Secondary Bifurcations in Symmetric Systems

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1. INTRODUCTION

Consider an autonomous parameter-dependent system of the form

$$\dot{x} = f(x,\lambda) \ , \tag{1.1}$$

with $f : \mathbb{R}^n \times \mathbb{R}^k \to \mathbb{R}^n$ sufficiently smooth. We will assume that the system is symmetric, that is to say we have

$$f(\gamma x,\lambda) - \gamma f(x,\lambda) \tag{1.2}$$

for all γ belonging to a compact group Γ of linear operators on $\mathbb{R}^n$; by standard theory we may assume that Γ is a closed subgroup of the orthogonal group $O(n)$, and hence also a Lie group. We are mainly interested in the case $\dim \Gamma > 0$, since otherwise most of our discussion becomes trivial. We want to study secondary bifurcations for (1.1); by this we mean bifurcations from non-zero equilibria and from non-constant periodic solutions.

When there is no symmetry (i.e. Γ is trivial) the bifurcation problem has been studied using a wide variety of methods which essentially all reduce to a combination of one of the following : the Liapunov-Schmidt method, reduction to center manifolds, Poincaré mappings and normal form theory. Using these methods one obtains for example easily all bifurcations which can appear generically in one-parameter problems (k=1); these are saddle-node and Hopf bifurcations at equilibria, and saddle-nodes of limit cycles and period-doublings at non-constant periodic solutions (see e.g. Guckenheimer and Holmes (1983)). In the symmetric case a lot of work has been done about steady-state and Hopf bifurcations at equilibria which are invariant under the full group Γ. In that case the methods mentioned above combine perfectly with a group-theoretic approach, especially with group representation theory; the outcome has been a by now well established equivariant bifurcation theory (see e.g. Vanderbauwhede (1982), Golubitsky, Stewart and Schaeffer (1988)).

The situation changes considerably when one wants to study bifurcations near equilibria which do not have the full Γ-symmetry, or near non-constant periodic solutions. Indeed, such solutions generate, by the group action, a compact invariant manifold filled with either equilibria or periodic solutions, and the corresponding "local" bifurcation problem takes a somewhat more global flavour : one has to study bifurcations near this invariant manifold, and not near a particular solution on it. As a consequence the classical methods - Liapunov-Schmidt, center manifold and Poincaré-mapping - are no longer directly applicable, since in their usual formulation the starting point is always a particular solution (equilibrium or periodic). So the first step towards a general bifurcation theory for this case seems to be to establish appropriate versions of the basic methods, adapted to and incorporating the symmetries involved in the problem.

In this paper we prove a relatively simple result which clearly indicates a possible approach. We show that near any group orbit any equivariant vector field decomposes into two equivariant vector fields; one of these is at each point tangent to the group orbit through that point, and therefore its flow is just a "drift" along group orbits; the second component of the decomposition leaves a normal section to the given group orbit invariant, and its bifurcations in this normal section generate, via the group action, the bifurcations of the original vector field. The idea of such a decomposition was first suggested by Chossat and Golubitsky (1987); the additional frequencies which are a consequence of the drift along group orbits have already been introduced before by Renardy (1982), Dangelmayr (1986), Iooss (1986) and Chossat (1986).

In the next section we give a precise formulation of the decomposition result and discuss its consequences for the bifurcation problem; in section 3 we prove theorem 1.

2. RESULTS AND DISCUSSION

Let Γ be a closed subgroup of O(n) and L(Γ) its Lie algebra, i.e. L(Γ) is the tangent space to Γ at the identity operator; all elements of L(Γ) are anti-symmetric linear operators on $\mathbb{R}^n$. We also remark that the group Γ acts on its Lie algebra by the action

$$(\gamma,\eta) \in \Gamma\times L(\Gamma) \mapsto \gamma\eta\gamma^{-1} \in L(\Gamma) . \tag{2.1}$$

Fix $x_0 \in \mathbb{R}^n$ and let $\Gamma x_0 = \{\gamma x_0 | \gamma \in \Gamma\}$ be the corresponding group orbit. One knows from general theory that Γx_0 is a smooth manifold, with tangent space at the point x_0 given by $L(\Gamma)x_0 = \{\eta x_0 | \eta \in L(\Gamma)\}$. Let Y be the orthogonal complemtent of $L(\Gamma)x_0$ in $\mathbb{R}^n$; both $L(\Gamma)x_0$ and Y are invariant under the action of the isotropy subgroup of x_0, which we denote by Σ_0 :

$$\Sigma_0 := \{\gamma \in \Gamma | \gamma x_0 = x_0\} . \tag{2.2}$$

By the tubular neighborhood theorem (see Bredon (1972) for a general theory, or Vanderbauwhede (1982) for a more direct treatment) there exists a Σ_0-invariant open neighborhood Ω of the origin in Y such that :

(i) $U = \{\gamma(x_0+y) | \gamma \in \Gamma, y \in \Omega\}$ is a Γ-invariant open neighborhood of Γx_0 in $\mathbb{R}^n$;
(ii) if $\gamma_1(x_0+y_1) = \gamma_2(x_0+y_2)$ for $\gamma_i \in \Gamma$ and $y_i \in Y$ (i=1,2) then $\gamma_1^{-1}\gamma_2 \in \Sigma_0$.

Our main result is then the following (see also Krupa (1988)) :

<u>Theorem 1</u>. The neighborhood Ω of the origin in Y can be chosen such that next to the properties (i)-(ii) above also the following holds :
Each Γ-equivariant vector field $f : U \to \mathbb{R}^n$ can be written in the form

$$f(\gamma(x_0+y)) = \gamma[\tilde{f}(y)+\tilde{\eta}(y)(x_0+y)] \tag{2.3}$$

where $\tilde{f} : \Omega \to Y$ and $\tilde{\eta} : \Omega \to L(\Gamma)$ have the same smoothness properties as f, and are Σ_0-equivariant :

$$\tilde{f}(\sigma y) = \sigma\tilde{f}(y) \quad , \quad \tilde{\eta}(\sigma y) = \sigma\tilde{\eta}(y)\sigma^{-1} \quad , \quad \forall\sigma \in \Sigma_0 . \tag{2.4}$$

<u>Remark 1</u>. In general the mappings $\tilde{f}$ and $\tilde{\eta}$ will not be uniquely determined by f; indeed, if $L(\Sigma_0)$ is the Lie algebra of Σ_0, and if $\zeta : \Omega \to L(\Sigma_0)$ is any (sufficiently smooth) Σ_0-equivariant mapping, then (2.3) and (2.4) remain valid if we replace $\tilde{f}$ and $\tilde{\eta}$ by the mappings $\tilde{f}_1 : \Omega \to Y$ and $\tilde{\eta}_1 : \Omega \to L(\Gamma)$ defined by

$$\tilde{f}_1(y) = \tilde{f}(y) - \tilde{\zeta}(y).(x_0+y)$$

$$\tilde{\eta}_1(y) = \tilde{\eta}(y) + \tilde{\zeta}(y) .$$

Therefore, if $L(\Sigma_0)$ is nontrivial (i.e. if $\dim \Sigma_0 > 0$) we may have non-uniqueness. Of course, $\tilde{f}(0)$ is uniquely determined by $f(x_0)$.

Remark 2. When $f : U\times\Lambda \to \mathbb{R}^n$ is a Γ-equivariant vectorfield depending on a parameter λ in a parameter space Λ, then (2.3) and (2.4) hold for each $\lambda \in \Lambda$, with $\tilde{f} : \Omega\times\Lambda \to Y$ and $\tilde{\eta} : \Omega\times\Lambda \to L(\Gamma)$ as smooth as f.

The result of theorem 1 can be formulated in a different way. Because of (2.4) the formula (2.3) says that any Γ-equivariant vectorfield f can be decomposed into two Γ-equivariant vectorfields $f_T : U \to \mathbb{R}^n$ and $f_N : U \to \mathbb{R}^n$, given by

$$f_T(\gamma(x_0+y)) := \gamma\tilde{\eta}(y)(x_0+y) \quad , \quad f_N(\gamma(x_0+y)) := \gamma\tilde{f}(y) . \tag{2.5}$$

For each $x = \gamma(x_0+y) \in U$ we have $f_T(x) = \bar{\eta}(x)x$, with $\bar{\eta}(x) := \gamma\tilde{\eta}(y)\gamma^{-1} \in L(\Gamma)$; this means that $f_T(x)$ is at each point $x \in U$ tangent to the group orbit, and hence the flow of f_T is simply a "drift" along group orbits. The flow of f_N leaves the normal section $S := \{x_0+y \,|\, y \in \Omega\}$ to the group orbit Γx_0 invariant; the flow on S is described by the reduced Σ_0-equivariant equation

$$\dot{y} = \tilde{f}(y) . \tag{2.6}$$

Using (2.3) it is then easily verified (see also Krupa (1988)) that the flow of f on U is given by

$$\tilde{x}(t;\gamma(x_0+y)) = \gamma\tilde{\gamma}(t;y)(x_0+\tilde{y}(t;y)) , \tag{2.7}$$

where $\tilde{y}(t;y)$ denotes the flow of (2.6), and $\tilde{\gamma} : \mathbb{R}\times\Omega \to \Gamma$ is the solution of the initial value problem

$$\dot{\gamma} = \gamma\tilde{\eta}(\tilde{y}(t;y)) \quad , \quad \gamma(0) = Id . \tag{2.8}$$

It follows that the flow of f on U may be understood as the flow of the Σ_0-equivariant vectorfield $\tilde{f}$ on S modulated by "drift along the (group) orbit". Let us now return to the bifurcation problem. Let $f : \mathbb{R}^n \times\mathbb{R}^k \to \mathbb{R}^n$ be Γ-equivariant, and suppose that for some parameter-value λ_0 the system (1.1) has an equilibrium solution $\tilde{x}_0(t) \equiv x_0$, or, more generally, a solution of "rotating wave" type :

$$\tilde{x}_0(t) = e^{\eta_0 t}x_0 \quad , \quad \forall t \in \mathbb{R} , \tag{2.9}$$

with $\eta_0 \in L(\Gamma)$ and $x_0 \in \mathbb{R}^n$. For $\lambda = \lambda_0$ the group orbit Γx_0 is invariant under the flow, and we want to discuss bifurcation near this invariant manifold. When we decompose f near Γx_0 then we have

$$\tilde{f}(0,\lambda_0) = 0 \quad \text{and} \quad \tilde{\eta}(0,\lambda_0) = \eta_0 . \tag{2.10}$$

i.e. for $\lambda = \lambda_0$ the vectorfield $\tilde{f}$ on Ω has an equilibrium y = 0. Now suppose that at $\lambda = \lambda_0$ an invariant manifold M bifurcates from Γx_0; by the equivariance we may assume that M is Γ-invariant. By (2.7) there exists an $\tilde{f}$-invariant manifold $M_{x_0} \subset S$ such that $M \cap S \subset M_{x_0}$ and $\Sigma_0(M_{x_0}) = M_{x_0}$. Conversely, each bifurcation from y = 0 for the reduced vectorfield $\tilde{f}$ will generate, via the group action, an invariant manifold bifurcating from Γx_0. So we have reduced the problem to that of the bifurcations from y = 0 for the vectorfield $\tilde{f}(y,\lambda)$. Theorem 1 may be extended by replacing the group orbit Γx_0 by a Γ-invariant manifold of the form ΓM, where M itself is a compact manifold. The torus of standing waves obtained by Hopf bifurcation in an O(2)-equivariant system is an example of such a ΓM. In this setting we require that a Γ-equivariant vector field on M decomposes into a Γ-equivariant vector field tangent to the sections γM and a Γ-equivariant vector field tangent to group orbits. This motivates the following theorem :

<u>Theorem 2</u>. Let $M \subset \mathbb{R}^n$ be a smooth and compact submanifold satisfying the following conditions :

(i) $T_x M \cap L(\Gamma)x = \{0\}$ for each $x \in M$;

(ii) all points $x \in M$ have the same isotropy subgroup Σ_0 ;

(iii) the sets $\Sigma_{x,M} := \{\gamma \in \Gamma \mid \gamma x \in M\}$ are independent of $x \in M$, and therefore form a closed subgroup Σ of Γ.

Then $\Gamma(M) := \{\gamma x \mid \gamma \in \Gamma,\ x \in M\}$ is a compact, Γ-invariant submanifold of $\mathbb{R}^n$. Moreover, if $\pi : N \to \Gamma(M)$ is the normal bundle of $\Gamma(M)$, and $Y := \pi^{-1}(M)$, then there exists a Σ-invariant open neighborhood Ω of M in Y such that the following holds :

(a) $U := \{\gamma y \mid \gamma \in \Gamma,\ y \in \Omega\}$ is an open Γ-invariant neighborhood of $\Gamma(M)$ in $\mathbb{R}^n$;

(b) if $\gamma_1 y_1 = \gamma_2 y_2$ with $\gamma_i \in \Gamma$ and $y_i \in \Omega$ (i=1,2), then $\gamma_1^{-1}\gamma_2 \in \Sigma$;

(c) each Γ-equivariant vectorfield $f : U \to \mathbb{R}^n$ can be written in the form

$$f(\gamma y) = \gamma[\tilde{f}(y) + \tilde{\eta}(y)y] \ , \tag{2.11}$$

with $\tilde{f} : \Omega \to T\Omega$ a Σ-equivariant vectorfield over Ω, and $\tilde{\eta} : \Omega \to L(\Gamma)$ a Σ-equivariant mapping, both with the same smoothness as f.

Hypotheses (i)-(iii) imply that, for $x \in M$, T_xM is complementary to $L(\Gamma)x$ in $T_x\Gamma M$ and the projection of f(x) to T_xM with kernel $L(\Gamma)x \oplus N_x(\Gamma M)$ defines a Σ-equivariant vector field. In particular $M = \{x_0\}$ corresponds to theorem 1.

If $\tilde{x}_0(t)$ is a periodic solution of (1.1) for $\lambda = \lambda_0$, and $\tilde{x}_0(t)$ is not a rotating wave, then one can apply theorem 2 with $M = \{\tilde{x}_0(t) | t \in \mathbb{R}\}$. As a result, the bifurcation problem near $\Gamma(M) = \{\gamma\tilde{x}_0(t) | \gamma \in \Gamma, \ t \in \mathbb{R}\}$ reduces to the bifurcation problem near M for the reduced Σ-equivariant vectorfield; moreover, $\tilde{x}_0(t)$ is, for $\lambda = \lambda_0$, still a periodic solution of the reduced equation

$$\dot{y} = \tilde{f}(y,\lambda) \ . \tag{2.12}$$

What are now the consequences of our theorems for the basic methods of bifurcation theory? First, it is sufficient to find a center manifold $\tilde{W}_c$ for $\tilde{f}$ containing M (=$\{x_0\}$ or =$\{\tilde{x}_0(t) | t \in \mathbb{R}\}$, depending on the case), since $W_c := \{\gamma x | \gamma \in \Gamma, \ x \in \tilde{W}_c\}$ is then a center manifold through $\Gamma(M)$. Also, one should construct a Poincaré mapping for $\tilde{x}_0(t)$ as a periodic solution of (2.12) (and not for the original equation (1.1)); see Chossat and Golubitsky (1987) for an example. By the way, the reduction of (1.1) tot (2.12) is in a sense already a kind of Poincaré mapping, although the result is not a mapping but a vectorfield.
Finally one can apply Liapunov-Schmidt methods to (2.12); since such methods concentrate on steady-state or periodic, this implies (via (2.7) and Floquet theory for (2.8)) that one will obtain solutions of the original equation (1.1) of the form

$$\tilde{x}(t) = e^{\eta t}y(t) \ , \tag{2.13}$$

with $\eta \in L(\Gamma)$ and y(t) a periodic function; if one implements this in the situation of theorem 1 and studies steady-state or Hopf bifurcation for (2.12), then one refinds some of the results of Renardy (1982) on bifurcation from rotating waves. When the starting point is a periodic solution $\tilde{x}_0(t)$ of (2.12), then one may study subharmonic bifurcation for (2.12) using the approach outlined in Vanderbauwhede (1987). For example, a period-doubling for (2.12) will result in an "invariant-manifold-doubling" for (1.1). The advantage of the Liapunov-Schmidt method is that one can work directly with the original equation, without explicitly making the reduction to (2.12). Indeed, the function x(t) as given by (2.12) will be a solution of (1.1) if and only if

$y(t)$ is a solution of the equation

$$\dot{y} = F(y,\eta,\lambda) = f(y,\lambda) - \eta y \ . \qquad (2.14)$$

Since one looks for periodic $y(t)$, one can apply the Liapunov-Schmidt method to (2.14), in which $\eta \in L(\Gamma)$ appears as a supplementary unknown. The problem is still Γ-equivariant, since

$$F(\gamma y,\ \gamma\eta\gamma^{-1},\lambda) = \gamma F(y,\eta,\lambda) \ , \qquad \forall\gamma \in \Gamma \ . \qquad (2.15)$$

We hope to report elsewhere on the details of this approach.

3. PROOF OF THEOREM 1

The proof of theorem 1 is based on the following lemma.

Lemma. Under the conditions of theorem 1, let K be a Σ_0-invariant complement of $L(\Sigma_0)$ in $L(\Gamma)$. Then there exists a Σ_0-invariant neighborhood Ω of the origin in Y and a unique smooth mapping $\eta^* : \Omega\times\mathbb{R}^n \to K$ such that

$$z - \eta^*(y,z)(x_0+y) \in Y \quad , \quad \forall(y,z) \in \Omega\times\mathbb{R}^n \ . \qquad (3.1)$$

Moreover, $\eta^*(y,z)$ is linear in its second argument, and also Σ_0-equivariant :

$$\eta^*(\sigma y,\sigma z) = \sigma\eta^*(y,z)\sigma^{-1} \quad , \quad \forall\sigma \in \Sigma_0 \ . \qquad (3.2)$$

P r o o f. Let P be the orthogonal projection in $\mathbb{R}^n$ onto $L(\Gamma)x_0 = Kx_0$; then $Y = \ker P$. Define $\Phi : Y\times\mathbb{R}^n\times K \to Kx_0$ by

$$\Phi(y,z,\eta) := P(z-\eta(x_0+y)) \ . \qquad (3.3)$$

This mapping is smooth, linear in $(z,\eta) \in \mathbb{R}^n\times K$, with $\Phi(0,0,0) = 0$ and $D_\eta\Phi(0,0,0).\zeta = -\zeta x_0$, such that $D_\eta\Phi(0,0,0)$ is an isomorphism between K and Kx_0. The result then follows from the implicit function theorem and the fact that $\Phi(\sigma y,\sigma z,\sigma\eta\sigma^{-1}) = \sigma\Phi(y,z,\eta)$ for $\sigma \in \Sigma_0$.

Using this lemma the proof of theorem 1 is almost immediate : one simply takes

$$\tilde{\eta}(y) := \eta^*(y, f(x_0+y)) \tag{3.4}$$

and

$$\tilde{f}(y) := f(x_0+y) - \tilde{\eta}(y)(x_0+y) . \tag{3.5}$$

The proof of theorem 2 uses essentially the same idea but requires some more technicalities.

REFERENCES

Bredon E. (1972) Introduction to Compact Transformation Groups. Academic Press, New York.

Chossat P. (1986) Bifurcation secondaire de solutions quasi-périodiques dans un problème de bifurcation de Hopf invariant par symétrie O(2). C.R. Acad. Sci. Paris 302, 539-541.

Chossat P. and Golubitsky M. (1987) Iterates of maps with symmetry SIAM J. Math. Anal. To appear.

Dangelmayr G. (1986) Steady-state mode interactions in the presence of O(2)-symmetry. Dynamics and Stability of Systems 1, 159-185.

Golubitsky M., Stewart I. and Schaeffer D. (1988) Singularities and Groups in Bifurcation Theory, Vol. II. Appl. Math. Sci. 69, Springer, New York. To appear.

Guckenheimer J. and Holmes P. (1983) Nonlinear oscillations, dynamical systems and bifurcations of vector fields. Appl. Math. Sci. 42, Springer, New York.

Iooss G. (1986) Secondary bifurcations of the Taylor vortices into wavy inflow and outflow boundaries. J. Fluid. Mech. 173, 273-288.

Krupa M. (1988) Bifurcation from critical group orbits. Thesis, University of Houston.

Renardy M. (1982) Bifurcation from rotating waves. Arch. Rat. Mech. Anal. 75, 49-84.

Vanderbauwhede A. (1982) Local bifurcation and symmetry. Res. Not. Math. 75, Pitman, London.

Vanderbauwhede A. (1987) Secondary bifurcations of periodic solutions in autonomous systems. In : C.M.S. Conference Proc. 8, 693-701.