$ and, for each t, G_t is g-equivalent to G. We assume, however, that, in (2.1), $\Lambda(\lambda,t) \equiv \lambda$ for these g-equivalences. Then $(d/dt)G_t|_{t=0}$ is the typical element of $\tilde{g}G$. It should be clear from (2.3) that $\tilde{g}G$ is a submodule of $\mathcal{E}^g_{n+1,n}$ and that generators for this module may be determined once generators for the modules of the X's and T's are known. Note that the module of the X's in (2.3) is just $\mathcal{E}^g_{n+1,n}$ and that the module of the T's is identified with $\mathfrak{M}^g_{n+1,n}$.

We observed in [5] that if certain assumptions were satisfied, then the symmetry group g could be removed from the calculation (2.2). In particular, we needed to know that the modules of the T's and X's are each finitely generated, that the module of the X's is free, and that the ring of polynomial invariant functions $\mathcal{P}^g_{n+1} \subset \mathcal{E}^g_{n+1}$ is a polynomial ring. As we saw in Section 1 these facts are true for the representation of S_3 on $\mathbb{R}^2$.

Let $G:\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ be a bifurcation problem with symmetry group $\Delta = S_3$. As in Section 1 we view $G:\mathbb{C} \times \mathbb{R} \to \mathbb{C}$. We see by Proposition 1.8(b) that

$$G(z,\lambda) = a(u,v,\lambda)z + b(u,v,\lambda)\bar{z}^2, \tag{2.4}$$

where $u = z\bar{z}$ and $v = \mathcal{R}e\, z^3$. The parameter λ is unaffected by the group action.

Let $\mathcal{E}_3$ be the ring of germs of C^∞ functions from $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}$ with coordinates (u,v,λ) on $\mathbb{R}^2 \times \mathbb{R}$. Let $\mathcal{E}_{3,2}$ be the module of germs of C^∞ mappings of $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ over the ring $\mathcal{E}_3$. Proposition 1.8 allows one to eliminate the group Δ from the singularity theory calculation implied by (2.2). More precisely, define

$$\Phi : \mathcal{E}^\Delta_{3,2} \to \mathcal{E}_{3,2} \quad \text{by} \quad \Phi(G) = (a,b), \tag{2.5}$$

where G has the form (2.4). Observe that Φ is an isomorphism. It follows directly that

$$\mathcal{E}^\Delta_{3,2}/\tilde{\Delta}G \cong \mathcal{E}_{3,2}/\Phi(\tilde{\Delta}G). \tag{2.6}$$

PROPOSITION 2.7. *Let G be as in* (2.4). *Then $\Phi(\tilde{\Delta}G)$ is generated by the six generators*

$$\begin{aligned} &\text{(i)} \quad (a,b),(ub,a),(ua+vb,0),(va+u^2b,0),\\ &\text{(ii)} \quad (2ua_u+3va_v, b+2ub_u+3vb_v),(ub+2va_u+3u^2a_v, 2vb_u+3u^2b_v). \end{aligned} \tag{2.8}$$

Proof: Let T_0, T_1, T_2, T_3 be the generators for the module $\mathfrak{M}^\Delta_{2,2}$ given in Proposition 1.8(c). Let G_1, G_2 be the generators for the module $\mathcal{E}^\Delta_{2,2}$. Then $T_i(z,G(z,\lambda))$, $i = 0,\cdots,3$, and $(d_xG)(G_i)$, $i = 1,2$, are generators of the module

$\tilde{\Delta}G$ (see (2.3)). It is an easy calculation to show that the span of the generators T_iG is the same as the span of the generators listed in (2.8)(i).

Using a suggestion of D. Sattinger we compute the last two generators as follows. Observe that the Jacobian of G may be computed using complex notation as

$$\delta G \equiv \begin{pmatrix} G_z & G_{\bar z} \\ \overline{G}_z & \overline{G}_{\bar z} \end{pmatrix} = d_x G. \tag{2.9}$$

Letting δG act on $\binom{w}{\bar w}$ one finds

$$\delta G(w) = G_z w + G_{\bar z}\bar w. \tag{2.10}$$

An easy computation yields

$$\begin{aligned} G_z &= a + ua_u + \tfrac{3}{2}u^2 b_v + \tfrac{3}{2}a_v z^3 + b_u \bar z^3, \\ G_{\bar z} &= (2b + \tfrac{3}{2}ua_v + ub_u + 3vb_v)\bar z + (a_u - \tfrac{3}{2}ub_v)z^2. \end{aligned} \tag{2.11}$$

Next, using (2.10) and (2.11), one computes

$$\begin{aligned} \delta G(z) &= (a + 2ua_u + 3va_v)z + (2b + 2ub_u + 3vb_v)\bar z^2, \\ \delta G(\bar z^2) &= (2ub + 2va_u + 3u^2 a_v)z + (a + 2vb_u + 3u^2 b_v)\bar z^2. \end{aligned} \tag{2.12}$$

One sees that the generators in (2.12) are equivalent to the ones in (2.8)(ii).

3. Analysis of Specific Singularities

We have shown that bifurcation problems commuting with the action of Δ have the form

$$G(x, y, \lambda) = a(u, v, \lambda)z + b(u, v, \lambda)\bar z^2, \tag{3.1}$$

where $u = x^2 + y^2$, $v = x^3 - 3xy^2$, and $z = x + iy$. We wish to analyze—up to Δ-equivalence—certain singularities which are present in the Bénard problem.

We assume that $0 = (0,0,0)$ is a bifurcation point for G and that the bifurcation parameter λ enters non-singularly in G; that is

$$a(0) = 0 \quad \text{and} \quad a_\lambda(0) \neq 0. \tag{3.2}$$

We consider two situations:

$$b(0) \neq 0, \tag{3.3}$$

$$b(0) = 0, \tag{3.4}$$

where the linear terms of a and b are nondegenerate, defined as follows. If we write

$$\begin{aligned} a(u,v,\lambda) &= Au + Bv + \alpha\lambda + \cdots, \\ b(u,v,\lambda) &= Cu + Dv + \beta\lambda + \cdots, \end{aligned} \tag{3.5}$$

then G in situation (3.4) is *non-degenerate* if all of the conditions (3.6) hold:

$$\begin{aligned} &\text{(i)} && A \neq 0, \\ &\text{(ii)} && \alpha \neq 0, \\ &\text{(iii)} && \alpha C - \beta A \neq 0, \\ &\text{(iv)} && AD - BC \neq 0. \end{aligned} \tag{3.6}$$

Note that (3.3) is the simplest case where bifurcation occurs while Chossat [3] has shown that (3.4) holds for the selfadjoint case. The non-degeneracy conditions (3.6) define the simplest case when (3.4) holds. Although we have analyzed more singular problems, we do not present the results here.

PROPOSITION 3.7. (A) *Assume* $a(0) = 0$, $b(0) \cdot a_\lambda(0) \neq 0$. *Then* G *is* Δ-*equivalent to*

$$N(z,\lambda) = \bar{z}^2 \pm \lambda z, \tag{3.8}$$

where $\operatorname{sgn}(\lambda z) = \operatorname{sgn}(b(0) \cdot a_\lambda(0))$.

(B) $\operatorname{codim} N = 0$, *i.e., all small perturbations of* N *are* Δ-*equivalent to* N.

THEOREM 3.9. (A) *Assume* $a(0) = b(0) = 0$ *and that* G *is non-degenerate* (*as in* (3.6)). *Then* G *is* Δ-*equivalent to*

$$N(z,\lambda) = (u \pm \lambda)z + (\pm u + \tilde{D}v)\bar{z}^2, \tag{3.10}$$

where

$$\begin{aligned} &\text{(i)} && \operatorname{sgn}(\lambda z) = \operatorname{sgn}(A\alpha) && \textit{in } (3.6)\text{(i), (ii)}, \\ &\text{(ii)} && \operatorname{sgn}(u\bar{z}^2) = \operatorname{sgn}(\alpha C - \beta A) \cdot \operatorname{sgn}(A\alpha) && \textit{in } (3.6)\text{(iii)}, \\ &\text{(iii)} && \operatorname{sgn}(\tilde{D}) \textit{ in } (3.10) = \operatorname{sgn}(AD - BC) && \textit{in } (3.6)\text{(iv)}. \end{aligned} \tag{3.11}$$

In fact, $\tilde{D} = (AD - BC)\alpha^2/(\alpha C - \beta A)^2$.

(B) $\operatorname{codim} N = 2$ *and a universal unfolding is given by*

$$F(z,\lambda,D,E) = (u \pm \lambda)z + (\pm u + Dv - E)\bar{z}^2, \tag{3.12}$$

where E *is near* 0 *and* D *is near* $\tilde{D}$.

The results are obtained in three basic steps. First one puts the lowest-order terms in the normal form N; thus $G = N + p$, where p represents the "higher-order" terms in G. Second, one shows that $\tilde{\Delta}G = \tilde{\Delta}N$ for all possible p. Then one applies Theorem 1.13 of [5] to show that G is Δ-equivalent to N thus proving part A. For the final step, let $\mathcal{E}_\lambda$ denote the space of germs of real-valued C^∞ functions depending on the one parameter λ and let $\partial G/\partial\lambda = a_\lambda z + b_\lambda \bar{z}^2$. One then computes a complementary vector subspace Q to

$$\Delta G = \tilde{\Delta}G + \mathcal{E}_\lambda\left\{ \frac{\partial G}{\partial \lambda} \right\} \tag{3.13}$$

in $\mathcal{E}^\Delta_{3,2}$. (Note that ΔG is not necessarily a submodule of $\mathcal{E}^\Delta_{3,2}$ over the ring $\mathcal{E}^\Delta_3$. As a result these computations are best done by first finding a complementary subspace to $\tilde{\Delta}G$.) The dimension of Q is defined to be the *codimension* of G.

For computational convenience we recall Nakayama's lemma (in the form we need). Let $\mathfrak{M}$ be the (maximal) ideal in $\mathcal{E}_3$ generated by the coordinate functions u, v, λ. Let S be a submodule of $\mathcal{E}_{3,2}$ over the ring $\mathcal{E}_3$ generated by $p_1, \cdots, p_k$. (For example, let $S = \Phi(\tilde{\Delta}G)$.) Let $q_1, \cdots, q_k$ be in $\mathfrak{M} \cdot S$, the submodule generated by $m \cdot s$, where $m \in \mathfrak{M}$ and $s \in S$. Then $p_1 + q_1, \cdots, p_k + q_k$ is another set of generators for S.

Let

$$\mathfrak{M}^k = \underbrace{\mathfrak{M}, \cdots, \mathfrak{M}}_{k\text{-times}}$$

be the ideal in $\mathcal{E}_3$ consisting of those functions whose Taylor expansion at 0 begins with terms of order at least k. We shall use the notation that m_k is some unspecified element of $\mathfrak{M}^k$.

Proof of Proposition 3.7: Since $b(0) \neq 0$, one sees that G is Δ-equivalent to $(1/b)G = \tilde{a}z + \bar{z}^2$, where $\tilde{a} = a/b$. One can rescale λ so that $|\tilde{a}_\lambda(0)| = 1$. Thus one may assume that $G = N + pz$, where $p \in \mathcal{E}^\Delta_3$ satisfies $p(0) = p_\lambda(0) = 0$. We now see that the first three terms in (2.8)(i) and the first term in (2.8)(ii) have the form (assuming $p = a_1 u + a_2 v + \lambda + \cdots$)

$$(a_1 u + a_2 v + \lambda + m_2, 1),\quad (u, m_1),\quad (v + m_2, 0),\quad (2a_1 u + 3a_2 v, 1). \tag{3.14}$$

Multiplying the fourth element by m_1 and subtracting from the second yields $(u + m_2, 0) \in \Phi(\tilde{\Delta}G)$. Moreover, subtracting the fourth generator from the first yields an equivalent set of generators

$$(\lambda - a_1 u - 2a_2 v + m_2, 0),\quad (u + m_2, 0),\quad (v + m_2, 0),\quad (2a_1 u + 3a_2 v, 1). \tag{3.15}$$

By Nakayama's lemma, the first three elements in (3.15) generate the submodule $(\mathfrak{M}, 0)$. It follows from the fourth generator of (3.15) that $(0, 1) \in \Phi(\tilde{\Delta}G)$. As a result, $\Phi(\tilde{\Delta}G) = (\mathfrak{M}, 0) \oplus (0, \mathcal{E}_3)$ which is independent of p. Hence (using Theorem 1.13 of [5]) G is Δ-equivalent to N. Finally, observe that $\Phi(\Delta G) = \mathcal{E}_{3,2}$ as $a_\lambda(0) \neq 0$.

In order to prove Theorem 3.9, we proceed through the three steps outlined above. We begin by computing the general Δ-equivalence up to first order. Our approach is to compute the lowest-order terms of

$$G'' = T(z,\lambda)G(Z(z,\lambda), \Lambda(\lambda)), \tag{3.16}$$

where

$$Z = \delta z + \epsilon \bar{z}^2, \qquad \Lambda = \sigma\lambda + \cdots \tag{3.17}$$

and

$$T = \rho T_0 + \tau T_1 + \phi T_2 + \psi T_3 \quad \text{(in the notation of Section 2)}. \tag{3.18}$$

Here δ, ϵ, ρ, τ, ϕ, ψ are functions of $u = z\bar{z}$, $v = \mathfrak{Re}\, z^3$, and λ. Note that $T(0,0)$ is invertible and Z and Λ are orientation preserving changes of coordinates; hence

$$\rho(0) \neq 0 \quad \delta(0) > 0 \quad \text{and} \quad \sigma > 0. \tag{3.19}$$

We use the following notation:

$$\begin{aligned} G(z,\lambda) &= az + b\bar{z}^2, \\ G(Z,\Lambda) &= a'z + b'\bar{z}^2, \\ G'' = TG(Z,\Lambda) &= a''z + b''\bar{z}^2. \end{aligned} \tag{3.20}$$

Since

$$G(Z,\Lambda) = a(Z,\Lambda)Z + b(Z,\Lambda)\bar{Z}^2, \tag{3.21}$$

one has

$$\begin{aligned} a' &= \delta a(Z,\Lambda) + 2\epsilon(\delta u + \epsilon v)b(Z,\Lambda), \\ b' &= \epsilon a(Z,\Lambda) + (\delta^2 - \epsilon^2 u)b(Z,\Lambda). \end{aligned} \tag{3.22}$$

Also one computes

$$\begin{aligned} a'' &= (\rho + \phi u + 2\psi v)a' + (\tau u + 2\phi v + \psi u^2)b', \\ b'' &= (\tau - \psi u)a' + (\rho - \phi u)b', \end{aligned} \tag{3.23}$$

$$\begin{aligned} u(Z) &= Z\bar{Z} = \delta^2 u + 2\delta\epsilon v + \epsilon^2 u^2, \\ v(Z) &= \tfrac{1}{2}(Z^3 + \bar{Z}^3) = \delta^3 v + 3\delta^2\epsilon u^2 + 3\delta\epsilon^2 uv + 2\epsilon^3 v^2 - 2\epsilon^3 u^3. \end{aligned} \tag{3.24}$$

Now one can compute G'' to first order. Let

$$G = (Au + Bv + \alpha\lambda)z + (Cu + Dv + \beta\lambda)\bar{z}^2 + \cdots, \tag{3.25}$$
$$G'' = (\tilde{A}u + \tilde{B}v + \tilde{\alpha}\lambda)z + (\tilde{C}u + \tilde{D}v + \tilde{\beta}\lambda)\bar{z}^2 + \cdots;$$

then

$$\begin{aligned}
\tilde{A} &= \rho\delta^3 A,\\
\tilde{B} &= \rho\delta^2(2\epsilon A + \delta^2 B),\\
\tilde{\alpha} &= \rho\delta\sigma\alpha,\\
\tilde{C} &= (\tau\delta + \rho\epsilon)\delta^2 A + \rho\delta^4 C,\\
\tilde{D} &= (\tau\delta + \rho\epsilon)(2\delta\epsilon A + \delta^3 B) + \rho\delta^3(2\epsilon C + \delta^2 D),\\
\tilde{\beta} &= (\tau\delta + \rho\epsilon)\sigma\alpha + \rho\delta^2\sigma\beta,
\end{aligned} \tag{3.26}$$

where ρ, δ, σ, τ, ϵ are constants and $\rho \neq 0$, $\delta > 0$, $\sigma > 0$.

LEMMA 3.27. (i) $\tilde{A}\tilde{D} - \tilde{B}\tilde{C} = \rho^2\delta^8(AD - BC)$,
(ii) $\tilde{\alpha}\tilde{C} - \tilde{\beta}\tilde{A} = \rho^2\sigma\delta^5(\alpha C - \beta A)$,
(iii) $\tilde{A}\tilde{\alpha} = \rho^2\delta^4\sigma A\alpha$.

The proof is a straightforward calculation. From (3.26) and Lemma 3.27 one sees that the four non-degeneracy assumptions (3.6) are invariants of Δ-equivalence.

Note. By using Lemma 3.27(i) and (iii) and the fact that $\tilde{\alpha} = \rho\delta\sigma\alpha$ one can recover the formula for $\tilde{D}$ in (3.11).

Proof of Theorem 3.9: We first show that the lowest-order terms in G can be put in the form (3.10) by a Δ-equivalence. Choose ϵ and τ by

$$\epsilon = -\delta^2 B/(2A) \quad \text{and} \quad \tau\delta + \rho\epsilon = -\rho\delta^2\beta/\alpha. \tag{3.28}$$

Then (3.26) becomes

$$\begin{aligned}
\tilde{A} &= \rho\delta^3 A,\\
\tilde{C} &= \rho\delta^4(\alpha C - \beta A)/\alpha,\\
\tilde{\alpha} &= \rho\delta\sigma\alpha,\\
\tilde{D} &= (*),\\
\tilde{\beta} &= \tilde{B} = 0.
\end{aligned} \tag{3.29}$$

One can now choose ρ so that $\tilde{A} = 1$, δ so that $\tilde{C} = \operatorname{sgn}(\alpha C - \beta A) \cdot \operatorname{sgn}(\alpha A)$ and σ so that $\tilde{\alpha} = \operatorname{sgn}(A\alpha)$. (Recall that δ and σ must be positive). Note that $\operatorname{sgn}(D)$ in (3.10) is determined by Lemma 3.27(i).

We may now assume that G has the form

$$G = (u + \lambda)z + (u + Dv)\bar{z}^2 + \cdots = N + P \tag{3.30}$$

since the cases with the other signs are identical. The second step is to prove that $\tilde{\Delta}G = \tilde{\Delta}N$ for all such G and for N as in (3.10). From (2.8) the generators for $\Phi(\tilde{\Delta}G)$ have the form (where m^k indicates an element of $\mathfrak{M}^k$). Note the change of basis from (2.8)(i) indicated by (3.31)(iii) and (iv):

$$\begin{aligned}
&\text{(i)} && (u + \lambda + \mathrm{m}^2, u + Dv + \mathrm{m}^2) = (a, b),\\
&\text{(ii)} && (u^2 + Duv + \mathrm{m}^3, u + \lambda + \mathrm{m}^2) = (ub, a),\\
&\text{(iii)} && (uv + Dv^2 + \mathrm{m}^3, -u^2 - Duv + \mathrm{m}^3) = (vb, -ub),\\
&\text{(iv)} && (\mathrm{m}^3, uv + Dv^2 + \mathrm{m}^3) = (-u^2b, vb),\\
&\text{(v)} && (u + \mathrm{m}^2, \tfrac{3}{2}u + 2Dv + \mathrm{m}^2),\\
&\text{(vi)} && (v + \mathrm{m}^2, v + \mathrm{m}^2).
\end{aligned} \tag{3.31}$$

One may use (i) and (ii) to eliminate λ from the higher-order terms in the generators (iii)–(vi). Thus we think of (iii)–(vi) as generators of a submodule of $\mathcal{E}_{2,2}$. Nakayama's lemma states that the ideal generated by $u + \mathrm{m}^2$ and $v + \mathrm{m}^2$ in $\mathcal{E}^2$ is equal to the maximal ideal $\mathfrak{M}$ generated by u and v. Thus we may replace (v) and (vi) by

(v′) $(u, \frac{3}{2}u + 2Dv + \mathrm{m}^2)$,

(vi′) $(v, v + \mathrm{m}^2)$.

Consequently (iii) and (vi) may be replaced by

(iii′) $(uv + Dv^2, -u^2 - Duv + \mathrm{m}^3)$,

(iv′) $(0, uv + Dv^2 + \mathrm{m}^3)$.

Here one shifts the third-order terms in the first coordinate to third-order terms in the second coordinate using (v′) and (vi′). Next replace (iii′) by $-$(iii′)$+$ v(v′) $+ Dv$(vi′) to obtain

(iii″) $(0, u^2 + (D + \frac{3}{2})uv + 3Dv^2 + \mathrm{m}^3)$.

Observing that the quadratic terms in (iii″) and (iv′) are relatively prime allows us to conclude—by using the following lemma—that $(0, \mathfrak{M}^3) \subset \mathfrak{M} \cdot \Phi(\tilde{\Delta}G)$.

LEMMA 3.32. *Let $\mathfrak{L}$ be the ideal in $\mathcal{E}_2$ generated by p and q, where $p = p_2 + \mathrm{m}^3$, $q = q_2 + \mathrm{m}^3$ and p_2 and q_2 have no common factors. Then $\mathfrak{M}^3 = \mathfrak{M}\mathfrak{L}$ and p_2, q_2 form a set of generators for $\mathfrak{L}$.*

Proof: This lemma is *known* (cf. [11]); we include a proof for completeness. The case where p_2 and q_2 have linear factors is the case of interest. Assume

$$p_2 = l_1 l_2 \quad \text{and} \quad q_2 = m_1 m_2. \tag{3.33}$$

We claim up_2, vp_2, uq_2, vq_2 are all linearly independent. If this is true, then $\mathfrak{M}\langle p_2, q_2\rangle = \mathfrak{M}^3$ since the space of homogeneous cubics has dimension 4 as a real vector space. By Nakayama, $\mathfrak{M}\mathfrak{L} = \mathfrak{M}\langle p_2, q_2\rangle$.

To prove the claim, assume

$$(\alpha x + \beta y)p_2 + (\gamma x + \delta y)q_2 \equiv 0. \tag{3.34}$$

Then the factors l_1 and l_2 must each divide $(\gamma x + \delta y)m_1 m_2$. Thus one of l_1 and l_2 must divide one of m_1 and m_2 implying that p_2 and q_2 have a common factor. As this contradicts our assumption, (3.34) is not possible unless $\alpha = \beta = \gamma = \delta = 0$.

We return to the proof of the theorem. Note that (v′) and (vi′) imply that $\mathfrak{M}^3\mathcal{E}_{3,2} \subset \mathfrak{M}\Phi(\tilde{\Delta}G)$. Next one checks that the six quadradic terms given by u(v′), v(v′), u(vi′), v(vi′), (iii″) and (iv′) are linearly independent over $\mathbb{R}$. Thus $\mathfrak{M}^2\mathcal{E}_{3,2} \subset \Phi(\tilde{\Delta}G)$. Finally, one has

$$\Phi(\tilde{\Delta}G) = \mathfrak{M}^2\mathcal{E}_{3,2} + \mathbb{R}\{(u+\lambda, u+Dv), (0, u+\lambda), (u, \tfrac{3}{2}u + 2Dv), (v, v)\}, \tag{3.35}$$

which is independent of the higher-order terms in G. So we conclude that $\tilde{\Delta}G = \tilde{\Delta}N$ and that G is Δ-equivalent to N.

To obtain part B of the theorem, note that

$$\Phi(\Delta G) = \Phi(\tilde{\Delta}G) + \mathbb{R}\{(1,0), (\lambda, 0)\}. \tag{3.36}$$

It is now easy to show that

$$\mathcal{E}_{3,2} = \Phi(\Delta G) + \mathbb{R}\{(0,1), (0,v)\}. \tag{3.37}$$

Apply the unfolding theorem in [5] to yield the result.

4. Bifurcation Problems Commuting with $O(3)$

We wish to show that the singularity theory of bifurcation problems commuting with the five-dimensional representation $\Gamma = O(3)$ on V is the same as for bifurcation problems commuting with the two-dimensional representation of $\Delta = S_3$ on D. We showed in Lemma 1.23 that the module $\mathcal{E}^{\Gamma}_{6,5}$ over the ring $\mathcal{E}^{\Gamma}_{6}$ is isomorphic to the module $\mathcal{E}^{\Delta}_{3,2}$ over the ring $\mathcal{E}^{\Delta}_{3}$, this isomorphism being given by restriction and denoted by Ψ.

PROPOSITION 4.1. *Let H_1 and H_2 be in $\mathcal{E}^{\Gamma}_{6,5}$ with G_1 and G_2 in $\mathcal{E}^{\Delta}_{3,2}$ their restrictions. Then H_1 and H_2 are Γ-equivalent if and only if G_1 and G_2 are Δ-equivalent.*

Proof: First we assume that G_1 and G_2 are Δ-equivalent; then

$$G_1(A,\lambda) = T(A,\lambda)\cdot G_2(X(A,\lambda),\Lambda(\lambda)), \tag{4.2}$$

where T is in $\mathfrak{M}^{\Delta}_{3,2}$ and X is in $\mathcal{E}^{\Delta}_{3,2}$.

Lemmas 1.23 and 1.24 show that there exist smooth extensions of both X and T to V which we denote by Z and S. Hence $S(A,\lambda)\cdot H_2(Z(A,\lambda),\Lambda(\lambda))$ is a smooth extension of $G_1(A,\lambda)$ to V. As state in Lemma 1.23 the extension of a given G is unique; hence it follows that

$$H_1(A,\lambda) = S(A,\lambda)\cdot H_2(Z(A,\lambda),\Lambda(\lambda)). \tag{4.3}$$

To prove that the equivalence of the H's imply the equivalence of the G's, we assume that (4.3) holds. If we knew that for each diagonal matrix A in D, $S(A,\lambda): D\to D$, then we could obtain (4.2) from (4.3) by restricting to D. This fact is proved as follows. Recall from the proof of Lemma 1.3 that $A\in D$ if and only if $\gamma_1\cdot A = \gamma_2\cdot A = A$. Now observe that if A and B are in D, then

$$S(A,B) = S(\gamma_i\cdot A,\gamma_i\cdot B) = \gamma_i\cdot S(A,B). \tag{4.4}$$

Hence $S(A,B)$ is in D.

The unfolding theory for H is the same as the unfolding theory for the restriction G if we can show

PROPOSITION 4.5. *$\Psi(\Gamma H) = \Delta G$, where Ψ is the isomorphism of the module $\mathcal{E}^{\Gamma}_{6,5}$ with $\mathcal{E}^{\Delta}_{3,2}$ induced by restriction.*

Proof: Recall that ΔG is really the tangent space to the orbit of all elements in $\mathcal{E}^{\Delta}_{3,2}$ which are Δ-equivalent with G. That is, we may represent an arbitrary element G_1 of ΔG by

$$G_1(A,\lambda) = \frac{d}{dt}T(A,\lambda,t)G(X(A,\lambda,t),\Lambda(\lambda,t))|_{t=0}, \tag{4.6}$$

where $T(A,\lambda,0) = I_2$, $X(A,\lambda,0) = A$, and $\Lambda(\lambda,0)=\lambda$. The proof of Proposition 4.1 shows that the extensions of T and X may be done smoothly in t. So one sees that the extension H_1 of G_1 is in ΓH and conversely. Thus $\Psi(\Gamma H) = \Delta G$.

We now have proved the following analogue of Proposition 3.7 and Theorem 3.9.

THEOREM 4.7. *Suppose $H = aA + b(A^2 - \frac{1}{3}\mathrm{tr}(A^2)I)$, where a and b are in $\mathcal{E}^{\Gamma}_6$, that is $a = a(u,v,\lambda)$, $b = b(u,v,\lambda)$, $u = \mathrm{tr}(A^2)$ and $v = \det A$.*

(I) *Assume a and b satisfy* (3.2) *and* (3.3), *then H is* Γ*-equivalent to*

$$N(A,\lambda) = \left(A^2 - \tfrac{1}{3}\operatorname{tr}(A^2)I\right) \pm \lambda A, \tag{4.8}$$

where $\operatorname{sgn}(\lambda A) = \operatorname{sgn}(b(0) \cdot a_\lambda(0))$.

Moreover, $\operatorname{codim} H = 0$.

(II) *Assume a and b satisfy* (3.2) *and* (3.6); *then H is* Γ*-equivalent to*

$$N(A,\lambda) = (u \pm \lambda)A + (\pm u + Dv)\left(A^2 - \tfrac{1}{3}\operatorname{tr}(A^2)I\right). \tag{4.9}$$

The signs are determined as in (3.11).

Moreover, $\operatorname{codim} H = 2$ *and a universal unfolding of N is*

$$F(A,\lambda,D,E) = (u \pm \lambda)A + (\pm u + Dv - E)\left(A^2 - \tfrac{1}{3}\operatorname{tr}(A^2)I\right). \tag{4.10}$$

5. The Bifurcation Diagrams

The results of this section give an analysis of the solution sets and of the linearized stability of each solution of the 2×2 canonical forms of Section 3. Because of the reduction from five to two dimensions discussed in Sections 1 and 4, this calculation leads to the same results for the five-dimensional problem, but some preliminary comments on this reduction are necessary in order to understand this fully.

The three-dimensional group $O(3)$ acting on the two-dimensional linear space D fills up the five-dimensional space V. Thus the orbit of a typical point in D is three-dimensional. However, not every point in D has a three-dimensional orbit. For example, the orbit of 0 consists of a single point. In general for $d \in D$ we define the *isotropy subgroup* of d as

$$\Sigma_d = \{\gamma \in O(3) \mid \gamma \cdot d = d\}; \tag{5.1}$$

that is, the subgroup of $O(3)$ which commutes with d when considering ordinary matrix multiplication. If the isotropy subgroup of d has dimension l, then the orbit of d has dimension $3 - l$. There are two possibilities for Σ_d when d is non-zero. If d has a double eigenvalue (that is, two equal entries), then $\Sigma_d \cong O(2) \oplus Z_2$. If d has distinct eigenvalues, then Σ_d consists of diagonal matrices in $O(3)$. Since such diagonal matrices must have ± 1 in each entry on the diagonal, we see that $\Sigma_d = Z_2 \oplus Z_2 \oplus Z_2$ is finite. Note that if d has a triple eigenvalue, then $d = 0$ since $\operatorname{tr} d = 0$.

In terms of the isomorphic representation of $O(3)$ on the spherical harmonics of order two, the harmonic polynomial associated with d has axial symmetry if and only if there is a copy of $SO(2)$ in Σ_d. Thus axisymmetric solutions correspond to solutions in V with a double eigenvalue. Also non-zero axisymmetric solutions have two-dimensional orbits (for fixed λ) in V while non-axisymmetric solutions have three-dimensional orbits (with 8-fold symmetry).

This classification of orbits into three types appears naturally in trying to solve an equivariant bifurcation problem $H : V \times \mathbb{R} \to V$. As in Sections 1 and 4, let $G = H \mid D \times \mathbb{R}$. According to Section 3, we may write

$$G(z,\lambda) = a(u,v,\lambda)z + b(u,v,\lambda)\bar{z}^2 = 0, \tag{5.2}$$

where $u = z\bar{z}$, $v = \mathfrak{R}e\, z^3$ and $z = x + iy$. In real coordinates, (5.2) becomes

$$G(x,y,\lambda) = \begin{pmatrix} x & x^2 - y^2 \\ y & -2xy \end{pmatrix}\begin{pmatrix} a \\ b \end{pmatrix} = 0, \tag{5.3}$$

where $u = x^2 + y^2$ and $v = x^3 - 3xy^2$. A point (x, y, λ) can satisfy (5.3) only if one of the three following possibilities obtains:

$$\text{(5.4)} \qquad \begin{aligned} &\text{(i)} \quad x = y = 0, \\ &\text{(ii)} \quad \det\begin{pmatrix} x & x^2 - y^2 \\ y & -2xy \end{pmatrix} = y^3 - 3x^2y = 0, \\ &\text{(iii)} \quad a = b = 0 \quad \text{and} \quad \text{(ii) does not hold.} \end{aligned}$$

Note that (5.4)(ii) consists of three lines $y = 0$, $y = \pm\sqrt{3}\,x$ which are invariant under rotation through 120°, that is, invariant under the action of Δ. We may therefore characterize case (ii) by the equations

$$y = 0, \qquad a + xb = 0, \tag{5.5}$$

obtaining the other two lines in (5.4)(ii) by symmetry.

Now in (1.7) we gave explicitly an isomorphism between the action of Δ on $\mathbb{R}^2$ in the coordinates (x, y) as above and the action of Δ on the diagonal matrices D. From this one sees that points of the form $y = 0$ correspond to diagonal matrices with double eigenvalues. Thus case (5.4)(ii) corresponds exactly to axisymmetric solutions while case (5.4)(iii) corresponds to non-axisymmetric solutions.

The above discussion concerned the relationship between the zeroes of H and of G. It is also possible to relate the eigenvalues of dH and dG at solutions, thereby relating the stability of the respective solutions. As we have noted many times, D is an invariant subspace for H and thus D is an invariant subspace for dH. So dH and dG share two eigenvalues. We claim that the remaining three eigenvalues of dH are also determined, though determined differently in each of the three cases.

We consider these cases separately. Fix λ and suppose that $d \in D$ with $H(d,\lambda)=0$. If d has three distinct eigenvalues (i.e., d corresponds to a non-axisymmetric solution), then its isotropy subgroup is discrete, hence the orbit through d is three-dimensional. Hence $H(\cdot,\lambda)$ vanishes identically on a three-dimensional manifold passing through d and dH has three zero eigenvalues at d.

If $d=0$, then both dH and dG at d are multiples of the identity matrix, that multiple being $a(0,0,\lambda)$. The only difference in the structure of the eigenvalues of dH and dG is the multiplicity of the eigenvalue $a(0,0,\lambda)$.

Finally, suppose d has a double eigenvalue (i.e., d corresponds to an axisymmetric solution). Then the isotropy subgroup Σ_d is $O(2)\oplus Z_2$. Hence the orbit of d under the action of $O(3)$ is two-dimensional at d and transverse to D. Consequently two of the three remaining eigenvalues of dH are zero. We show that the two eigenvalues of dG are real in case (ii) and that one of them is a double eigenvalue for dH, thus accounting for the fifth eigenvalue of dH. Before proving this statement we make a general comment.

Let g be a group acting linearly in $\mathbb{R}^n$ with $G \in \mathcal{E}^g_{n,n}$. By definition, G commutes with the action of g, that is, $G(\gamma \cdot x) = \gamma \cdot G(x)$ for γ in g. The chain rule states that

$$(dG)_{\gamma\cdot x} \circ \gamma \cdot v = \gamma \cdot (dG)_x(v) \tag{5.6}$$

for all $v \in \mathbb{R}^n$. In particular, if Σ is the isotropy subgroup of g corresponding to x and γ is in Σ, then

$$(dG)_x \circ \gamma \cdot v = \gamma \cdot (dG)_x(v). \tag{5.7}$$

So $(dG)_x$ commutes with γ for all γ in Σ.

We now return to the two-dimensional representation of Δ and analyse a point $(x,0)$ in $\mathbb{R}^2$. The isotropy subgroup in this case is $\Sigma = Z_2$, where the non-trivial element γ in Σ is generated by complex conjugation. In real coordinates, $\gamma = \left(\begin{smallmatrix}1 & 0\\ 0 & -1\end{smallmatrix}\right)$. According to (5.7), $(dG)|_{y=0}$ commutes with this γ. Hence $(dG)|_{y=0}$ has the same eigenvectors as γ, which proves

Lemma 5.8. *$(dG)|_{y=0}$ is diagonal.*

Note. The same trick shows that if $T \in \mathfrak{M}^{\Delta}_{2,2}$, then $T|_{y=0}$ is diagonal since $T(\gamma\cdot(x,0))\gamma = \gamma T(x,0)$. Since $\gamma\cdot(x,0)=(x,0)$, $T(x,0)$ also commutes with γ.

Returning now to the five-dimensional representation of Γ, let $d \in D$ be a point with axial symmetry (that is, d has a double eigenvalue as a matrix in D and corresponds to a point $y=0$ in the real representation of Δ). Let H be in $\mathfrak{M}^{\Gamma}_{6,5}$. We claim that d itself is an eigenvector for $(dH)_d$. First we prove that if $\tilde d \in D$ and $\Sigma_{\tilde d} \supset \Sigma_d$, then $\tilde d$ is a multiple of d. Our assumption on d implies that $\Sigma_d \cong O(2)\oplus Z_2$. If $\Sigma_{\tilde d} = O(3)$, then $\tilde d = 0$ and the result holds. If not, then

$\Sigma_{\tilde{d}} = \Sigma_d$ so that $\tilde{d}$ has a double eigenvalue also. It follows that the two equal entries in d and $\tilde{d}$ occur in the same positions; otherwise one would not have $\Sigma_{\tilde{d}} = \Sigma_d$.

From (5.7) it follows that

$$(dH)_d(d) = (dH)_d(\gamma \cdot d) = \gamma(dH)_d(d)$$

for $\gamma \in \Sigma_d$. Thus $\gamma \in \Sigma_{\tilde{d}}$, where $\tilde{d} = (dH)(d)$. So $\tilde{d}$ is a multiple of d and the claim holds.

Now $(dH)_d \mid D = (dG)_d$ since D is an invariant subspace for H. Lemma 5.8 implies that there is an eigenvector $e \neq d$ (with eigenvalue denoted by a) for $(dG)_d$ since $(dG)_d$ is diagonal. Hence e is an eigenvector for $(dH)_d$. Next choose $\gamma \in \Sigma_d$ such that $\gamma(D) \not\subset D$. (The existence of γ follows from the fact that $\dim \Sigma_d = 1$.) It follows that $\gamma \cdot e \notin D$ (as $\gamma \cdot d = d$). One applies (5.7) once again with $v = e$ to see that $(dH)_d(\gamma \cdot e) = \gamma \cdot (dH)_d(e) = a\gamma \cdot e$. Thus $\gamma \cdot e$ is an eigenvector for dH with the same eigenvalue as that of e. We have now found the fifth eigenvalue of $(dH)_d$.

We note that one could have used an argument involving Clebsch-Gordon coefficients for $SO(2) \subset SO(3)$ to obtain the same result.

PROPOSITION 5.9. *A solution of $G = 0$ is (linearly) stable if and only if as a solution of $H = 0$ it has (linearized) orbital stability.*

We shall use this fact to label solutions of $H = 0$ in our diagrams as follows:

	s :	both eigenvalues of dG have positive real part (stable),
(5.10)	$-$:	the eigenvalues of dG have opposite sign (negative degree),
	u :	both eigenvalues of dG have negative real parts (unstable).

Of course the case "$-$" is also unstable. We normalize our labelling by the convention that the trivial solution is stable below criticality, i.e., for $\lambda < 0$. It may be necessary to multiply the equation $G = 0$ by an overall minus sign to achieve this. (Similarly, a possible multiplication by -1 was also required to derive the normal form (3.10).) Recall that dG at the trivial solution is a multiple of the identity, so that the degree is always positive.

Our intention is to make the stability assignments listed in (5.10) for the bifurcation problems considered in Section 3 by computing with the normal forms. To do this, we must show that these assignments are invariants of Δ-equivalence. We cannot make this argument in general, however we can prove this invariance for the specific cases of Section 3.

We begin our discussion with a general remark about g-equivalence. Suppose g acts linearly on $\mathbb{R}^n$ and H_1 and H_2 are equivariant with respect to this action. To show that the eigenvalues of dH_1 and dH_2 are invariants of g-equivalence we observe that one has only to consider g-equivalences of the form $H_2 = TH_1$. For if $H_2(x) = H_1(X(x))$, then

$$H_2(x) = (dX)_x\left[(dX)_x^{-1}H_1(X(x))\right]. \tag{5.11}$$

Note that the term within the brackets in (5.11) is just the formula for transforming a vector field by the diffeomorphism $X(x)$. More precisely, consider $\dot{y} = H(y)$ and $y = X(x)$. In the x-coordinates one obtains $\dot{x} = (dX)_x^{-1}H_1(X(x))$. Clearly the linearization of this vector field at a zero is an invariant of changes of coordinates. Hence to show that the linearized stability is an invariant of g-equivalence one must show this for $H_2 = TH_1$. Moreover, we need only compute dH_2 and dH_1 on solutions. Then one has $\{H_2 = 0\} = \{H_1 = 0\}$. Hence $dH_2 = TdH_1$ on solutions.

We now return to the specific case of Δ-equivalence. We let G_1 and G_2 be in $\mathcal{E}_{3,2}^{\Delta}$ and assume that $G_1 = TG_2$. For the trivial solution $x = y = 0$, $(dG_1)_{(0,0,\lambda)} = T(0,0,\lambda)(dG_2)_{(0,0,\lambda)}$. From Proposition 1.8(c) one sees that T has the form

$$\rho_0T_0 + \rho_1T_1 + \rho_2T_2 + \rho_3T_3, \quad \text{where} \quad \rho_i = \rho_i(u,v,\lambda) \tag{5.12}$$

are invariant functions and the T_i are generators for the module $\mathfrak{M}_{2,2}^{\Delta}$. Note that $T(0) = \rho_0(0)I$ where, as stated in Sections 1 and 3,

$$\rho_0(0) > 0. \tag{5.13}$$

It follows from (5.13) that the sign of the eigenvalues of dG are invariant along the trivial solution.

For solutions corresponding to axisymmetric solutions (in the five-dimensional space) one can assume $y = 0$. From Lemma 5.8 it follows that $(dG)_1$, T, and $(dG)_2$ are diagonal matrices on $y = 0$. Moreover (5.13) asserts that (for (x,λ) near $(0,0)$) T has positive entries. Thus linearized stability is an invariant of Δ-equivalence along axisymmetric solutions.

Finally, observe that if $G_1 = TG_2$, then

$$\det(dG_1) = \det T \det(dG_2) \tag{5.14}$$

on a solution. Since $\det T \sim \rho(0)^2 > 0$ for (x,y,λ) near 0, the sign of the determinant is an invariant of Δ-equivalence and so the assignment of a "$-$" along a solution branch is always an invariant of Δ-equivalence.

Thus the only stability assignments which can be confused by a Δ-equivalence are "s" and "u" for a non-axisymmetric solution. It turns out that this one bad case can be eliminated by *ad hoc* arguments for the normal forms we consider here, as we show below.

Note that (5.6) implies that $\det dG$ and $\operatorname{tr} dG$ are invariant functions in $\mathcal{E}_3^{\Delta}$. These functions can be evaluated explicitly as:

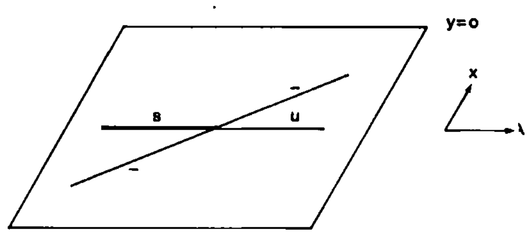

Figure 1. $N = \bar{z}^2 - \lambda z = 0.$

LEMMA 5.15. *For non-axisymmetric solutions, that is,* $a = b = 0$, *one has*

$$\det(dG) = (u^3 - v^2)[a_u b_v - a_v b_u],$$
$$\operatorname{tr}(dG) = 2ua_u + 3va_v + 2vb_u + 3u^2 b_v. \tag{5.16}$$

Note that $u^3 - v^2 = y^2(y^2 - 3x^2)^2 \geqq 0$.

We now begin the discussion of the bifurcation diagrams. In the figures below we indicate non-axisymmetric solution branches by dashed lines. We draw only the axisymmetric solutions in the plane $\{y = 0\}$, as the other branches are obtained by rotating this plane by 120° and 240°.

The bifurcation diagram associated with the normal form $N(z,\lambda) = \bar{z}^2 - \lambda z$ is shown in Figure 1. Using the notation of (5.3), we have in this case $a = -\lambda$ and $b = 1$. There are no non-axisymmetric solutions; hence the stability assignments of the figure are invariants of Δ-equivalence. Moreover the eigenvalues of dG along the axisymmetric solution branch are x and $-3x$. For $x \neq 0$, one of these is negative and one positive, so the degree of the solution branch is negative, as indicated in the figure. The normal form N has codimension 0; thus any small perturbation will only produce a Δ-equivalent problem.

In Figure 2 we show the bifurcation diagram associated with the normal form $N(z,\lambda) = (u \pm \lambda)z + (\pm u + Dv)\bar{z}^2$, where $D = 0$. We consider the case

$$a = u - \lambda \quad \text{and} \quad b = u + Dv. \tag{5.17}$$

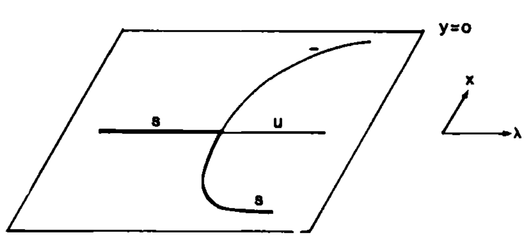

Figure 2. $N = (u - \lambda)z + (u + Dv)\bar{z}^2 = 0.$

The choice of the sign of λ in a corresponds to supercritical bifurcation which Chossat proved occurs in the selfadjoint Bénard problem. In any event, the choice of $+\lambda$ is entirely analogous. We have also chosen $+u$ for b. This choice determines which of the two families of axisymmetric solutions which bifurcates from the trivial solution is stable. Again, the other choice of sign is analogous.

There are no non-trivial non-axisymmetric solutions for this normal form. To see this observe that $b = 0$ yields (in real coordinates)

$$u + Dv = x^2(1 + Dx) + y^2(1 - 3Dx) = 0.$$

For x, y near 0 the only solution is $x = y = 0$, the trivial solution. So in this case the stability assignments are invariants of Δ-equivalence. A simple calculation shows that along the non-trivial solution branch in Figure 2 the eigenvalues of dG are $2x^2 + O(x^3)$ and $-3x^3 + O(x^4)$, which suffices to verify the stability assignments of the figure.

We now compute the bifurcation diagrams associated with the universal unfolding $F(z, \lambda, D, E) = (u - \lambda)z + (u + Dv - E)\bar{z}^2$, the diagrams obtained by perturbing the problem of Figure 2. Thus we have

$$a = u - \lambda \quad \text{and} \quad b = u + Dv - E. \tag{5.18}$$

Not surprisingly, taking $E \neq 0$ makes the bifurcation of the axisymmetric solutions transcritical, as in Figure 1. It is less obvious that taking $E \neq 0$ can lead to secondary bifurcation of non-axisymmetric solutions, depending on the sign of E. The occurrence of non-axisymmetric solutions may be demonstrated by writing (5.4)(iii) explicitly from (5.20). Thus $a = 0$, $b = 0$ yield

$$\lambda = x^2 + y^2, \qquad x^2(1 + Dx) + y^2(1 - 3Dx) = E. \tag{5.19}$$

Provided $E > 0$, (5.19) has real solutions in a neighborhood of the origin. From (5.16) one can compute the stability of the non-axisymmetric solutions. In particular,

$$\begin{aligned} \det(dG) &= D(u^3 - v^2), \\ \operatorname{tr}(dG) &= 2u + 2v + 3Du^2. \end{aligned} \tag{5.20}$$

Since $u^3 - v^2 > 0$ for non-axisymmetric solutions and $\operatorname{tr}(dG) > 0$ for (x, y, λ) near $(0, 0, 0)$, one knows that the non-axisymmetric solutions have negative degree if $D < 0$ and are stable if $D > 0$. Here for the first time we encounter a stability assignment which is not *a priori* an invariant of Δ-equivalence. We now present an ad hoc argument to justify our claim that in this particular case the stability assignment for a non-axisymmetric solution is in fact an invariant of

Δ-equivalence. Consider

$$G(z,\lambda) = (Au + Bv + \alpha\lambda)z + (Cu + Dv + \beta\lambda - E)\bar{z}^2 + \cdots = 0. \tag{5.21}$$

From (5.16) we see that for non-axisymmetric solutions of (5.17) one has

$$\begin{aligned} \det(dG) &= (u^3 - v^2)\left[AD - BC + O(|z|)\right], \\ \operatorname{tr}(dG) &= 2Au + O(|z|^3). \end{aligned} \tag{5.22}$$

From Lemma 3.27 we see that the sign of $AD - BC$ is an invariant of Δ-equivalence and from (3.26) we see that the sign of A is an invariant of those Δ-equivalences for which $\rho(0) > 0$ (as described above), thus proving our assertion. We give these bifurcation diagrams in Figure 3.

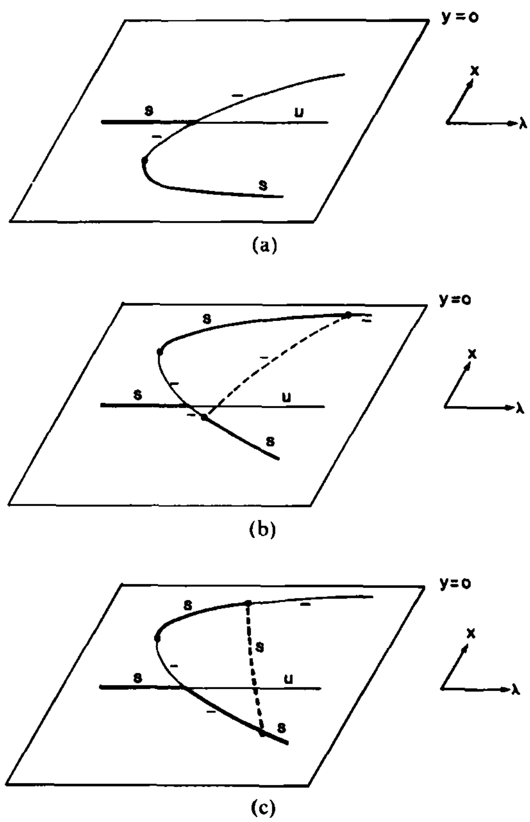

Figure 3. (a) $F = (u - \lambda)z + (u + Dv - E)\bar{z}^2 = 0$, $E > 0$, and $D > 0$.

The only point remaining is to describe how the non-axisymmetric solutions are fit into these figures. Observe that when $E > 0$ but small, the solutions to the second equation of (5.19) (which are near the origin) form a circle-like curve in the x, y-plane with radius approximately $\sqrt{E}$. This would be exactly true for $D = 0$; however, the curve is only approximately a circle for $D \neq 0$. It follows that in x, y, λ-space this curve intersects the $y = 0$ plane at two points (x_-, λ_-) and (x_+, λ_+), where $x_- < 0$ and $x_+ > 0$. The information we need is which of λ_+ and λ_- is bigger. Of course, from (5.19) one has $x_+ = \lambda_+^2$ and $x_- = \lambda_-^2$. Now compute

$$\lambda_- - \lambda_+ = \mathrm{ED}\frac{x_+ - x_-}{(1 + Dx_-)(1 + Dx_+)}. \tag{5.23}$$

Thus

$$\operatorname{sgn}(\lambda_- - \lambda_+) = \operatorname{sgn}(D). \tag{5.24}$$

The zero sets of $F = 0$ are now proved to be those in Figure 3. When $D < 0$, there exists for $\lambda_- < \lambda < \lambda_+$ two distinct stable axisymmetric solutions and no stable non-axisymmetric solutions. For $D > 0$, there is always one stable solution for $\lambda > 0$, and this stable solution (unique modulo the symmetry group) is non-axisymmetric for $\lambda_+ < \lambda < \lambda_-$. As we have shown, the stability of the non-axisymmetric solution is given by the sign of D; the stability for the axisymmetric solutions are calculated directly from $(dF)|_{y=0}$.

6. Implications for the Bénard Problem in a Spherical Shell

We begin this section with a brief review of the formulation of the Bénard problem in the Boussinesq approximation. We refer to [3], [10] for more detail.

The problem is posed in a three-dimensional annular region

$$\Omega = \{x \in \mathbb{R}^3 : \eta R_0 < |x| < R_0\},$$

where we suppose η near 0.3. (But see the remarks below concerning other values of η.) After subtraction of the conduction solution the equations become

$$v_t = -\nabla p + \Delta v + R\mathbf{g}(\mathbf{r})\theta - (v \cdot \nabla)v, \tag{6.1a}$$

$$\theta_t = \frac{1}{P}\{\Delta\theta + R\nabla T_0 \cdot v\} - (v \cdot \nabla)\theta. \tag{6.1b}$$

Here v, p, and θ measure velocity, pressure, and temperature, respectively, R and P are the Rayleigh and Prandtl numbers, respectively; the gravity vector $\mathbf{g}(\mathbf{r})$ and the equilibrium temperature gradient ∇T_0 have the form (0.1). Of course (6.1) must be supplemented by appropriate boundary conditions on $\partial\Omega$: a typical choice would be (homogeneous) Dirichlet conditions for θ and either Dirichlet or free surface conditions for v.

We consider only steady state solutions of (6.1), which we regard as a bifurcation problem with R as the bifurcation parameter. The zero solution of (6.1) is stable for small R but loses stability as R is increased.

We study the first bifurcation from the trivial solution with the standard Lyapunov-Schmidt reduction. Let ψ_i, $i = 1, \cdots, n$, be a basis for the kernel of the linearization of (6.1) at the first bifurcation point, say $R = R^*$. The reduction is based on looking for a solution of (6.1) in the form

$$u = \sum_{i=1}^{n} x_i\psi_i + W(x,\lambda), \tag{6.2}$$

where $\lambda = R - R^*$ and $\langle\psi_i, W\rangle = 0$ for $i = 1, \cdots, n$. (Here x denotes the n-vector of unknown coefficients in (6.2), *not* a spatial coordinate.) It is not difficult to show that

$$W(x,\lambda) = O(|x|^2). \tag{6.3}$$

The Lyapunov-Schmidt method leads to a parametrization of steady state solutions of (6.1) near the bifurcation point by the solutions of a certain $n \times n$ system of equations

$$G(x,\lambda) = 0. \tag{6.4}$$

The reduced mapping $G: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ provides the starting point for the singularity theory methods. With minor modifications (the center manifold theorem) the reduction may be performed so that a solution is stable if and only if the eigenvalues of dG are all positive; we shall suppose this done here.

Because of the symmetry of (6.1) with respect to $O(3)$, the bifurcations of this equation have high multiplicity. As observed by Sattinger [8], the symmetry group $O(3)$ acts on the kernel of the linearization, typically irreducibly. The dimension of the kernel (or representation of $O(3)$) tends to ∞ as $\eta \to 1$. We consider $\eta = 0.3$ in this paper so that at the first bifurcation the kernel will have dimension five. See [3], [10] for calculations verifying that this does occur for $\eta = 0.3$.

The velocity field and temperature distribution of the bifurcating solutions are graphed in Young [10]. The flows typically involve convection upwards at the equator and downwards at the two poles. Of course by (6.3) the solution u in (6.2) is to lowest order just a linear combination of the eigenfunctions ψ_i. It is of great importance to realize that a linear combination of eigenfunctions, say $\Sigma x_i\psi_i$, and its negative, $-\Sigma x_i\psi_i$, are *not* related to one another through operations in the symmetry group. This is in contrast to rolls in the planar Bénard problem where a periodic eigenfunction and its negative are simply translates through half a period of one another; it is analogous to hexagons. In the spherical case, ψ involves upwelling fluid at the equator, a connected one-dimensional set, while $-\psi$ involves upwelling fluid at the two poles, a two-component zero-dimensional set. No symmetry operation can transform one of these to the other.

We now begin the application of singularity theory methods to the Bénard problem.

Our earlier results show that the reduced equations (6.4) may be written in the form (3.1). Chossat [3] proved that in the selfadjoint case $b(0) = 0$, so that Theorem 3.9 gives the relevant normal form. We assume that the non-degeneracy conditions (3.6) are satisfied.

The first application is to correct a statement of Chossat [3] that the stability of the bifurcating solution depends on the sign of a coefficient. Rather it may be seen from Figure 2 that there are two distinct solution branches emerging from the bifurcation point, one stable and the other unstable, and as remarked above, *these branches are not related through any symmetry operation*. In particular, there is always a stable solution emerging from the bifurcation point. If one changes the sign of $\alpha C - \beta A$, the coefficient which Chossat refers to and which appears in our condition (iii) of (3.6), this interchanges the stable and unstable solutions, but it does not eliminate either. Thus the coefficient effects the choice between upwelling at the equator versus upwelling at the poles, as discussed above.

The unfolding (3.12) allows one to discuss perturbations from the selfadjoint case such as the one considered in the numerical simulation of Young [10], provided the perturbation is not too great. If Figure 3(a) obtains, the bifurcation will be modified from Figure 2 in that the non-trivial solution will be stable for values of R lower than predicted by the linear theory. There will also be hysteresis effects, i.e., the jump in the solution will occur at different values of R when this parameter is increased or decreased. If Figure 3(b) or (c) obtains, the solution set will have an even richer structure, involving an initial bifurcation with circulation in the opposite sense of its eventual pattern and involving possible non-axisymmetric flows.

Young [10] finds good agreement of the bifurcation point with the predictions of the linear theory and makes no mention of hysteresis in the bifurcation. Of course his scan in Rayleigh number was rather coarse. However, his results suggest that the perturbation from the selfadjoint case is rather small. Thus we shall assume (3.12) is applicable. Also, Young only sees axisymmetric solutions with upwelling at the equator and does not find non-axisymmetric solutions *near the bifurcation point*. Thus it seems likely that in his case $E < 0$, although it is quite possible that the secondary bifurcations indicated in Figure 3(b) or (c) occur in a small range of Rayleigh numbers between the fairly widely spaced points Young investigated. Young considers the case $\beta_2 = \gamma_1 = 0$ (notation of (0.2)) of heating from below and equal density for $|x| < \eta R_0$ and $\eta R_0 < |x| < R_0$.

It would be desirable to investigate a perturbation from the selfadjoint case of the opposite sign, say $\beta_1 = \gamma_2 = 0$, to see whether the secondary bifurcations of Figure 3(b) or (c) appear here.

We conclude with some speculative remarks on Young's data in [10]. Young finds that there are non-axisymmetric solutions of (6.1) which appear at Rayleigh number roughly twice that of the bifurcation point R^* which remain stable as far as his calculations go ($\sim 5.5\ R^*$). Since these remain stable over so large a range and since they can coexist with stable axisymmetric solutions, Figure 3 does not seem like the appropriate diagram to match this data. However, it is most interesting to note that the zero set of (3.10) includes non-axisymmetric solutions which are some distance from the origin. (We were unable to find a normal form not possessing these solutions.) Although at present there is not enough data to test this, we conjecture that these non-axisymmetric solutions are related to those seen by Young. It seems that terms of higher order than in (3.12) would be required for an accurate match of the experimental data.

Acknowledgment. We are grateful to Pascal Chossat and Gerard Iooss for suggesting that the study of the Bénard problem might profit from our theory. In particular, it was their questioning whether Chossat's calculations had enumerated all of the relevant terms in the Taylor expansion of H in (0.3) which led to this investigation. We thank Leon Green for describing to us the presentation of the five-dimensional representation of $O(3)$ we used in Section 1. We are also grateful to David Sattinger for his careful reading of our original manuscript. The presentation given here profitted greatly from his observations.

Parts of this research were done during visits to the Courant Institute (NYU), the Université de Nice, the University of Minnesota, and the University of Warwick. We are indebted to these institutions for their support.

The research of the first author was supported in part by N.S.F. Grant MCS-7905799, that of the second author by N.S.F. Grant MCS-7902010 and the United States Army Contract DAA-29-78-G0127.

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Received September, 1980.
Received revised version April, 1981.