

**SINGULARITY THEORY OF STRATEGY FUNCTIONS
UNDER DIMORPHISM EQUIVALENCE**

DISSERTATION

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ABSTRACT

We study dimorphisms by applying adaptive dynamics theory and singularity theory based on a new type of equivalence relation called dimorphism equivalence. Dimorphism equivalence preserves ESS singularities, CvSS singularities, and dimorphisms for strategy functions. Specifically, we classify and compute normal forms and universal unfoldings for strategy functions with low codimension singularities up to dimorphism equivalence. These calculations lead to the classification of local mutual invasibility plots that can be seen in systems of two parameters. This problem is complicated because the allowable coordinate changes are restricted to those that preserve dimorphisms and the singular nature of strategy functions; hence the singularity theory applied in this thesis is not a standard one.

Dedicated to
Martin Golubitsky ,
and
Chunxue Cao,
for their encouragement

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CHAPTER 1

INTRODUCTION

The application of game theory to biology has a long history. *Evolutionary Game Theory* studies the evolution of phenotypic traits and was originated by Maynard-Smith and Price [17] in 1973. Since then, there has been an explosion of interest in evolutionary game theory by mathematicians and other scientists. *Adaptive Dynamics* is a set of techniques and methods that studies the long-term consequences of phenotypes by small mutations in the genotypes. In the past twenty years, adaptive dynamics has been studied by many people, including Dieckmann and Law [5], Geritz *et al.* [10], Diekmann [7], Dieckmann and Metz *et al.* [4], Derole and Rinaldi [3], Polechova and Barton [18], and Waxman and Gavrillets [19]. Golubitsky and Vutha [14] applied singularity theory and adaptive dynamics theory to study the ESS and CvSS singularities of strategy functions. This thesis expands their research to further study the dimorphisms of strategy functions.

1.1 Background of Adaptive Dynamics Theory

In this section, we briefly introduce the background of evolutionary game theory and adaptive dynamics theory.

Evolutionary Game Theory

In evolutionary theory, changes in the environment are often reflected by the changes in the residents' ability to reproduce. Organisms that can adapt better normally have higher reproductive rates. In biology, the individual's ability to adapt (or reproduce) is called *fitness*. Mathematical models define fitness in terms of reasonable biological assumptions that are encoded in strategy functions. The simplest game in evolution is a two-player single trait game. In this case, a *strategy function* is a real-valued function $f(x, y)$ where x and y are the *strategies* or (*phenotypes*) of the players (or organisms). A strategy function $f(x, y)$ represents the fitness advantage of a mutant with phenotype y when competing against a resident with phenotype x . In game theory, $f(x, y) > 0$ means that the mutant has a fitness advantage over the resident. Since any strategy has 0 advantage against itself, we define

Definition 1.1. The smooth function $f: \mathbf{R} \times \mathbf{R} \rightarrow \mathbf{R}$ is a *strategy function* if $f(x, x) = 0$ for all x .

Remark 1.2. Since a strategy function f vanishes along the diagonal (x, x) , we can see by direct calculation that certain derivatives of f also vanish along the diagonal. For example,

$$\begin{aligned}f_x + f_y &= 0 \\f_{xx} + 2f_{xy} + f_{yy} &= 0 \\&\dots\end{aligned}$$

at (x, x) for all x .

Adaptive Dynamics Theory

Adaptive dynamics applies a game theoretic approach to study the evolution of heritable phenotypes (or strategies), such as beak lengths of birds. There are two fundamental ideas when applying adaptive dynamics:

- (i) The resident population is assumed to be in a dynamical equilibrium when new mutants appear.
- (ii) The eventual fate of the mutants can be inferred from the mutant's initial growth rate.

The evolution of strategies is modeled using strategy functions. The idea is that an environment contains players (or organisms) playing all possible strategy values (phenotypes), and that a given strategy (phenotype) evolves based on the interactions with nearby strategies or mutations. In adaptive dynamics theory, strategies evolve through a series of advantageous interactions against mutant strategies.

In fact, adaptive dynamics assumes that the resident strategy x increases when a mutant strategy $y > x$ has an advantage over x , and decreases when a mutant strategy $y < x$ has an advantage over x . Note that we can apply Taylor's Theorem at (x, x) to obtain:

$$f(x, y) = f(x, x) + (y - x)f_y(x, x) + \mathcal{O}(y - x)^2 = (y - x)f_y(x, x) + \mathcal{O}(y - x)^2$$

In this expression, it is obvious that when $f_y(x, x) > 0$, for any y near x , we have

$$f(x, y) > 0 \quad \text{if and only if} \quad y > x$$

Therefore, we can see a proportional relationship between the rate of change of strategies in evolution and the mutation gradient $f_y(x, x)$ of the fitness function f . Dieckmann and Law [5] have applied this approach and obtained the *canonical equation of adaptive dynamics*:

$$\frac{dx}{dt} = \alpha(x) f_y(x, x) \quad (1.1.1)$$

where $\alpha(x) > 0$ depends on the resident strategy x .

From the canonical equation of adaptive dynamics, we see that (1.1.1) has an equilibrium at x_0 if and only if $f_y(x_0, x_0) = 0$. Therefore, we define

Definition 1.3. A strategy x_0 is a *singular strategy* if $f_y(x_0, x_0) = 0$.

An Example of a Fitness Function

In the simplest case, we assume that the resident population initially consists of asexual organisms that all possess the same phenotype x (also known as a *monomorphic population*). We also assume that y is the phenotype of the rare mutations and that the fitness function is $f(x, y)$.

Mutations that differ from the resident are randomly and recurrently generated, and these can be thought of attempting to *invade* the resident. In this context, f can be defined in terms of a population growth rate k (see (1.1.5)).

We now introduce k for monomorphic populations, show how k leads to growth rates for a system of two populations with different phenotypes (see (1.1.4)), and show that the stability of the populations are given in terms of k (see Theorem 1.4).

To define k , suppose that w is the size of a monomorphic population with phenotype x . Assume that the growth rate of the population is determined by the population and the trait value. That is:

$$\dot{w} = k(w, x)w \quad (1.1.2)$$

where $k(w, x)$ is smooth, and \cdot is the derivative of w with respect to time.

Since the population is monomorphic, we can assume that there is a zero of $k(w, x)$ for each x (defined locally near a given x_0) denoted by $(p(x), x)$ where $p(x) > 0$ is smooth. That is,

$$k(p(x), x) \equiv 0 \tag{1.1.3}$$

This equilibrium is assumed to be stable for (1.1.2) near the given x_0 , that is,

$$k_w(p(x_0), x_0) < 0$$

When there is more than one phenotype in the system, the growth rate for each phenotype is assumed to be determined by the trait value of this phenotype and the total population of all phenotypes. In particular, when there are two phenotypes, suppose that the resident x has a population u and the mutant y has a population v , then the population equations have the form

$$\begin{aligned} \dot{u} &= uk(u + v, x) \\ \dot{v} &= vk(v + u, y) \end{aligned} \tag{1.1.4}$$

Note that $(p(x), 0, x, y)$ is an equilibrium of (1.1.4) where the mutant population is zero and the resident population is $p(x)$. In this case, the growth rate of the mutant is $k(p(x), y)$ and the fitness function of rare mutations with phenotype y in a resident population with phenotype x is

$$f(x, y) = k(p(x), y) \tag{1.1.5}$$

where x is near x_0 . Equation (1.1.3) implies that

$$f(x, x) = k(p(x), x) = 0$$

Theorem 1.4. *The equilibrium $(p(x), 0, x, y)$ of (1.1.4) is unstable if $f(x, y) > 0$ and stable if $f(x, y) < 0$.*

Proof. The Jacobian $J(u, v, x, y)$ for the system (1.1.4) is

$$\begin{pmatrix} k(u + v, x) + uk_w(u + v, x) & uk_w(u + v, x) \\ vk_w(v + u, y) & k(v + u, y) + vk_w(v + u, y) \end{pmatrix} \quad (1.1.6)$$

At the equilibrium point $(p(x), 0, x, y)$, the Jacobian becomes

$$J(u, v, x, y)|_{(p(x), 0, x, y)} = \begin{pmatrix} p(x)k_w(p(x), x) & p(x)k_w(p(x), x) \\ 0 & k(p(x), y) \end{pmatrix} \quad (1.1.7)$$

We know from the assumptions of $k(w, x)$ that

$$p(x) > 0 \quad k_w(p(x), x) < 0$$

Therefore, we have

$$p(x)k_w(p(x), x) < 0$$

Thus, the stability of the equilibrium $(p(x), 0, x, y)$ is determined by the sign of $k(p(x), y)$. The proof is complete by recalling (1.1.5). \square

Remark 1.5. Theorem 1.4 shows that the stability of the equilibrium is determined by the signs of the fitness function $f(x, y)$. This is consistent with the definition of fitness function; that is, $f(x, y) < 0$ indicates that the mutant has no advantage against the resident. This is the same as saying that the resident population stays in a stable equilibrium. Moreover, the population with phenotype y will gradually die out, and the system will remain monomorphic.

On the contrary, if $f(x, y) > 0$ then the mutant has an advantage over the resident. Moreover, the resident population is unstable and will be invaded by the mutant. Thus, the population with phenotype y will increase.

1.2 Important Concepts in Adaptive Dynamics

Next we introduce three important concepts from adaptive dynamics.

ESS

An *evolutionarily stable strategy* (ESS) is a resident phenotype (i.e. strategy \bar{x}) such that no mutant with phenotype y near \bar{x} can invade the resident. That is, $f(\bar{x}, y) \leq 0$ for all y near \bar{x} . Considering that $f(x, x) = 0$ for all x , it is necessary to have that $f_y(\bar{x}, \bar{x}) = 0$ and $f_{yy}(\bar{x}, \bar{x}) \leq 0$. Specifically:

Definition 1.6. \bar{x} is an *ESS* if

$$f_x(\bar{x}, \bar{x}) = 0 \quad f_{yy}(\bar{x}, \bar{x}) < 0$$

Remark 1.7. Remark 1.2 implies that $f_x(\bar{x}, \bar{x}) = 0$ if and only if $f_y(\bar{x}, \bar{x}) = 0$.

CvSS

A *convergence stable strategy* (CvSS) is a phenotype (i.e. strategy \bar{x}) such that it is a linearly stable equilibrium in x for the canonical equation of adaptive dynamics (1.1.1). If a phenotype \bar{x} is a CvSS singularity, it is actually a local attractor in adaptive dynamics. This means, any nearby resident strategy will follow the canonical equation of adaptive dynamics and evolve towards to direction of \bar{x} . Recall that \bar{x} is a linearly stable equilibrium of (1.1.1) if

$$f_y(\bar{x}, \bar{x}) = 0 \quad \frac{d}{dx}(\alpha(x)f_y(x, x))|_{x=\bar{x}} < 0$$

By Remark 1.2, we have

$$\begin{aligned} \frac{d}{dx}(\alpha(x)f_y(x, x))|_{x=\bar{x}} &= \alpha'(\bar{x})f_y(\bar{x}, \bar{x}) + \alpha(\bar{x})(f_{xy}(\bar{x}, \bar{x}) + f_{yy}(\bar{x}, \bar{x})) \\ &= \frac{1}{2}\alpha(\bar{x})(f_{yy}(\bar{x}, \bar{x}) - f_{xx}(\bar{x}, \bar{x})) \end{aligned}$$

Since $\alpha(x) > 0$ for all x , we have

Definition 1.8. \bar{x} is a *CvSS* if

$$f_x(\bar{x}, \bar{x}) = 0 \quad f_{yy}(\bar{x}, \bar{x}) - f_{xx}(\bar{x}, \bar{x}) < 0$$

Dimorphism and Region of Coexistence

Adaptive dynamics theory attempts to predict phenotypic evolutionary change. In the long run of evolution, either mutations die out or they do not die out. If the mutant with phenotype y has no advantage over the resident with phenotype x (i.e. $f(x, y) < 0$), the mutant will die out. On the other hand, if the mutant with phenotype y has an advantage over the resident with phenotype x (i.e. $f(x, y) > 0$), then the mutant's population will grow and they will not die out. However, in some situations, the mutant cannot outcompete the resident once the resident is rare and we get coexistence of two subpopulations with different phenotypes. That is, the system becomes a dimorphic population. This happens when $f(x, y) > 0$ and $f(y, x) > 0$ and such a pair of phenotypes is called a *dimorphism*. So we have

Definition 1.9. Let $f(x, y)$ be a strategy function. A pair of strategies x, y that satisfies

$$f(x, y) > 0 \quad f(y, x) > 0$$

is called a *dimorphism* of f . The set of all dimorphisms of f is called the *region of coexistence*.

Remark 1.10. There is a sufficient condition for the local existence of dimorphisms: If $f_x(\bar{x}, \bar{x}) = 0$ and $f_{yy}(\bar{x}, \bar{x}) + f_{xx}(\bar{x}, \bar{x}) > 0$, then there exists pairs of strategies (x, y) near (\bar{x}, \bar{x}) such that both $f(x, y) > 0$ and $f(y, x) > 0$.

Proof. Let $g(x) = f(x, 2\bar{x} - x)$, we claim that $g(x) > 0$ when x is close to \bar{x} . We know

$$\begin{aligned} g'(x) &= f_x(x, 2\bar{x} - x) - f_y(x, 2\bar{x} - x) \\ g''(x) &= f_{xx}(x, 2\bar{x} - x) - 2f_{xy}(x, 2\bar{x} - x) + f_{yy}(x, 2\bar{x} - x) \\ &\dots \end{aligned}$$

where $'$ is derivative with respect to x . Then, we have

$$g(x) = g(\bar{x}) + g'(\bar{x})(x - \bar{x}) + \frac{1}{2}g''(\bar{x})(x - \bar{x})^2 + h.o.t,$$

From Remark 1.2, we know

$$g(\bar{x}) = f(\bar{x}, \bar{x}) = 0$$

$$g'(\bar{x}) = f_x(\bar{x}, \bar{x}) - f_y(\bar{x}, \bar{x}) = 0$$

$$g''(\bar{x}) = 2(f_{xx}(\bar{x}, \bar{x}) + f_{yy}(\bar{x}, \bar{x})) > 0$$

Therefore, we can conclude that when x is close to \bar{x}

$$g(x) = \frac{1}{2}g''(\bar{x})(x - \bar{x})^2 + h.o.t > 0$$

Note that $(x, 2\bar{x} - x)$ and $(2\bar{x} - x, x)$ are symmetrical about (\bar{x}, \bar{x}) . With similar calculations we also have

$$g(2\bar{x} - x) = g''(\bar{x})(\bar{x} - x)^2 + h.o.t > 0$$

That is to say,

$$f(x, 2\bar{x} - x) > 0 \quad f(2\bar{x} - x, x) > 0$$

Hence, $(x, 2\bar{x} - x)$ are pairs of dimorphisms around the singular point (\bar{x}, \bar{x}) when x is close to \bar{x} . □

Remark 1.11. In some of the literature of adaptive dynamics, dimorphism is also referred to as *mutual invasibility*. Diekmann [7] discussed the consequence of mutual invasibility. It is found that

- If there is a dimorphism (x, y) near a singular strategy \bar{x} that is both an ESS and a CvSS, then both strategies x and y will evolve towards \bar{x} , and this dimorphism (x, y) is called a *converging dimorphism*.

- If there is a dimorphism (x, y) near a singular strategy \bar{x} that is a CvSS, but not an ESS, then we can observe an interesting phenomenon called *evolutionary branching*. When this happens, (x, y) is called a *diverging dimorphism*. Some detailed discussion about evolutionary branching can be found in [15, 9, 16].

In this thesis we will keep track of dimorphism pairs, but not explicitly of converging and diverging dimorphisms.

1.3 Background of Singularity Theory

In this section, we discuss the general ideas of singularity theory and how it helps in studying ESS singularities, CvSS singularities and dimorphisms in the context of adaptive dynamics. In addition, we introduce mutual invasibility plots which show how ESS, CvSS, and dimorphisms can interact. At the end, we introduce the classification of singularities by topological codimension.

Standard Singularity Theory and its Applications

Singularity theory studies how a certain local property of a class of functions changes as parameters are varied. The important first step when trying to apply singularity theory is to determine transformations of functions that preserve the property that one is trying to study. We will use a series of examples to explain this.

1. In the simplest case, the property to be studied is the zero sets of functions $f : \mathbf{R}^n \rightarrow \mathbf{R}^n$ near a given zero. We define:

Definition 1.12. Two functions $f, g : \mathbf{R}^n \rightarrow \mathbf{R}^n$ are *contact equivalent* if there exists a function $S : \mathbf{R}^n \rightarrow \mathbf{R}$ and a coordinate change $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ such that

$$g(x) = S(x)f(\Phi(x))$$

where $S(x) > 0$ and $\Phi : \mathbf{R}^n \rightarrow \mathbf{R}^n$ is a near identity diffeomorphism.

It is easy to see that the contact equivalence (S, Φ) transforms the zero set of g to the zero set of f .

2. Golubitsky and Vutha [14] study the ESS and CvSS singularities of strategy functions. They develop an equivalence relation called strategy equivalence (see Definition 3.1) to preserve these singularities among strategy functions.
3. This thesis studies the transformations that preserve dimorphisms as well as ESS and CvSS singularities of strategy functions. We construct a modified version of strategy equivalence called dimorphism equivalence (See Definition 3.3) such that it preserves these three properties of strategy functions.

In singularity theory, a *singularity* is a transition point where the property one studies changes. For example, in the study of the zero set of a function f , a singularity is a point x such that $f(x) = 0$ and $(Df)_x$ is singular. That is, the transition point when the number of zeroes of f changes. In the case of studying dimorphism, ESS, and CvSS simultaneously, a singularity will be a transition point where at least one of these three properties changes.

Once one has established the most general equivalence relation that preserves these properties, it is natural to ask when is a function f equivalent to a specific function h on a neighborhood of a singularity? This question is called the *recognition problem*. The specific function h is called the *normal form* and is usually the ‘simplest representative’ from the whole equivalence class of h . A major result of singularity theory is that the recognition problem can be solved by examining a finite number of derivatives of f at the singularity point. For example:

Theorem 1.13. *Assume that a strategy function $f(x, y)$ has a singularity at $(0, 0)$. Then f is dimorphism equivalent to*

$$h = (x - y)^4 + (x + y)(x - y)$$

if and only if at $(0, 0)$

$$f_x = 0 \quad f_{xy} = 0$$

$$f_{xx} > 0 \quad f_{xx}(f_x^4 + 6f_{x^2y^2} + f_y^4) - 4(f_{x^2y} + f_{xy^2})(3f_{xy^2} + f_y^3) > 0$$

In the theorem for a recognition problem, the equalities are called *defining conditions*, the inequalities are called *non-degeneracy conditions*.

Singularity theory studies the key properties of small perturbations of a given function f by applying universal unfolding theory. An *unfolding* of a function f is a parametrized family $F(x, \alpha)$ such that $F(x, 0) = f$ where $\alpha \in \mathbf{R}^k$ is a parameter. A *versal unfolding* of f contains all small perturbations of f up to the general equivalence. A *universal unfolding* of f is a versal unfolding with the smallest k . Once a singularity is identified in a normal form h , we can find all possible small perturbations of h in its universal unfolding $H(x, \alpha)$ up to equivalence. For example:

Theorem 1.14. *A universal unfolding of $h = (x - y)^4 + (x + y)(x - y)$ is*

$$H = ((x - y)^2 + a)(x - y)^2 + (x + y)(x - y)$$

where a is a parameter near 0.

The techniques needed to prove Theorem 1.14 follow from those needed to solve the recognition problem in Theorem 1.13. With universal unfolding theory, we are able to classify all small perturbations of a strategy function f up to dimorphism equivalence.

Geometry and Mutual Invasibility Plots

Another benefit of universal unfolding theory is that it helps study the geometry of small perturbations around singularities. For a given strategy function we use mutual invasibility plots (MIPs) to illustrate ESS and CvSS singularities, and regions of coexistence. For example, Figure 1.1 shows the two possible perturbations of the strategy function $h = (x-y)^4 + (x+y)(x-y)$ up to diffeomorphism equivalence by plotting the data for the universal unfolding $H = ((x-y)^2 + a)(x-y)^2 + (x+y)(x-y)$ for $a > 0$ and $a < 0$. In these plots, the blue curve is $H(x, y, a) = 0$, the red curve is $H(y, x, a) = 0$, and the black curve is the line $y = x$. In each region, we can see a pair of signs and they stand for $\text{sgn}(H(x, y, a))$ and $\text{sgn}(H(y, x, a))$. The centered red dot stands for a singularity that is both ESS and CvSS. Plots like these are called *mutual invasibility plots* (MIPs).

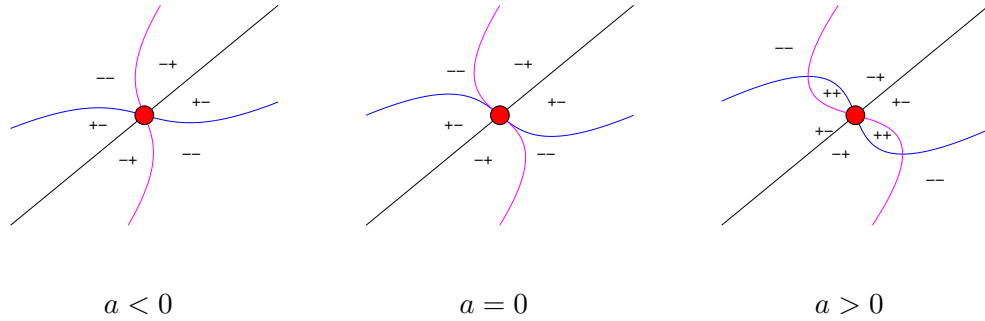


Figure 1.1: MIPs of the universal unfolding function $H = ((x-y)^2 + a)(x-y)^2 + (x+y)(x-y)$. In this example the transition from $a < 0$ to $a > 0$ causes the emergence of two regions of coexistence (in the right plot). Note that the evolutionary and convergence stability of the singularity $(0,0)$ do not change when the parameter a is varied.

In Figure 1.1, we see that regions of coexistence can emerge from the perturbation of a singular strategy function. The MIP for the singular strategy function h is the middle plot of Figure 1.1. There is no region of coexistence in the middle plot. However, if we perturb the parameter to $a > 0$ (as in the right plot), the strategy function has two regions of coexistence; whereas, when $a < 0$ (as in the left plot), the strategy function has no region of coexistence. We can think of this singular strategy as creating regions of coexistence.

Remark 1.15. In the adaptive dynamics literature, there is another useful plot called a *pairwise invasibility plot* (PIP). For any strategy function f , a PIP contains the curve $f(x, y) = 0$ and $\text{sgn}(f)$ in each region bounded by $f(x, y) = 0$.

Classification of Singularities

In this thesis we solve the recognition problems, find the universal unfoldings, and plot the MIPs (for all perturbations up to dimorphism equivalence) for singularities of topological codimension ≤ 2 . In specific, Theorem 5.1 classifies all such singularities, Theorem 6.9 gives the universal unfoldings for these singularities, and Chapter 8 contains MIPs for all possible perturbations. The results for topological codimension one are much simpler and they are presented in Chapter 2.

Codimension

Next, we explain what codimension is. For a strategy function f with a given singularity, the universal unfolding $F(x, \alpha)$ contains all perturbations of f up to dimorphism equivalence. The C^∞ *codimension* of f is the number of parameters in the universal unfolding F . The parameters in universal unfoldings are often called *unfolding parameters*.

Suppose that the perturbations of f associated with a parameter β , that is $F(\cdot, \beta)$,

have constant C^∞ codimension and the functions are not equivalent for different β . Then we call the parameter β a *modal parameter*. Golubitsky and Schaeffer [12] define the *topological codimension* of a function f as its C^∞ codimension minus the number of modal parameters in a universal unfolding of f . Therefore, as is standard in singularity theory, we can classify singularities of strategy functions up to a given C^∞ or topological codimension. We choose ‘topological codimension’ because this is the number of parameters that are needed in an application for the particular singularity to be able to occur generically.

1.4 Structure of the Thesis

In Chapter 2, we summarize the major results of this thesis and provide an application of our theory. In Chapter 3 we review Golubitsky and Vutha’s [14] strategy equivalence and introduce dimorphism equivalence which preserves ESS and CvSS singularities and dimorphisms. In Chapter 4, given a strategy function f , we present a sufficient condition to determine all small perturbations η so that $f + \eta$ is dimorphism equivalent to f . The result is stated in the modified tangent space constant theorem (Theorem 4.7). In Chapter 5, we apply Theorem 4.7 to solve the recognition problems under dimorphism equivalence for singularities of topological codimension ≤ 2 . For each of these singularities we find the normal form and its defining and non-degeneracy conditions. See Table 5.1 for details. In Chapter 6, we discuss universal unfoldings of strategy functions up to dimorphism equivalence. An important theorem in singularity theory shows the existence of universal unfoldings and provides methods to calculate them. We apply these methods in the context of dimorphism equivalence and find the universal unfoldings for singularities up to topological codimension two. See Table 6.1 for details. In Chapter 7, we solve the problem of when is an unfolding universal for a given strategy function f . This result is useful when

determining whether a universal unfolding of a degenerate singularity is contained in a given application. We illustrate the standard methods with a few examples. In Chapter 8, we study the geometry of the unfolding space for each singularity up to topological codimension two. For these singularities, we determine the MIPs of all possible perturbations up to diffeomorphism equivalence.

CHAPTER 2

MAJOR RESULTS AND APPLICATIONS

In this chapter, we present our major results and explain why these results are important. In addition, we apply our theory to study the famous Hawk-Dove game.

2.1 Major Results

As discussed in Chapter 1, we apply adaptive dynamics theory and singularity theory to study certain local properties (ESS singularities, CvSS singularities, and dimorphisms) of strategy functions. That is, we assume x_0 is a singular strategy and we determine when we have ESS singularities, CvSS singularities, and dimorphisms in a universal unfolding of that singularity. We study all singularities up to topological codimension two. In this section we only list the results for singularities up to topological codimension one. The results for singularities of codimension two are more complicated and are shown in Theorem 5.1 (normal forms) and Theorem 6.9 (universal unfoldings). Note that all results use the (u, v) coordinates

$$u = x + y$$

$$v = x - y$$

We also denote $w = v^2$. The (u, v) coordinates simplify the calculations in the proofs of many of our theorems. We show in Chapter 4 that a general strategy function f

can be rewritten as

$$f = p(u, w)w + q(u, w)v$$

In Table 2.1, we enumerate the important information of the singularities of topological codimension zero and one. For each singularity, we include defining and non-degeneracy conditions, normal forms, and universal unfoldings of the normal form.

	Def	ND	TC	Normal Form h	Universal Unfolding H
(a)	—	p q_u	0	$\epsilon(w + \mu_0 uv)$ $\epsilon = \text{sgn}(p) \quad \mu_0 = \frac{q_u}{p}$	$\epsilon(w + \mu uv)$
(b)	p	q_u $p_w q_u - p_u q_w$	1	$\epsilon(\delta w^2 + uv)$ $\epsilon = \text{sgn}(q_u)$ $\delta = \text{sgn}(p_w q_u - p_u q_w)$	$\epsilon((\delta w + a)w + uv)$
(c)	q_u	p q_{uu}	1	$(\epsilon + \mu_0 u)w + \delta u^2 v$ $\epsilon = \text{sgn}(p) \quad \delta = \text{sgn}(q_{uu})$ $\mu_0 = \delta \frac{6q_{uu} p_u - 2p q_u^3}{3q_{uu}^2}$	$(\epsilon + \mu u)w + (a + \delta u^2)v$

Table 2.1: The normal forms, defining and non-degeneracy conditions, and universal unfoldings of singularities up to topological codimension one for strategy functions f with a singular strategy at $(0, 0)$. Derivatives in the table are evaluated at the singularity $(0, 0)$. Information about singularities of topological codimension two can be found in Table 5.1 and Table 6.1. (Def = defining conditions. ND = non-degeneracy conditions. TC = topological codimension.)

From Table 2.1, we see that singularity theory provides an easy way for us to identify singularities. To determine whether f is diffeomorphism equivalent to the normal

form h , we only need to calculate certain derivatives of the strategy function f evaluated at the singularity point. Once we have identified the singularity, we draw the MIPs of the universal unfolding for all possible parameter values up to dimorphism equivalence.

Figure 2.1 to Figure 2.4 contain all possible mutual invasibility plots up to dimorphism equivalence for the universal unfolding H of singularities of topological codimension zero and one. The type (ESS or CvSS) and the stability of each singular point is indicated by a symbol with a certain color and shape as shown in Table 2.2. The subscripts have the following meaning: $+$ stands for asymptotic stability, $-$ stands for unstable, and 0 stands for a transition between stable and unstable strategies (that is, the singularity).

●	CvSS ₊ ESS ₊
■	CvSS ₊ ESS ₋
●	CvSS ₋ ESS ₊
■	CvSS ₋ ESS ₋
●	CvSS ₀ ESS ₊
■	CvSS ₀ ESS ₋
◆	CvSS ₊ ESS ₀
◆	CvSS ₋ ESS ₀

Table 2.2: Markers for different singularity types on the MIPs.

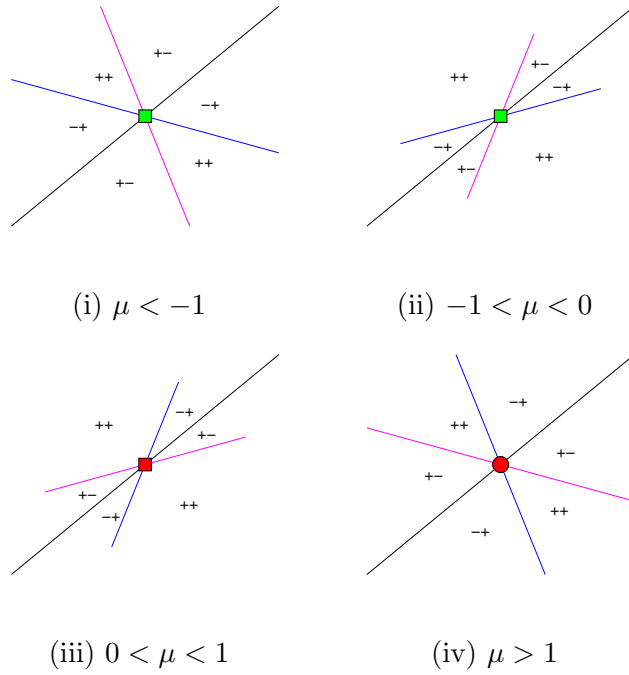


Figure 2.1: The MIPs of strategy function $F = (w + \mu uv)$ for different values of μ .

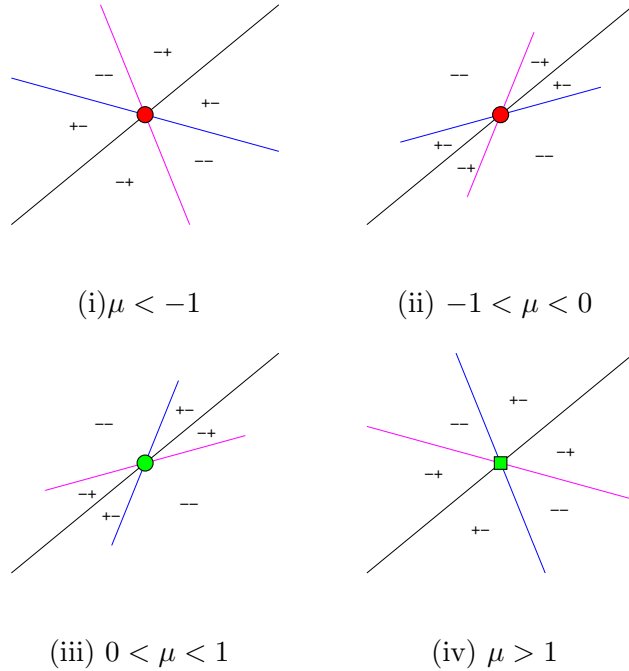
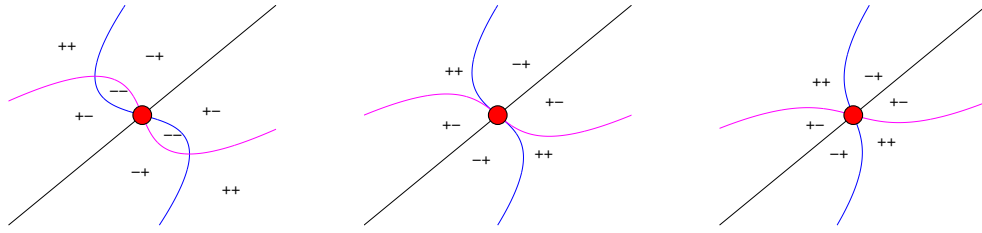
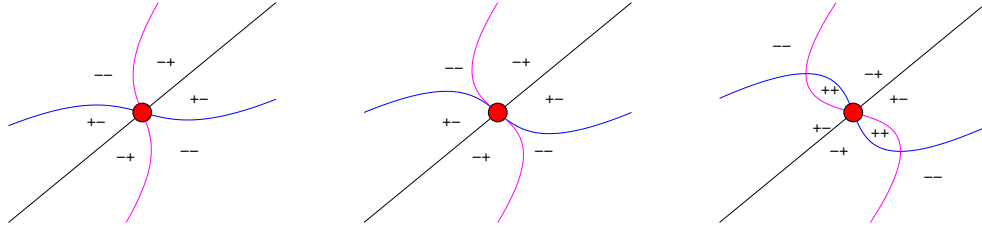


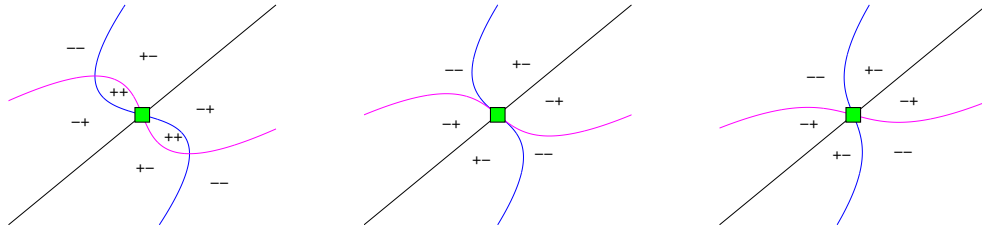
Figure 2.2: The MIPs of strategy function $F = -(w + \mu uv)$ for different values of μ .



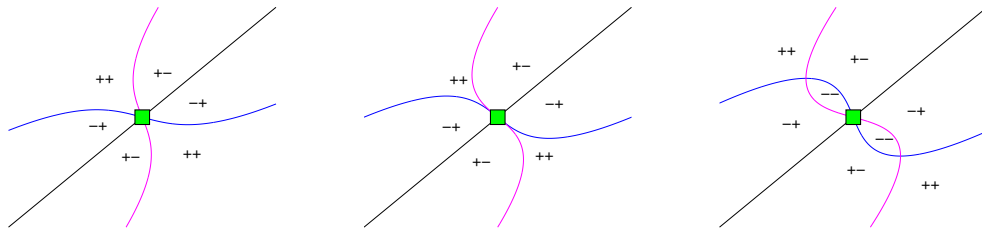
(a) $F = (w + a)w + uv$: (i) $a < 0$, (ii) $a = 0$, (iii) $a > 0$



(b) $F = (-w + a)w + uv$: (i) $a < 0$, (ii) $a = 0$, (iii) $a > 0$

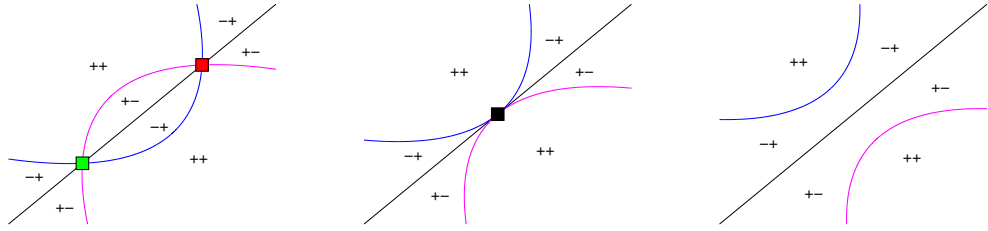


(c) $F = -(w + a)w - uv$: (i) $a < 0$, (ii) $a = 0$, (iii) $a > 0$

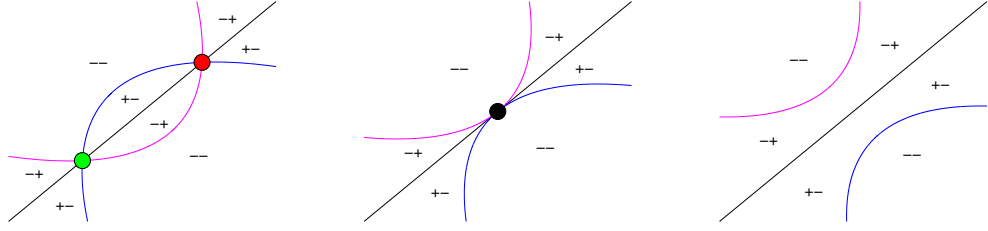


(d) $F = -(-w + a)w - uv$: (i) $a < 0$, (ii) $a = 0$, (iii) $a > 0$

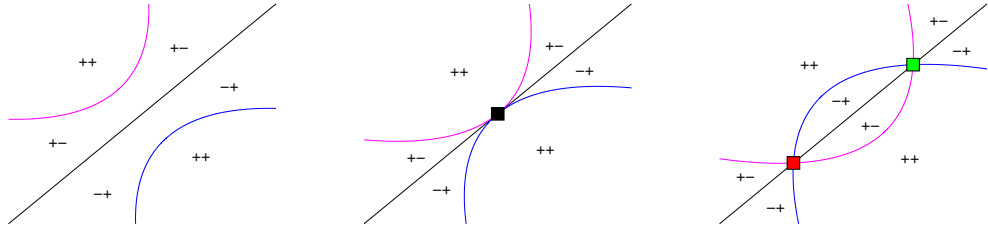
Figure 2.3: MIPs of $F = \epsilon((\delta w + a)w + uv)$ when: (a) $\epsilon = 1, \delta = 1$; (b) $\epsilon = 1, \delta = -1$; (c) $\epsilon = -1, \delta = 1$; (d) $\epsilon = -1, \delta = -1$. In each scenario of (ϵ, δ) , we can see emergence of new regions at one direction of the parameter change.



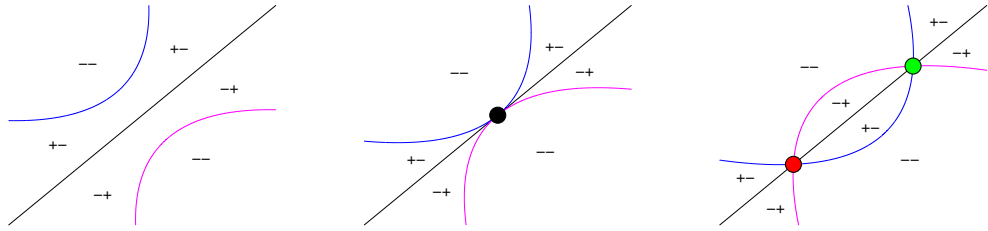
(a) $F = (1 + \mu u)w + (a + u^2)v$ when $\mu = 0$: (L) $a < 0$, (M) $a = 0$, (R) $a > 0$



(b) $F = (-1 + \mu u)w + (a + u^2)v$ when $\mu = 0$: (L) $a < 0$, (M) $a = 0$, (R) $a > 0$



(c) $F = (1 + \mu u)w + (a - u^2)v$ when $\mu = 0$: (L) $a < 0$, (M) $a = 0$, (R) $a > 0$



(d) $F = (-1 + \mu u)w + (a - u^2)v$ when $\mu = 0$: (L) $a < 0$, (M) $a = 0$, (R) $a > 0$

Figure 2.4: MIPs of $F = (\epsilon + \mu u)w + (a + \delta u^2)v$ when: (a) $\epsilon = 1$, $\delta = 1$; (b) $\epsilon = -1$, $\delta = 1$; (c) $\epsilon = 1$, $\delta = -1$; (d) $\epsilon = -1$, $\delta = -1$. In each scenario, we see emergence of additional singularity in one direction of parameter change while no singularities in the other direction.

The MIPs of the universal unfolding of singularity (a) up to dimorphism equivalence are given in Figures 2.1 and 2.2. We see that dimorphisms only exist when $p > 0$. Once $\text{sgn}(p)$ is fixed, the difference in the modal parameter μ can change the stability of ESS or CvSS. Up to dimorphism equivalence, there are eight different types of MIPs for the topological codimension zero singularity.

The MIPs of the universal unfolding of singularity (b) up to dimorphism equivalence are given in Figure 2.3. We see that as we vary the parameter a across 0, there is a creation of new regions. In addition, $\text{sgn}(q_u)$ determines the ESS or CvSS singularity types and corresponding stabilities. If a strategy function has this singularity, with the help of singularity theory and these MIPs, we can predict whether we can find pairs of dimorphisms in the perturbed strategy function and if the answer is yes, where we can obtain pairs of dimorphisms in the parameter space.

Figure 2.4 contains the MIPs of all universal unfolding of singularity (c) up to dimorphism equivalence when the modal parameter $\mu = 0$. The figures show that as we vary the parameter a , the number and stabilities of ESS and CvSS singularities change, but the existence of regions of coexistence does not. Note that $\mu = 0$ is a special case of all strategy functions with singularity (c), we point out that $\mu = 0$ is a standard representative for the all parameter values of μ . We discuss the influence of different values of the modal parameter μ on the topology of the MIPs in Chapter 8.

2.2 Application of the Theory

This thesis provides the theoretical support and general methodology to simultaneously study ESS singularities, CvSS singularities, and dimorphisms of strategy functions. It is usually extremely difficult to study the properties of a general strategy function f and its perturbations. By developing an equivalence relation (i.e.

dimorphism equivalence) that preserves ESS singularities, CvSS singularities, and dimorphisms, we find the conditions for a class of strategy functions to be dimorphism equivalent to singularities up to topological codimension two. This theory hugely simplifies the calculations in identifying the singularities. The universal unfolding theory allows us to study the key properties of all perturbations of a given strategy function. Since a class of strategy functions that are dimorphism equivalent share the same key properties in their corresponding perturbations up to dimorphism equivalence, we only need to study a representative (usually the normal form) from this class. The MIPs in Chapter 8 show all key properties for the universal unfoldings of each singularity up to topological codimension 2. Within each plot, ESS singularities, CvSS singularities, and dimorphisms are given specifically. Therefore, with the help of our theory and the MIPs, we can easily tell whether dimorphisms exist if one is studying an evolutionary problem with a specific fitness function.

In particular, to apply our theory, we look at the famous Hawk-Dove example in evolutionary theory.

The Hawk Dove Game

Dieckmann and Metz [6] considered generalizations of the classical Hawk-Dove game that lead to strategy functions. Golubitsky and Vutha [14] study this game in the context of strategy equivalence and find different types of ESS and CvSS singularities as parameters are varied.

The classical Hawk-Dove game has two players A and B who can play either a hawk strategy or a dove strategy with payoffs given in Table 2.3. Here $V > 0$ is a reward and $C \geq 0$ is a cost.

In fact, [6] considers a game where A plays hawk with probability x and B plays

	Hawk	Dove
Hawk	$\frac{1}{2}(V - C)$	V
Dove	0	$\frac{1}{2}V$

Table 2.3: The hawk-dove game

hawk with probability y . Dieckmann and Metz [6] show that the advantage for B in this game is given by the strategy function

$$f(x, y) = (y - x)(V - Cx) \tag{2.2.1}$$

Dieckmann and Metz [6] also consider variations of (2.2.1) that lead to parametrized families of strategy functions, which are based on various ecological assumptions (See [6] for details). Their most complicated game has the form

$$f(x, y) = \ln\left(\frac{1 + Q(x, y)}{1 + Q(x, x)}\right) \tag{2.2.2}$$

where

$$\begin{aligned} P(x) &= r_0 + r_1(x - x_0) + r_2(x - x_0)^2 \\ A(x, y) &= \frac{1}{2}\sqrt{P(x)P(y)} \\ B(x, y) &= V(1 - x + y) - Cxy \\ Q(x, y) &= A(x, y)B(x, y)/R \end{aligned} \tag{2.2.3}$$

Based on certain biological explanation of the parameters, we also assume that in (2.2.3)

$$R > 0 \quad C > 0 \quad V > 0$$

Example 2.1. By applying our singularity theory results we show that the fitness

function (2.2.2) of the generalized Hawk-Dove example has pairs of dimorphisms. Specifically, assume

$$V = \frac{3}{16}C \quad x_0 = \frac{1}{4} \quad r_0 = 1 \quad r_1 = 1 \quad r_2 = 0 \quad (2.2.4)$$

Then the strategy function $f(x, y)$ has a singularity at $(x, y) = (\frac{1}{4}, \frac{1}{4})$ and is dimorphism equivalent to

$$h(x, y) = (x - y)^2 + \frac{14}{13}(x + y)(x - y) \quad (2.2.5)$$

This normal form is a special case of singularity (a) in Table 2.1 and its MIPs can be found in Figure 2.1 (iv) where we see regions of coexistence.

With direct calculation, Golubitsky and Vutha [14] show that at $(x, y) = (x_0, x_0)$

$$f_x(x_0, x_0) = -\frac{Q_y(x_0, x_0)}{1 + Q(x_0, x_0)}$$

If x_0 is a singular strategy of $f(x, y)$, then

$$\begin{aligned} f_{xy}(x_0, x_0) &= \frac{Q_{xy}(x_0, x_0)}{1 + Q(x_0, x_0)} \\ f_{xx}(x_0, x_0) - f_{yy}(x_0, x_0) &= -2\frac{Q_{xy}(x_0, x_0) + Q_{yy}(x_0, x_0)}{1 + Q(x_0, x_0)} \end{aligned}$$

Based on the definition of $Q(x, y)$ in (2.2.3) and the parameter values (2.2.4), we can compute at $(x, y) = (\frac{1}{4}, \frac{1}{4})$

$$\frac{Q_y}{1 + Q} = 0 \quad \frac{Q_{xy}}{1 + Q} = -\frac{39C}{4C + 64R} \quad \frac{Q_{xy} + Q_{yy}}{1 + Q} = -\frac{21C}{2C + 32R}$$

Thus, $(\frac{1}{4}, \frac{1}{4})$ is a singularity of the Dieckmann-Metz f since $f_x(\frac{1}{4}, \frac{1}{4}) = 0$. To identify the type of this singularity, we use $(u = x + y, v = x - y)$ coordinates. Recall that f can be written as

$$f = p(u, w)w + q(u, w)v$$

Note that in u, v coordinates $(x, y) = (x_0, x_0)$ corresponds to $(u, v) = (2x_0, 0)$ and we have

$$f_x = q \quad f_{xy} = -2p \quad f_{xx} - f_{yy} = 4q_u$$

Based on these calculation, we know that at $(x, y) = (\frac{1}{4}, \frac{1}{4})$

$$\begin{aligned}
 q &= f_x &= -\frac{Q_y}{1+Q} &= 0 \\
 p &= -\frac{1}{2}f_{xy} &= -\frac{1}{2}\frac{Q_{xy}}{1+Q} &= \frac{39C}{8C+128R} > 0 \\
 q_u &= \frac{1}{4}(f_{xx} - f_{yy}) &= -\frac{1}{2}\frac{Q_{xy} + Q_{yy}}{1+Q} &= \frac{42C}{8C+128R} > 0
 \end{aligned}$$

Thus, by looking up Table 2.1 we see that f is dimorphism equivalent to the strategy function $w + \mu uv$ where

$$\mu = \frac{q_u}{p} = \frac{14}{13}$$

Therefore, this specific Hawk-Dove game contains a singularity that is dimorphism equivalent to the strategy function in (2.2.5).

CHAPTER 3

DIMORPHISM EQUIVALENCE

In this chapter, we review the strategy equivalence (see Definition 3.1) developed by Golubitsky and Vutha [14]. Strategy equivalence preserves ESS and CvSS singularities of strategy functions. However, we show in Example 3.2 that strategy equivalence does not always preserve pairs of dimorphisms. We then introduce a special type of strategy equivalence that does preserve pairs of dimorphism (Theorem 3.4) and is called dimorphism equivalence (Definition 3.3). To develop dimorphism equivalence, we combine strategy equivalence with concepts from singularity theory with symmetry.

3.1 Strategy Equivalence

Golubitsky and Vutha [14] study strategy functions by defining a form of equivalence relation that preserves ESS and CvSS singularities. In particular

Definition 3.1. Two strategy functions f and \hat{f} are *strategy equivalent* if

$$\hat{f}(x, y) = S(x, y)f(\Phi(x, y))$$

where

1. $S(x, y) > 0$ for all x, y .
2. $\Phi \equiv (\Phi_1, \Phi_2)$ where $\Phi_i: \mathbf{R}^2 \rightarrow \mathbf{R}$, $\det(d\Phi)_{x,y} > 0$ for all x, y .

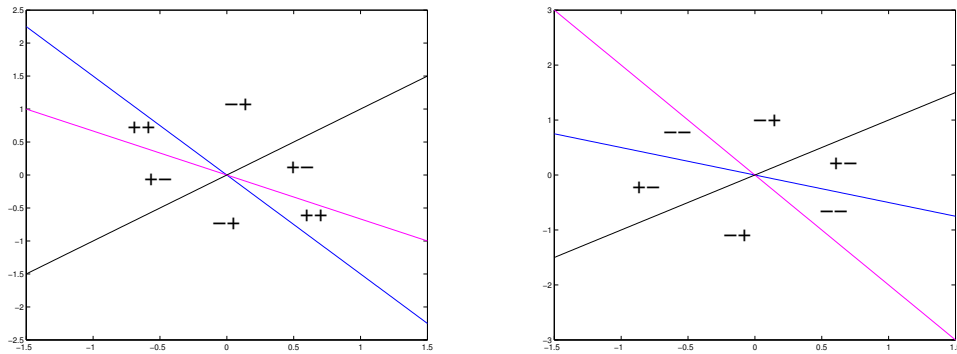
3. $\Phi(x, x) = (\phi(x), \phi(x))$ for every x where $\phi : \mathbf{R} \rightarrow \mathbf{R}$.
4. $\Phi_{1,y}(x, x) = 0$ for every x .

However, this equivalence relation is not strong enough to study dimorphisms in adaptive dynamics. The following is an example of two strategy functions that are strategy equivalent, but one has regions of coexistence, whereas the other does not.

Example 3.2. Consider the strategy functions

$$f(x, y) = -(y - x)(2x + y) \quad \text{and} \quad g(x, y) = -(y - x)(x + 2y). \quad (3.1.1)$$

Theorem 5.5 in Golubitsky and Vutha [14] shows that $f(x, y)$ and $g(x, y)$ are strategy equivalent to the normal form $h(x, y) = -(y - x)y$. However, we see that f and g have different dimorphism properties; more precisely, f has regions of coexistence, whereas g does not. See the mutual invasibility plots in Figure 3.1.



(a) MIP of f in Example 3.2

(b) MIP of g in Example 3.2

Figure 3.1: Strategy functions f and g are strategy equivalent, but have different dimorphism properties because f has regions of coexistence, whereas g does not.

Example 3.2 implies the need for a new equivalence relation to preserve regions of coexistence. Recall that $f(x, y) > 0$ indicates a fitness advantage of the mutant playing strategy y when competing against the resident playing strategy x , whereas $f(y, x) > 0$ indicates a fitness advantage of the mutant playing strategy x when competing against the resident playing the strategy y . We wish to preserve the regions of coexistence for f under a new equivalence relation, that is, regions where $f(x, y) > 0$ and $f(y, x) > 0$.

Definition 3.3. Two strategy functions f and \hat{f} are *dimorphism equivalent* if

$$\hat{f}(x, y) = S(x, y)f(\Phi(x, y)),$$

where

1. $S(x, y) > 0$ for all (x, y) .
2. $\Phi(x, y) = (\varphi(x, y), \varphi(y, x))$ where $\varphi: \mathbf{R}^2 \rightarrow \mathbf{R}$.
3. $(d\Phi)_{x,x} = c(x)I_2$ where $c(x) > 0$.

Moreover, we call the dimorphism equivalence *strong* if $c(x) \equiv 1$.

The remainder of this section is devoted to proving:

Theorem 3.4. *If the strategy functions $f(x, y)$ and $\hat{f}(x, y)$ are dimorphism equivalent, that is,*

$$\hat{f}(x, y) = S(x, y)f(\Phi(x, y)),$$

where (S, Φ) satisfies the assumptions in Definition 3.3, then the diffeomorphism Φ maps the regions of coexistence of $\hat{f}(x, y)$ to those of $f(x, y)$.

Remark 3.5. In fact, we can show that dimorphism equivalence will preserve both $\text{sgn}(f(x, y))$ and $\text{sgn}(f(y, x))$. There are four regions based on the signs of $f(x, y)$ and $f(y, x)$ and dimorphism equivalence preserves all of these regions.

3.2 Motivation of Dimorphism Equivalence

One way to preserve dimorphisms is to use the same strategy equivalence simultaneously on two pairs of strategy functions $(f(x, y), \hat{f}(x, y))$ and $(f(y, x), \hat{f}(y, x))$. This idea suggests considering vector functions of the form

$$F(x, y) = \begin{pmatrix} f(x, y) \\ f(y, x) \end{pmatrix} \quad (3.2.1)$$

where $F : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ and interpreting dimorphism equivalence as a symmetry condition on (3.2.1).

Specifically, define the interchange symmetry $\sigma(x, y) = (y, x)$. Note that every $F(x, y)$ in (3.2.1) is σ -equivariant, that is

$$F(\sigma(x, y)) = \sigma F(x, y). \quad (3.2.2)$$

In this section we show that specializing σ -equivalence (see [13] for details) to include the restrictions of strategy equivalence leads to Definition 3.3 of dimorphism equivalence, and allows us to prove Theorem 3.4. We begin by defining

Definition 3.6. Suppose $F(x, y)$ and $\hat{F}(x, y)$ are σ -equivariant. Then $F(x, y)$ and $\hat{F}(x, y)$ are called *modified σ -equivalent* if there exists a diffeomorphism $\Phi : \mathbf{R}^2 \rightarrow \mathbf{R}^2$ and a 2×2 smooth matrix T such that

$$\hat{F}(x, y) = T(x, y)F(\Phi(x, y)), \quad (3.2.3)$$

where

$$T(x, y) = \begin{pmatrix} S(x, y) & 0 \\ 0 & S(y, x) \end{pmatrix} \quad \Phi(x, y) = (\varphi(x, y), \varphi(y, x)) \quad (3.2.4)$$

and $\det(T(x, y)) \neq 0$, $\det(d\Phi)_{x,y} \neq 0$ near the given point.

Remark 3.7. This is a modification of σ -equivalence because it follows from (3.2.4) that $T(x, y)$ and $\Phi(x, y)$ satisfy

$$T(\sigma(x, y))\sigma = \sigma T(x, y) \quad \Phi(\sigma(x, y)) = \sigma\Phi(x, y)$$

Modified σ -equivalence is sufficient to preserve pairs of dimorphisms near singular points. To preserve ESS and CvSS singularities, we include the restrictions of strategy equivalence to this modified σ -equivalence. Note that a certain equivalence between $f(x, y)$ and $\hat{f}(x, y)$ can be induced by the modified σ -equivalence. This is because

$$\begin{aligned} \hat{F}(x, y) &= \begin{pmatrix} \hat{f}(x, y) \\ \hat{f}(y, x) \end{pmatrix} \\ &= T(x, y)F(\Phi(x, y)) \\ &= \begin{pmatrix} S(x, y) & 0 \\ 0 & S(y, x) \end{pmatrix} \begin{pmatrix} f(\varphi(x, y), \varphi(y, x)) \\ f(\varphi(y, x), \varphi(x, y)) \end{pmatrix} \\ &= \begin{pmatrix} S(x, y)f(\varphi(x, y), \varphi(y, x)) \\ S(y, x)f(\varphi(y, x), \varphi(x, y)) \end{pmatrix} \end{aligned}$$

Hence

$$\hat{f}(x, y) = S(x, y)f(\varphi(x, y), \varphi(y, x)). \quad (3.2.5)$$

In order for $\hat{f}(x, y)$ to be strategy equivalent to $f(x, y)$ under the equivalence relation (S, Φ) , it follows from Definition 3.1 that the following conditions have to be satisfied

$$S(x, y) > 0 \quad \det(d\Phi)_{x,y} > 0 \quad \varphi_y(x, x) = 0$$

Therefore, to preserve ESS singularities, CvSS singularities, and dimorphisms, the equivalence (S, Φ) should satisfy

1. $S(x, y) > 0$ for all x, y .
2. $\Phi(x, y) = (\varphi(x, y), \varphi(y, x))$.

3. $\det(d\Phi)_{x,y} > 0$ for all x, y .

4. $\varphi_y(x, x) = 0$ for all x .

Remark 3.8. Note that

$$(d\Phi)_{x,y} = \begin{pmatrix} \varphi_x(x, y) & \varphi_y(x, y) \\ \varphi_y(y, x) & \varphi_x(y, x) \end{pmatrix}.$$

So $\varphi_y(x, x) = 0$ implies that

$$(d\Phi)_{x,x} = \begin{pmatrix} \varphi_x(x, x) & \varphi_y(x, x) \\ \varphi_y(x, x) & \varphi_x(x, x) \end{pmatrix} = \begin{pmatrix} \varphi_x(x, x) & 0 \\ 0 & \varphi_x(x, x) \end{pmatrix} = \varphi_x(x, x) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

The local singularity theory only requires that all conditions are satisfied near a given point. So we can replace the conditions $\varphi_y(x, x) = 0$ and $\det(d\Phi)_{x,y} > 0$ with a single requirement that $(d\Phi)_{x,x} = c(x)I_2$ where $c(x) > 0$.

In this way we have constructed an equivalence relation (S, Φ) that preserves ESS singularities, CvSS singularities, and dimorphisms. We obtain the formal definition of dimorphism equivalence as in Definition 3.3.

3.3 Proof of Theorem 3.4

We showed that dimorphism equivalence is a modification of strategy equivalence in the previous section. It follows that,

Lemma 3.9. *If $f(x, y)$ and $\hat{f}(x, y)$ are dimorphism equivalent, then they are strategy equivalent too.*

Moreover, we have the following lemma showing that $f(y, x)$ and $\hat{f}(y, x)$ are also strategy equivalent under the same dimorphism equivalence.

Lemma 3.10. *If two strategy functions $f(x, y)$ and $\hat{f}(x, y)$ are diffeomorphism equivalent, suppose*

$$\hat{f}(x, y) = S(x, y)f(\Phi(x, y)),$$

then $f(y, x)$ and $\hat{f}(y, x)$ are strategy equivalent under the same equivalence (S, Φ) .

Proof. Define

$$g(x, y) = f(y, x)$$

$$\hat{g}(x, y) = \hat{f}(y, x)$$

We want to show that $g(x, y)$ is strategy equivalent to $\hat{g}(x, y)$ under (S, Φ) .

Note that $\hat{f}(x, y) = S(x, y)f(\Phi(x, y))$. By switching the variables x, y , we have

$$\begin{aligned} \hat{f}(y, x) &= S(y, x)f(\Phi(y, x)) \\ &= S(y, x)f(\varphi(y, x), \varphi(x, y)) \\ &= S(y, x)g(\varphi(x, y), \varphi(y, x)) \end{aligned} \tag{3.3.1}$$

Equation (3.3.1) can be written as

$$\hat{g}(x, y) = S(y, x)g(\Phi(x, y))$$

Note that $S(y, x) > 0$ and $\Phi(x, y)$ satisfies the conditions in Definition 3.1, hence $\hat{g}(x, y)$ is strategy equivalent to $g(x, y)$. That is, $\hat{f}(y, x)$ is strategy equivalent to $f(y, x)$. \square

Golubitsky and Vutha [14] show that coordinate changes that are defined in strategy equivalence preserve the zero sets of strategy functions. Hence, with Lemma 3.9 and Lemma 3.10 we know

Lemma 3.11. *If two strategy functions $f(x, y)$ and $\hat{f}(x, y)$ are diffeomorphism equivalent, that is*

$$\hat{f}(x, y) = S(x, y)f(\Phi(x, y)),$$

where (S, Φ) satisfies all assumptions in Definition 3.3, then

1. Φ maps the zero set of $\hat{f}(x, y)$ to the zero set of $f(x, y)$;

2. Φ maps the zero set of $\hat{f}(y, x)$ to the zero set of $f(y, x)$.

Proof. Lemma 3.9 tells us that $f(x, y)$ and $\hat{f}(x, y)$ are strategy equivalent, so the diffeomorphism Φ preserves the zero set of $\hat{f}(x, y)$. That is, Φ maps the zero set of $\hat{f}(x, y)$ to the zero set of $f(x, y)$.

Lemma 3.10 tells us that $f(y, x)$ and $\hat{f}(y, x)$ are also strategy equivalent under the same (S, Φ) , so the diffeomorphism Φ preserves the zero set of $\hat{f}(y, x)$. That is, Φ maps the zero set of $\hat{f}(y, x)$ to the zero set of $f(y, x)$. \square

Proof of Theorem 3.4 Note that $S(x, y) > 0$ and $S(y, x) > 0$, it follows directly from Lemma 3.11 that Φ maps the region

$$\{(x, y) : \hat{f}(x, y) > 0, \hat{f}(y, x) > 0\}$$

to the region

$$\{(x, y) : f(\varphi(x, y), \varphi(y, x)) > 0, f(\varphi(y, x), \varphi(x, y)) > 0\}.$$

That is, diffeomorphism equivalence preserves the regions of coexistence for strategy functions.

CHAPTER 4

THE RESTRICTED TANGENT SPACE

In this chapter, we answer the question: when is $f + \eta$ diffeomorphism equivalent to f for a small perturbation η ? WLOG, we assume that f and η are both strategy functions and $(0, 0)$ is the singularity for both of them.

First, by applying singularity theory, we reduce the question that is set in an infinite dimensional functional space to a question that is set in a finite dimensional polynomial ring. To do so, we introduce the *diffeomorphism equivalence restricted tangent space*: the subspace of all strategy function perturbations η that satisfies $f + t\eta$ is diffeomorphism equivalent to f for all small t . We find the general form for the strategy function η in Proposition 4.2. By applying the form of η we prove the tangent space constant theorem (see Theorem 4.7) in the context of diffeomorphism equivalence.

We find that the calculations are more easily done in the coordinate system

$$\begin{aligned} u &= x + y \\ v &= x - y \end{aligned} \tag{4.0.1}$$

and we set $w = v^2$.

4.1 A Change of Coordinates

The idea of considering σ -symmetry (where $\sigma(x, y) = (y, x)$) when studying dimorphisms motivates the coordinate change in Lemma 4.1. Note that under (u, v) coordinates in (4.0.1), we have $\sigma(u, v) = (u, -v)$.

Lemma 4.1. *Every smooth real-valued function $a(u, v)$ can be written in the form*

$$a(u, v) = b(u, w) + c(u, w)v \quad (4.1.1)$$

where b and c are smooth functions.

Proof. For the proof see the discussion in [11, p.108]. It follows that every smooth real-valued function a can be written as a pair of smooth σ -invariant functions b, c as in (4.1.1). \square

Let f be a strategy function. Since $f(x, x) = 0$ or in (u, v) coordinates $f(u, 0) = 0$, it follows from Lemma 4.1 that f can be defined as

$$f = b(u, w) + q(u, w)v$$

where $b(u, 0) = 0$. By Taylor's Theorem f has the form

$$f = p(u, w)w + q(u, w)v,$$

where $p, q : \mathbf{R}^2 \rightarrow \mathbf{R}$ are smooth functions. In the remainder of this thesis we will identify strategy functions f with the notation $[p, q] \in \mathcal{E}^2$ where \mathcal{E} is the space of smooth real-valued functions in (u, w) coordinates.

4.2 Dimorphism Equivalence Restricted Tangent Space

Assume that $f + t\eta$ is dimorphism equivalent to f for all small t . That is, suppose there exists a smooth t -dependent dimorphism equivalence such that

$$f(x, y) + t\eta(x, y) = S(x, y, t)f(\Phi(x, y, t)) \quad (4.2.1)$$

where

$$\begin{aligned} S(x, y, 0) = 1 \quad \varphi(x, y, 0) = x \quad \varphi(0, 0, t) = 0 \quad S(x, y, t) > 0 \\ \Phi(x, y, t) = (\varphi(x, y, t), \varphi(y, x, t)) \quad \varphi_y(x, x, t) = 0 \quad \varphi_x(x, x, t) > 0 \end{aligned} \quad (4.2.2)$$

Differentiating both sides of (4.2.1) with respect to t and evaluating at $t = 0$ gives

$$\eta(x, y) = \dot{S}(x, y, 0)f(x, y) + \dot{\varphi}(x, y, 0)f_x(x, y) + \dot{\varphi}(y, x, 0)f_y(x, y) \quad (4.2.3)$$

where \cdot is the derivative with respect to t and (S, Φ) satisfies all the requirements in (4.2.2).

We define the *dimorphism equivalence restricted tangent space* $RT(f)$ of f to be the space of all $\eta(x, y)$ that satisfies (4.2.3) where (S, Φ) satisfies all assumptions in (4.2.2).

Proposition 4.2. Let $f = pw + qv$, the dimorphism equivalence restricted tangent space $RT(f)$ is

$$\langle [p, q], [q, wp], [wp_u, wq_u], [wp + 2w^2p_w, 2w^2q_w] \rangle + \mathbf{R}\{\dots\}$$

where $\{\dots\}$ indicates the \mathbf{R} -module generated by

$$u^{j-1}[jp + up_u + 2jwp_w, uq_u + 2jwq_w]$$

for $j = 1, 2, \dots$

Proof. Dimorphism equivalences are generated by two kinds of equivalence

1. $\hat{f}(x, y) = S(x, y)f(x, y)$ where $S(x, y) > 0$.
2. $\hat{f}(x, y) = f(\varphi(x, y), \varphi(y, x))$ where $\varphi(0, 0) = 0$, $\varphi_y(x, x) = 0$, $\varphi_x(x, x) > 0$.

Consider the first type of equivalences in (u, v) coordinates, we can assume that

$$S = S_e(u, w) + S_o(u, w)v$$

where $S_e(0, 0) > 0$. Then the tangent vectors given by this type of equivalencies can be computed by

$$\begin{aligned} \left. \frac{d}{dt} S(x, y, t) f(x, y) \right|_{t=0} &= \dot{S}(x, y, 0) f(x, y) \\ &= (\dot{S}_e + \dot{S}_o v)(pw + qv) \\ &= \dot{S}_e(pw + qv) + \dot{S}_o(qw + wpv) \end{aligned}$$

where \cdot is the derivative with respect to t . Since \dot{S}_e and \dot{S}_o are arbitrary σ -invariant functions we see that

$$\langle [p, q], [q, wp] \rangle \subset RT(f)$$

Consider the second type of equivalences in (u, v) coordinates, we can assume

$$\varphi(x, y, t) = \varphi_e(u, w, t) + \varphi_o(u, w, t)v.$$

Since $\varphi(y, x, t) = \varphi_e(u, w, t) - \varphi_o(u, w, t)v$, we compute

$$\begin{aligned} u(\Phi(x, y, t)) &= \varphi(x, y, t) + \varphi(y, x, t) = (\varphi_e + \varphi_o v) + (\varphi_e - \varphi_o v) = 2\varphi_e \\ v(\Phi(x, y, t)) &= \varphi(x, y, t) - \varphi(y, x, t) = (\varphi_e + \varphi_o v) - (\varphi_e - \varphi_o v) = 2\varphi_o v \\ w(\Phi(x, y, t)) &= v(\Phi(x, y, t))^2 = (2\varphi_o v)^2 = 4\varphi_o^2 w \end{aligned} \tag{4.2.4}$$

Then the tangent vectors given by this type of equivalencies can be computed by

$$\left. \frac{d}{dt} f(\varphi(x, y, t), \varphi(y, x, t)) \right|_{t=0} \tag{4.2.5}$$

Using (4.2.4) we calculate the derivative in (4.2.5) as

$$\begin{aligned} &\frac{d}{dt} f(\varphi(x, y, t), \varphi(y, x, t)) \\ &= \frac{d}{dt} (p(u(\Phi), v(\Phi)^2)v(\Phi)^2 + q(u(\Phi), v(\Phi)^2)v(\Phi)) \\ &= \frac{d}{dt} (4p(2\varphi_e, 4\varphi_o^2 w)\varphi_o^2 w + 2q(2\varphi_e, 4\varphi_o^2 w)\varphi_o v) \\ &= 8p_1\varphi_o^2 w\dot{\varphi}_e + 32p_2w^2\varphi_o^3\dot{\varphi}_o + 8pw\varphi_o\dot{\varphi}_o + 4q_1\varphi_o v\dot{\varphi}_e + 16q_2wv\varphi_o^2\dot{\varphi}_o + 2qv\dot{\varphi}_o \\ &= \dot{\varphi}_e(8p_1\varphi_o^2 w + 4q_1\varphi_o v) + \dot{\varphi}_o(32p_2w^2\varphi_o^3 + 8pw\varphi_o + 16q_2wv\varphi_o^2 + 2qv) \end{aligned}$$

where p_i, q_i are the derivative of p, q respect to the i^{th} component ($i = 1, 2$) and \cdot is the derivative with respect to t .

Note that the diffeomorphism Φ is the identity at $t = 0$, we have

$$x = \varphi(x, y, 0) = \varphi_e(u, w, 0) + \varphi_o(u, w, 0)v$$

Then we have

$$\varphi_e(u, w, 0) = \frac{1}{2}u \quad \varphi_o(u, w, 0) = \frac{1}{2} \quad (4.2.6)$$

Using (4.2.6) and evaluating at $t = 0$, we obtain

$$\left. \frac{d}{dt} f(\varphi(x, y, t), \varphi(y, x, t)) \right|_{t=0} = 2\dot{\varphi}_e(p_u w + q_u v) + 4\dot{\varphi}_o(p_w w^2 + p w + q_w w v + \frac{1}{2} q v) \quad (4.2.7)$$

Note that

$$\varphi_y(x, x, t) = (\varphi_e)_u(2x, 0, t) - \varphi_o(2x, 0, t)$$

It follows from $\varphi_y(x, x, t) = 0$ that

$$\varphi_o(u, 0, t) = (\varphi_e)_u(u, 0, t)$$

Hence, we can write

$$\varphi_e(u, w, t) = \frac{1}{2}u + g(u, w, t)t \quad \varphi_o(u, w, t) = \frac{1}{2} + h(u, w, t)t.$$

Therefore

$$(\varphi_e)_u(u, 0, t) = \frac{1}{2} + g_u(u, 0, t)t \quad \varphi_o(u, 0, t) = \frac{1}{2} + h(u, 0, t)t.$$

Then we have

$$g_u(u, 0, t) = h(u, 0, t)$$

In addition,

$$\dot{\varphi}_e = \frac{d}{dt} \varphi_e(u, w, t)|_{t=0} = g(u, w, 0) = g(u, 0, 0) + \hat{g}(u, w, 0)w$$

$$\dot{\varphi}_o = \frac{d}{dt}\varphi_o(u, w, t)|_{t=0} = h(u, w, 0) = h(u, 0, 0) + \hat{h}(u, w, 0)w$$

where $h(u, 0, 0) = g_u(u, 0, 0)$ and \hat{g}, \hat{h} are arbitrary.

Now we redefine the functions g, \hat{g}, \hat{h} such that $g : \mathbf{R} \rightarrow \mathbf{R}, \hat{g}, \hat{h} : \mathbf{R}^2 \rightarrow \mathbf{R}$. It follows that the RHS of (4.2.7) contains

$$2(g(u) + w\hat{g}(u, w))(p_u w + q_u v) + 4(g'(u) + w\hat{h}(u, v))(p_w w^2 + p w + q_w w v + \frac{1}{2}q v)$$

In $[\dots, \dots]$ notation, we have

$$\begin{aligned} & 2\hat{g}[p_u w, q_u w] + 2\hat{h}[2p w + 2p_w w^2, q w + 2q_w w^2] \\ & + 2g(u)(p_u w + q_u v) + 2g'(u)(2p_w w^2 + 2p w + 2q_w w v + q v) \end{aligned}$$

Since $\hat{g}(u, w), \hat{h}(u, w), g(u)$ are arbitrary σ invariant functions, we see that

$$\langle [p_u w, q_u w], [2p w + 2p_w w^2, q w + 2q_w w^2] \rangle \subset RT(f)$$

Moreover, the dimorphism equivalence restricted tangent space includes the vectors

$$2g(u)(p_u w + q_u v) + 2g'(u)(2p_w w^2 + 2p w + 2q_w w v + q v)$$

for arbitrary $g(u)$. Therefore, now we have

$$\langle [p, q], [q, w p], [w p_u, w q_u], [2w p + 2w^2 p_w, q w + 2w^2 q_w] \rangle \subset RT(f)$$

Note that we can simplify $\langle [p, q], [2p w + 2p_w w^2, q w + 2q_w w^2] \rangle$ into

$$\langle [p, q], [p w + 2p_w w^2, 2q_w w^2] \rangle$$

Therefore, we have proved the ideal part of expression of $RT(f)$ as follows

$$\langle [p, q], [q, w p], [w p_u, w q_u], [w p + 2w^2 p_w, 2w^2 q_w] \rangle \subset RT(f)$$

Next we continue to find the vector part of $RT(f)$. Note that in the definition of $RT(f)$, we require that $\varphi(0, 0, t) = 0$ for all small t . Then we have

$$\frac{\partial \varphi}{\partial t}(0, 0, 0) = \dot{\varphi}_e(0, 0, 0) = g(0) = 0.$$

Therefore, g satisfies the following

$$\begin{aligned} g(u) &= k_1u + k_2u^2 + \cdots + k_ju^j + \cdots \\ g'(u) &= k_1 + 2k_2u + \cdots + jk_ju^{j-1} + \cdots \end{aligned}$$

It follows that $(g(u), g'(u))$ can be any element in the vector space spanned by

$$(u, 1) \quad (u^2, 2u) \quad (u^3, 3u) \quad \cdots$$

It follows from (4.2.7) that the dimorphism equivalence restricted tangent space includes the vectors

$$u^j(p_uw + q_uv) + ju^{j-1}(2p_w w^2 + 2pw + 2q_w wv + qv)$$

where $j = 1, 2, \dots$. Hence the $\{\cdots\}$ notation in the statements of Proposition 4.2 is

$$u^{j-1}[jp + up_u + 2jwp_w, uq_u + 2jq_w]$$

for $j = 1, 2, \dots$

□

Remark 4.3. Note that in Definition 3.3, we define the strongly dimorphism equivalence. If we try to calculate the restricted tangent space of a strategy functions f in the context of strongly dimorphism equivalence, we would go through similar calculations except that we will not have the vector terms in $\{\cdots\}$. That is, the strongly dimorphism equivalence restricted tangent space of a strategy function $f = pw + qv$ is

$$\langle [p, q], [q, wp], [wp_u, wq_u], [wp + 2w^2p_w, 2w^2q_w] \rangle$$

Remark 4.4. In the definition of $RT(f)$, we require that $\varphi(0, 0, t) = 0$, that is, the singularity is always at $(0, 0)$. However, a general perturbation of a strategy function f does not always have this restriction. If this condition does not hold, we have the definition of *dimorphism equivalence tangent space*, denoted by $T(f)$: the space of

$\eta(x, y)$ that satisfies (4.2.1) where (S, Φ) satisfies all assumptions in (4.2.2) except the one $\varphi(0, 0, t) = 0$. To calculate the dimorphism equivalence tangent space $T(f)$, we eliminate the restriction $g(0) = 0$. It lead to an additional vector $\mathbf{R}\{[p_u, q_u]\}$. We will discuss this in detail in Chapter 6.

With Proposition 4.2 we can compute the dimorphism equivalence restricted tangent space of a strategy function. Define

$$\mathcal{I}(f) = \langle [p, q], [q, wp], [wp_u, wq_u], [wp + 2w^2p_w, 2w^2q_w] \rangle \quad (4.2.8)$$

Remark 4.5. In the proof of Proposition 4.2, we showed that there is an alternative expression for $\mathcal{I}(f)$ as below

$$\mathcal{I}(f) = \langle [p, q], [q, wp], [wp_u, wq_u], [2pw + 2p_w w^2, qw + 2q_w w^2] \rangle \quad (4.2.9)$$

Remark 4.6. Note that Remark 4.3 shows that $\mathcal{I}(f)$ is in fact the restricted tangent space of f in the context of strongly dimorphism equivalence.

Recall that a strategy s is a singular strategy if $f_y(s, s) = 0$. Under the (u, v) coordinates, since we assume that $f(x, y) = p(u, w)w + q(u, w)v$, we call a strategy s a *singular strategy* if $q(2s, 0) = 0$.

4.3 Modified Tangent Space Constant Theorem

The definition of $RT(f)$ implies that if $f + t\eta$ is dimorphism equivalent to f for all small t , then $\eta \in RT(f)$. In the converse direction, we have Theorem 4.7.

Theorem 4.7 (Modified Tangent Space Constant Theorem). *Let f be a strategy function. If*

$$\mathcal{I}(f + tg) = \mathcal{I}(f) \text{ for all } t \in [0, 1] \quad (4.3.1)$$

Then $f + tg$ is strongly dimorphism equivalent to f for all $t \in [0, 1]$.

Remark 4.8. The proof of this theorem is a modification of the proof of an analogous theorem in bifurcation theory (see [12] Theorem 2.2).

Proof. In this proof, we use the alternative form of $\mathcal{I}(f)$ as in (4.2.9). Suppose $\mathcal{I}(f + t_0g) = \mathcal{I}(f)$ for some $t_0 \neq 0$ and $f = p^f w + q^f v$, $g = p^g w + q^g v$. Let

$$F(x, y, t) = f(x, y) + tg(x, y) \quad (4.3.2)$$

We prove the statement in following steps. We show in order

- (a) There exist smooth functions $A(u, w, t)$, $B(u, w, t)$, $C(u, w, t)$ and $D(u, w, t)$ such that

$$\begin{aligned} g &= [p^g, q^g] \\ &= A[p^F, q^F] + B[q^F, wp^F] + C[wp_u^F, wq_u^F] \\ &\quad + D[2wp^F + 2w^2p_w^F, wq^F + 2w^2q_w^F] \end{aligned} \quad (4.3.3)$$

- (b) $F(x, y, t)$ is strongly diffeomorphism equivalent to $f(x, y)$ for each t sufficiently close to 0.
- (c) $F(x, y, t)$ is strongly diffeomorphism equivalent to $f(x, y)$ for all $t \in [0, 1]$.

We show (a). Note that for each fixed t , $\mathcal{I}(f + tg) = \mathcal{I}(f)$ implies that there exist functions A, B, C, D such that (4.3.3) holds. In (a), we imply that A, B, C, D can be chosen to vary smoothly in t .

Choose t_0 near 0 so that

$$\mathcal{I}(f + t_0g) = \mathcal{I}(f)$$

In particular, each generator of $\mathcal{I}(f + t_0g)$ can be written as a linear combination

of generators of $\mathcal{I}(f)$. Therefore, there exist functions $A_i(u, w)$, $B_i(u, w)$, $C_i(u, w)$, $D_i(u, w)$ for $1 \leq i \leq 4$ such that

$$\begin{aligned}
[p^f + t_0 p^g, q^f + t_0 q^g] &= A_1[p^f, q^f] + B_1[q^f, wp^f] + C_1[wp_u^f, wq_u^f] \\
&\quad + D_1[2wp^f + 2w^2 p_w^f, wq^f + 2w^2 q_w^f] \\
[q^f + t_0 q^g, w(p^f + t_0 p^g)] &= A_2[p^f, q^f] + B_2[q^f, wp^f] + C_2[wp_u^f, wq_u^f] \\
&\quad + D_2[2wp^f + 2w^2 p_w^f, wq^f + 2w^2 q_w^f] \\
[w(p_u^f + t_0 p_u^g), w(q_u^f + t_0 q_u^g)] &= A_3[p^f, q^f] + B_3[q^f, wp^f] + C_3[wp_u^f, wq_u^f] \\
&\quad + D_3[2wp^f + 2w^2 p_w^f, wq^f + 2w^2 q_w^f] \\
[2w(p^f + t_0 p^g) + 2w^2(p_w^f + t_0 p_w^g), \\
w(q^f + t_0 q^g) + 2w^2(q_w^f + t_0 q_w^g)] &= A_4[p^f, q^f] + B_4[q^f, wp^f] + C_4[wp_u^f, wq_u^f] \\
&\quad + D_4[2wp^f + 2w^2 p_w^f, wq^f + 2w^2 q_w^f]
\end{aligned} \tag{4.3.4}$$

Rearranging terms in (4.3.4), we obtain a matrix equation

$$\begin{pmatrix} [p^g, q^g] \\ [q^g, wp^g] \\ [wp_u^g, wq_u^g] \\ [2wp^g + 2w^2 p_w^g, wq^g + 2w^2 q_w^g] \end{pmatrix} = Q \begin{pmatrix} [p^f, q^f] \\ [q^f, wp^f] \\ [wp_u^f, wq_u^f] \\ [2wp^f + 2w^2 p_w^f, wq^f + 2w^2 q_w^f] \end{pmatrix} \tag{4.3.5}$$

where

$$Q = \frac{1}{t_0} \begin{pmatrix} A_1 - 1 & B_1 & C_1 & D_1 \\ A_1 & B_1 - 1 & C_1 & D_1 \\ A_1 & B_1 & C_1 - 1 & D_1 \\ A_1 & B_1 & C_1 & D_1 - 1 \end{pmatrix}$$

is a 4×4 matrix whose entries are smooth functions.

Next, for any strategy function h , let

$$z(h) = \begin{pmatrix} [p^h, q^h] \\ [q^h, wp^h] \\ [wp_u^h, wq_u^h] \\ [2wp^h + 2w^2 p_w^h, wq^h + 2w^2 q_w^h] \end{pmatrix}$$

Using this notation, rewrite (4.3.5) as

$$z(g) = Qz(f) \tag{4.3.6}$$

By (4.3.2), $f = F - tg$. Therefore,

$$z(f) = z(F) - tz(g) \tag{4.3.7}$$

Substituting (4.3.7) into (4.3.6) and rearranging, we find

$$(I + tQ)z(g) = Qz(F) \tag{4.3.8}$$

Observe that (4.3.8) is a system of four equations with smooth dependence on t . Since I is invertible, $I + tQ$ is also an invertible 4×4 matrix for small t . Thus $(I + tQ)^{-1}$ is a 4×4 matrix whose entries are smooth functions in t . The invertibility of $I + tQ$ and (4.3.8) imply

$$z(g) = (I + tQ)^{-1}Qz(F) \tag{4.3.9}$$

(a) follows from equating the first components on each side of the matrix equation (4.3.9).

We show (b). Specifically, we construct a strongly dimorphism equivalence mapping $(S(x, y, t), \Phi(x, y, t))$ varying smoothly in t and satisfying that

$$\begin{aligned} S(x, y, t)F(\Phi(x, y, t), t) &= f(x, y) \\ S(x, y, 0) &= 1 \\ \Phi(x, y, 0) &= (x, y) \\ \varphi_x(x, x, t) &= 1 \\ \varphi_y(x, x, t) &= 0 \end{aligned} \tag{4.3.10}$$

The functions Φ and S are found by solving certain ODE's. From direct calculation in Proposition 4.2, we can write (S, Φ) in the following form

$$\begin{aligned} S(x, y, t) &= S_e(u, w, t) + S_o(u, w, t)v \\ \varphi(x, y, t) &= \varphi_e(u, w, t) + \varphi_o(u, w, t)v \end{aligned}$$

Now, we consider the following two sets of system of differential equations

$$\begin{aligned}
\dot{\varphi}_e(u, w, t) &= -2\varphi_o^2 w C(2\varphi_e, 4\varphi_o^2 w) \\
\dot{\varphi}_o(u, w, t) &= -4\varphi_o^3 w D(2\varphi_e, 4\varphi_o^2 w) \\
\varphi_e(u, w, 0) &= \frac{1}{2}u \\
\varphi_o(u, w, 0) &= \frac{1}{2}
\end{aligned} \tag{4.3.11}$$

and

$$\begin{aligned}
\dot{S}_e(u, w, t) &= -(S_e A(2\varphi_e, 4\varphi_o^2 w) + S_o B(2\varphi_e, 4\varphi_o^2 w) 2\varphi_o w) \\
\dot{S}_o(u, w, t) &= -(S_o A(2\varphi_e, 4\varphi_o^2 w) + S_0 B(2\varphi_e, 4\varphi_o^2 w)) \\
S_e(u, w, 0) &= 1 \\
S_o(u, w, 0) &= 0
\end{aligned} \tag{4.3.12}$$

where A, B, C, D are the coefficients in (4.3.3).

Suppose there exists solutions $(\varphi_e, \varphi_o, S_e, S_o)$ of the ODEs (4.3.11) and (4.3.12) on a neighborhood of $K \times L \times M \times N$ and denote

$$[p, q]_{\Phi} = p(\Phi)w(\Phi) + q(\Phi)v(\Phi)$$

Then we have

$$\begin{aligned}
& \frac{d}{dt}(S(x, y, t)F(\Phi, t)) \\
= & \frac{d}{dt}((S_e + S_o v)(p^F(\Phi)w(\Phi) + q^F(\Phi)v(\Phi))) \\
= & \frac{d}{dt}[(S_e + S_o v)](p^F(\Phi)w(\Phi) \\
& + q^F(\Phi)v(\Phi)) + (S_e + S_o v)\frac{d}{dt}[(p^F(\Phi)w(\Phi) + q^F(\Phi)v(\Phi))] \\
= & (\dot{S}_e + \dot{S}_o v)(p^F(\Phi)w(\Phi) + q^F(\Phi)v(\Phi)) \\
& + (S_e + S_o v)\frac{d}{dt}(p^F(2\varphi_e, 4\varphi_o^2, t)4\varphi_o^2 w + q^F(2\varphi_e, 4\varphi_o^2, t)2\varphi_o v) \\
= & \dot{S}_e[p^F, q^F]_{\Phi} + \dot{S}_o v[p^F, q^F]_{\Phi} + (S_e + S_o v)[(p_u^F(2\varphi_e, 4\varphi_o^2, t)(2\dot{\varphi}_e)4\varphi_o^2 \\
& + p_w^F(2\varphi_e, 4\varphi_o^2, t)(8\varphi_o\dot{\varphi}_o)4\varphi_o^2 + p^F(2\varphi_e, 4\varphi_o^2, t)8\varphi_o\dot{\varphi}_o + p_t^F(2\varphi_e, 4\varphi_o^2, t)4\varphi_o^2)w \\
& + (q_u^F(2\varphi_e, 4\varphi_o^2, t)(2\dot{\varphi}_e)2\varphi_o + q_w^F(2\varphi_e, 4\varphi_o^2, t)(8\varphi_o\dot{\varphi}_o)2\varphi_o \\
& + q^F(2\varphi_e, 4\varphi_o^2, t)2\dot{\varphi}_o + q_t^F(2\varphi_e, 4\varphi_o^2, t)2\varphi_o)v] \\
= & \dot{S}_e[p^F, q^F]_{\Phi} + \frac{\dot{S}_o}{2\varphi_o}[q^F, wp^F]_{\Phi} + (S_e + S_o v)\left(\frac{\dot{\varphi}_e}{2\varphi_o^2 w}[wp_u^F, wq_u^F]_{\Phi} \right. \\
& \left. + \frac{\dot{\varphi}_o}{4\varphi_o^3 w}[2wp^F + 2w^2 p_w^F, wq^F + 2w^2 q_w^F]_{\Phi} + [p_t^F, q_t^F]_{\Phi}\right)
\end{aligned} \tag{4.3.13}$$

Note that in part (a), we have

$$g = [p^g, q^g] = A[p^F, q^F] + B[q^F, wp^F] + C[wp_u^F, wq_u^F] + D[2wp^F + 2w^2 p_w^F, wq^F + 2w^2 q_w^F].$$

Therefore,

$$\begin{aligned}
& (S_e + S_o v)g(\Phi) \\
= & (S_e + S_o v)[p^g, q^g]_{\Phi} \\
= & (S_e + S_o v)(A(\Phi)[p^F, q^F]_{\Phi} + B(\Phi)[q^F, wp^F]_{\Phi} + \\
& C(\Phi)[wp_u^F, wq_u^F]_{\Phi} + D(\Phi)[wp^F + 2w^2 p_w^F, 2w^2 q_w^F]_{\Phi}) \\
= & (S_e + S_o v)A(\Phi)[p^F, q^F]_{\Phi} + (S_e + S_o v)B(\Phi)[q^F, wp^F]_{\Phi} \\
& + (S_e + S_o v)(C(\Phi)[wp_u^F, wq_u^F]_{\Phi} + D(\Phi)[wp^F + 2w^2 p_w^F, 2w^2 q_w^F]_{\Phi}) \\
= & S_e A(\Phi)[p^F, q^F]_{\Phi} + S_o A(\Phi)[q^F, wp^F]_{\Phi} + S_0 B(\Phi)[q^F, wp^F]_{\Phi} + S_o B(\Phi)2\varphi_o w[p^F, q^F]_{\Phi} \\
& + (S_e + S_o v)(C(\Phi)[wp_u^F, wq_u^F]_{\Phi} + D(\Phi)[wp^F + 2w^2 p_w^F, 2w^2 q_w^F]_{\Phi}) \\
= & (S_e A(\Phi) + S_o B(\Phi)2\varphi_o w)[p^F, q^F]_{\Phi} + (S_o A(\Phi) + S_0 B(\Phi))[q^F, wp^F]_{\Phi} \\
& + (S_e + S_o v)C(\Phi)[wp_u^F, wq_u^F]_{\Phi} + (S_e + S_o v)D(\Phi)[wp^F + 2w^2 p_w^F, 2w^2 q_w^F]_{\Phi}
\end{aligned}$$

The right-hand side of (4.3.13) can be simplified considering the fact that $(\varphi_e, \varphi_o, S_0, S_1)$ are the solutions to the ODEs (4.3.11) and (4.3.12). Then we have

$$\begin{aligned}
& \frac{d}{dt}(S(x, y, t)F(\Phi, t)) \\
= & \dot{S}_e[p^F, q^F]_{\Phi} + \frac{\dot{S}_o}{2\varphi_o}[q^F, wp^F]_{\Phi} \\
& + (S_e + S_o v)\left(\frac{\dot{\varphi}_e}{2\varphi_o^2 w}[wp_u^F, wq_u^F]_{\Phi} + \frac{\dot{\varphi}_o}{4\varphi_o^3 w}[2wp^F + 2w^2 p_w^F, 2w^2 q_w^F]_{\Phi} + [p_t^F, q_t^F]_{\Phi}\right) \\
= & -(S_e A(\Phi) + S_o B(\Phi)2\varphi_o w)[p^F, q^F]_{\Phi} - (S_o A(\Phi) + S_0 B(\Phi))[q^F, wp^F]_{\Phi} \\
& + (S_e + S_o v)(-C(\Phi)[wp_u^F, wq_u^F]_{\Phi} - D(\Phi)[2wp^F + 2w^2 p_w^F, 2w^2 q_w^F]_{\Phi} + [p^g, q^g]_{\Phi}) \\
= & 0
\end{aligned}$$

Hence,

$$S(x, y, t)F(\Phi(x, y, t), t) = S(x, y, 0)F(\Phi(x, y, 0), t) = f(x, y)$$

In other words, $F(x, y, t) = f(x, y) + tg(x, y)$ is strongly diffeomorphism equivalent to $f(x, y)$ when t is small enough around 0.

Now we show that $(\varphi_e, \varphi_o, S_0, S_1)$ can be chosen to satisfy all conditions in (4.3.10).

Note that all conditions are explicitly expressed in the ODEs (4.3.11) and (4.3.12) except $\varphi_x(x, x, t) = 1$ and $\varphi_y(x, x, t) = 0$. In Proposition 4.2, we showed that

$$\varphi_y(x, x, t) = (\varphi_e)_u(2x, 0, t) - \varphi_o(2x, 0, t)$$

Therefore with ODEs (4.3.11) we have

$$\begin{aligned} \frac{d}{dt}(\varphi_x(x, x, t)) &= \frac{d}{dt}((\varphi_e)_u(2x, 0, t) + \varphi_o(2x, 0, t)) \\ &= (\dot{\varphi}_e)_u(2x, 0, t) + \dot{\varphi}_o(2x, 0, t) = 0 + 0 = 0 \end{aligned}$$

$$\begin{aligned} \frac{d}{dt}(\varphi_y(x, x, t)) &= \frac{d}{dt}((\varphi_e)_u(2x, 0, t) - \varphi_o(2x, 0, t)) \\ &= (\dot{\varphi}_e)_u(2x, 0, t) - \dot{\varphi}_o(2x, 0, t) = 0 - 0 = 0 \end{aligned}$$

Hence

$$\begin{aligned} \varphi_x(x, x, t) &= (\varphi_e)_u(2x, 0, t) + \varphi_o(2x, 0, t) \\ &= (\varphi_e)_u(2x, 0, 0) + \varphi_o(2x, 0, 0) \\ &= \frac{1}{2} + \frac{1}{2} = 1 \\ \varphi_y(x, x, t) &= (\varphi_e)_u(2x, 0, t) - \varphi_o(2x, 0, t) \\ &= (\varphi_e)_u(2x, 0, 0) - \varphi_o(2x, 0, 0) \\ &= \frac{1}{2} - \frac{1}{2} = 0 \end{aligned}$$

This means that the solution $(\varphi_e, \varphi_o, S_0, S_1)$ are the solutions to the ODEs (4.3.11) and (4.3.12) and satisfy all the conditions in (4.3.10).

The standard existence theorem for ODEs with smooth dependence on parameters implies that there exists intervals $K \times L \times M \times N$ where the ODEs (4.3.11) and (4.3.12) have unique solutions. Therefore, (b) holds.

We show (c). Define t_1 and t_2 in $[0, 1]$ to be equivalent if $f + t_1g$ is strongly

dimorphism equivalent to $f + t_2g$. We claim that (b) implies equivalence classes of t 's in $[0, 1]$ are open. To verify this, assume $h = f + t_0g$ for some $t_0 \in [0, 1]$. Then,

$$\mathcal{I}(h + sg) = \mathcal{I}(f + (s + t_0)g) = \mathcal{I}(f) = \mathcal{I}(h)$$

for all s sufficiently near 0. By part (b), $h + sg$ is strongly dimorphism equivalent to h for all s near 0. Thus $f + t_0g$ is strongly dimorphism equivalent to $f + tg$ for all t near t_0 and the equivalence class for t 's are open.

It follows from compactness of $[0, 1]$ that there is exactly one equivalence class. Hence, part (c) holds and $f + tg$ is strongly dimorphism equivalent to f for all $t \in [0, 1]$. □

CHAPTER 5

RECOGNITION OF LOW CODIMENSION SINGULARITIES

This chapter is devoted to proving Theorem 5.1. Note that strategy functions f have singularities at $(0, 0)$ and all derivatives of f are computed at $(0, 0)$ unless otherwise indicated.

Theorem 5.1. *Suppose $f = p(u, w)w + q(u, w)v$ is a strategy function with a singularity at the origin. Then f is diffeomorphism equivalent to the normal form in Table 5.1 if and only if the corresponding defining conditions and non-degeneracy conditions in Table 5.1 are satisfied.*

In this theorem, we apply the modified tangent space constant theorem (Theorem 4.7) to solve the recognition problem for low codimension singularities under diffeomorphism equivalence. To prove this theorem, we use Nakayama's Lemma (cf. [12, Chapter 2]), which we now state.

Remark 5.2. In this thesis \mathcal{E} is the ring of functions $z(u, w)$ (defined on a neighborhood of the origin) and \mathcal{M} is the maximal ideal in \mathcal{E} generated by u, w .

Lemma 5.3. *[Nakayama's Lemma]. Let $\mathcal{I}, \mathcal{J} \subset \mathcal{E}^2$ be a finitely generated \mathcal{E} -modules. Then*

$$\mathcal{I} \subset \mathcal{J} \text{ if and only if } \mathcal{I} \subset \mathcal{J} + \mathcal{M}\mathcal{I}$$

	Def	ND	TC	Normal Form h
(a)	—	p q_u	0	$\epsilon(w + \mu_0 uv)$ $\epsilon = \text{sgn}(p) \quad \mu_0 = \frac{q_u}{p}$
(b)	p	q_u $p_w q_u - p_u q_w$	1	$\epsilon(\delta w^2 + uv)$ $\epsilon = \text{sgn}(q_u) \quad \delta = \text{sgn}(p_w q_u - p_u q_w)$
(c)	q_u	p q_{uu}	1	$(\epsilon + \mu_0 u)w + \delta u^2 v$ $\epsilon = \text{sgn}(p) \quad \delta = \text{sgn}(q_{uu}) \quad \mu_0 = \delta \frac{6q_{uu}p_u - 2pq_{u^3}}{3q_{uu}^2}$
(d)	p $p_w q_u - p_u q_w$	q_u Δ	2	$\epsilon(\delta w^3 + uv)$ $\epsilon = \text{sgn}(q_u) \quad \delta = \text{sgn}(\Delta)$
(e)	q_u q_{uu}	p q_{u^3}	2	$\epsilon w + (\delta u^3 + \mu_0 u^5)v$ $\epsilon = \text{sgn}(p) \quad \delta = \text{sgn}(q_{u^3})$ $\mu_0 = \epsilon \frac{-720p_{u^2}q_{u^3}^2 + 60p_u q_{u^3} q_{u^4} - 15pq_{u^4}^2 + 12pq_{u^3} q_{u^5}}{40q_{u^3}^3}$
(f)	p q_u	p_u, q_w, q_{uu} $p_u^2 - 2q_w q_{uu}$	2	$\epsilon(uw + (\alpha_0 w + \beta_0 u^2)v)$ $\epsilon = \text{sgn}(p_u) \quad \alpha_0 = \frac{q_w}{p_u} \quad \beta_0 = \frac{q_{uu}}{2p_u}$

Table 5.1: Solution to recognition problems for singularities up to topological codimension two for strategy functions f with a singular strategy at $(0, 0)$. (Def = defining conditions. ND = non-degeneracy conditions. TC = topological codimension. $\Delta = q_u^2(q_u p_{ww} - p_u q_{ww}) + q_w^2(q_u p_{uu} - p_u q_{uu}) - 2q_u q_w(q_u p_{uw} - p_u q_{uw})$)

Proof of Theorem 5.1:

- (a) First, we show that f is dimorphism equivalent to $\epsilon(w + \mu_0 uv)$ if the corresponding defining and non-degeneracy conditions in Table 5.1 are satisfied.

We can write the strategy function $f = pw + qv$ into the form

$$p = p \quad \text{and} \quad q = uq_1 + wq_2$$

where $p(0,0) \neq 0$ and $q_1(0,0) \neq 0$. We calculate $\mathcal{I}(f)$. With formula (4.2.8), we have

$$\begin{aligned} [p, q] &= [p, uq_1 + wq_2] \\ [q, wp] &= [uq_1 + wq_2, wp] \\ [wp_u, wq_u] &= [wp_u, w(q_1 + u(q_1)_u + w(q_2)_u)] \\ [wp + 2w^2p_w, 2w^2q_w] &= [wp + 2w^2p_w, 2w^2(u(q_1)_w + q_2 + w(q_2)_w)] \end{aligned}$$

We claim that $[\mathcal{M}, \mathcal{M}^2] \subset \mathcal{I}(f)$. With Nakayama's Lemma, we need to prove

$$[\mathcal{M}, \mathcal{M}^2] \subset \mathcal{I}(f) + [\mathcal{M}^2, \mathcal{M}^3]$$

Let $\mathcal{J} = \mathcal{I}(f) + [\mathcal{M}^2, \mathcal{M}^3]$. We have

$$\begin{aligned} \mathcal{M}[uq_1 + wq_2, wp] \in \mathcal{J} &\Rightarrow [0, w\mathcal{M}] \in \mathcal{J} \\ [0, w\mathcal{M}], w[p, uq_1 + wq_2] \in \mathcal{J} &\Rightarrow [w, 0] \in \mathcal{J} \\ [w, 0], [0, w\mathcal{M}], [wp_u, w(q_1 + u(q_1)_u + w(q_2)_u)] \in \mathcal{J} &\Rightarrow [0, w] \in \mathcal{J} \\ [w, 0], [0, w], [uq_1 + wq_2, wp] \in \mathcal{J} &\Rightarrow [u, 0] \in \mathcal{J} \\ [u, 0], [0, w\mathcal{M}], u[p, uq_1 + wq_2] \in \mathcal{J} &\Rightarrow [0, u^2] \in \mathcal{J} \end{aligned}$$

Thus we have proved the claim that $[\mathcal{M}, \mathcal{M}^2] \subset \mathcal{I}(f)$. Therefore, $\mathcal{J} = \mathcal{I}$. The above calculation shows that

$$\langle [w, 0], [u, 0], [0, w], [0, u^2] \rangle \subset \mathcal{I}(f).$$

Denote

$$\mathcal{K} = \langle [w, 0], [u, 0], [0, w], [0, u^2] \rangle$$

We know that

$$\mathcal{I}(f) = \mathcal{K} + \langle [p(0,0), uq_1(0,0)] \rangle$$

Since for any strategy function h inside the module \mathcal{K} , $f + th$ satisfies all the defining and non-degeneracy conditions of the theorem. Thus we know that

$$\mathcal{I}(f + th) = \mathcal{I}(f)$$

for all $h \in \mathcal{K}$, $t \in [0, 1]$. According to Theorem 4.7, we know that $f + h$ is dimorphism equivalent to f . So we can assume that f is in the following form:

$$f = p_{00}w + q_{10}uv$$

where

$$p_{00} = p(0, 0) \neq 0 \quad q_{10} = q_u(0, 0) \neq 0$$

Next, we will try to further simplify the form of the strategy function f with specific dimorphism equivalence transformation.

As discussed earlier, for any $h \in \mathcal{K}$ we have

$$\mathcal{I}(f) = \mathcal{I}(f + h)$$

Thus in the following calculation, we can modulo \mathcal{K} , that is, we will ignore all terms in \mathcal{K} during the dimorphism transformation.

Let (S, Φ) satisfy that

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

where $S_e, S_o, \varphi_o, \varphi_e \in \mathcal{E}$. In Proposition 4.2, we show

$$u(\Phi) = 2\varphi_e \quad v(\Phi) = 2\varphi_o v \quad w(\Phi) = 4\varphi_o^2 w$$

$$\varphi_e(0, 0) = 0 \quad (\varphi_e)_u(u, 0) = \varphi_o(u, 0)$$

Now consider the case when

$$S_e = \frac{\epsilon}{p_{00}}$$

$$S_o = 0$$

$$\varphi_e = \frac{1}{2}u$$

$$\varphi_o = \frac{1}{2}$$

By the definition of dimorphism equivalence, we know f is dimorphism equivalent to $g = Sf(\Phi)$. That is,

$$\begin{aligned}
g &= (S_e + S_o v)(p(\Phi)w(\Phi) + q(\Phi)v(\Phi)) \\
&= (S_e p(\Phi)w(\Phi) + S_o q(\Phi)v(\Phi)v) + (S_e q(\Phi)v(\Phi) + S_o p(\Phi)w(\Phi)v) \\
&= (S_e p(\Phi)(4\varphi_o^2) + S_o q(\Phi)(2\varphi_o))w + (S_e q(\Phi)(2\varphi_o) + S_o p(\Phi)(4\varphi_o^2))v \\
&= (\epsilon + p^h(u, w))w + \left(\epsilon \frac{q_{10}}{p_{00}}u + q^h(u, w)\right)v
\end{aligned}$$

where $[p^h, q^h] \in \mathcal{K}$. Therefore, we have shown that f is dimorphism equivalent to

$$\epsilon(w + \mu_0 uv)$$

where at $(0, 0)$

$$\epsilon = \text{sgn}(p) \quad \mu_0 = \frac{q_u}{p}$$

At the end, we prove that if $f = pw + qv$ is dimorphism equivalent to

$$h = \epsilon(w + \mu_0 uv)$$

where $\mu_0 \neq 0$ and the singularity is at $(0, 0)$, then the corresponding defining and non-degeneracy conditions in Table 5.1 hold. The dimorphism equivalence indicates that there exists a dimorphism equivalence transformation (S, Φ) such that $f = Sh(\Phi)$ and we can write

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

where at $(0, 0)$

$$S_e > 0 \quad \varphi_e = 0 \quad (\varphi_e)_u = \varphi_o > 0$$

Then we have

$$\begin{aligned}
f = pw + qv &= (S_e + S_o v)\epsilon(4\varphi_o^2 w + \mu_0 4\varphi_e \varphi_o v) \\
&= \epsilon[(4\varphi_o^2 S_e + 4\varphi_e \varphi_o S_o \mu_0)w + (4\varphi_e \varphi_o S_e \mu_0 + 4\varphi_o^2 w S_o)v]
\end{aligned}$$

Therefore,

$$p = \epsilon(4\varphi_o^2 S_e + 4\varphi_e \varphi_o S_o)\mu_0 \quad q = \epsilon(4\varphi_e \varphi_o S_e \mu_0 + 4\varphi_o^2 w S_o)$$

So we know that at $(u, v) = (0, 0)$

$$p = 4\epsilon\varphi_o^2 S_e \neq 0 \quad q = 0 \quad q_u = 4(\varphi_e)_u \varphi_o S_e \epsilon \mu_0 \neq 0$$

Now we have completely proved part (a).

- (b) First, we show that f is dimorphism equivalent to $\epsilon(\delta w^2 + uv)$ if the corresponding defining and non-degeneracy conditions in Table 5.1 are satisfied.

Since $p(0, 0) = q(0, 0) = 0$, using Taylor Theorem, we can write

$$p(u, w) = p_{10}u + p_{01}w + r(u, w) \quad q(u, w) = q_{10}u + q_{01}w + s(u, w)$$

where

$$r(u, w), s(u, w) \in \mathcal{M}^2 \quad p_{10} = p_u \quad p_{01} = p_w \quad q_{10} = q_u \quad q_{01} = q_w$$

The assumptions indicate that $q_{10} \neq 0$ and $p_{01}q_{10} - p_{10}q_{01} \neq 0$. We claim that $\mathcal{I}(f)$ is independent with all $r(u, w), s(u, w)$.

Now we show that $[\mathcal{M}^2, \mathcal{M}^2] \subset \mathcal{I}(f)$. Recall the formula (4.2.8)

$$\mathcal{I}(f) = \langle [p, q], [q, wp], [wp_u, wq_u], [wp + 2w^2 p_w, 2w^2 q_w] \rangle$$

To apply Nakayama's Lemma, we need to show

$$[\mathcal{M}^2, \mathcal{M}^2] \subset \mathcal{I}(f) + [\mathcal{M}^3, \mathcal{M}^3].$$

Denote $\mathcal{J} = \mathcal{I}(f) + [\mathcal{M}^3, \mathcal{M}^3]$, we have

$$\begin{aligned}
u[p_{10}u + p_{01}w + r(u, w), q_{10}u + q_{01}w + s(u, w)] &\in \mathcal{J} \\
w[p_{10}u + p_{01}w + r(u, w), q_{10}u + q_{01}w + s(u, w)] &\in \mathcal{J} \\
u[q_{10}u + q_{01}w + s(u, w), p_{10}uw + p_{01}w^2 + wr(u, w)] &\in \mathcal{J} \\
w[q_{10}u + q_{01}w + s(u, w), p_{10}uw + p_{01}w^2 + wr(u, w)] &\in \mathcal{J} \\
u[p_{10}w + r_u(u, w), q_{10}w + s_u(u, w)] &\in \mathcal{J} \\
w[p_{10}w + r_u(u, w), q_{10}w + s_u(u, w)] &\in \mathcal{J}
\end{aligned}$$

Note that $[\mathcal{M}^3, \mathcal{M}^3] \subset \mathcal{J}$, so

$$\begin{aligned}
[p_{10}u^2 + p_{01}uw, q_{10}u^2 + q_{01}uw] &\in \mathcal{J} \\
[p_{10}uw + p_{01}w^2, q_{10}uw + q_{01}w^2] &\in \mathcal{J} \\
[q_{10}u^2 + q_{01}uw, 0] &\in \mathcal{J} \\
[q_{10}uw + q_{01}w^2, 0] &\in \mathcal{J} \\
[p_{10}uw, q_{10}uw] &\in \mathcal{J} \\
[p_{10}w^2, q_{10}w^2] &\in \mathcal{J}
\end{aligned} \tag{5.0.1}$$

Since $[\mathcal{M}^2, \mathcal{M}^2]$ has a basis $\{[u^2, 0], [uw, 0], [w^2, 0], [0, u^2], [0, uw], [0, w^2]\}$, we claim that the vectors in (5.0.1) are independent according to the this basis.

Note that the corresponding matrix is

$$T = \begin{pmatrix} p_{10} & p_{01} & 0 & q_{10} & q_{01} & 0 \\ 0 & p_{10} & p_{01} & 0 & q_{10} & q_{01} \\ q_{10} & q_{01} & 0 & 0 & 0 & 0 \\ 0 & q_{10} & q_{01} & 0 & 0 & 0 \\ 0 & p_{10} & 0 & 0 & q_{10} & 0 \\ 0 & 0 & p_{10} & 0 & 0 & q_{10} \end{pmatrix} \tag{5.0.2}$$

Since the determinant of the matrix T is

$$\det(T) = q_{10}^4(p_{10}q_{01} - p_{01}q_{10}) \neq 0$$

Therefore, we have proved that $[\mathcal{M}^2, \mathcal{M}^2] \subset \mathcal{J}$. By applying Nakayama's Lemma, we have proved $[\mathcal{M}^2, \mathcal{M}^2] \subset \mathcal{I}(f)$ if the corresponding defining and non-degeneracy conditions in Table 5.1 are satisfied. Hence, we know

$$\begin{aligned}
\mathcal{I}(f) &= \mathcal{I}(f) + [\mathcal{M}^2, \mathcal{M}^2] \\
&= \langle [p, q], [q, wp], [wp_u, wq_u], [wp + 2w^2p_w, 2w^2q_w] \rangle + [\mathcal{M}^2, \mathcal{M}^2] \\
&= \langle [p_{10}u + p_{01}w, q_{10}u + q_{01}w], [q_{10}u + q_{01}w, 0], [p_{10}w, q_{10}w] \rangle + [\mathcal{M}^2, \mathcal{M}^2] \\
&= \mathcal{I}(((p_{10}u + p_{01}w)w + (q_{10}u + q_{01}w)v)) + [\mathcal{M}^2, \mathcal{M}^2] \\
&= \mathcal{I}(((p_{10}u + p_{01}w)w + (q_{10}u + q_{01}w)v))
\end{aligned}$$

This proves the claim that $\mathcal{I}(f)$ is independent with $r(u, w), s(u, w)$. So by Theorem 4.7, we know $f(x, y) = p(u, w)w + q(u, w)v$ is dimorphism equivalent to

$$(p_{10}u + p_{01}w)w + (q_{10}u + q_{01}w)v$$

where $p_{10} = p_u, p_{01} = p_w, q_{10} = q_u \neq 0, q_{01} = q_w$ and $p_{01}q_{10} - p_{10}q_{01} \neq 0$.

Next we will further simplify the function $((p_{10}u + p_{01}w)w + (q_{10}u + q_{01}w)v)$ with a specific dimorphism equivalence transformation modulo $\mathcal{M}^2 \times \mathcal{M}^2$. From here, we can assume

$$f(x, y) = (p_{10}u + p_{01}w)w + (q_{10}u + q_{01}w)v$$

where $q_{10} \neq 0, p_{01}q_{10} - p_{10}q_{01} \neq 0$.

Let (S, Φ) satisfy that

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

where $S_e, S_o, \varphi_o, \varphi_e \in \mathcal{E}$. Note that in Proposition 4.2, we showed

$$u(\Phi) = 2\varphi_e \quad v(\Phi) = 2\varphi_o v \quad w(\Phi) = 4\varphi_o^2 w$$

$$\varphi_e(0, 0) = 0 \quad (\varphi_e)_u(u, 0) = \varphi_o(u, 0)$$

Now consider the case when

$$\begin{aligned}
S_e &= \frac{\delta(p_{01}q_{10} - p_{10}q_{01})}{\epsilon q_{10}^3} \\
S_o &= -\frac{p_{10}(\sqrt{\delta(p_{01}q_{10} - p_{10}q_{01})})}{\epsilon q_{10}^3} \\
\varphi_e &= \frac{\epsilon q_{10}(\sqrt{\delta(p_{01}q_{10} - p_{10}q_{01})}u - q_{01}w)}{2\delta(p_{01}q_{10} - p_{10}q_{01})} \\
\varphi_o &= \frac{\epsilon q_{10}}{2\sqrt{\delta(p_{01}q_{10} - p_{10}q_{01})}}
\end{aligned}$$

where

$$\epsilon = \text{sgn}(q_{10}) \quad \delta = \text{sgn}(p_{01}q_{10} - p_{10}q_{01})$$

Therefore, f is dimorphism equivalent to

$$\begin{aligned}
g(x, y) &= (S_e + S_o v)f(\Phi(x, y)) \\
&= (S_e + S_o v)((p_{10}u(\Phi(x, y)) + p_{01}w(\Phi(x, y)))w(\Phi(x, y)) \\
&\quad + (q_{10}u(\Phi(x, y)) + q_{01}w(\Phi(x, y)))v(\Phi(x, y))) \\
&= (S_e(p_{10}u(\Phi) + p_{01}w(\Phi)) + S_o(q_{10}u(\Phi) + q_{01}w(\Phi)))w(\Phi) \\
&\quad + (S_o(p_{10}u(\Phi) + p_{01}w(\Phi)) + S_e(q_{10}u(\Phi) + q_{01}w(\Phi)))v(\Phi) \\
&= ((p_{10}S_e + q_{10}S_o)u(\Phi) + (p_{01}S_e + q_{01}S_o)w(\Phi))w(\Phi) + \\
&\quad (q_{10}S_e u(\Phi) + q_{01}S_e w(\Phi))v(\Phi) + h.o.t \\
&= ((p_{10}S_e + q_{10}S_o)2\varphi_o + (p_{01}S_e + q_{01}S_o)4\varphi_o^2 w)4\varphi_o^2 w + \\
&\quad (q_{10}S_e 2\varphi_o + q_{01}S_e 4\varphi_o^2 w)2\varphi_o v + h.o.t \\
&= \epsilon((\delta w + p^h(u, w)) + (u + q^h(u, w))v)
\end{aligned}$$

where $[p^h, q^h] \in [\mathcal{M}^2, \mathcal{M}^2]$. Thus, we have shown that f is dimorphism equivalent to

$$g = \epsilon(\delta w^2 + uv)$$

as long as $p = 0$, $q = 0$, $q_u \neq 0$ and $p_w q_u - p_u q_w \neq 0$.

At the end, we prove that if $f = pw + qv$ is dimorphism equivalent to

$$h = \epsilon(\delta w^2 + uv)$$

with the singularity at $(0, 0)$, then the corresponding defining and non-degeneracy conditions in Table 5.1 hold. The dimorphism equivalence indicates that there exists a dimorphism equivalence transformation (S, Φ) such that $f = Sh(\Phi)$ and we can write

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

where at $(0, 0)$

$$S_e > 0 \quad \varphi_e = 0 \quad (\varphi_e)_u = \varphi_o > 0$$

Then we have

$$f = pw + qv = \epsilon(16\delta\varphi_o^4 w S_e + 4\varphi_e \varphi_o S_o)w + \epsilon(4\varphi_e \varphi_o S_e + 16\delta\varphi_o^4 w^2 S_o)v$$

Therefore,

$$p = \epsilon(16\delta\varphi_o^4 w S_e + 4\varphi_e \varphi_o S_o) \quad q = \epsilon(4\varphi_e \varphi_o S_e + 16\delta\varphi_o^4 w^2 S_o)$$

So we know that when $(u, v) = (0, 0)$

$$p = 0 \quad q = 0 \quad q_u = 4\epsilon(\varphi_e)_u \varphi_o S_e \neq 0$$

and

$$\begin{aligned} p_w q_u - p_u q_w &= (16\delta\varphi_o^4 S_e + 4(\varphi_e)_w \varphi_o S_o)((4\varphi_e)_u \varphi_o S_e) \\ &\quad - (4(\varphi_e)_u \varphi_o S_o)(4(\varphi_e)_w \varphi_o S_e) \\ &= 16\delta\varphi_o^4 S_e \neq 0 \end{aligned}$$

Now we have completely proved part (b).

(c) First, we show that f is dimorphism equivalent to

$$(\epsilon + \mu_0 u)w + \delta u^2 v$$

if the corresponding defining and non-degeneracy conditions in Table 5.1 are satisfied.

Define

$$\mathcal{K} = \langle [u^3, 0], [w, 0], [0, u^5], [0, w^2], [0, uw] \rangle$$

We claim $\mathcal{K} \subset \mathcal{I}(f)$.

We use Nakayama's Lemma to prove this claim. So we only need to show

$$\mathcal{K} \subset \mathcal{I}(f) + \mathcal{MK}.$$

Denote $\mathcal{J} = \mathcal{I}(f) + \mathcal{MK}$. We assume that

$$\begin{aligned} p(u, w) &= p_{00} + p_{10}u + p_{01}w + p_{20}u^2 + p_{30}u^3 + \hat{p}(u, w) \\ q(u, w) &= q_{01}w + q_{20}u^2 + q_{11}uw + q_{02}w^2 + q_{30}u^3 + q_{40}u^4 + q_{50}u^5 + \hat{q}(u, w) \end{aligned}$$

where

$$p_{ij} = \frac{p_{u^i w^j}}{i!j!}(0, 0) \quad q_{ij} = \frac{q_{u^i w^j}}{i!j!}(0, 0) \quad p_{00} \neq 0 \quad q_{20} \neq 0 \quad [\hat{p}(u, w), \hat{q}(u, w)] \in \mathcal{MK}$$

From the construction of \mathcal{MK} , we know that for any choice of

$$[\hat{p}(u, w), \hat{q}(u, w)] \in \mathcal{MK}$$

we have

$$\begin{aligned} &[\hat{p}(u, w), \hat{q}(u, w)], [\hat{q}(u, w), w\hat{p}(u, w)], [w\hat{q}_u(u, w), w\hat{q}_u(u, w)], \\ &[w\hat{p}(u, w) + 2w^2\hat{q}_w(u, w), 2w^2\hat{q}_w(u, w)] \in \mathcal{MK}. \end{aligned} \quad (5.0.3)$$

Then by definition of $\mathcal{I}(f)$, we obtain

$$\begin{aligned}
& [p_{00} + p_{10}u + p_{01}w + p_{20}u^2 + p_{30}u^3, \\
& q_{20}u^2 + q_{01}w + q_{02}w^2 + q_{11}uw + q_{30}u^3 + q_{40}u^4 + q_{50}u^5] \in \mathcal{J} \\
& [q_{20}u^2 + q_{01}w + q_{30}u^3, p_{00}w + p_{10}uw + p_{01}w^2] \in \mathcal{J} \\
& [p_{10}w, 2q_{20}uw + q_{11}w^2] \in \mathcal{J} \\
& [p_{00}w, 2q_{01}w^2] \in \mathcal{J}
\end{aligned}$$

Therefore,

$$\begin{aligned}
w[q_{20}u^2 + q_{01}w + q_{30}u^3, p_{00}w + p_{10}uw + p_{01}w^2] &= [0, p_{00}w^2] \in \mathcal{J} \\
[p_{00}w, 2q_{01}w^2] - 2q_{01}b[0, w^2] &= [p_{00}w, 0] \in \mathcal{J} \\
[p_{10}w, 2q_{20}uw + q_{11}w^2] - q_{11}[0, w^2] - p_{10}[w, 0] &= [0, 2q_{20}uw] \in \mathcal{J} \\
u[q_{20}u^2 + q_{01}w + q_{30}u^3, p_{00}w + p_{10}uw + p_{01}w^2] \\
- q_{01}u[w, 0] - (p_{00} + p_{10}u + p_{01}w)[0, uw] &= [q_{20}u^3, 0] \in \mathcal{J} \\
u^3[p_{00} + p_{10}u + p_{01}w + p_{20}u^2 + p_{20}u^3, \\
q_{20}u^2 + q_{01}w + q_{02}w^2 + q_{11}uw + q_{30}u^3 + q_{40}u^4 + q_{50}u^5] &= [0, q_{20}u^5] \in \mathcal{J}
\end{aligned}$$

This proves the claim that $\mathcal{K} \subset \mathcal{I}(f)$ as long as

$$q = q_u = 0 \quad p \neq 0 \quad q_{uu} \neq 0$$

Therefore, by definition, we have

$$\begin{aligned}
\mathcal{I}(f) &= \langle [p, q], [q, wp], [wp_u, wq_u], [wp + 2w^2p_w, 2w^2q_w] \rangle \\
&= \mathcal{K} + \langle [p_{00} + p_{10}u + p_{20}u^2, q_{20}u^2 + q_{01}w + q_{30}u^3 + q_{40}u^4], [q_{20}u^2, p_{00}w] \rangle \\
&= \mathcal{K} + \mathbf{R}\{[p_{00} + p_{10}u + p_{20}u^2, q_{20}u^2 + q_{01}w + q_{30}u^3 + q_{40}u^4], \\
& \quad [p_{00}u + p_{10}u^2, q_{20}u^3 + q_{30}u^4], [p_{00}u^2, q_{20}u^4], [q_{20}u^2, p_{00}w]\}
\end{aligned}$$

The expression of $\mathcal{I}(f)$ indicates that for any $h \in \mathcal{K}$, it can not change the form of \mathcal{I} , that is

$$\mathcal{I}(f + th) = \mathcal{I}(f)$$

holds for all $h \in \mathcal{K}$, $t \in [0, 1]$. Therefore, according to the Theorem 4.7 we know that f is dimorphism equivalent to the function

$$(p_{00} + p_{10}u + p_{20}u^2)w + (q_{20}u^2 + q_{01}w + q_{30}u^3 + q_{40}u^4)v$$

In the following, we can just work on this simplified function. Next, we use a specific dimorphism equivalence transformation to simplify f modulo \mathcal{K} .

Let (S, Φ) satisfy that

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

where $S_e, S_o, \varphi_e, \varphi_o \in \mathcal{E}$. Note that in Proposition 4.2, we showed

$$u(\Phi) = 2\varphi_e \quad v(\Phi) = 2\varphi_o v \quad w(\Phi) = 4\varphi_e^2 w$$

$$\varphi_e(0, 0) = 0 \quad (\varphi_e)_u(u, 0) = \varphi_o(u, 0)$$

Now consider the case when

$$\begin{aligned} S_e &= \frac{q_{20}^2}{\epsilon p_{00}^3} - \frac{\delta q_{30}}{p_{00}^2} u \\ &\quad + \frac{5p_{00}p_{20}q_{20}^2\epsilon - 5p_{00}p_{10}q_{30}q_{20}\epsilon - 6p_{00}^2q_{40}q_{20}\epsilon + 6p_{00}^2q_{30}^2\epsilon - 5q_1q_{20}^3\epsilon}{p_{00}^3q_{20}^2} u^2 \\ S_o &= -\frac{q_1q_{20}}{\delta p_{00}^3} \\ \varphi_e &= \frac{\epsilon p_{00}}{2\delta q_{20}} u + \frac{\epsilon p_{00}(-p_{00}p_{20}q_{20}^2 + p_{00}p_{10}q_{30}q_{20} + p_{00}^2q_{40}q_{20} - p_{00}^2q_{30}^2 + q_1q_{20}^3)}{2\delta q_{20}^5} u^3 \\ \varphi_o &= \frac{\epsilon p_{00}}{2\delta q_{20}} \end{aligned}$$

where

$$\epsilon = \text{sgn}(p_{00}) \quad \delta = \text{sgn}(q_{20})$$

Therefore, f is dimorphism equivalent to

$$\begin{aligned}
Sf(\Phi) &= (S_e + S_o v)(p(\Phi)w(\Phi) + q(\Phi)v(\Phi)) \\
&= (S_e + S_o v)((p_{00} + p_{10}2\varphi_e + p_{20}4\varphi_e^2)4\varphi_o^2 w \\
&\quad + (q_{20}4\varphi_e^2 + q_{01}4\varphi_o^2 w + q_{30}8\varphi_e^3 + q_{40}16\varphi_e^4)2\varphi_o v) \\
&= \left(\epsilon + \frac{\delta(p_{10}q_{20} - p_{00}q_{30})}{q_{20}^2}\right)u + p^h(u, w)w + (\delta u^2 + q^h(u, w))v
\end{aligned}$$

where $[p^h, q^h] \in \mathcal{K}$. Therefore, we have shown that f is dimorphism equivalent to

$$(\epsilon + \mu_0 u)w + \delta u^2 v$$

where

$$\epsilon = \text{sgn}(p) \quad \delta = \text{sgn}(q_{uu}) \quad \mu_0 = \delta \frac{\frac{1}{2}q_{uu}p_u - \frac{1}{6}pq_{uuu}}{\frac{1}{4}q_{uu}^2}$$

At the end, we prove that if $f = pw + qv$ is dimorphism equivalent to

$$(\epsilon + \mu_0 u)w + \delta u^2 v$$

with the singularity at $(0, 0)$, then the corresponding defining and non-degeneracy conditions in Table 5.1 hold. The dimorphism equivalence indicates that there exists a dimorphism equivalence transformation (S, Φ) such that $f = Sh(\Phi)$ and we can write

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

where at $(0, 0)$

$$S_e > 0 \quad \varphi_e = 0 \quad (\varphi_e)_u = \varphi_o > 0$$

Then we have

$$f = pw + qv = [4e^2(\epsilon + 2\mu_0\varphi_e)S_e + 8\delta\varphi_e^2\varphi_o S_o]w + [4\varphi_o^2(\epsilon + 2\mu_0\varphi_e)wS_o + 8\delta\varphi_e^2\varphi_o S_e]v$$

Therefore,

$$p = 4\varphi_o^2(\epsilon + 2\mu_0\varphi_e)S_e + 8\delta\varphi_e^2\varphi_o S_o \quad q = 4\varphi_o^2(\epsilon + 2\mu_0\varphi_e)wS_o + 8\delta\varphi_e^2\varphi_o S_e$$

So we know that at $(0, 0)$

$$p = 4\epsilon\varphi_o^2 \neq 0 \quad q = 0 \quad q_u = 0 \quad q_{uu} = 8\delta(\varphi_e)_u^2 \varphi_o S_e \neq 0$$

Now we have completely proved part (c).

(d) First, we show that f is dimorphism equivalent to

$$\epsilon(\delta w^3 + uv)$$

if the corresponding defining and non-degeneracy conditions in Table 5.1 are satisfied.

Since at $(0, 0)$ we have

$$q = 0 \quad p = 0 \quad p_w q_u - p_u q_w = 0 \quad q_u \neq 0$$

Then we can write

$$\begin{aligned} p(u, w) &= p_{10}u + p_{01}w + p_{20}u^2 + p_{11}uw + p_{02}w^2 + \hat{p}^h(u, w) \\ q(u, w) &= q_{10}u + q_{01}w + q_{20}u^2 + q_{11}uw + q_{02}w^2 + \hat{q}^h(u, w) \end{aligned}$$

where

$$p_{ij} = \frac{p_{u^i w^j}}{i!j!}(0, 0) \quad q_{ij} = \frac{q_{u^i w^j}}{i!j!}(0, 0) \quad p_{10}q_{01} - p_{01}q_{10} = 0 \quad q_{10} \neq 0$$

and \hat{p}^h, \hat{q}^h are order 3 respect to u, w .

Let (S, Φ) satisfy that

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

where $S_e, S_o, \varphi_o, \varphi_e \in \mathcal{E}$. Note that in Proposition 4.2, we showed

$$u(\Phi) = 2\varphi_e \quad v(\Phi) = 2\varphi_o v \quad w(\Phi) = 4\varphi_o^2 w$$

$$\varphi_e(0, 0) = 0 \quad (\varphi_e)_u(u, 0) = \varphi_o(u, 0)$$

Now consider the case when

$$\begin{aligned} \varphi_e &= \frac{1}{2}u - \frac{q_{01}}{2q_{10}}w \\ \varphi_o &= \frac{1}{2} \\ S_e &= \frac{\epsilon}{q_{10}} \\ S_o &= -\frac{\epsilon p_{10}}{q_{10}^2} + \frac{\epsilon(p_{10}q_{20} - q_{10}p_{20})}{q_{10}^3}u \\ &\quad + \frac{\epsilon[2q_{01}(p_{10}q_{20} - q_{10}p_{20}) + q_{10}(p_{10}q_{11} - q_{10}p_{11})]}{q_{10}^4}w \end{aligned}$$

where $\epsilon = \text{sgn}(q_{10})$. Therefore, f is dimorphism equivalent to $g = Sf(\Phi)$ and

$$\begin{aligned} g &= (S_e + S_o v)(p(\Phi)w(\Phi) + q(\Phi)v(\Phi)) \\ &= (S_e p(\Phi)w(\Phi) + S_o q(\Phi)v(\Phi)v) + (S_e q(\Phi)v(\Phi) + S_o p(\Phi)w(\Phi)v) \\ &= \left[\frac{\epsilon}{q_{10}^4} (q_{10}^2(q_{10}p_{02} - p_{10}q_{02}) + q_{01}^2(q_{10}p_{20} - p_{10}q_{20}) \right. \\ &\quad \left. - q_{10}q_{01}(q_{10}p_{12} - p_{10}q_{11}))w^2 + \tilde{p}(u, w) \right]w + (\epsilon u + \tilde{q}(u, w))v \end{aligned}$$

where \tilde{p} is order 3 and \tilde{q} is order 2 respect to u, w . Let

$$\begin{aligned} \tilde{p}_{22} &= \frac{1}{q_{10}^4} (q_{10}^2(q_{10}p_{02} - p_{10}q_{02}) + q_{01}^2(q_{10}p_{20} - p_{10}q_{20}) - q_{10}q_{01}(q_{10}p_{12} - p_{10}q_{11})) \\ \delta &= \text{sgn}(\tilde{p}_{22}) \end{aligned}$$

We know that f is dimorphism equivalent to

$$g = (\epsilon \tilde{p}_{22} w^2 + \tilde{p}(u, w))w + (\epsilon u + \tilde{q}(u, w))v$$

Note that in the assumption, we have

$$q_u^2(q_u p_{ww} - p_u q_{ww}) + q_w^2(q_u p_{uu} - p_u q_{uu}) - 2q_u q_w (q_u p_{uw} - p_u q_{uw}) \neq 0$$

That means

$$\tilde{p}_{22} \neq 0$$

Therefore, we can further apply the dimorphism equivalence transformation (S, Φ) satisfying that

$$\begin{aligned} S &= \sqrt{\delta \tilde{p}_{22}} \\ \varphi &= \frac{1}{2}(\delta \tilde{p}_{22})^{-\frac{1}{4}} \left[\left(u - \frac{\epsilon}{2}(\delta \tilde{p}_{22})^{-\frac{3}{4}} \tilde{q}_{ww}(0, 0) w^2 \right) + v \right] \end{aligned}$$

Then we know that g is dimorphism equivalent to

$$\begin{aligned} \tilde{g} = Sg(\Phi) &= \sqrt{\delta \tilde{p}_{22}} [(\epsilon \tilde{p}_{22}(\delta \tilde{p}_{22})^{-1} + \tilde{p}(\Phi))(\delta \tilde{p}_{22})^{-\frac{1}{2}} w \\ &\quad + (\epsilon(\delta \tilde{p}_{22})^{-\frac{1}{4}} \left(u - \frac{\epsilon}{2}(\delta \tilde{p}_{22})^{-\frac{3}{4}} \tilde{q}_{ww}(0, 0) w^2 \right) + \tilde{q}(\Phi))(\delta \tilde{p}_{22})^{-\frac{1}{4}} v] \\ &= (\epsilon \delta w^2 + \tilde{p}^h) w + \left(\epsilon u - \frac{1}{2}(\delta \tilde{p}_{22})^{-\frac{3}{4}} \tilde{q}_{ww}(0, 0) w^2 + (\delta \tilde{p}_{22})^{\frac{1}{4}} \tilde{q}(\Phi) \right) v \end{aligned}$$

Let

$$\tilde{q}^h = -\frac{1}{2}(\delta \tilde{p}_{22})^{-\frac{3}{4}} \tilde{q}_{ww}(0, 0) w^2 + (\delta \tilde{p}_{22})^{\frac{1}{4}} \tilde{q}(\Phi)$$

From the form of Φ and the fact that \tilde{q} is order 2 respect to u, w , we know

$$\tilde{q}_{ww}^h = -\frac{1}{2}(\delta \tilde{p}_{22})^{-\frac{3}{4}} \tilde{q}_{ww}(0, 0) + (\delta \tilde{p}_{22})^{\frac{1}{4}} \cdot \frac{1}{2} \tilde{q}_{ww}(0, 0) \cdot (\delta \tilde{p}_{22})^{-1} = 0$$

Thus, we have shown that given any strategy function $f = pw + qv$ that satisfies the corresponding defining and non-degeneracy conditions in Table 5.1 is dimorphism equivalent to

$$(\epsilon \delta w^2 + \tilde{p}^h) w + (\epsilon u + \tilde{q}^h) v \tag{5.0.4}$$

where

$$\epsilon = \text{sgn}(q_u)$$

$$\delta = \text{sgn}(q_u^2(q_u p_{ww} - p_u q_{ww}) + q_w^2(q_u p_{uu} - p_u q_{uu}) - 2q_u q_w(q_u p_{uw} - p_u q_{uw}))$$

and

$$\tilde{p}^h \in O(u^3 + w^3) \quad \tilde{q}^h \in O(u^2 + w^2)$$

with additional assumption that

$$\tilde{q}_{ww}^h = 0$$

Next, we will show that any strategy function in the form of (5.0.4) is dimorphism equivalent to $\epsilon(\delta w^3 + uw)$. Let

$$f_t = (\epsilon\delta w^2 + t\tilde{p}^h)w + (\epsilon u + t\tilde{q}^h)v$$

where $t \in [0, 1]$. To apply Theorem 4.7, we will show that

$$\mathcal{I}(f_0) = \mathcal{I}(f_t)$$

With the formula (4.2.8) and that

$$f_t = (\delta w^2 + t\tilde{p}^h)w + (\epsilon u + t\tilde{q}^h)v$$

we have

$$\begin{aligned} [p, q] &= [\delta\epsilon w^2 + t\tilde{p}^h, \epsilon u + t\tilde{q}^h] \\ [q, wp] &= [\epsilon u + t\tilde{q}^h, \delta\epsilon w^3 + tw\tilde{p}^h] \\ [wp_u, wq_u] &= [tw\tilde{p}_u^h, w(\epsilon + t\tilde{q}_u^h)] \\ [wp + 2w^2p_w, 2w^2q_w] &= [5\delta\epsilon w^3 + tw\tilde{p}^h + 2tw^2\tilde{p}_w^h, 2tw^2\tilde{q}_w^h] \end{aligned}$$

We claim that $[\mathcal{M}^3, \mathcal{M}^2] \subset \mathcal{I}(f_t)$. With Nakayama's Lemma, we need to prove

$$[\mathcal{M}^3, \mathcal{M}^2] \subset \mathcal{I}(f_t) + [\mathcal{M}^4, \mathcal{M}^3]$$

Let $\mathcal{J} = \mathcal{I}(f_t) + \mathcal{M}^4 \times \mathcal{M}^3$. Since $\tilde{p}^h \in O(u^3 + w^3)$, $\tilde{q}^h \in O(u^2 + w^2)$, we know

$$\begin{aligned} [5\delta\epsilon w^3 + tw\tilde{p}^h + 2tw^2\tilde{p}_w^h, 2tw^2\tilde{q}_w^h] \in \mathcal{J} &\Rightarrow [w^3, 0] \in \mathcal{J} \\ \mathcal{M}[\epsilon u + t\tilde{q}^h, \delta\epsilon w^3 + tw\tilde{p}^h] \in \mathcal{J} &\Rightarrow [u\mathcal{M}, 0] \in \mathcal{J} \\ [w^3, 0], [u\mathcal{M}, 0] \in \mathcal{J} &\Rightarrow [\mathcal{M}^3, 0] \in \mathcal{J} \\ \mathcal{M}[\delta\epsilon w^2 + t\tilde{p}^h, \epsilon u + t\tilde{q}^h] \in \mathcal{J} &\Rightarrow [0, u\mathcal{M}] \in \mathcal{J} \\ \mathcal{M}[tw\tilde{p}_u^h, w(\epsilon + t\tilde{q}_u^h)] \in \mathcal{J} &\Rightarrow [0, w\mathcal{M}] \in \mathcal{J} \\ [0, u\mathcal{M}], [0, w\mathcal{M}] \in \mathcal{J} &\Rightarrow [0, \mathcal{M}^2] \in \mathcal{J} \end{aligned}$$

Now, we have shown that $[\mathcal{M}^3, \mathcal{M}^2] \in \mathcal{I}(f_t)$. Therefore, we know

$$\mathcal{J} = \mathcal{I}(f_t)$$

Then we have

$$[u\mathcal{M}, 0] \in \mathcal{I}(f_t)$$

and

$$\begin{aligned} \mathcal{I}(f_t) &= [\mathcal{M}^3, \mathcal{M}^2] + [u\mathcal{M}, 0] \\ &\quad + [p, q] + [q, wp] + [wp_u, wq_u] + [wp + 2w^2p_w, 2w^2q_w] \\ &= [\mathcal{M}^3, \mathcal{M}^2] + [u\mathcal{M}, 0] + [\delta w^2, \epsilon u] + [\epsilon u + t\tilde{q}_{ww}^h(0, 0)w^2, 0] + [0, \epsilon w] \end{aligned}$$

Note that we have an assumption that $\tilde{q}_{ww}^h(0, 0)$, then we know

$$\mathcal{I}(f_t) = [\mathcal{M}^3, \mathcal{M}^2] + [u\mathcal{M}, 0] + [\delta\epsilon w^2, \epsilon u] + [\epsilon u, 0] + [0, \epsilon w]$$

With similar deduction, we can also show that

$$\mathcal{I}(f_0) = [\mathcal{M}^3, \mathcal{M}^2] + [u\mathcal{M}, 0] + [\delta\epsilon w^2, \epsilon u] + [\epsilon u, 0] + [0, \epsilon w]$$

Therefore, we have proved

$$\mathcal{I}(f_t) = \mathcal{I}(f_0)$$

for any $t \in [0, 1]$. According to Theorem 4.7, we can conclude that $f = f_1$ is dimorphism equivalent to f_0 . That is, f is dimorphism equivalent to $\epsilon(\delta w^3 + uv)$ where

$$\begin{aligned} \epsilon &= \text{sgn}(q_u) \\ \delta &= \text{sgn}(q_u^2(q_u p_{ww} - p_u q_{ww}) + q_w^2(q_u p_{uu} - p_u q_{uu}) - 2q_u q_w(q_u p_{uw} - p_u q_{uw})) \end{aligned}$$

At the end, we prove that if $f = pw + qv$ is dimorphism equivalent to

$$\epsilon(\delta w^3 + uv)$$

with the singularity at $(0, 0)$, then the corresponding defining and non-degeneracy conditions in Table 5.1 hold. The dimorphism equivalence indicates that there exists a dimorphism equivalence transformation (S, Φ) such that $f = Sh(\Phi)$ and we can write

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

where at $(0, 0)$

$$S_e > 0 \quad \varphi_e = 0 \quad (\varphi_e)_u = \varphi_o > 0$$

Then we have

$$\begin{aligned} f = pw + qv &= \epsilon(S_e + S_o v)(\delta 64 \varphi_o^6 w^3 + 4 \varphi_e \varphi_o v) \\ &= (64 \delta \epsilon \varphi_o^6 w^2 S_e + 4 \epsilon \varphi_e \varphi_o S_o)w + (4 \epsilon \varphi_e \varphi_o S_e + 64 \delta \epsilon \varphi_o^6 w^3 S_o)v \end{aligned}$$

Therefore,

$$p = 64 \delta \epsilon \varphi_o^6 w^2 S_e + 4 \epsilon \varphi_e \varphi_o S_o \quad q = 4 \epsilon \varphi_e \varphi_o S_e + 64 \delta \epsilon \varphi_o^6 w^3 S_o$$

So we know at $(0, 0)$

$$p = 0 \quad q = 0 \quad q_u = 4 \epsilon (\varphi_e)_u \varphi_o S_e \neq 0$$

$$p_w q_u - p_u q_w = 4 (\varphi_e)_w \varphi_o S_o \cdot 4 (\varphi_e)_u \varphi_o S_e - 4 (\varphi_e)_u \varphi_o S_o \cdot 4 (\varphi_e)_w \varphi_o S_e = 0$$

$$q_u^2 (q_u p_{ww} - p_u q_{ww}) + q_w^2 (q_u p_{uu} - p_u q_{uu}) - 2 q_u q_w (q_u p_{uw} - p_u q_{uw}) = 8192 \delta (\varphi_e)_u^3 \varphi_o^9 S_e^4 \neq 0$$

Now we have completely proved part (d).

(e) First, we show that f is dimorphism equivalent to

$$\epsilon w + (\delta u^3 + \mu_0 u^5)v$$

if the corresponding defining and non-degeneracy conditions in Table 5.1 are satisfied.

According to the assumptions, we know at $(0, 0)$

$$q = q_u = q_{uu} = 0 \quad p \neq 0 \quad q_{uuu} \neq 0$$

Thus we can write the strategy function $f = pw + qv$ into the form

$$p = p \quad q = wq_1 + u^3q_2$$

where

$$p(0, 0) \neq 0 \quad q_2(0, 0) \neq 0$$

First we calculate $\mathcal{I}(f)$. With formula (4.2.8), we have

$$\begin{aligned} [p, q] &= [p, wq_1 + u^3q_2] \\ [q, wp] &= [wq_1 + u^3q_2, wp] \\ [wp_u, wq_u] &= [wp_u, w(w(q_1)_u + 3u^2q_2 + w^3(q_3)_u)] \\ [wp + 2w^2p_w, 2w^2q_w] &= [wp + 2w^2p_w, 2w^2(q_1 + w(q_1)_w + u^3(q_2)_w)] \end{aligned}$$

We claim that $[\mathcal{M}^5, \mathcal{M}^8] \subset \mathcal{I}(f)$. With Nakayama's Lemma, we need to prove

$$[\mathcal{M}^5, \mathcal{M}^8] \subset \mathcal{I}(f) + [\mathcal{M}^6, \mathcal{M}^9]$$

Let $\mathcal{J} = \mathcal{I}(f) + [\mathcal{M}^6, \mathcal{M}^9]$. We have

$$\begin{aligned}
\mathcal{M}^5[wq_1 + u^3q_2, wp] \in \mathcal{J} &\Rightarrow [0, w\mathcal{M}^5] \in \mathcal{J} \\
\mathcal{M}^4[wp + 2w^2p_w, 2w^2(q_1 + w(q_1)_w + u^3(q_2)_w)] \in \mathcal{J} &\Rightarrow [w\mathcal{M}^4, 0] \in \mathcal{J} \\
w\mathcal{M}^3[wq_1 + u^3q_2, wp] \in \mathcal{J} &\Rightarrow [0, w^2\mathcal{M}^3] \in \mathcal{J} \\
\mathcal{M}^3[wp + 2w^2p_w, 2w^2(q_1 + w(q_1)_w + u^3(q_2)_w)] \in \mathcal{J} &\Rightarrow [w\mathcal{M}^3, 0] \in \mathcal{J} \\
w\mathcal{M}^2[wq_1 + u^3q_2, wp] \in \mathcal{J} &\Rightarrow [0, w^2\mathcal{M}^2] \in \mathcal{J} \\
\mathcal{M}^2[wp + 2w^2p_w, 2w^2(q_1 + w(q_1)_w + u^3(q_2)_w)] \in \mathcal{J} &\Rightarrow [w\mathcal{M}^2, 0] \in \mathcal{J} \\
w\mathcal{M}[wq_1 + u^3q_2, wp] \in \mathcal{J} &\Rightarrow [0, w^2\mathcal{M}] \in \mathcal{J} \\
\mathcal{M}[wp + 2w^2p_w, 2w^2(q_1 + w(q_1)_w + u^3(q_2)_w)] \in \mathcal{J} &\Rightarrow [w\mathcal{M}, 0] \in \mathcal{J} \\
w[wq_1 + u^3q_2, wp] \in \mathcal{J} &\Rightarrow [0, w^2] \in \mathcal{J} \\
[0, w^2], [wp + 2w^2p_w, 2w^2(q_1 + w(q_1)_w + u^3(q_2)_w)] \in \mathcal{J} &\Rightarrow [w, 0] \in \mathcal{J} \\
[w, 0], [0, w^2], w[p, wq_1 + u^3q_2] \in \mathcal{J} &\Rightarrow [0, wu^3] \in \mathcal{J} \\
[w, 0], [0, w^2], [wp_u, w(w(q_1)_u + 3u^2q_2 + w^3(q_3)_u)] \in \mathcal{J} &\Rightarrow [0, wu^2] \in \mathcal{J} \\
[w, 0], [0, wu^2], u^2[wq_1 + u^3q_2, wp] \in \mathcal{J} &\Rightarrow [u^5, 0] \in \mathcal{J} \\
[u^5, 0], [0, wu^2], u^5[p, wq_1 + u^3q_2] \in \mathcal{J} &\Rightarrow [0, u^8] \in \mathcal{J}
\end{aligned}$$

Thus we have proved the claim that $[\mathcal{M}^5, \mathcal{M}^8] \subset \mathcal{I}(f)$. Let

$$\mathcal{K} = \langle [w, 0], [u^5, 0], [0, w^2], [0, u^2w], [0, u^8] \rangle \subset \mathcal{I}(f)$$

The calculation indicates that as long as f satisfies the corresponding defining and non-degeneracy conditions in Table 5.1, we always have $\mathcal{K} \subset \mathcal{I}(f)$. That is, any strategy function h inside the module \mathcal{K} satisfies that

$$\mathcal{I}(f + th) = \mathcal{I}(f)$$

for all $t \in [0, 1]$. According to Theorem 4.7, we know that $f + h$ is dimorphism equivalent to f . So we can assume that f is in the following form modulo \mathcal{K} :

$$\begin{aligned}
f = &(p_{00} + p_{10}u + p_{20}u^2 + p_{30}u^3 + p_{40}u^4)w + (q_{01}w + q_{11}uw \\
&+ q_{30}u^3 + q_{40}u^4 + q_{50}u^5 + q_{60}u^6 + q_{70}u^7)v
\end{aligned}$$

where

$$p_{ij} = \frac{p u^i w^j}{i!j!}(0, 0) \quad q_{ij} = \frac{q u^i w^j}{i!j!}(0, 0) \quad p_{00} \neq 0 \quad q_{30} \neq 0$$

Next, we will try to further simplify the form of the strategy function f with specific dimorphism equivalence transformations. As discussed earlier, all terms in the module \mathcal{K} can not change the form $\mathcal{I}(f)$. Thus we can modulo \mathcal{K} when performing the calculation, that is, we will ignore any terms in \mathcal{K} during the dimorphism transformation.

Let (S, Φ) satisfy that

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

and consider the case

$$\begin{aligned} \varphi_e &= \frac{1}{2} \left(\sqrt{\frac{p_{00}\epsilon}{q_{30}\delta}} u + \epsilon \delta \frac{p_{10}q_{30} - p_{00}q_{40}}{q_{30}^2} u^2 \right) \\ \varphi_o &= \frac{1}{2} \left(\sqrt{\frac{p_{00}\epsilon}{q_{30}\delta}} + 2\epsilon \delta \frac{p_{10}q_{30} - p_{00}q_{40}}{q_{30}^2} u \right) \\ S_e &= \frac{q_{30}\delta}{p_{00}^2} + \frac{-5p_{10}q_{30} + 4p_{00}q_{40}}{\sqrt{p_{00}^5 q_{30} \epsilon \delta}} u \\ &\quad + \epsilon \frac{16p_{10}^2 q_{30}^2 - 28p_{00}p_{10}q_{30}q_{40} + 13p_{00}^2 q_{40}^2 - p_{00}^2 q_{30}q_{50}}{p_{00}^3 q_{30}^2} u^2 \\ S_o &= -q_{01} \sqrt{\frac{q_{30}\delta}{p_{00}^5 \epsilon}} + \epsilon \frac{4p_{10}q_{01}q_{30} - p_{00}q_{11}q_{30} - 2p_{00}q_{01}q_{40}}{p_{00}^3 q_{30}} u \end{aligned}$$

where $\epsilon = \text{sgn}(p_{00})$, $\delta = \text{sgn}(q_{30})$.

By the definition of dimorphism equivalence, we know f is dimorphism equivalent to $g = Sf(\Phi)$. That is,

$$\begin{aligned}
g &= (S_e + S_o v)(p(\Phi)w(\Phi) + q(\Phi)v(\Phi)) \\
&= (S_e p(\Phi)w(\Phi) + S_o q(\Phi)v(\Phi)v) + (S_e q(\Phi)v(\Phi) + S_o p(\Phi)w(\Phi)v) \\
&= (S_e p(\Phi)(4\varphi_o^2) + S_o q(\Phi)(2\varphi_o))w + (S_e q(\Phi)(2\varphi_o) + S_o p(\Phi)(4\varphi_o^2))v \\
&= (\epsilon + \tilde{p}_{30}u^3 + \tilde{p}_{40}u^4 + \tilde{p}^h(u, w))w \\
&\quad + (\delta u^3 + \tilde{q}_{50}u^5 + \tilde{q}_{60}u^6 + \tilde{q}_{70}u^7 + \tilde{q}^h(u, w))v
\end{aligned}$$

where

$$\tilde{q}_{50} = \epsilon \frac{-p_{20}q_{30}^2 + p_{10}q_{30}q_{40} - p_{00}q_{40}^2 + p_{00}q_{30}q_{50}}{q_{30}^3} \quad [\tilde{p}^h, \tilde{q}^h] \in \mathcal{K}$$

Next we take another dimorphism equivalence to reduce terms $\tilde{p}_{30}, \tilde{p}_{40}, \tilde{q}_{60}, \tilde{q}_{70}$.

Let (S, Φ) satisfy that

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

and consider

$$\begin{aligned}
\varphi_e &= \frac{1}{2}(u + (-4\epsilon\tilde{p}_4 + 5\delta\tilde{q}_7)u^4 + (-\frac{1}{2}\epsilon\tilde{p}_4 + \frac{1}{2}\delta\tilde{q}_7)u^5) \\
\varphi_o &= \frac{1}{2}(1 + 4(-4\epsilon\tilde{p}_4 + 5\delta\tilde{q}_7)u^3 + 5(-\frac{1}{2}\epsilon\tilde{p}_4 + \frac{1}{2}\delta\tilde{q}_7)u^4) \\
S_e &= 1 + (-7\epsilon\tilde{p}_3 + 8\delta\tilde{q}_6)u^3 + (-\epsilon\tilde{p}_3 + \delta\tilde{q}_6)u^4 \\
S_o &= 0
\end{aligned}$$

By the definition of dimorphism equivalence, we know g is dimorphism equivalent to $\tilde{g} = Sg(\Phi)$. That is,

$$\begin{aligned}
\tilde{g} &= (S_e + S_o v)(p(\Phi)w(\Phi) + q(\Phi)v(\Phi)) \\
&= (S_e p(\Phi)w(\Phi) + S_o q(\Phi)v(\Phi)v) + (S_e q(\Phi)v(\Phi) + S_o p(\Phi)w(\Phi)v) \\
&= (S_e p(\Phi)(4\varphi_o^2) + S_o q(\Phi)(2\varphi_o))w + (S_e q(\Phi)(2\varphi_o) + S_o p(\Phi)(4\varphi_o^2))v \\
&= (\epsilon + p^h(u, w))w + (\delta u^3 + \tilde{p}_{50}u^5 + q^h(u, w))v
\end{aligned}$$

where

$$\tilde{q}_{50} = \epsilon \frac{-p_{20}q_{30}^2 + p_{10}q_{30}q_{40} - p_{00}q_{40}^2 + p_{00}q_{30}q_{50}}{q_{30}^3} \quad [p^h, q^h] \in \mathcal{K}$$

This shows that g is dimorphism equivalent to \tilde{g} , so f is dimorphism equivalent to

$$\epsilon w + (\delta u^3 + \mu_0 u^5)v$$

where

$$\begin{aligned} \epsilon &= \text{sgn}(p) & \delta &= \text{sgn}(q_{u^3}) \\ \mu_0 &= \epsilon \frac{-\frac{1}{12}p_u^2 q_{u^3}^2 + \frac{1}{144}p_u q_{u^3} q_{u^4} - \frac{1}{576}p q_{u^4}^2 + \frac{1}{720}p q_{u^3} q_{u^5}}{\frac{1}{216}q_{u^3}^3} \end{aligned}$$

At the end, we prove that if $f = pw + qv$ is dimorphism equivalent to

$$\epsilon w + (\delta u^3 + \mu_0 u^5)v$$

with the singularity at $(0, 0)$, then the corresponding defining and non-degeneracy conditions in Table 5.1 hold. The dimorphism equivalence indicates that there exists a dimorphism equivalence transformation (S, Φ) such that $f = Sh(\Phi)$ and we can write

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

where

$$S_e > 0 \quad \varphi_e = 0 \quad (\varphi_e)_u = \varphi_o > 0$$

at $(0, 0)$. Then we have

$$\begin{aligned} f = pw + qv &= [4\epsilon\varphi_o^2 S_e + 2(8\delta\varphi_e^3 + 32\mu_0\varphi_e^5)\varphi_o S_o]w \\ &\quad + [4\varphi_o^2 \epsilon w S_o + 2(8\delta d^3 + 32\mu_0\varphi_e^5)\varphi_o w S_e]v \end{aligned}$$

Therefore,

$$p = 4\epsilon\varphi_o^2 S_e + 2(8\delta\varphi_e^3 + 32\mu_0\varphi_e^5)\varphi_o S_o \quad q = 4\varphi_o^2 \epsilon w S_o + 2(8\delta\varphi_e^3 + 32\mu_0\varphi_e^5)\varphi_o w S_e$$

So we know at $(0, 0)$

$$p = 4\epsilon\varphi_o^2 S_e \neq 0 \quad q = 0 \quad q_u = 0 \quad q_{uu} = 0 \quad q_{uuu} = 96\delta(\varphi_e)_u^3 \neq 0$$

Now we have completely proved part (e).

(f) First, we show that $f = pw + qv$ is dimorphism equivalent to

$$\epsilon(uw + (\alpha_0 w + \beta_0 u^2)v)$$

if the corresponding defining and non-degeneracy conditions in Table 5.1 are satisfied.

According to the assumptions, we can perform the dimorphism equivalence

(S, Φ) that satisfies

$$S = \frac{\epsilon}{p_u(0, 0)} \quad \Phi = \text{Identity}$$

where $\epsilon = \text{sgn}(p_u(0, 0))$. Then by the definition of dimorphism equivalence, we know f is dimorphism equivalent to $\bar{f} = Sf(\Phi)$. That is,

$$\bar{f} = Sf(\Phi) = \frac{\epsilon}{p_u(0, 0)}(pw + qv) = \epsilon \frac{p}{p_u(0, 0)}w + \epsilon \frac{q}{p_u(0, 0)}v$$

Define $\bar{f} = \bar{p}w + \bar{q}v$, that is

$$\bar{p} = \epsilon \frac{p}{p_u(0, 0)} \quad \bar{q} = \epsilon \frac{q}{p_u(0, 0)}$$

Now we assume that $\bar{f} = \bar{p}w + \bar{q}v$ satisfies the following form:

$$\begin{aligned} \bar{p} &= \bar{p}_{10}u + \bar{p}_{01}w + \bar{p}_{20}u^2 + \bar{p}_{11}uw + \bar{p}_{30}u^3 + \bar{p}^h(u, w) \\ \bar{q} &= q_{01}w + q_{20}u^2 + q_{11}uw + q_{02}w^2 + q_{30}u^3 + q_{21}u^2w + q_{40}u^4 + \bar{q}^h(u, w) \end{aligned}$$

where

$$\bar{p}_{ij} = \frac{\epsilon}{p_u(0, 0)} \frac{p_{u^i w^j}}{i!j!}(0, 0) \quad \bar{q}_{ij} = \frac{\epsilon}{p_u(0, 0)} \frac{q_{u^i w^j}}{i!j!}(0, 0) \quad \bar{p}^h \in \mathcal{M}^2 \quad \bar{q}^h \in \mathcal{M}^3$$

and

$$\begin{aligned}\bar{p}_{uu}^h(0,0) &= \bar{p}_{uw}^h(0,0) = \bar{p}_{u^3}^h(0,0) = 0 \\ \bar{q}_{u^3}^h(0,0) &= \bar{q}_{u^2w}^h(0,0) = \bar{q}_{u^4}^h(0,0) = 0\end{aligned}$$

One thing to point out here is that with direct calculation, we see

$$\bar{p}_{10} = \epsilon$$

Also with the assumptions in the theorem, we know that

$$\bar{q}_{20} \neq 0 \quad \bar{q}_{01} \neq 0 \quad \bar{p}_{10}^2 - 4\bar{q}_{20}\bar{q}_{01} = 1 - 4\bar{q}_{20}\bar{q}_{01} \neq 0$$

Let (S, Φ) satisfy that

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

where $S_e, S_o, \varphi_o, \varphi_e \in \mathcal{E}$. Note that in Proposition 4.2, we showed

$$u(\Phi) = 2\varphi_e \quad v(\Phi) = 2\varphi_o v \quad w(\Phi) = 4\varphi_o^2 w$$

$$\varphi_e(0,0) = 0 \quad (\varphi_e)_u(u,0) = \varphi_o(u,0)$$

Now consider the case

$$\begin{aligned}\varphi_e &= \frac{1}{2} \left(u - \frac{-2\bar{q}_{01}^2 \bar{q}_{20} \bar{p}_{20} + 2\bar{q}_{01} \bar{q}_{20}^2 \bar{p}_{01} - \bar{q}_{20} \bar{p}_{01} + \bar{q}_{01}^2 \bar{q}_{30} \epsilon + \bar{q}_{01} \bar{q}_{20} \bar{q}_{11} \epsilon}{\bar{q}_{20} \epsilon (4\bar{q}_{01} \bar{q}_{20} - 1)} w \right. \\ &\quad \left. - \frac{2\bar{q}_{01} \bar{q}_{20}^2 \bar{p}_{20} - 2\bar{q}_{20}^3 \bar{p}_{01} - \bar{q}_{20} \bar{p}_{20} - 3\bar{q}_{01} \bar{q}_{20} \bar{q}_{30} \epsilon + \bar{q}_{20}^2 \bar{q}_{11} \epsilon + \bar{q}_{30} \epsilon}{\bar{q}_{20} \epsilon (4\bar{q}_{01} \bar{q}_{20} - 1)} u^2 \right) \\ \varphi_o &= \frac{1}{2} \left(1 - \frac{2\bar{q}_{01} \bar{q}_{20}^2 \bar{p}_{20} - 2\bar{q}_{20}^3 \bar{p}_{01} - \bar{q}_{20} \bar{p}_{20} - 3\bar{q}_{01} \bar{q}_{20} \bar{q}_{30} \epsilon + \bar{q}_{20}^2 \bar{q}_{11} \epsilon + \bar{q}_{30} \epsilon}{\bar{q}_{20} \epsilon (4\bar{q}_{01} \bar{q}_{20} - 1)} 2u \right) \\ S_e &= 1 + \left[\frac{4(2\bar{q}_{01} \bar{q}_{20}^2 \bar{p}_{20} - 2\bar{q}_{20}^3 \bar{p}_{01} - \bar{q}_{20} \bar{p}_{20} - 3\bar{q}_{01} \bar{q}_{20} \bar{q}_{30} \epsilon + \bar{q}_{20}^2 \bar{q}_{11} \epsilon + \bar{q}_{30} \epsilon)}{\bar{q}_{20} \epsilon (4\bar{q}_{01} \bar{q}_{20} - 1)} - \frac{\bar{q}_{30}}{\bar{q}_{20}} \right] u \\ S_o &= - \frac{2\bar{q}_{01} \bar{q}_{20} \bar{p}_{20} + 2\bar{q}_{20}^2 \bar{p}_{01} - \bar{q}_{01} \bar{q}_{30} \epsilon - \bar{q}_{20} \bar{q}_{11} \epsilon}{\bar{q}_{20} (4\bar{q}_{01} \bar{q}_{20} - 1)}\end{aligned}$$

By the definition of dimorphism equivalence, we know \bar{f} is dimorphism equivalent to $g = Sf(\Phi)$. That is,

$$\begin{aligned}
g &= (S_e + S_o v)(p(\Phi)w(\Phi) + q(\Phi)v(\Phi)) \\
&= (S_e p(\Phi)w(\Phi) + S_o q(\Phi)v(\Phi)v) + (S_e q(\Phi)v(\Phi) + S_o p(\Phi)w(\Phi)v) \\
&= (S_e p(\Phi)(4\varphi_o^2) + S_o q(\Phi)(2\varphi_o))w + (Rq(\Phi)(2\varphi_o) + S_o p(\Phi)(4\varphi_o^2))v \\
&= (\epsilon u + \tilde{p}_{11}uw + \tilde{p}_{30}u^3 + \tilde{p}^h)w \\
&\quad + (\bar{q}_{01}w + \bar{q}_{20}u^2 + \bar{q}_{02}w^2 + \bar{q}_{21}u^2w + \bar{q}_{40}u^4 + \bar{q}^h)v
\end{aligned}$$

where $\tilde{p}^h \in \mathcal{M}^2$, $\bar{q}^h \in \mathcal{M}^3$, and

$$\begin{aligned}
\tilde{p}_{uu}^h(0, 0) &= \tilde{p}_{uw}^h(0, 0) = \tilde{p}_3^h(0, 0) = 0 \\
\bar{q}_{u^3}^h(0, 0) &= \bar{q}_{u^2w}^h(0, 0) = \bar{q}_4^h(0, 0) = 0
\end{aligned}$$

Now we have shown that a general strategy function f that satisfies the corresponding defining and non-degeneracy conditions in Table 5.1 is dimorphism equivalent to the following strategy function:

$$g = (\epsilon u + \tilde{p}_{11}uw + \tilde{p}_{30}u^3 + \tilde{p}^h)w + (\bar{q}_{01}w + \bar{q}_{20}u^2 + \bar{q}_{02}w^2 + \bar{q}_{21}u^2w + \bar{q}_{40}u^4 + \bar{q}^h)v$$

Next, let (S, Φ) satisfy that

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

and consider

$$\begin{aligned}
\varphi_e &= \frac{1}{2} \left(u - \frac{2\bar{q}_{01}^2 \bar{q}_{20}^2 \bar{p}_{30} - 2\bar{q}_{01} \bar{q}_{20}^3 \bar{p}_{11} - \bar{q}_{01} \bar{q}_{20} \bar{p}_{30} - 3\bar{q}_{01}^2 \bar{q}_{20} \bar{q}_{40} \epsilon + \bar{q}_{01} \bar{q}_{20}^2 \bar{q}_{21} \epsilon + \bar{q}_{01} \bar{q}_{40} \epsilon + \bar{q}_{20}^3 \bar{q}_{2} \epsilon}{2\bar{q}_{01} \bar{q}_{20} \epsilon (4\bar{q}_{01} \bar{q}_{20} - 1)} u^3 \right) \\
\varphi_o &= \frac{1}{2} \left(1 - \frac{2\bar{q}_{01}^3 \bar{q}_{20} \bar{p}_{30} - 2\bar{q}_{01}^2 \bar{q}_{20}^2 \bar{p}_{11} + \bar{q}_{01} \bar{q}_{20} \bar{p}_{11} - \bar{q}_{01}^3 \bar{q}_{40} \epsilon - \bar{q}_{01}^2 \bar{q}_{20} \bar{q}_{21} \epsilon + 3\bar{q}_{01} \bar{q}_{20}^2 \bar{q}_{2} \epsilon - \bar{q}_{20} \bar{q}_{2} \epsilon}{\bar{q}_{01} \bar{q}_{20} \epsilon (4\bar{q}_{01} \bar{q}_{20} - 1)} w \right. \\
&\quad \left. - \frac{2\bar{q}_{01}^2 \bar{q}_{20}^2 \bar{p}_{30} - 2\bar{q}_{01} \bar{q}_{20}^3 \bar{p}_{11} - \bar{q}_{01} \bar{q}_{20} \bar{p}_{30} - 3\bar{q}_{01}^2 \bar{q}_{20} \bar{q}_{40} \epsilon + \bar{q}_{01} \bar{q}_{20}^2 \bar{q}_{21} \epsilon + \bar{q}_{01} \bar{q}_{40} \epsilon + \bar{q}_{20}^3 \bar{q}_{2} \epsilon}{2\bar{q}_{01} \bar{q}_{20} \epsilon (4\bar{q}_{01} \bar{q}_{20} - 1)} 3u^2 \right) \\
S_e &= 1 + \left[\frac{3(2\bar{q}_{01}^3 \bar{q}_{20} \bar{p}_{30} - 2\bar{q}_{01}^2 \bar{q}_{20}^2 \bar{p}_{11} + \bar{q}_{01} \bar{q}_{20} \bar{p}_{11} - \bar{q}_{01}^3 \bar{q}_{40} \epsilon - \bar{q}_{01}^2 \bar{q}_{20} \bar{q}_{21} \epsilon + 3\bar{q}_{01} \bar{q}_{20}^2 \bar{q}_{2} \epsilon - \bar{q}_{20} \bar{q}_{2} \epsilon)}{\bar{q}_{01} \bar{q}_{20} \epsilon (4\bar{q}_{01} \bar{q}_{20} - 1)} - \frac{q_2}{\bar{q}_{01}} \right] w \\
&\quad - \frac{-10\bar{q}_{01}^2 \bar{q}_{20}^2 \bar{p}_{30} + 10\bar{q}_{01} \bar{q}_{20}^3 \bar{p}_{11} + 5\bar{q}_{01} \bar{q}_{20} \bar{p}_{30} + 23\bar{q}_{01}^2 \bar{q}_{20} \bar{q}_{40} \epsilon - 5\bar{q}_{01} \bar{q}_{20}^2 \bar{q}_{21} \epsilon - 7\bar{q}_{01} \bar{q}_{40} \epsilon - 5\bar{q}_{20}^3 \bar{q}_{2} \epsilon}{2\bar{q}_{01} \bar{q}_{20} \epsilon (4\bar{q}_{01} \bar{q}_{20} - 1)} u^2 \\
S_o &= - \frac{2\bar{q}_{01}^2 \bar{q}_{20} \bar{p}_{30} + 2\bar{q}_{01} \bar{q}_{20}^2 \bar{p}_{11} - \bar{q}_{01}^2 \bar{q}_{40} \epsilon - \bar{q}_{01} \bar{q}_{20} \bar{q}_{21} \epsilon - \bar{q}_{20}^2 \bar{q}_{2} \epsilon}{\bar{q}_{01} \bar{q}_{20} (4\bar{q}_{01} \bar{q}_{20} - 1)} u
\end{aligned}$$

By the definition of dimorphism equivalence, we know g is dimorphism equivalent to $\tilde{g} = Sg(\Phi)$. That is,

$$\begin{aligned}
\tilde{g} &= (S_e + S_o v)(p(\Phi)w(\Phi) + q(\Phi)v(\Phi)) \\
&= (S_e p(\Phi)w(\Phi) + S_o q(\Phi)v(\Phi)v) + (S_e q(\Phi)v(\Phi) + S_o p(\Phi)w(\Phi)v) \\
&= (S_e p(\Phi)(4\varphi_o^2) + S_o q(\Phi)(2\varphi_o))w + (Rq(\Phi)(2\varphi_o) + S_o p(\Phi)(4\varphi_o^2))v \\
&= (\epsilon u + p^h)w + (\bar{q}_{01}w + \bar{q}_{20}u^2 + q^h)v
\end{aligned}$$

where $p^h \in \mathcal{M}^2$, $q^h \in \mathcal{M}^3$, and

$$\begin{aligned}
p_{uu}^h(0, 0) &= p_{uw}^h(0, 0) = p_{u^3}^h(0, 0) = 0 \\
q_{u^3}^h(0, 0) &= q_{u^2w}^h(0, 0) = q_{u^4}^h(0, 0) = 0
\end{aligned}$$

Therefore, we have successfully applied a series of specific dimorphism equivalence transformations to reduce the form of the strategy function f into the following:

$$\tilde{g} = (\epsilon u + p^h)w + (\bar{q}_{01}w + \bar{q}_{20}u^2 + q^h)v \quad (5.0.5)$$

where $p^h \in \mathcal{M}^2$, $q^h \in \mathcal{M}^3$, and

$$\begin{aligned}
p_{uu}^h(0, 0) &= p_{uw}^h(0, 0) = p_{u^3}^h(0, 0) = 0 \\
q_{u^3}^h(0, 0) &= q_{u^2w}^h(0, 0) = q_{u^4}^h(0, 0) = 0 \\
\epsilon &= \text{sgn}(p_u) \quad \bar{q}_{01} = \epsilon \frac{q_w}{p_u} \quad \bar{q}_{20} = \epsilon \frac{q_{uu}}{2p_u}
\end{aligned}$$

To continue, we can assume that the general strategy function f is in the form of (5.0.5). We want to show that f is dimorphism equivalence to

$$f_0 = \epsilon uw + (\bar{q}_{01}w + \bar{q}_{20}u^2)v$$

where

$$\epsilon = \text{sgn}(p_u) \quad \bar{q}_{01} = \epsilon \frac{q_w}{p_u} \quad \bar{q}_{20} = \epsilon \frac{q_{uu}}{2p_u}$$

First we calculate $\mathcal{I}(f)$. With formula (4.2.8), we have

$$\begin{aligned}
[p, q] &= [\epsilon u + p^h, \bar{q}_{01}w + \bar{q}_{20}u^2 + q^h] \\
[q, wp] &= [\bar{q}_{01}w + \bar{q}_{20}u^2 + q^h, w(\epsilon u + p^h)] \\
[wp_u, wq_u] &= [w(\epsilon + p_u^h), w(2\bar{q}_{20}u + q_u^h)] \\
[wp + 2w^2p_w, 2w^2q_w] &= [w(\epsilon u + p^h) + 2w^2p_w^h, 2w^2(\bar{q}_{01} + q_w^h)]
\end{aligned}$$

We claim that $[\mathcal{M}^4, \mathcal{M}^5] + w[\mathcal{M}^2, \mathcal{M}^3] + w^2[\mathcal{M}^0, \mathcal{M}] \subset \mathcal{I}(f)$. With Nakayama's Lemma, we need to prove

$$\begin{aligned}
&[\mathcal{M}^4, \mathcal{M}^5] + w[\mathcal{M}^2, \mathcal{M}^3] + w^2[\mathcal{M}^0, \mathcal{M}] \\
&\quad \subset \mathcal{I}(f) + [\mathcal{M}^5, \mathcal{M}^6] + w[\mathcal{M}^3, \mathcal{M}^4] + w^2[\mathcal{M}, \mathcal{M}^2]
\end{aligned}$$

Let $\mathcal{J} = \mathcal{I}(f) + [\mathcal{M}^5, \mathcal{M}^6] + w[\mathcal{M}^3, \mathcal{M}^4] + w^2[\mathcal{M}, \mathcal{M}^2]$. Then we have

$$\begin{aligned}
\bar{q}_{01}w[w(\epsilon + p_u^h), w(2\bar{q}_{20}u + q_u^h)] &\in \mathcal{J} \Rightarrow [\epsilon\bar{q}_{01}w^2, 2\bar{q}_{01}\bar{q}_{20}u^2] \in \mathcal{J} \\
\epsilon w[\bar{q}_{01}w + \bar{q}_{20}u^2 + q^h, w(\epsilon u + p^h)] &\in \mathcal{J} \Rightarrow [\epsilon\bar{q}_{01}w^2 + \epsilon\bar{q}_{20}u^2w, \\
&\epsilon^2uw^2] \in \mathcal{J} \\
&[\epsilon\bar{q}_{01}w^2, 2\bar{q}_{01}\bar{q}_{20}u^2], \\
[\epsilon\bar{q}_{01}w^2 + \epsilon\bar{q}_{20}u^2w, \epsilon^2uw^2] &\in \mathcal{J} \Rightarrow [\epsilon\bar{q}_{20}u^2w, \\
&(\epsilon^2 - 2\bar{q}_{01}\bar{q}_{20})uw^2] \in \mathcal{J} \\
u[w(\epsilon u + p^h) + 2w^2p_w^h, 2w^2(\bar{q}_{01} + q_w^h)] &\in \mathcal{J} \Rightarrow [\epsilon u^2w, 2\bar{q}_{01}uw^2] \in \mathcal{J} \\
[\epsilon\bar{q}_{20}u^2w, (\epsilon^2 - 2\bar{q}_{01}\bar{q}_{20})uw^2], \\
[\epsilon u^2w, 2\bar{q}_{01}uw^2] &\in \mathcal{J} \Rightarrow [u^2w, 0], [0, uw^2] \in \mathcal{J} \\
w^2[\epsilon u + p^h, \bar{q}_{01}w + \bar{q}_{20}u^2 + q^h] &\in \mathcal{J} \Rightarrow [0, w^3] \in \mathcal{J} \\
[0, w^3], [0, uw^2] &\in \mathcal{J} \Rightarrow [0, \langle w^2 \rangle \mathcal{M}] \in \mathcal{J} \\
[0, uw^2], w[w(\epsilon + p_u^h), w(2\bar{q}_{20}u + q_u^h)] &\in \mathcal{J} \Rightarrow [\langle w^2 \rangle, 0] \in \mathcal{J} \\
[w^2, 0], [u^2w, 0] &\in \mathcal{J} \Rightarrow [\langle w \rangle \mathcal{M}^2, 0] \in \mathcal{J} \\
[u^2w, 0], u^2[w(\epsilon + p_u^h), w(2\bar{q}_{20}u + q_u^h)] &\in \mathcal{J} \Rightarrow [0, u^3w] \in \mathcal{J} \\
[0, u^3w], [0, \langle w^2 \rangle \mathcal{M}] &\in \mathcal{J} \Rightarrow [0, \langle w \rangle \mathcal{M}^3] \in \mathcal{J} \\
[u^2w, 0], \\
u^2[\bar{q}_{01}w + \bar{q}_{20}u^2 + q^h, w(\epsilon u + p^h)] &\in \mathcal{J} \Rightarrow [u^4, 0] \in \mathcal{J} \\
[u^4, 0], u^3[\epsilon u + p^h, \bar{q}_{01}w + \bar{q}_{20}u^2 + q^h] &\in \mathcal{J} \Rightarrow [0, u^5] \in \mathcal{J} \\
[u^4, 0], [\langle w \rangle \mathcal{M}^2, 0], [\langle w^2 \rangle, 0] &\in \mathcal{J} \Rightarrow [\mathcal{M}^4, 0] \in \mathcal{J} \\
[0, u^5], [0, \langle w \rangle \mathcal{M}^3], [0, \langle w^2 \rangle \mathcal{M}] &\in \mathcal{J} \Rightarrow [0, \mathcal{M}^5] \in \mathcal{J}
\end{aligned}$$

Thus we have proved the claim that

$$\mathcal{K} = [\mathcal{M}^4 + \langle w \rangle \mathcal{M}^2 + \langle w^2 \rangle, \mathcal{M}^5 + \langle w \rangle \mathcal{M}^3 + \langle w^2 \rangle \mathcal{M}] \subset \mathcal{I}(f).$$

The calculation indicates that as long as f satisfies the corresponding defining and non-degeneracy conditions in Table 5.1, we always have $\mathcal{K} \subset \mathcal{I}(f)$. That is,

any strategy function h inside the module \mathcal{K} will satisfy that $\mathcal{I}(f + th) = \mathcal{I}(f)$ for all $t \in [0, 1]$. According to Theorem 4.7, we know that $f + h$ is dimorphism equivalent to f . Note that p^h, q^h satisfies that $p^h \in \mathcal{M}^2, q^h \in \mathcal{M}^3$ and

$$\begin{aligned} p_{uu}^h(0, 0) &= p_{uw}^h(0, 0) = p_{u^3}^h(0, 0) = 0 \\ q_{u^3}^h(0, 0) &= q_{u^2w}^h(0, 0) = q_{u^4}^h(0, 0) = 0 \end{aligned}$$

It indicates that $[p^h, q^h] \in \mathcal{K}$ Therefore f is dimorphism equivalent to

$$h = \epsilon(uw + (\alpha_0w + \beta_0u^2)v)$$

where at $(0, 0)$

$$\begin{aligned} \epsilon &= \text{sgn}(p_u) & \alpha &= \frac{q_w}{p_u} & \beta_0 &= \frac{q_{uu}}{2p_u} & q &= q_u = p = 0 \\ p_u &\neq 0 & q_{uu} &\neq 0 & q_w &\neq 0 & p_u^2 - 2q_{uu}q_w &\neq 0 \end{aligned}$$

At the end, we prove that if $f = pw + qv$ is dimorphism equivalent to

$$\epsilon(uw + (\alpha_0w + \beta_0u^2)v)$$

with the singularity at $(0, 0)$, then the the corresponding defining and non-degeneracy conditions in Table 5.1 hold. The dimorphism equivalence indicates that there exists a dimorphism equivalence transformation (S, Φ) such that $f = Sh(\Phi)$ and we can write

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

where at $(0, 0)$

$$S_e > 0 \quad \varphi_e = 0 \quad (\varphi_e)_u = \varphi_o > 0$$

Then we have

$$\begin{aligned} f = pw + qv &= \epsilon[8\varphi_e\varphi_o^2S_e + 2(4\alpha\varphi_o^2w + 4\beta_0\varphi_e^2)\varphi_oS_o]w \\ &\quad + \epsilon[8\varphi_e\varphi_o^2wS_o + 2(4\alpha\varphi_o^2w + 4\beta_0\varphi_e^2)\varphi_oS_e]v \end{aligned}$$

Therefore,

$$\begin{aligned} p &= 8\epsilon\varphi_e\varphi_o^2S_e + 2\epsilon(4\alpha\varphi_o^2w + 4\beta_0\varphi_e^2)\varphi_oS_o \\ q &= 8\epsilon\varphi_e\varphi_o^2wS_o + 2\epsilon(4\alpha\varphi_o^2w + 4\beta_0\varphi_e^2)\varphi_oS_e \end{aligned}$$

So we know at $(0, 0)$

$$p = 0 \quad q = 0 \quad q_u = 0$$

$$q_{uu} = 16\epsilon\beta_0(\varphi_e)_u^2\varphi_oS_e \neq 0 \quad p_u = 8\epsilon(\varphi_e)_u\varphi_o^2S_e \neq 0$$

$$p_u^2 - 2q_{uu}q_w = 64(\varphi_e)_u^3\varphi_o^4S_e^2 - 32\beta_0(\varphi_e)_u^2\varphi_oS_e \cdot 8\alpha\varphi_o^3 = 64(\varphi_e)_u^2\varphi_o^4S_e^2(1 - 4\alpha\beta_0) \neq 0$$

Now we have completely proved part (f).

□

Remark 5.4. In the proof of Theorem 5.1 we calculate the strong dimorphism equivalence restricted tangent space $\mathcal{I}(f)$ for strategy functions f satisfying certain defining and non-degeneracy conditions. The result enables us to prove that these f are dimorphism equivalent to the given normal forms h . When we find universal unfoldings of h in Chapter 6 we will need to know $RT(h)$.

Lemma 5.5. *The restricted tangent spaces for the normal forms in Theorem 5.1 are given in Table 5.2.*

Proof. (a) It follows from Proposition 4.2 that the dimorphism equivalence restricted tangent space of the strategy function $h = \epsilon(w + \mu_0uv)$ is

$$\begin{aligned} RT(h) &= \langle [0, w], [0, u^2], [u, 0], [w, 0] \rangle + \mathbf{R}\{[1, \mu_0u]\} \\ &= [\mathcal{M}, \mathcal{M}^2] + w[\mathcal{M}^0, \mathcal{M}^0] + \mathbf{R}\{[1, \mu_0u]\} \end{aligned}$$

	Normal Form h	TC	$RT(h)$
(a)	$\epsilon(w + \mu_0 wv)$	0	$[\mathcal{M}, \mathcal{M}^2] + w[\mathcal{M}^0, \mathcal{M}^0] + \mathbf{R}\{[1, \mu_0 u]\}$
(b)	$\epsilon(\delta w^2 + uv)$	1	$[\mathcal{M}, \mathcal{M}]$
(c)	$(\epsilon + \mu_0 u)w + \delta u^2 v$	1	$[\mathcal{M}^2, \mathcal{M}^4] + w[\mathcal{M}^0, \mathcal{M}^0]$ $+ \mathbf{R}\{[\epsilon + 2\mu_0 u, 2\delta u^2], [\epsilon + \mu_0 u, \delta u^2], [\epsilon u, \delta u^3]\}$
(d)	$\epsilon(\delta w^3 + uv)$	2	$[\mathcal{M}^2, \mathcal{M}] + \mathbf{R}\{[u, 0]\}$
(e)	$\epsilon w + (\delta u^3 + \mu_0 u^5)v$	2	$[\mathcal{M}^3, \mathcal{M}^6] + w[\mathcal{M}^0, \mathcal{M}^0] + \mathbf{R}\{[\epsilon, \delta u^3 + \mu_0 u^5],$ $[\epsilon, 3\delta u^3 + 5\mu_0 u^5], [\epsilon u^2, \delta u^5], [u, 0], [0, u^4]\}$
(f)	$\epsilon(uw + (\alpha_0 w + \beta_0 u^2)v)$	2	$[\mathcal{M}^2, \mathcal{M}^3] + w[\mathcal{M}^0, \mathcal{M}] + \mathbf{R}\{[u, \alpha_0 w + \beta_0 u^2]\}$

Table 5.2: The dimorphism equivalence restricted tangent space for the singularities up to topological codimension two. (TC = topological codimension. $RT(h)$ = dimorphism equivalence restricted tangent space of h .)

- (b) It follows from Proposition 4.2 that the dimorphism equivalence restricted tangent space of the strategy function $h = \epsilon(\delta w^2 + uv)$ is

$$RT(h) = \langle [0, w], [0, u], [u, 0], [w, 0] \rangle = [\mathcal{M}, \mathcal{M}]$$

(c) It follows from Proposition 4.2 that the dimorphism equivalence restricted tangent space of the strategy function $h = (\epsilon + \mu_0 u)w + \delta u^2 v$ is

$$\begin{aligned}
RT(h) &= \mathcal{I}(h) + \mathbf{R}\{[\epsilon + 2\mu_0 u, 2\delta u^2], [2\epsilon u + 3\mu_0 u^2, 2\delta u^3], [3\epsilon u^2, 2\delta u^4]\} \\
&= \langle [u^3, 0], [w, 0], [0, u^5], [0, uw], [0, w^2] \rangle \\
&\quad + \mathbf{R}\{[\epsilon + 2\mu_0 u, 2\delta u^2], [\epsilon + \mu_0 u, \delta u^2], [\epsilon u, \delta u^3], [u^2, 0], [0, u^4], [0, w]\} \\
&= \langle [u^2, 0], [w, 0], [0, u^4], [0, w] \rangle \\
&\quad + \mathbf{R}\{[\epsilon + 2\mu_0 u, 2\delta u^2], [\epsilon + \mu_0 u, \delta u^2], [\epsilon u, \delta u^3]\} \\
&= [\mathcal{M}^2, \mathcal{M}^4] + w[\mathcal{M}^0, \mathcal{M}^0] \\
&\quad + \mathbf{R}\{[\epsilon + 2\mu_0 u, 2\delta u^2], [\epsilon + \mu_0 u, \delta u^2], [\epsilon u, \delta u^3]\}
\end{aligned}$$

(d) It follows from Proposition 4.2 that the dimorphism equivalence restricted tangent space of the strategy function $h = \epsilon(\delta w^3 + uv)$ is

$$RT(h) = \mathcal{I}(h) + \mathbf{R}[5\delta w^2, u] = [\mathcal{M}^2, \mathcal{M}] + \mathbf{R}\{[u, 0]\}$$

(e) It follows from Proposition 4.2 that the dimorphism equivalence restricted tangent space of the strategy function $h = \epsilon w + (\delta u^3 + \mu_0 u^5)v$ is

$$\begin{aligned}
RT(h) &= \mathcal{I}(h) + \mathbf{R}\{[\epsilon, 3\delta u^3 + 5\mu_0 u^5], [2\epsilon u, 3\delta u^4 + 5\mu_0 u^6], \\
&\quad [3\epsilon u^2, 3\delta u^5 + 5\mu_0 u^7], [4\epsilon u^3, 3\delta u^6], [5\epsilon u^4, 3\delta u^7]\} \\
&= \langle [w, 0], [u^5, 0], [0, w^2], [0, u^2 w], [0, u^8] \rangle \\
&\quad + \mathbf{R}\{[\epsilon, \delta u^3 + \mu_0 u^5], [\epsilon u, \delta u^4 + \mu_0 u^6], [\epsilon u^2, \delta u^5 + \mu_0 u^7], [\epsilon u^3, \delta u^6], \\
&\quad [\epsilon u^4, \delta u^7], [\delta u^3, \epsilon w], [\delta u^4, \epsilon uw], [\epsilon, 3\delta u^3 + 5\mu_0 u^5], [2\epsilon u, 3\delta u^4 + 5\mu_0 u^6], \\
&\quad [3\epsilon u^2, 3\delta u^5 + 5\mu_0 u^7], [4\epsilon u^3, 3\delta u^6], [5\epsilon u^4, 3\delta u^7]\} \\
&= \langle [w, 0], [u^3, 0], [0, w], [0, u^6] \rangle \\
&\quad + \mathbf{R}\{[\epsilon, \delta u^3 + \mu_0 u^5], [\epsilon, 3\delta u^3 + 5\mu_0 u^5], [\epsilon u^2, \delta u^5], [u, 0], [0, u^4]\} \\
&= [\mathcal{M}^3, \mathcal{M}^6] + w[\mathcal{M}^0, \mathcal{M}^0] \\
&\quad + \mathbf{R}\{[\epsilon, \delta u^3 + \mu_0 u^5], [\epsilon, 3\delta u^3 + 5\mu_0 u^5], [\epsilon u^2, \delta u^5], [u, 0], [0, u^4]\}
\end{aligned}$$

(f) It follows from Proposition 4.2 that the dimorphism equivalence restricted tangent space of the strategy function $h = \epsilon(uw + (\alpha_0w + \beta_0u^2)v)$ is

$$\begin{aligned}
RT(h) &= \mathcal{I}(h) + \mathbf{R}\{[3u^2, 2\beta_0u^3 + 4\alpha_0uw], \\
&\quad [4u^3, 2\beta_0u^4 + 6\alpha_0u^2w], [2u, 2\beta_0u^2 + 2\alpha_0w]\} \\
&= [\mathcal{M}^4, \mathcal{M}^5] + w[\mathcal{M}^2, \mathcal{M}^3] + w^2[\mathcal{M}^0, \mathcal{M}] + \mathbf{R}\{[u, \beta_0u^2 + \alpha_0w], \\
&\quad [uw, \beta_0u^2w + \alpha_0w^2], [u^2, \beta_0u^3 + \alpha_0uw], [u^3, \beta_0u^4 + \alpha_0u^2w], \\
&\quad [\alpha_0w + \beta_0u^2, uw], [\alpha_0uw + \beta_0u^3, u^2w], [w, 2\beta_0uw], [uw, 2\beta_0u^2w], \\
&\quad [uw, 2\alpha_0w^2], [3u^2, 2\beta_0u^3 + 4\alpha_0uw], [4u^3, 2\beta_0u^4 + 6\alpha_0u^2w]\} \\
&= [\mathcal{M}^2, \mathcal{M}^3] + w[\mathcal{M}^0, \mathcal{M}] + \mathbf{R}\{[u, \alpha_0w + \beta_0u^2]\}
\end{aligned}$$

□

CHAPTER 6

UNIVERSAL UNFOLDINGS UNDER DIMORPHISM EQUIVALENCE

In this chapter we develop the universal unfolding theory in the context of dimorphism equivalence. First we define universal unfolding and codimension (see Definition 6.4). Then we define the dimorphism equivalence tangent space and state the universal unfolding theorem in singularity theory (see Theorem 6.7). This theorem is a special case of a general theorem proved by Damon [2]. At the end, we apply the universal unfolding theorem to calculate the universal unfoldings for singularities up to topological codimension two in Theorem 6.9.

6.1 Preliminary Definitions

In this section, we define a few concepts in universal unfolding theory of strategy functions. All of these definitions can be found in [14].

Definition 6.1. Let f be a C^∞ strategy function $\mathbf{R}^2 \rightarrow \mathbf{R}$ defined on a neighborhood of the origin. Then $F : \mathbf{R}^2 \times \mathbf{R}^k \rightarrow \mathbf{R}$ is a k -parameter unfolding of f if

$$F(x, y, 0) = f(x, y) \quad F(x, x, \alpha) = 0$$

where the parameter $\alpha \in \mathbf{R}^k$.

Suppose $F(x, y, \alpha)$ ($\alpha \in \mathbf{R}^k$) and $H(x, y, \beta)$ ($\beta \in \mathbf{R}^l$) are unfoldings of f . We

say that the perturbations of the H unfolding are contained in the F unfolding if for every $\beta \in \mathbf{R}^l$, there exists $A(\beta) \in \mathbf{R}^k$ such that $F(\cdot, \cdot, A(\beta))$ is diffeomorphism equivalent to $H(\cdot, \cdot, \beta)$. We formalize this in Definition 6.2.

Definition 6.2. Let $F(x, y, \alpha)$ be a k -parameter unfolding of f and let $H(x, y, \beta)$ be an l -parameter unfolding of f . We say that H *factors through* F if there exists maps $S : \mathbf{R}^2 \times \mathbf{R}^l \rightarrow \mathbf{R}$, $\Phi : \mathbf{R}^2 \times \mathbf{R}^l \rightarrow \mathbf{R}^2$, and $A : \mathbf{R}^l \rightarrow \mathbf{R}^k$ such that

$$H(x, y, \beta) = S(x, y, \beta)F(\Phi(x, y, \beta), A(\beta))$$

where

1. $S(x, y, 0) = 1$
2. $\Phi(x, y, 0) = (x, y)$
3. $\Phi(x, y, \beta) = (\varphi(x, y, \beta), \varphi(y, x, \beta))$ where $\varphi : \mathbf{R}^2 \times \mathbf{R}^l \rightarrow \mathbf{R}$
4. $(d\Phi)_{x,x,\beta} = c(x, \beta)I_2$ where $c(x, \beta) > 0$
5. $A(0) = 0$

Remark 6.3. We do not require that $\Phi(0, 0, \beta) = (0, 0)$; that is, when β is nonzero, the equivalence need not preserve the origin.

There are unfoldings F that contains all perturbations of f , up to diffeomorphism equivalence. These unfoldings are characterized in Definition 6.4.

Definition 6.4. An unfolding F of f is *versal* if every unfolding of f factors through F . A versal unfolding depending on the minimum number of parameters is called *universal*. That minimum number is called the C^∞ *codimension* of f . In addition, the *topological codimension* of f is the C^∞ codimension of f minus the number of modal parameters in the universal unfolding F .

Suppose $F(x, y, \alpha)$ is a universal unfolding of a strategy function f . This implies that every one-parameter unfolding of f can factor through F . For any strategy function perturbation $\eta(x, y)$, consider the one-parameter unfolding

$$H(x, y, t) = f(x, y) + t\eta(x, y)$$

Since H factors through F , we can write

$$H(x, y, t) = S(x, y, t)F(\Phi(x, y, t), A(t)) \quad (6.1.1)$$

where S, Φ, A satisfy the conditions in Definition 6.2.

On differentiating (6.1.1) with respect to t and evaluating at $t = 0$, we have

$$\begin{aligned} \eta(x, y) &= \dot{S}(x, y, 0)f(x, y) + \dot{\varphi}(x, y, 0)F_x(x, y, 0) \\ &+ \dot{\varphi}(y, x, 0)F_y(x, y, 0) + \sum_{i=1}^k \dot{A}_i F_{\alpha_i}(x, y, 0) \end{aligned} \quad (6.1.2)$$

where $A(t) = (A_1(t), \dots, A_k(t))$ and \cdot is differentiation with respect to t .

6.2 Dimorphism Equivalence Tangent Space

Definition 6.5. The *dimorphism equivalence tangent space* of a strategy function f , denoted by $T(f)$, is the set of all $\eta(x, y)$ of the form given in the first three terms on the right hand side of (6.1.2), where (S, Φ) satisfies all assumptions in Definition 6.2.

Remark 6.6. The only difference between the definition of $T(f)$ and $RT(f)$ is the fact stated in Remark 6.3 that Φ in the definition of $T(f)$ does not need to fix the origin for different parameter values, but Φ in the definition of $RT(f)$ needs to fix the origin for all parameters. Therefore, with similar calculation performed in Proposition 4.2, we can have

$$T(f) = RT(f) + \mathbf{R}\{[p_u, q_u]\} \quad (6.2.1)$$

Thus, $RT(f)$ has finite codimension if and only if $T(f)$ has finite codimension.

The structure of $T(f)$ leads to a necessary condition for F to be a universal unfolding of f . Specifically, if an unfolding F is versal, then

$$\mathcal{E}^2 = T(f) + \mathbf{R}\{F_{\alpha_1}(x, y, 0), \dots, F_{\alpha_k}(x, y, 0)\} \quad (6.2.2)$$

An important result in singularity theory states that (6.2.2) is also a sufficient condition for F to be a universal unfolding. See §9 in [2].

Theorem 6.7. *Let F be a k -parameter unfolding of f . Then F is a versal unfolding of f if and only if*

$$\mathcal{E}^2 = T(f) + \mathbf{R}\left\{\frac{\partial G}{\partial \alpha_1}(x, y, 0), \dots, \frac{\partial G}{\partial \alpha_k}(x, y, 0)\right\} \quad (6.2.3)$$

Corollary 6.8. An unfolding F of f is universal if and only if the sum in (6.2.3) is a direct sum. The number of parameters in F equals the codimension of $T(f)$. In particular, if f has C^∞ codimension k and $z_1, \dots, z_k \in \mathcal{E}^2$ are chosen so that

$$\mathcal{E}^2 = T(f) \oplus \mathbf{R}\{z_1, \dots, z_k\}$$

then

$$F(x, y, \alpha) = f(x, y) + \alpha_1 z_1(x, y) + \dots + \alpha_k z_k(x, y)$$

is a universal unfolding of f .

6.3 Universal Unfoldings of Low Codimension Singularities

In this section, we calculate the universal unfolding for each singularity up to topological codimension two.

Theorem 6.9. *Table 6.1 gives a universal unfolding for each normal form in Theorem 5.1.*

	Normal Form h	TC	$T(h)$	Universal Unfolding H
(a)	$\epsilon(w + \mu_0 uv)$	0	$[\mathcal{M}, \mathcal{M}^2] + w[\mathcal{M}^0, \mathcal{M}^0] + \mathbf{R}\{[1, \mu_0 u], [0, \mu_0]\}$	$\epsilon(w + \mu uv)$
(b)	$\epsilon(\delta w^2 + uv)$	1	$[\mathcal{M}, \mathcal{M}] + \mathbf{R}\{[0, 1]\}$	$\epsilon((a + \delta w)w + uv)$
(c)	$(\epsilon + \mu_0 u)w + \delta u^2 v$	1	$[\mathcal{M}^2, \mathcal{M}^4] + w[\mathcal{M}^0, \mathcal{M}^0] + \mathbf{R}\{[\epsilon + 2\mu_0 u, 2\delta u^2], [\epsilon + \mu_0 u, \delta u^2], [\epsilon u, \delta u^3], [\mu_0, 2\delta u]\}$	$(\epsilon + \mu u)w + (a + \delta u^2)v$
(d)	$\epsilon(\delta w^3 + uv)$	2	$[\mathcal{M}^2, \mathcal{M}] + \mathbf{R}\{[0, 1], [u, 0]\}$	$\epsilon((a + bw + \delta w^2)w + uv)$
(e)	$\epsilon w + (\delta u^3 + \mu_0 u^5)v$	2	$[\mathcal{M}^3, \mathcal{M}^6] + w[\mathcal{M}^0, \mathcal{M}^0] + \mathbf{R}\{[\epsilon, \delta u^3 + \mu_0 u^5], [\epsilon, 3\delta u^3 + 5\mu_0 u^5], [\epsilon u^2, \delta u^5], [u, 0], [0, u^4], [0, u^2]\}$	$\epsilon w + (a + bu + \delta u^3 + \mu u^5)v$
(f)	$\epsilon(uw + (\alpha_0 w + \beta_0 u^2)v)$	2	$[\mathcal{M}^2, \mathcal{M}^3] + w[\mathcal{M}^0, \mathcal{M}] + \mathbf{R}\{[u, \alpha_0 w + \beta_0 u^2], [1, 2\beta_0 u]\}$	$\epsilon((a + u)w + (b + \alpha w + \beta u^2)v)$

Table 6.1: The universal unfoldings and dimorphism equivalence tangent space for the low codimension singularities in Theorem 6.7. (Def = defining conditions. ND = non-degeneracy conditions. TC = topological codimension.)

Proof. (a) Equation (6.2.1) shows that the dimorphism equivalence tangent space of the strategy function $h = \epsilon(w + \mu_0 uv)$ is

$$\begin{aligned} T(h) &= RT(h) + \mathbf{R}\{[0, \mu_0]\} \\ &= [\mathcal{M}, \mathcal{M}^2] + w[\mathcal{M}^0, \mathcal{M}^0] + \mathbf{R}\{[1, \mu_0 u], [0, \mu_0]\} \end{aligned}$$

Therefore, the C^∞ codimension of $h = \epsilon(w + \mu_0 uv)$ is 1 and a complementary space is

$$\mathbf{R}\{[0, u]\}$$

So a universal unfolding of h is

$$\epsilon(w + \mu uv)$$

where μ is a modal parameter near μ_0 . Since μ is a modal parameter, the topological codimension of $h = \epsilon(w + \mu_0 uv)$ is 0.

(b) Equation (6.2.1) shows that the dimorphism equivalence tangent space of the strategy function $h = \epsilon(\delta w^2 + uv)$ is

$$T(h) = RT(h) + \mathbf{R}\{[0, 1]\} = [\mathcal{M}, \mathcal{M}] + \mathbf{R}\{[0, 1]\}$$

Therefore, the C^∞ codimension of $\epsilon(\delta w^2 + uv)$ is 1 and a complementary space is

$$\mathbf{R}\{[1, 0]\}$$

So a universal unfolding of h is

$$\epsilon((a + \delta w)w + uv)$$

where a is an unfolding parameter near 0. Thus, the topological codimension of $h = \epsilon(\delta w^2 + uv)$ is 1.

(c) Equation (6.2.1) shows that the dimorphism equivalence tangent space of the strategy function $h = (\epsilon + \mu_0 u)w + \delta u^2 v$ is

$$\begin{aligned} T(h) &= RT(h) + \mathbf{R}\{[\mu_0, 2\delta u]\} \\ &= [\mathcal{M}^2, \mathcal{M}^4] + w[\mathcal{M}^0, \mathcal{M}^0] \\ &\quad + \mathbf{R}\{[\epsilon + 2\mu_0 u, 2\delta u^2], [\epsilon + \mu_0 u, \delta u^2], [\epsilon u, \delta u^3], [\mu_0, 2\delta u]\} \end{aligned}$$

It shows that the C^∞ codimension of $T(h)$ is 2. A complementary space is

$$\mathbf{R}\{[0, 1], [u, 0]\}$$

So a universal unfolding of h is

$$(\epsilon + \mu u)w + (a + \delta u^2)v$$

where μ is a modal parameter near μ_0 and a is an unfolding parameter near 0. Since μ is a modal parameter, the topological codimension of $h = (\epsilon + \mu_0 u)w + \delta u^2 v$ is 1.

(d) Equation (6.2.1) shows that the dimorphism equivalence tangent space of the strategy function $h = \epsilon(\delta w^3 + uv)$ is

$$T(h) = RT(h) + \mathbf{R}[p_u, q_u] = [\mathcal{M}^2, \mathcal{M}] + \mathbf{R}\{[u, 0], [0, 1]\}$$

It shows that the C^∞ codimension of $T(h)$ is 2 and a complementary space is

$$\mathbf{R}\{[1, 0], [w, 0]\}$$

So a universal unfolding of $h = \epsilon(\delta w^3 + uv)$ is

$$\epsilon((a + bw + \delta w^2)w + uv)$$

where a, b are unfolding parameters near 0. Thus, the topological codimension of $h = \epsilon(\delta w^3 + uv)$ is 2.

(e) Equation (6.2.1) shows that the dimorphism equivalence tangent space of the strategy function $h = \epsilon w + (\delta u^3 + \mu_0 u^5)v$ is

$$\begin{aligned} T(h) &= RT(h) + \mathbf{R}[p_u, q_u] \\ &= [\mathcal{M}^3, \mathcal{M}^6] + w[\mathcal{M}^0, \mathcal{M}^0] \\ &\quad + \mathbf{R}\{[\epsilon, \delta u^3 + \mu_0 u^5], [\epsilon, 3\delta u^3 + 5\mu_0 u^5], [\epsilon u^2, \delta u^5], [u, 0], [0, u^4], [0, u^2]\} \end{aligned}$$

It shows that the C^∞ codimension of $T(h)$ is 3. A complementary space is

$$\mathbf{R}\{[0, 1], [0, u], [0, u^5]\}$$

So a universal unfolding of $h = \epsilon w + (\delta u^3 + \mu_0 u^5)v$ is

$$\epsilon w + (a + bu + \delta u^3 + \mu u^5)v$$

where a, b are unfolding parameters near 0 and μ is a modal parameter near μ_0 . Since μ is a modal parameter, the topological codimension of $h = (\epsilon + \mu_0 u)w + \delta u^2 v$ is 2.

(f) Equation (6.2.1) shows that the dimorphism equivalence tangent space of the strategy function $h = \epsilon(uw + (\alpha_0w + \beta_0u^2)v)$ is

$$\begin{aligned} T(h) &= RT(h) + \mathbf{R}[p_u, q_u] \\ &= [\mathcal{M}^2, \mathcal{M}^3] + w[\mathcal{M}^0, \mathcal{M}] + \mathbf{R}\{[u, \alpha_0w + \beta_0u^2], [1, 2\beta_0u]\} \end{aligned}$$

It shows that the C^∞ codimension of $T(h)$ is 4. A complementary space is

$$\mathbf{R}\{[0, w], [0, u^2], [0, 1], [1, 0]\}$$

So a universal unfolding of $\epsilon(uw + (\alpha_0w + \beta_0u^2)v)$ is

$$\epsilon((a + u)w + (b + \alpha w + \beta u^2)v)$$

where a, b are unfolding parameters near 0 and α, β are modal parameter near α_0, β_0 . Since α, β are modal parameters, the topological codimension of $h = (\epsilon + \mu_0u)w + \delta u^2v$ is 2.

□

CHAPTER 7

THE RECOGNITION PROBLEM FOR UNIVERSAL UNFOLDINGS

In this chapter, we consider the following question that is called the *recognition problem for universal unfoldings*: Let $F(x, y, \alpha)$ be an unfolding of a strategy function $f(x, y)$, where f is dimorphism equivalent to a normal form h . When is F a universal unfolding of f ? This situation usually arises in applications. We use the approach developed by Golubitsky and Schaeffer [12] to answer this question for low codimension bifurcation problems. The results are summarized in Theorem 7.7 for the strategy function singularities up to topological codimension one.

Let $\gamma = (S, \Phi)$ be a dimorphism equivalence. That is, S and Φ satisfy conditions in Definition 3.3. Denote

$$\gamma(h)(x, y) = S(x, y)h(\Phi(x, y))$$

Lemma 7.1. Suppose $f = \gamma(h)$. Then

$$T(f) = \gamma(T(h))$$

Proof. Define a smooth curve of dimorphism equivalences δ_t at h as

$$\delta_t(h) = S(x, y, t)h(\Phi(x, y, t))$$

where S, Φ vary smoothly in t . Assume that

$$S(x, y, 0) = 1 \quad \Phi(x, y, 0) = (x, y)$$

In other words, δ_0 is the identity map. Then $g = \frac{d}{dt}\delta_t(h)|_{t=0}$ is a typical member of $T(h)$. With direct calculation,

$$\begin{aligned}\gamma(g) &= \gamma \frac{d}{dt}\delta_t(h)|_{t=0} = \frac{d}{dt}\gamma(\delta_t(h))|_{t=0} \\ &= \frac{d}{dt}\gamma\delta_t\gamma^{-1}\gamma(h)|_{t=0} \\ &= \frac{d}{dt}\gamma\delta_t\gamma^{-1}(f)|_{t=0}\end{aligned}$$

Let $\hat{\delta}_t = \gamma\delta_t\gamma^{-1}$. Then $\hat{\delta}_0$ is the identity map and $\hat{\delta}_t$ is a smooth curve of diffeomorphisms at f such that

$$\gamma(g) = \frac{d}{dt}\hat{\delta}_t(f)|_{t=0}$$

In other words, $\gamma(g) \in T(f)$ and therefore

$$\gamma(T(h)) \subset T(f)$$

Interchanging the roles of f and h shows that

$$T(f) \subset \gamma(T(h))$$

and the equality holds as claimed. □

Define the *pullback mapping* Φ^* as

$$\Phi^*(f)(x, y) := f(\Phi(x, y))$$

Note that in this thesis, we consistently use the coordinates (u, v) instead of (x, y) . Therefore, we assume that from now on, the strategy functions are respective to the coordinates (u, v) .

Lemma 7.2. Let $\mathcal{I} = \langle [p_1, q_1], \dots, [p_k, q_k] \rangle$. Then $\Phi^*(\mathcal{I})$ is the module

$$\langle \Phi^*([p_1, q_1]), \dots, \Phi^*([p_k, q_k]) \rangle$$

Proof. Follows from the fact that if f_1, f_2 are strategy functions, then

$$\Phi^*(f_1 + f_2) = \Phi^*(f_1) + \Phi^*(f_2)$$

□

Remark 7.3. If \mathcal{I}, \mathcal{J} are modules, then Lemma 7.2 implies that

$$\Phi^*(\mathcal{I} + \mathcal{J}) = \Phi^*(\mathcal{I}) + \Phi^*(\mathcal{J})$$

Next, we show that the modules $w^s[\mathcal{M}^k, \mathcal{M}^k]$ and $w^s[\mathcal{M}^k, \mathcal{M}^{k+1}]$ are invariant under pullback maps which come from dimorphism equivalences that fix the origin $(0, 0)$.

Definition 7.4. An module \mathcal{I} is *intrinsic* if for every dimorphism equivalence $\gamma = (S, \Phi)$ such that $\Phi(0) = 0$, we have $\gamma(\mathcal{I}) = \mathcal{I}$.

Proposition 7.5. The modules $w^s[\mathcal{M}^k, \mathcal{M}^k]$ and $w^s[\mathcal{M}^k, \mathcal{M}^{k+1}]$ are intrinsic, where k, s are non-negative intergers.

Proof. Note that under the (u, v) coordinate, we can write

$$S = S_e + S_o v \quad \Phi = (\varphi_e + \varphi_o v, \varphi_e - \varphi_o v)$$

where $S_e, S_o, \varphi_e, \varphi_o \in \mathcal{E}$. Assume that

$$f = pw + qv = [p, q] \in [\mathcal{M}^k, \mathcal{M}^k]$$

then we see that

$$\begin{aligned} S \cdot f &= (S_e + S_o v)(pw + qv) \\ &= (S_e p + S_o q)w + (S_e q + S_o p w)v \\ &= [S_e p + S_o q, S_e q + S_o p w] \\ f \circ \Phi &= p(2\varphi_e, 4\varphi_o^2 w)4\varphi_o^2 w + q(2\varphi_e, 4\varphi_o^2 w)2\varphi_o v \\ &= [p(2\varphi_e, 4\varphi_o^2 w)4\varphi_o^2, q(2\varphi_e, 4\varphi_o^2 w)2\varphi_o] \end{aligned}$$

Since $S_o, S_e, \varphi_e, \varphi_o \in \mathcal{M}^0$, $p, q \in \mathcal{M}^k$, we have

$$S \cdot f \in [\mathcal{M}^k, \mathcal{M}^k] \quad f \circ \Phi \in [\mathcal{M}^k, \mathcal{M}^k]$$

Similarly, if we assume that

$$f = [p, q] = pw + qv \in w^s[\mathcal{M}^k, \mathcal{M}^{k+1}]$$

then we have

$$S \cdot f \in w^s[\mathcal{M}^k, \mathcal{M}^{k+1}] \quad f \circ \Phi \in w^s[\mathcal{M}^k, \mathcal{M}^k]$$

Hence, the proof is done. □

Remark 7.6. Proposition 7.5 implies that the modules

$$J_1 = [M^{k_0}, M^{k_0}] + \cdots + w^s[M^{k_s}, M^{k_s}] + [M^{l_0}, M^{l_0+1}] + \cdots + w^t[M^{l_t}, M^{l_t+1}]$$

are intrinsic for any finite set of nonnegative integers s, t, k_i, l_j where $i = 0, \dots, s$ and $j = 0, \dots, t$.

Let h be a normal form of f ($= [p, q]$). We now calculate necessary conditions for F to be a universal unfolding of f when $f = \gamma(h)$. We do this in the following way:

(a) Write

$$T(h) = \mathcal{J} \oplus V_h$$

where \mathcal{J} is intrinsic.

(b) Using Lemma 7.1 and the fact that \mathcal{J} is intrinsic, write

$$T(f) = \mathcal{J} \oplus V_f$$

(c) By Theorem 6.7, F is a k -parameter universal unfolding of f (where k is the C^∞ codimension of h) if and only if

$$\mathcal{E}^2 = \mathcal{J} \oplus V_f \oplus \mathbf{R}\{F_{\alpha_1}, \dots, F_{\alpha_k}\}$$

- (d) A complementary space to \mathcal{J} always consists of $\dim(V_f) + k$ dimensions. We can choose a basis for V_f in terms of $[p, q]$ and its derivatives. Then we solve the problem by writing the Taylor coefficients of this basis and F_{α_j} in the monomials that are not in \mathcal{J} . It follows that F is a universal unfolding of f if and only if this matrix has a nonzero determinant.

Theorem 7.7. (a) Suppose f is dimorphism equivalent to $h = \epsilon(w + \mu_0 v)$ where $\mu_0 \neq 0$, and $F = P(u, w, \alpha)w + Q(u, w, \alpha)v$ is a 1-parameter unfolding of $f = p(u, w)w + q(u, w)v$. Then F is a universal unfolding of f if and only if

$$\det \begin{pmatrix} p & 0 & q_u \\ p_u & q_u & q_{uu} \\ P_\alpha & Q_\alpha & Q_{\alpha u} \end{pmatrix} \neq 0 \quad (7.0.1)$$

at $u = v = \alpha = 0$.

(b) Suppose f is dimorphism equivalent to $h = \epsilon(\delta w^2 + uv)$, and $F = P(u, w, \alpha)w + Q(u, w, \alpha)v$ is a 1-parameter unfolding of $f = p(u, w)w + q(u, w)v$. Then F is a universal unfolding of f if and only if

$$p_u Q_\alpha - q_u P_\alpha \neq 0$$

at $u = v = \alpha = 0$.

(c) Suppose f is dimorphism equivalent to $h = (\epsilon + \mu_0 u)w + \delta u^2 v$, and $F = P(u, w, \alpha, \beta)w + Q(u, w, \alpha, \beta)v$ is a two-parameter unfolding of $f = p(u, w)w +$

$q(u, w)v$. Then F is a universal unfolding of f if and only if

$$\det \begin{pmatrix} 0 & p & 0 & 0 & 0 & 3q_{uu} \\ p & p_u & 0 & 0 & q_{uu} & q_u^3 \\ p_u & p_{uu} & 0 & q_{uu} & q_u^3 & q_u^4 \\ 0 & 2p_u & 0 & 0 & 2q_{uu} & 3q_u^3 \\ p_\alpha & p_{\alpha u} & q_\alpha & q_{\alpha u} & q_{\alpha u}^2 & q_{\alpha u}^3 \\ p_\beta & p_{\beta u} & q_\beta & q_{\beta u} & q_{\beta u}^2 & q_{\beta u}^3 \end{pmatrix} \neq 0 \quad (7.0.2)$$

at $u = v = \alpha = 0$.

Proof. (a) We prove the statement when $\epsilon = 1$. The proof of the case $\epsilon = -1$ is similar. As shown in the proof of Theorem 6.9

$$T(h) = [\mathcal{M}, \mathcal{M}^2] + w[\mathcal{M}^0, \mathcal{M}^0] + \mathbf{R}\{[1, \mu_0 u], [0, \mu_0]\}$$

The module $[\mathcal{M}, \mathcal{M}^2] + w[\mathcal{M}^0, \mathcal{M}^0]$ is intrinsic. We want to write

$$T(f) = ([\mathcal{M}, \mathcal{M}^2] + w[\mathcal{M}^0, \mathcal{M}^0]) \oplus V_f$$

Since f is diffeomorphism equivalent to h , we know at $(0, 0)$

$$q = 0 \quad p \neq 0 \quad q_u \neq 0$$

Therefore,

$$T(f) = [\mathcal{M}, \mathcal{M}^2] + w[\mathcal{M}^0, \mathcal{M}^0] + \mathbf{R}\{[p, q_u u], [p_u, q_u + q_{uu} u]\}$$

The universal unfolding Theorem 6.7 implies that F is a one-parameter universal unfolding of f if and only if

$$\mathbf{R}\{[p, q_u u], [p_u, q_u + q_{uu} u], [P_\alpha, Q_\alpha + Q_{\alpha u} u]\}$$

spans $\mathbf{R}\{[1, 0], [0, 1], [0, u]\}$. That is, (7.0.1) is satisfied.

- (b) We prove the statement when $\epsilon = 1$. The proof of the case $\epsilon = -1$ is similar. As shown in the proof of Theorem 6.9

$$T(h) = [\mathcal{M}, \mathcal{M}] + \mathbf{R}\{[0, 1]\}$$

The module $[\mathcal{M}, \mathcal{M}]$ is intrinsic. We want to write

$$T(f) = [\mathcal{M}, \mathcal{M}] \oplus V_f$$

Since f is dimorphism equivalent to h , we know at $(0, 0)$

$$q = 0, p = 0 \quad q_u \neq 0 \quad p_w q_u - p_u q_w \neq 0$$

Therefore,

$$T(f) = [\mathcal{M}, \mathcal{M}] + \mathbf{R}\{[p_u, q_u]\}$$

The universal unfolding Theorem 6.7 implies that F is a one-parameter universal unfolding of f if and only if

$$\mathbf{R}\{[p_u, q_u], [P_\alpha, Q_\alpha]\}$$

spans $\mathbf{R}\{[1, 0], [0, 1]\}$. That is

$$\det \begin{pmatrix} p_u & q_u \\ P_\alpha & Q_\alpha \end{pmatrix} = p_u Q_\alpha - q_u P_\alpha \neq 0$$

- (c) We prove the statement when $\epsilon = 1$. The proof of the case $\epsilon = -1$ is similar. As shown in the proof of Theorem 6.9

$$\begin{aligned} T(h) &= [\mathcal{M}^2, \mathcal{M}^4] + w[\mathcal{M}^0, \mathcal{M}^0] \\ &\quad + \mathbf{R}\{[1, \delta u^2 + \mu_0 u^3], [0, 2\delta u + 3\mu_0 u^2], [1, 2\delta u^2 + 3\mu_0 u^3], [u, \delta u^3]\} \end{aligned}$$

We can see that the module $[\mathcal{M}^3, \mathcal{M}^4] + w[\mathcal{M}^0, \mathcal{M}^0]$ is intrinsic. We want to write

$$T(f) = ([\mathcal{M}^3, \mathcal{M}^4] + w[\mathcal{M}^0, \mathcal{M}^0]) \oplus V_f$$

Since f is dimorphism equivalent to h , we know

$$q \neq 0 \quad p = 0 \quad q_u = 0 \quad q_{uu} \neq 0$$

Therefore, with direct calculation, we have

$$\begin{aligned} T(f) &= RT(f) + \mathbf{R}\{[p_u, q_u]\} \\ &= \mathcal{I}(f) + \mathbf{R}\{u^{j-1}[up_u + 2jp_w w + jp, uq_u + 2jq_w w]\} \end{aligned}$$

where $j = 0, 1, 2, \dots$. Hence we have

$$\begin{aligned} T(f) &= [\mathcal{M}^3, \mathcal{M}^4] + w[\mathcal{M}^0, \mathcal{M}^0] \\ &\quad + \mathbf{R}\{[pu^2, 0], [pu, \frac{q_{uu}}{2}u^3], [p + p_u u, \frac{q_{uu}}{2}u^2 + \frac{q_u^3}{6}u^3], \\ &\quad [2p_u u, q_{uu}u^2 + \frac{q_u^3}{2}u^3], [p_u + p_{uu}u, q_{uu}u + \frac{q_u^3}{2}u^2 + \frac{q_u^4}{6}u^3]\} \end{aligned}$$

The universal unfolding Theorem 6.7 implies that F is a two-parameter universal unfolding of f if and only if

$$\begin{aligned} \mathbf{R}\{[pu^2, 0], [pu, \frac{q_{uu}}{2}u^3], [p + p_u u, \frac{q_{uu}}{2}u^2 + \frac{q_u^3}{6}u^3], [p_u + p_{uu}u, q_{uu}u + \frac{q_u^3}{2}u^2 + \frac{q_u^4}{6}u^3], \\ [2p_u u, q_{uu}u^2 + \frac{q_u^3}{2}u^3], [p_\alpha + p_{\alpha u}u, q_\alpha + q_{\alpha u}u + \frac{q_{\alpha u^2}}{2}u^2 + \frac{q_{\alpha u^3}}{6}u^3], \\ [p_\beta + p_{\beta u}u, q_\beta + q_{\beta u}u + \frac{q_{\beta u^2}}{2}u^2 + \frac{q_{\beta u^3}}{6}u^3]\} \end{aligned}$$

spans $\mathbf{R}\{[1, 0], [u, 0], [u^2, 0], [0, 1], [0, u], [0, u^2], [0, u^3]\}$. That is, (7.0.2) is satisfied.

□

CHAPTER 8

GEOMETRY OF UNIVERSAL UNFOLDINGS

In this chapter, we list the mutual invasibility plots of all possible perturbations up to dimorphism equivalence for a strategy function of at most two parameters. First we review the four important evolutionary scenarios that is preserved by dimorphism equivalence. Then we describe the different types of singular transitions under dimorphism equivalence and introduce the transition variety (See Definition 8.6). For each singularity up to topological codimension two, we calculate the transition variety and draw the mutual invasibility plots for all small perturbations up to dimorphism equivalence.

8.1 Singularities of a Strategy Function

Geritz et al. [10] discussed four different evolutionary scenarios in a strategy function. We have already discussed ESS and CvSS in this thesis. Here we introduce the other two characteristics of singular strategies called MIS and NIS defined in Golubitsky and Vutha [14].

Definition 8.1. A singular strategy x_0 of a strategy function f is a *neighborhood invader strategy* (NIS) if

$$f_y(x_0, x_0) = 0 \quad f_{xx}(x_0, x_0) > 0$$

Remark 8.2. The concept of NIS is first introduced by Apaloo [1]. If x_0 is an NIS of f , then x_0 can always invade other local strategies when itself is initially rare.

Definition 8.3. A singular strategy x_0 of a strategy function f is *mutual invasibility* (MIS) if

$$f_y(x_0, x_0) = 0 \quad f_{yy}(x_0, x_0) + f_{xx}(x_0, x_0) > 0 \quad (8.1.1)$$

Remark 1.10 shows that (8.1.1) is a sufficient condition for the existence of pairs of dimorphisms around the singular strategy x_0 .

Dieckmann and Metz [4] discuss the evolutionary influence of these four types of singular behaviors. These four singularities are summarized in Figure 8.1, which is a modification of a picture in Geritz et al. [10]. This figure classifies the stabilities of the singular strategies ESS, CvSS, NIS, and MIS.

Lemma 8.4. *Dimorphism equivalence preserves the stabilities of ESS, CvSS, NIS, and MIS for all strategy functions.*

We recall that Golubitsky and Vutha [14] define

Definition 8.5. Two strategy functions f and \hat{f} are *strongly strategy equivalent* if

$$\hat{f}(x, y) = S(x, y)f(\Phi(x, y))$$

where the following conditions are satisfied near a given point

1. $S(x, y) > 0$ for all x, y
2. $\Phi \equiv (\Phi_1, \Phi_2)$ where $\Phi_i: \mathbf{R}^2 \rightarrow \mathbf{R}$, $\det(d\Phi)_{x,y} > 0$ for all x, y
3. $\Phi_1(x, x) = \Phi_2(x, x)$ for every x
4. $(d\Phi)_{x,x} = \alpha(x)I$ where $\alpha(x) > 0$

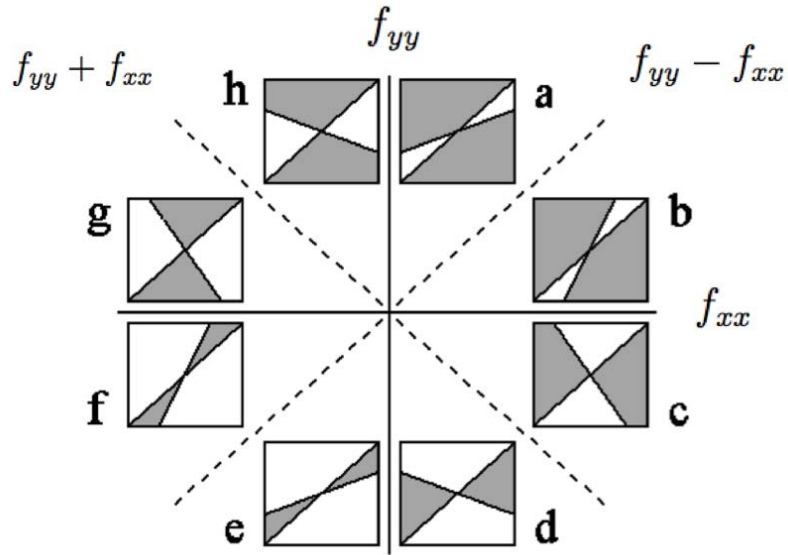


Figure 8.1: Classification of singular points modified from Figure 2 in [10]. Based on the evolutionary stability, convergence stability, existence of dimorphisms, and resistance to invasion, we can divide (f_{xx}, f_{yy}) -space into 8 separate regions. Note that each region has a sample plot. In this plot, the curves are that of $f(x, y) = 0$ and the shaded areas are those satisfying $f(x, y) > 0$.

Proposition 6.6 in [14] shows that if two strategy functions f and \hat{f} are strongly strategy equivalent, then \hat{f} has the same ESS, CvSS, NIS, and MIS singularity types at $\Phi^{-1}(x_0, x_0)$ as f has at (x_0, x_0) .

Proof. The proof of Lemma 8.4 follows by noting that every dimorphism equivalence is a strong strategy equivalence. □

8.2 Transition Varieties

Suppose $F(x, y, \alpha)$, where $\alpha \in \mathbf{R}^k$, is a universal unfolding of $f(x, y)$. In $F(x, y, \alpha)$, the classification of small perturbations proceeds by determining parameter values where singularity types change. In the parameter space of $F(x, y, \alpha)$, there are six varieties where such changes occur. These varieties are based on degeneracies of evolutionary stability, convergence stability, existence of dimorphisms, neighborhood invaders, bifurcation, and tangency.

The first four types of degeneracies correspond to the four evolutionary scenarios in the previous section. Bifurcation points occur at certain parameter values in phase space where the off-diagonal zero set of $F(x, y, \alpha)$ is singular; that is, at points where $F = F_x = F_y = 0$. Tangency points in parameter space occur at certain parameter values when $F(x, y, \alpha)$ and $F(y, x, \alpha)$ become tangent to each other. Note that we are studying dimorphism equivalence, so we consider a pair of universal unfolding strategy functions $(F(x, y, \alpha), F(y, x, \alpha))$. Denote

$$F(x, y) = (x - y)G(x, y)$$

Then we define

$$\begin{aligned}
 \mathcal{E} &= \{\alpha \in \mathbf{R}^k : \exists x \text{ such that } F_y = F_{yy} = 0 \text{ at } (x, x, \alpha)\} (\text{ESS variety}) \\
 \mathcal{C} &= \{\alpha \in \mathbf{R}^k : \exists x \text{ such that } F_y = F_{yy} - F_{xx} = 0 \text{ at } (x, x, \alpha)\} (\text{CvSS variety}) \\
 \mathcal{D} &= \{\alpha \in \mathbf{R}^k : \exists x \text{ such that } F_y = F_{yy} + F_{xx} = 0 \text{ at } (x, x, \alpha)\} (\text{MIS variety}) \\
 \mathcal{N} &= \{\alpha \in \mathbf{R}^k : \exists x \text{ such that } F_y = F_{xx} = 0 \text{ at } (x, x, \alpha)\} (\text{NIS variety}) \\
 \mathcal{B} &= \{\alpha \in \mathbf{R}^k : \exists x, y \text{ such that } F = F_x = F_y = 0 \text{ at } (x, y, \alpha) \text{ where } x \neq y\} \\
 &\quad (\text{Bifurcation variety}) \\
 \mathcal{T} &= \{\alpha \in \mathbf{R}^k : \exists x, y \text{ such that } G(x, y) = G(y, x) = \\
 &\quad G_x(x, y)G_y(y, x) - G_y(x, y)G_x(y, x) = 0 \text{ at } (x, y, \alpha) \text{ where } x \neq y\} \\
 &\quad (\text{Tangency variety})
 \end{aligned}$$

(8.2.1)

Definition 8.6. *The transition variety is the union of all the degenerate varieties in (8.2.1). That is*

$$\mathcal{TV} = \mathcal{E} \cup \mathcal{C} \cup \mathcal{D} \cup \mathcal{N} \cup \mathcal{B} \cup \mathcal{T}$$

We can simplify the calculation of transition varieties. Since the universal unfoldings of strategy functions vanish on the diagonal, we can define

$$F(x, y, \alpha) = P(u, w, \alpha)w + Q(u, w, \alpha)v.$$

Therefore, with direct calculation, we obtain

$$\begin{aligned} \mathcal{E} &= \{\alpha \in \mathbf{R}^k : \exists u \text{ such that } -Q = 2(P - Q_u) = 0 \text{ at } (u, 0, \alpha)\} \\ \mathcal{C} &= \{\alpha \in \mathbf{R}^k : \exists u \text{ such that } -Q = -4Q_u = 0 \text{ at } (u, 0, \alpha)\} \\ \mathcal{D} &= \{\alpha \in \mathbf{R}^k : \exists u \text{ such that } -Q = 4P = 0 \text{ at } (u, 0, \alpha)\} \\ \mathcal{N} &= \{\alpha \in \mathbf{R}^k : \exists u \text{ such that } -Q = 2(P + Q_u) = 0 \text{ at } (u, 0, \alpha)\} \\ \mathcal{B} &= \{\alpha \in \mathbf{R}^k : \exists u, w \text{ such that} \\ &\quad Pv + Q = P_uv + Q_u = 2P_w w + P + 2Q_w v = 0 \text{ at } (u, v, \alpha) \text{ where } v \neq 0\} \\ \mathcal{T} &= \{\alpha \in \mathbf{R}^k : \exists u, w \text{ such that} \\ &\quad Pv = Q = 2w(P_u Q_w - P_w Q_u) - P Q_u = 0 \text{ at } (u, v, \alpha) \text{ where } v \neq 0\} \end{aligned} \tag{8.2.2}$$

Remark 8.7. Note that we study ESS singularities, CvSS singularities, and dimorphisms in this thesis. We showed that dimorphism equivalence preserves these three properties. In addition, we find that dimorphism equivalence preserves NIS as well. Since Geritz et al. [10] point out that NIS is also one of the most important evolutionary scenarios, we would like to consider the transition of stability of NIS as part of the transition varieties too. That is, in the MIPs of the singularities, we identify singularities with different stabilities of NIS as different singularities. This is just something we have the option to keep track of along the way of drawing MIPs under dimorphism equivalence.

8.3 Mutual Invasibility Plots

We classify persistent perturbations of low codimension singularities for strategy functions and their universal unfoldings. A persistent perturbation is described by a mutual invasibility plot.

Topological Codimension Zero.

(a) Theorem 6.9 shows that the universal unfolding of $f = \epsilon(w + \mu_0 uv)$ is

$$F = \epsilon(w + \mu uv)$$

where μ is a modal parameter near $\mu_0 \neq 0$. Note that

$$P = \epsilon \quad Q = \epsilon \mu u$$

Therefore, the transition variety of F contains

$$\mathcal{E} = \{\mu = \epsilon\} \quad \mathcal{C} = \{\mu = 0\} \quad \mathcal{D} = \emptyset \quad \mathcal{N} = \{\mu = -\epsilon\} \quad \mathcal{B} = \emptyset \quad \mathcal{T} = \emptyset$$

The MIPs are displayed in Figure 2.1 and Figure 2.2 of Chapter 2. We see that μ is a modal parameter. Thus, when $\mu_0 \neq \pm 1, 0$ we have a topological codimension zero singularity. However, the two special cases $\mu_0 = \pm 1$ generate a topological codimension one singularity.

Topological Codimension One

(b) Theorem 6.9 shows that the universal unfolding of $f = \epsilon(\delta w^2 + uv)$ is

$$F = \epsilon((\delta w + a)w + uv)$$

where a is near 0. Note that

$$P = \epsilon(\delta w + a) \quad Q = \epsilon u$$

Therefore, the transition variety of F contains

$$\mathcal{E} = \emptyset \quad \mathcal{C} = \emptyset \quad \mathcal{D} = \{a = 0\} \quad \mathcal{N} = \emptyset \quad \mathcal{B} = \emptyset \quad \mathcal{T} = \emptyset$$

The MIPs are displayed in Figure 2.3 of Chapter 2. Since we have one unfolding parameter a , this is a topological codimension one singularity.

(c) Theorem 6.9 shows that the universal unfolding of $f = (\epsilon + \mu_0 u)w + \delta u^2 v$ is

$$F = (\epsilon + \mu u)w + (a + \delta u^2)v$$

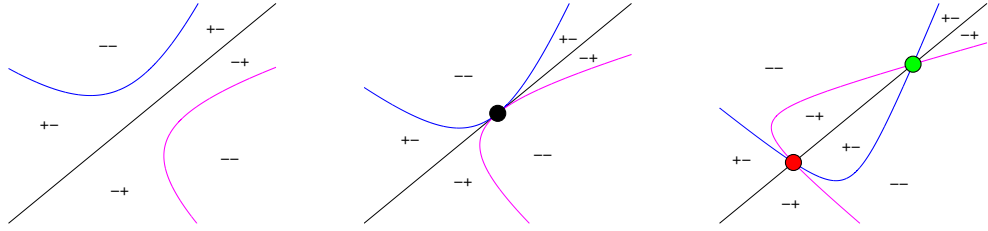
Note that

$$P = \epsilon + \mu u \quad Q = a + \delta u^2$$

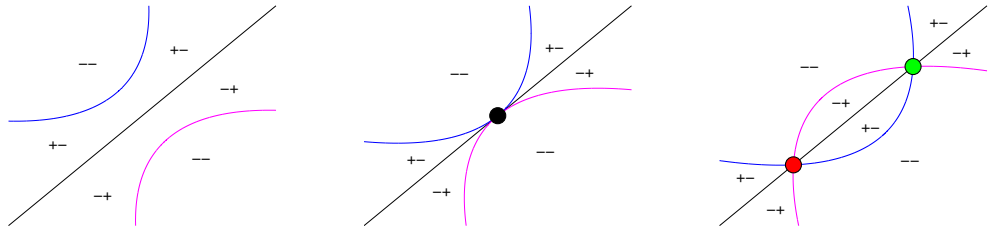
Therefore, the transition variety of F contains

$$\mathcal{E} = \emptyset \quad \mathcal{C} = \{a = 0\} \quad \mathcal{D} = \emptyset \quad \mathcal{N} = \emptyset \quad \mathcal{B} = \emptyset \quad \mathcal{T} = \emptyset$$

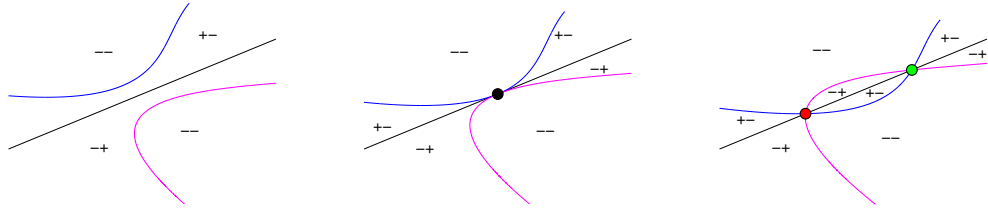
Figure 8.2 consists of MIPs of F for different μ , we can see that values of μ do not change the existence of dimorphisms and the stabilities of singularities. Even though strategy functions with distinct μ values are not dimorphism equivalent, we still have an intuitive idea that the MIPs with different μ share the same singular and dimorphism properties. In particular, the MIPs when $\mu = 0$ are displayed in Figure 2.4 of Chapter 2. Since μ is a modal parameter and a is an unfolding parameter, we have a topological codimension one singularity.



(a) $F = (-1 + \mu u)w + (a - u^2)v$ when $\mu < 0$: (i) $a < 0$, (ii) $a = 0$, (iii) $a > 0$



(b) $F = (-1 + \mu u)w + (a - u^2)v$ when $\mu = 0$: (i) $a < 0$, (ii) $a = 0$, (iii) $a > 0$



(c) $F = (-1 + \mu u)w + (a - u^2)v$ when $\mu > 0$: (i) $a < 0$, (ii) $a = 0$, (iii) $a > 0$

Figure 8.2: MIPs of $F = (-1 + \mu u)w + (a - u^2)v$ when: (a) $\mu < 0$; (b) $\mu = 0$; (c) $\mu > 0$. Note that the modal parameter μ only rotates the axis of the parabola F . However, it does not change the paired signs of each regions and the stability of the singularities.

Topological Codimension Two

(d) Theorem 6.9 shows that the universal unfolding of $f = \epsilon(\delta w^3 + uv)$ is

$$F = \epsilon((a + bw + \delta w^2)w + uv)$$

Note that

$$P = \epsilon(a + bw + \delta w^2) \quad Q = \epsilon u$$

The transition variety of F contains

$$\mathcal{E} = \emptyset \quad \mathcal{C} = \emptyset \quad \mathcal{D} = \{a = 0\} \quad \mathcal{N} = \emptyset \quad \mathcal{B} = \emptyset \quad \mathcal{I} = \{b^2 - 4\epsilon\delta a = 0\}$$

Note that

$$\begin{aligned} F(y, x) &= \epsilon((a + bw + \delta w^2)w - uv) \\ &= \epsilon((a + bw + \delta w^2)w - uv) \\ -F(x, y) &= \epsilon(-(a + bw + \delta w^2)w - uv) \\ &= \epsilon((-a - bw - \delta w^2)w - uv) \\ -F(y, x) &= \epsilon(-(a + bw + \delta w^2)w + uv) \\ &= \epsilon((-a - bw - \delta w^2)w + uv) \end{aligned}$$

Considering this symmetry and the fact that a, b are unfolding parameters near 0, we only need to study the case when $\epsilon = 1, \delta = 1$. Figure 8.3 is the transition variety of F . Figure 8.4 contains the MIPs of F for different a, b close to 0 up to diffeomorphism equivalence when $\epsilon = 1, \delta = 1$. We can see the emergence of regions of coexistence when the unfolding parameters are perturbed in Figure 8.4. Since we have two unfolding parameters a, b , this is a topological codimension two singularity.

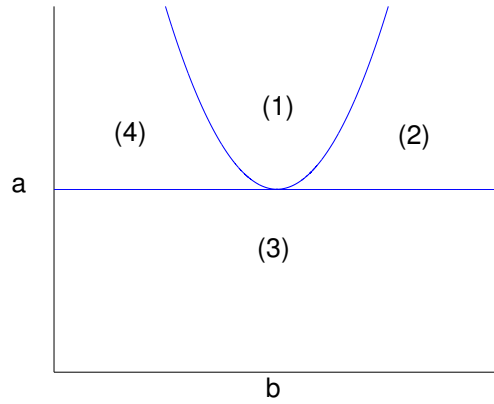


Figure 8.3: The transition variety for $F = (a + bw + w^2)w + uv$ are $a = 0$ and $b^2 - 4a = 0$.

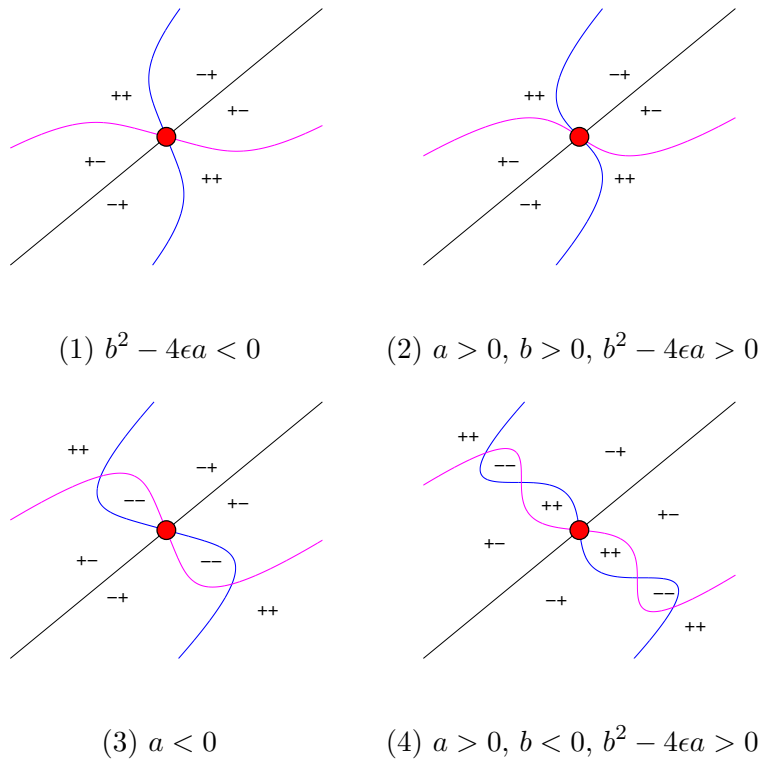


Figure 8.4: MIPs of $F = (a + bw + w^2)w + uv$ for different parameter values.

(e) Theorem 6.9 shows that the universal unfolding of $f = \epsilon w + (\delta u^3 + \mu_0 u^5)v$ is

$$F = \epsilon w + (a + bu + \delta u^3 + \mu u^5)v$$

Note that

$$P = \epsilon \quad Q = a + bu + \delta u^3 + \mu u^5$$

The transition varieties of F contains

$$\mathcal{E} = \emptyset \quad \mathcal{C} = \{27a^2 + 4\delta b^3 + o(\mu b^3) = 0\} \quad \mathcal{D} = \emptyset \quad \mathcal{N} = \emptyset \quad \mathcal{B} = \emptyset \quad \mathcal{T} = \emptyset$$

Note that

$$\begin{aligned} F(y, x) &= \epsilon w - (a + bu + \delta u^3 + \mu u^5)v \\ &= \epsilon w + (-a - bu - \delta u^3 - \mu u^5)v \\ -F(x, y) &= -\epsilon w - (a + bu + \delta u^3 + \mu u^5)v \\ &= -\epsilon w + (-a - bu - \delta u^3 - \mu u^5)v \\ -F(y, x) &= -\epsilon w + (a + bu + \delta u^3 + \mu u^5)v \\ &= -\epsilon w + (a + bu + \delta u^3 + \mu u^5)v \end{aligned}$$

Considering this symmetry and the fact that a, b are unfolding parameters near 0, we only need to study the case when $\epsilon = 1, \delta = 1$ for all μ . The choice of μ will be discussed in Remark 8.8. For the ease of displaying, we only show the transition variety when $\mu = 0$ in Figure 8.5. Figure 8.6 contains MIPs of F when $\mu = 0, \epsilon = 1, \delta = 1$.

Remark 8.8. If $\mu \neq 0$, then the transition variety will be tilted to the left or the right, so it is a modified cusp. But the MIPs will be similar in the sense of the existence of dimorphisms and the stabilities of singularities for those two regions in the parameter space.

Since μ is a modal parameter and a, b are two unfolding parameters, (e) is a topological codimension two singularity.

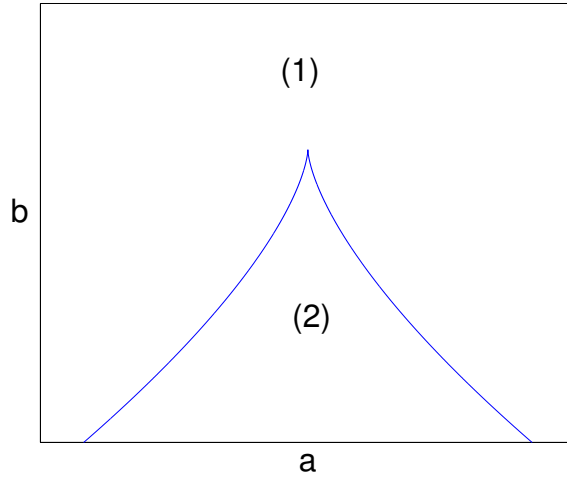


Figure 8.5: The transition variety of $F = \epsilon w + (a + bu + \delta u^3 + \mu u^5)v$ when $\mu = 0$ is $27a^2 + 4\delta b^3 = 0$.

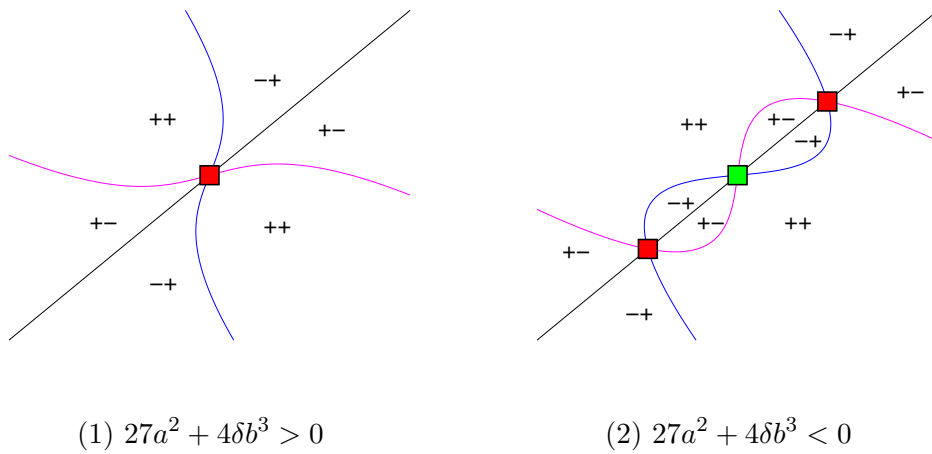


Figure 8.6: MIPs of $F = w + (a + bu + u^3)v$ for different parameter values.

(f) Theorem 6.9 shows that the universal unfolding of $f = \epsilon(uw + (\alpha_0w + \beta_0u^2)v)$ is

$$F = \epsilon((a + u)w + (b + \alpha w + \beta u^2)v)$$

where a, b are unfolding parameters near 0 and α, β are modal parameter near α_0, β_0 . Note that

$$P = \epsilon(a + u) \quad Q = \epsilon(b + \alpha w + \beta u^2)$$

The transition variety of F contains

$$\begin{aligned} \mathcal{E} &= \{\epsilon a^2 \beta + b(1 - 2\beta)^2 = 0\} & \mathcal{C} &= \{b = 0\} \\ \mathcal{D} &= \{\epsilon a^2 \beta + b = 0\} & \mathcal{N} &= \{\epsilon a^2 \beta + b(1 + 2\beta)^2 = 0\} \\ \mathcal{B} &= \{\epsilon a^2 \beta - b(4\alpha\beta - 1) = 0\} & \tilde{\mathcal{T}} &= \{\alpha = 0\} \end{aligned}$$

Note that in Theorem 6.9 we have assumptions

$$\alpha \neq 0 \quad \beta \neq 0 \quad 4\alpha\beta - 1 \neq 0$$

So we know

$$\tilde{\mathcal{T}} = \emptyset$$

From the definition of the varieties, we see a few degeneracies of the transition variety when modal parameter α, β are varied. In summary, we have

- (i) If $\beta = \pm \frac{1}{2}$, \mathcal{E} or \mathcal{N} would degenerate from parabola to a straight line;
- (ii) If $\beta = \pm 1$, \mathcal{E} or \mathcal{N} would overlap with \mathcal{D} ;
- (iii) If $\alpha + \beta = \pm 1$, \mathcal{E} or \mathcal{N} would overlap with \mathcal{B} .

In addition, under the symmetry $\sigma(x, y) = (y, x)$ we have

$$\begin{aligned} F(y, x) &= \epsilon((a + u)w - (b + \alpha w + \beta u^2)v) \\ &= \epsilon((a + \epsilon u)w + (-b - \alpha w - \beta u^2)v) \end{aligned}$$

Moreover,

$$\begin{aligned} -F(x, y) &= \epsilon(-(a + u)w - (b + \alpha w + \beta u^2)v) \\ &= \epsilon((-a - \epsilon u)w + (-b - \alpha w - \beta u^2)v) \end{aligned}$$

So we know

$$\begin{aligned} -F(y, x) &= \epsilon(-(a + u)w + (b + \alpha w + \beta u^2)v) \\ &= \epsilon((-a - \epsilon u)w + (b + \alpha w + \beta u^2)v) \end{aligned}$$

Considering that a, b are unfolding parameters near 0, we only need to study the perturbation of $f = \epsilon(uw + (\alpha_0 w + \beta_0 u^2)v)$ when $\epsilon = 1, \alpha_0 > 0$ since the other 3 cases can be obtained through symmetries. In addition, if we assume $\epsilon = 1, \alpha_0 > 0$, then the universal unfolding becomes

$$F = (a + u)w + (b + \alpha w + \beta u^2)v$$

where $\alpha > 0$. Note that when $\epsilon = 1, \alpha > 0$, we see

$$\begin{aligned} -F(-x, -y) &= \epsilon(-(a - u)w + (b + \alpha w + \beta u^2)v) \\ &= \epsilon((-a + u)w + (b + \alpha w + \beta u^2)v) \end{aligned} \tag{8.3.1}$$

Since a, b are unfolding parameters close to 0, Equation (8.3.1) indicates that a special symmetry exists when we change the parameter from a to $-a$.

Next, we work on the transition variety and MIPs of the universal unfolding $F = \epsilon((a + u)w + (b + \alpha w + \beta u^2)v)$ for different parameter values.

First we divide the modal parameter space (α, β) into regions with the same transition variety. Figure 8.7 is the separation of the modal parameters space.

Second, we look at the transition variety F for different a, b close to 0 when $\epsilon = 1, \alpha > 0$. Figure 8.8 is the transition variety plot (relative to unfolding parameter a, b) for each numbered region in Figure 8.7.

Last, we draw the MIPs of all small perturbations for each scenario of Figure 8.8. The MIPs can be found from Figure ?? to Figure 8.19.

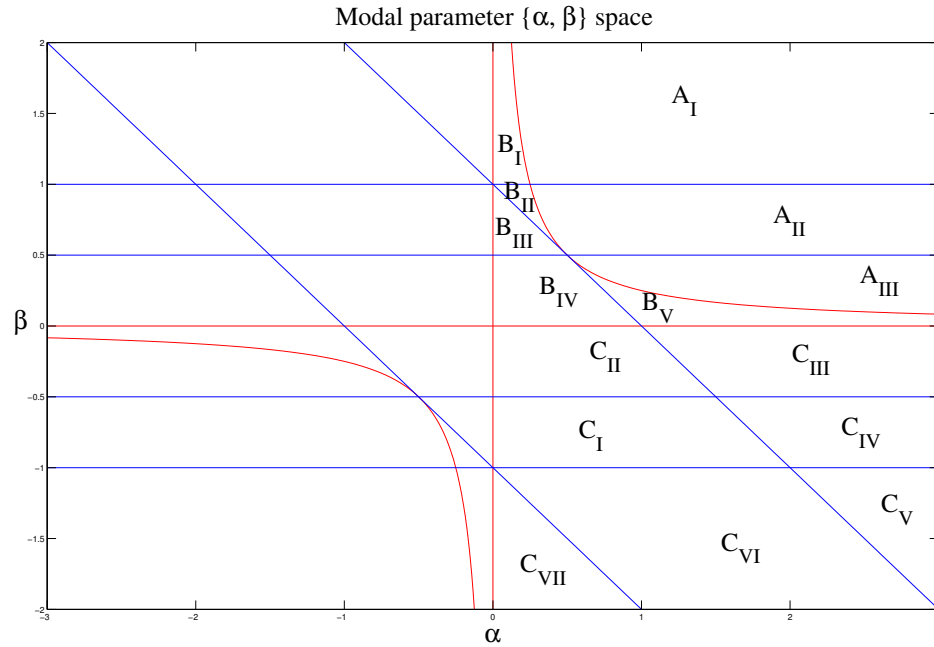


Figure 8.7: The modal parameter space (α, β) for $F = \epsilon((a + u)w + (b + \alpha w + \beta u^2)v)$. Red curves are the non-degeneracy conditions in Theorem 6.9; blue curves are the degenerate cases (i), (ii), (iii) summarized in previous paragraphs. (Note that only the $\alpha > 0$ half plane is numbered.)

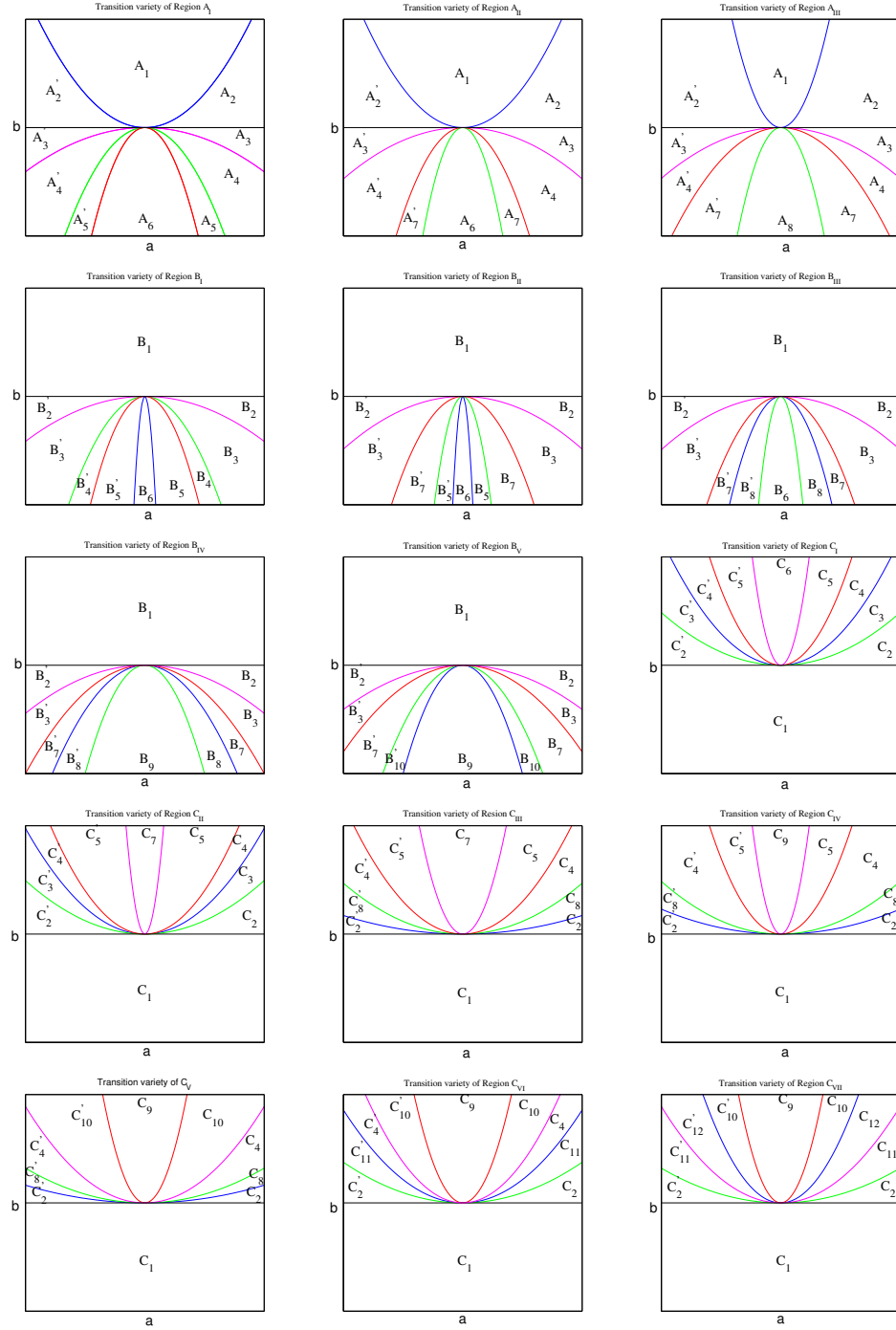


Figure 8.8: Transition varieties of $F = \epsilon((a + u)w + (b + \alpha w + \beta u^2)v)$ for different regions in the modal parameters space (α, β) when $\epsilon = 1$, $\alpha > 0$. Blue is variety \mathcal{B} ; Red is variety \mathcal{D} ; Green is variety \mathcal{E} ; Magenta is variety \mathcal{N} ; Black is variety \mathcal{C} .

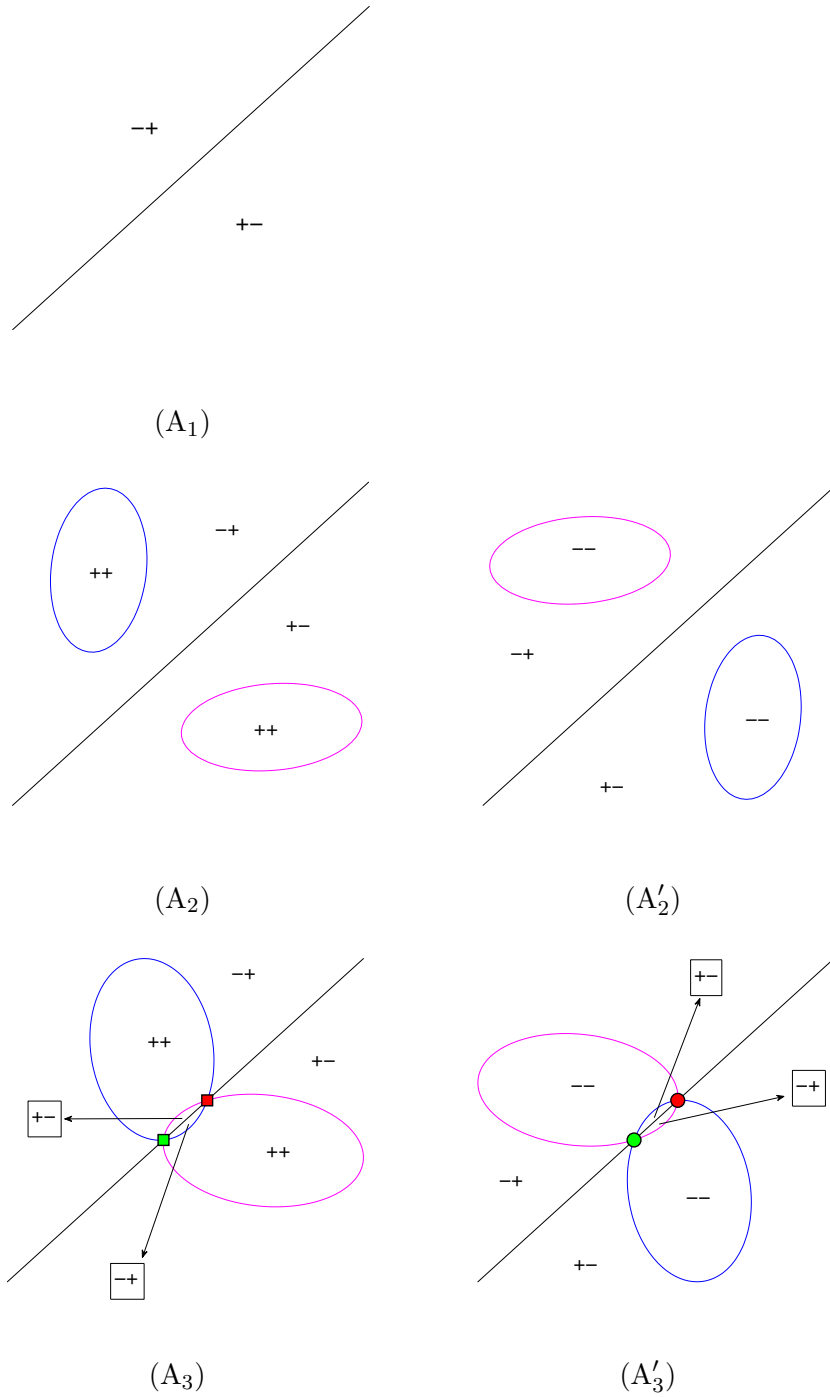


Figure 8.9: MIPs for all the non-degenerate perturbation of $F = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions corresponding to A_1 to A_3 .

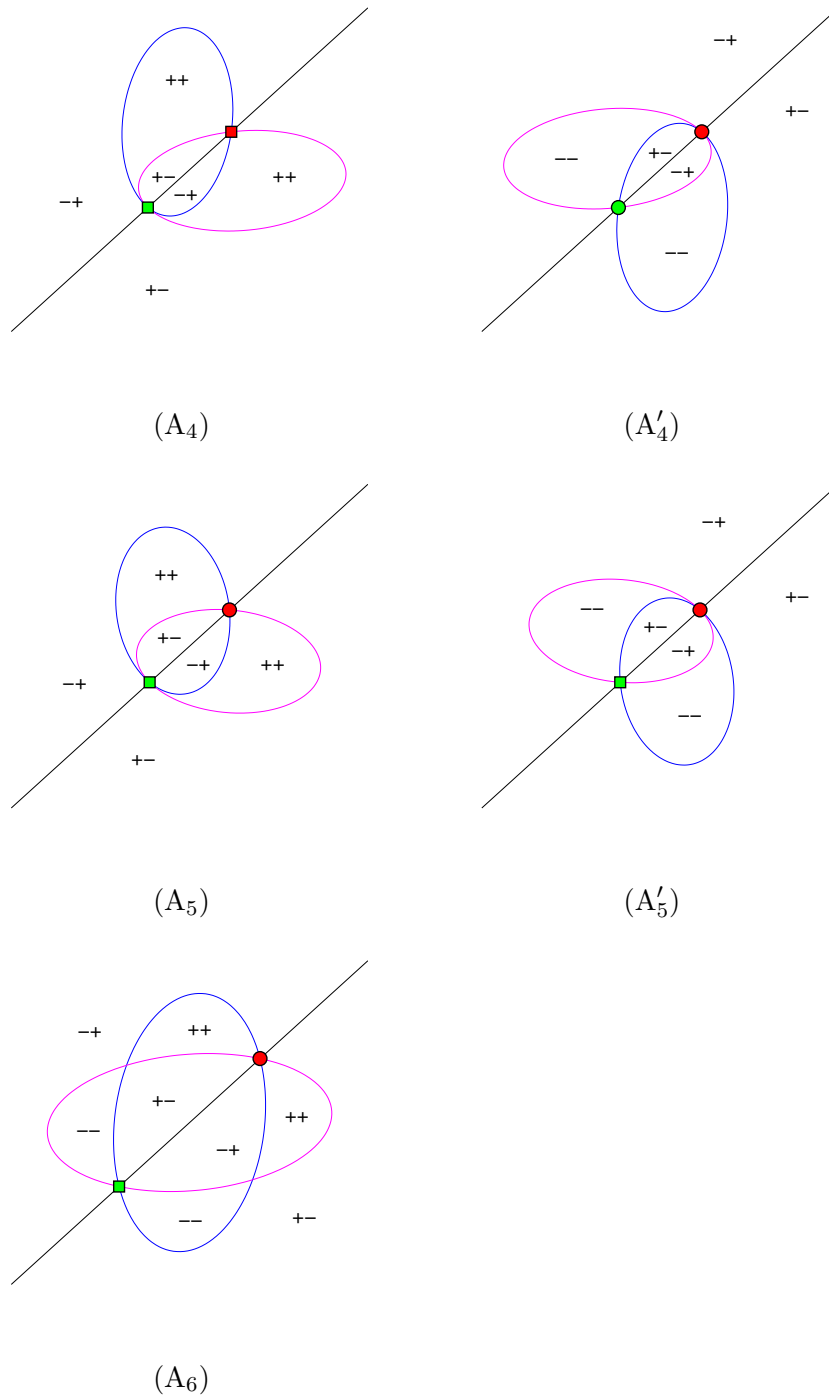


Figure 8.10: MIPs for all the non-degenerate perturbation of $F = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions corresponding to A_4 to A_6 .

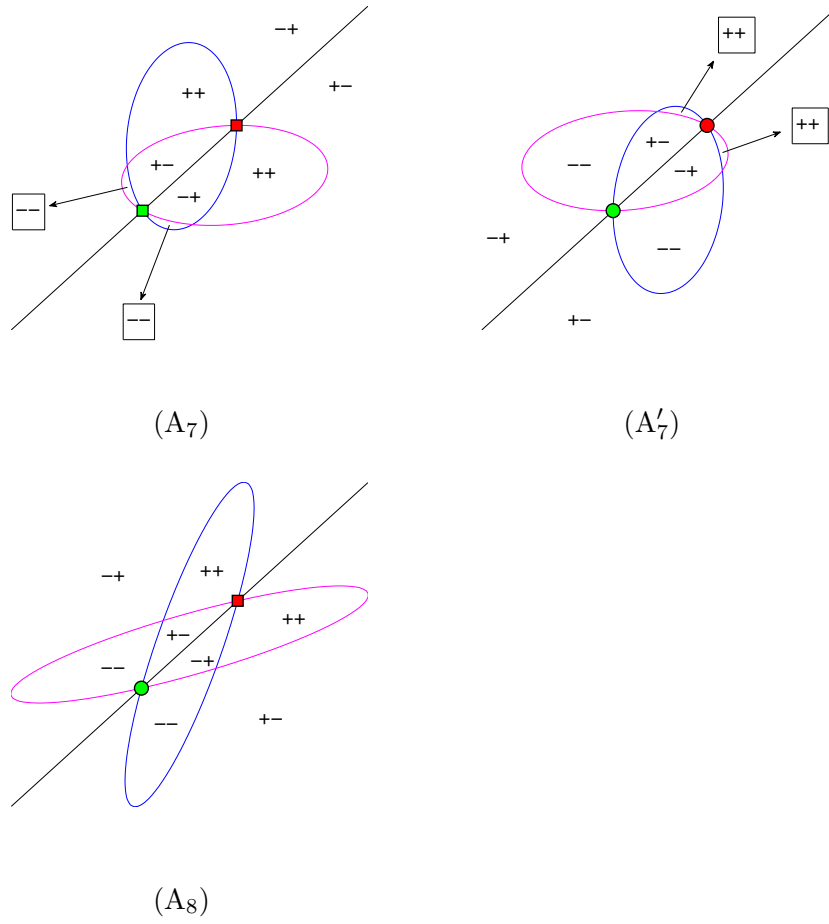
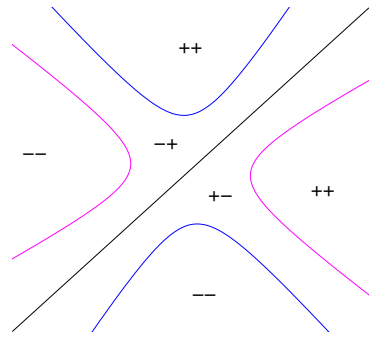
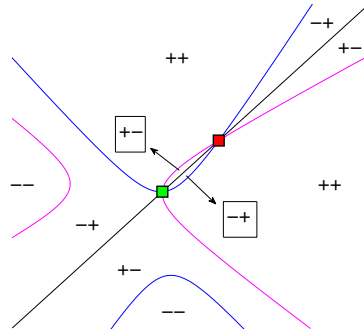


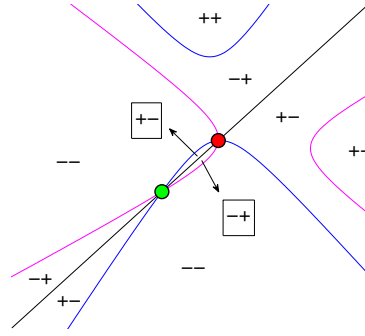
Figure 8.11: MIPs for all the non-degenerate perturbation of $F = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions corresponding to A₇ to A₈.



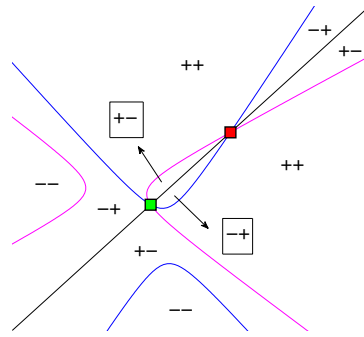
(B₁)



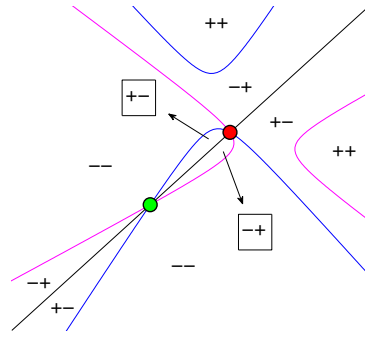
(B₂)



(B'₂)



(B₃)



(B'₃)

Figure 8.12: MIPs for all the non-degenerate perturbation of $F = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions corresponding to B₁ to B₃.

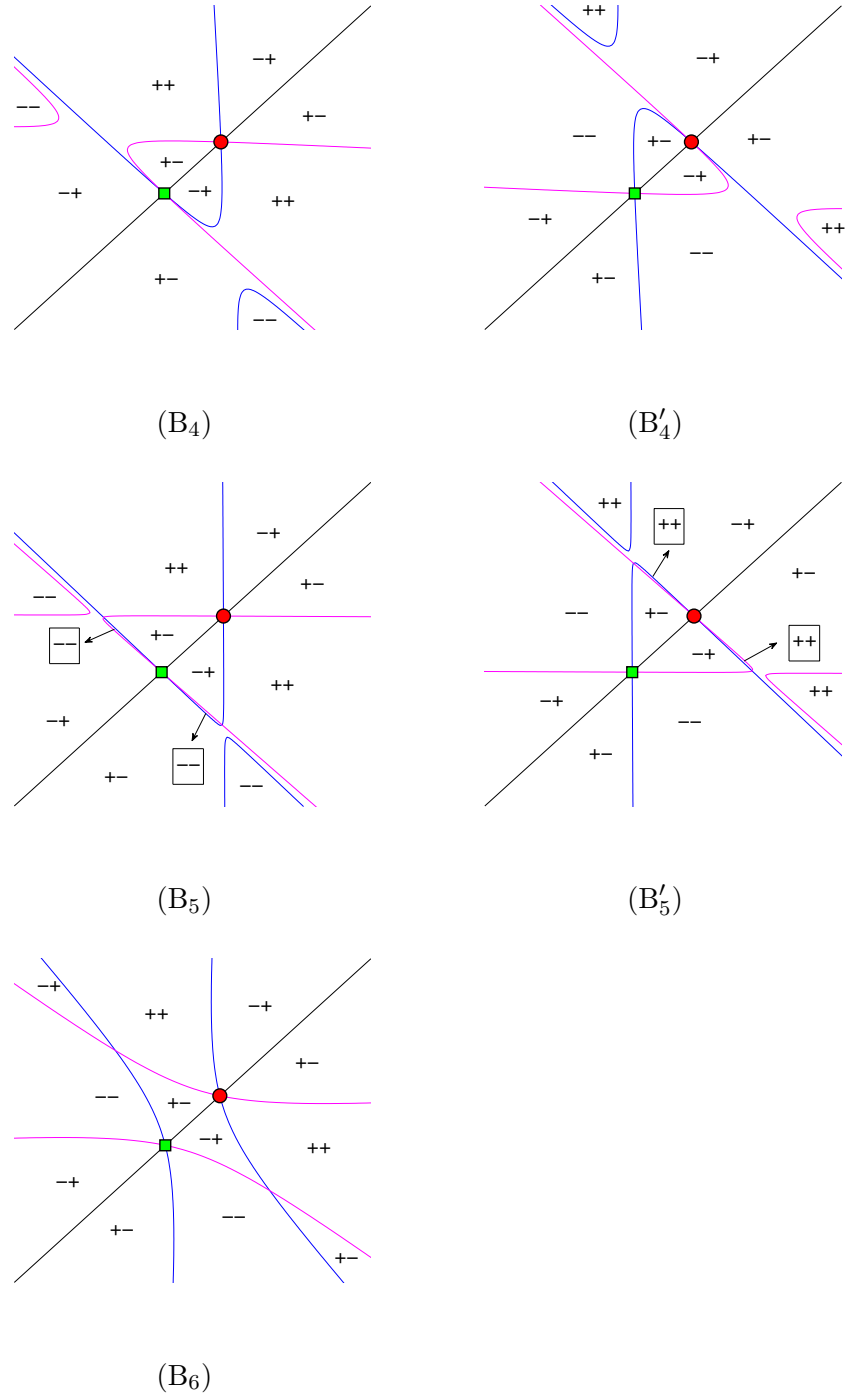


Figure 8.13: MIPs for all the non-degenerate perturbation of $F = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions corresponding to B_4 to B_6 .

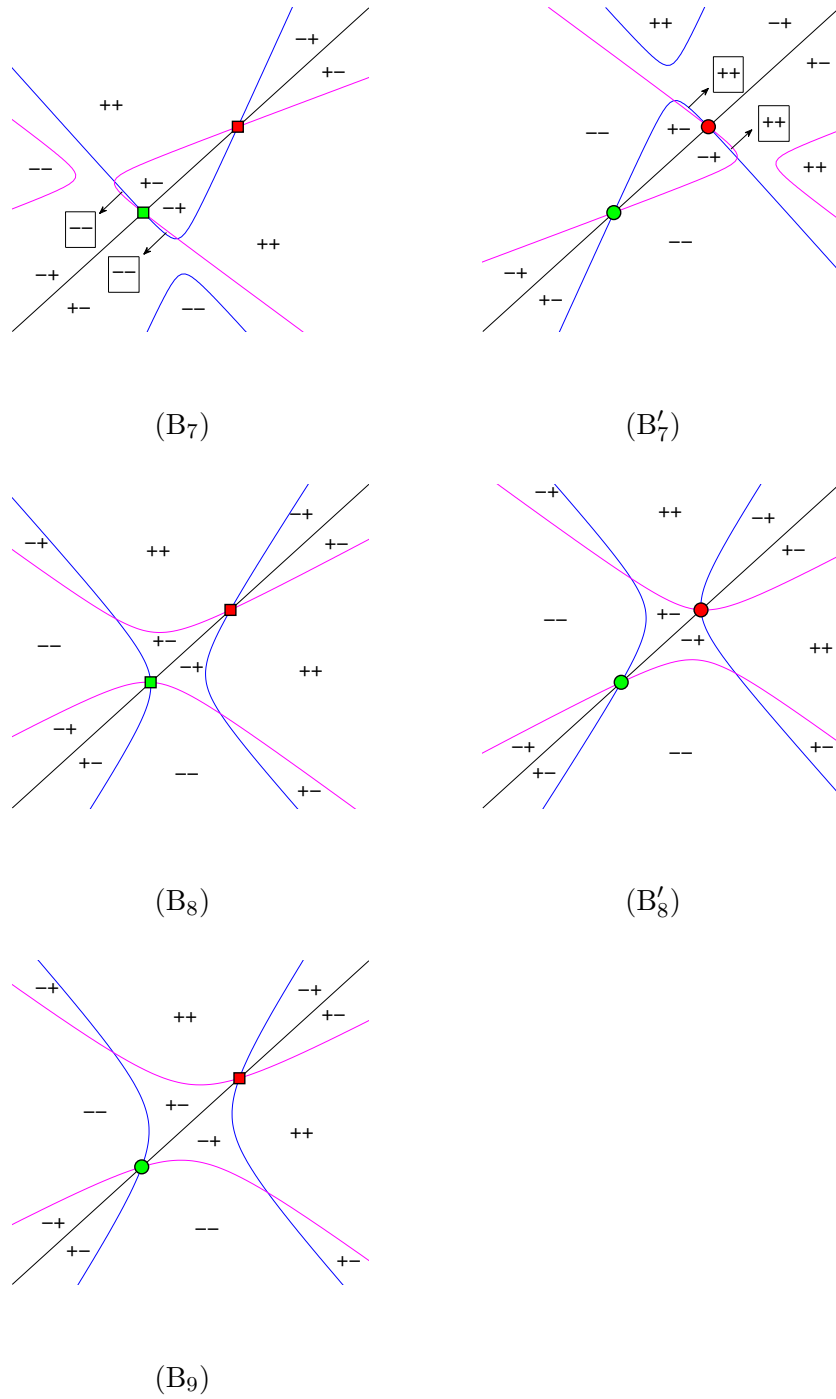
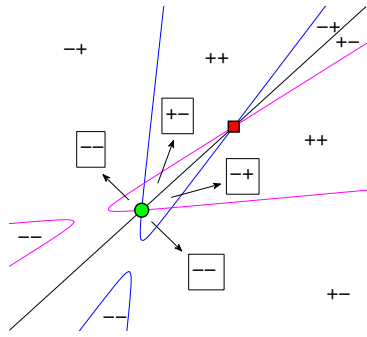
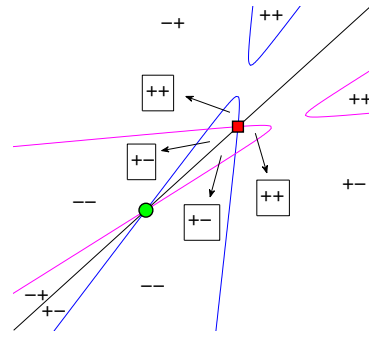


Figure 8.14: MIPs for all the non-degenerate perturbation of $F = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions corresponding to B₇ to A₉.

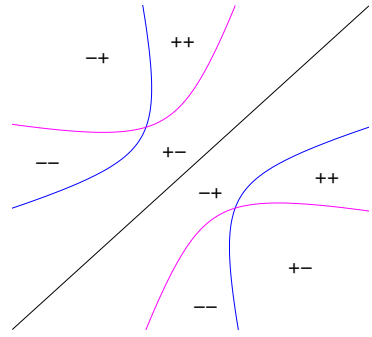


(B₁₀)

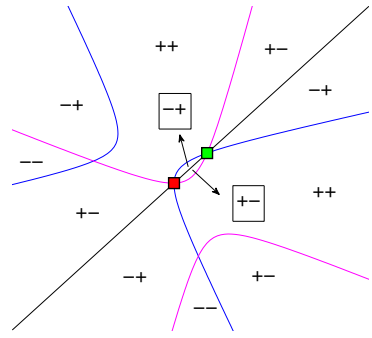


(B'₁₀)

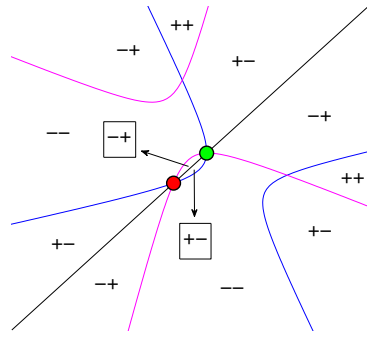
Figure 8.15: MIPs for all the non-degenerate perturbation of $F = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions corresponding to B₁₀.



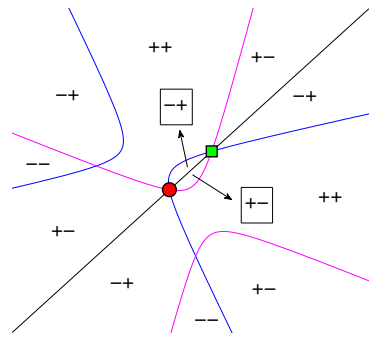
(C₁)



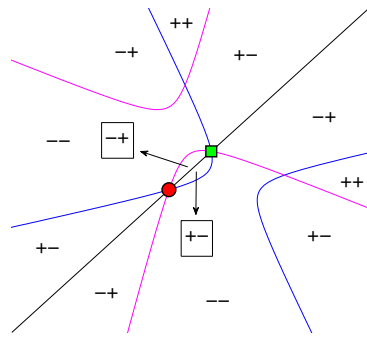
(C₂)



(C'₂)



(C₃)



(C'₃)

Figure 8.16: MIPs for all the non-degenerate perturbation of $F = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions corresponding to C₁ to C₃.

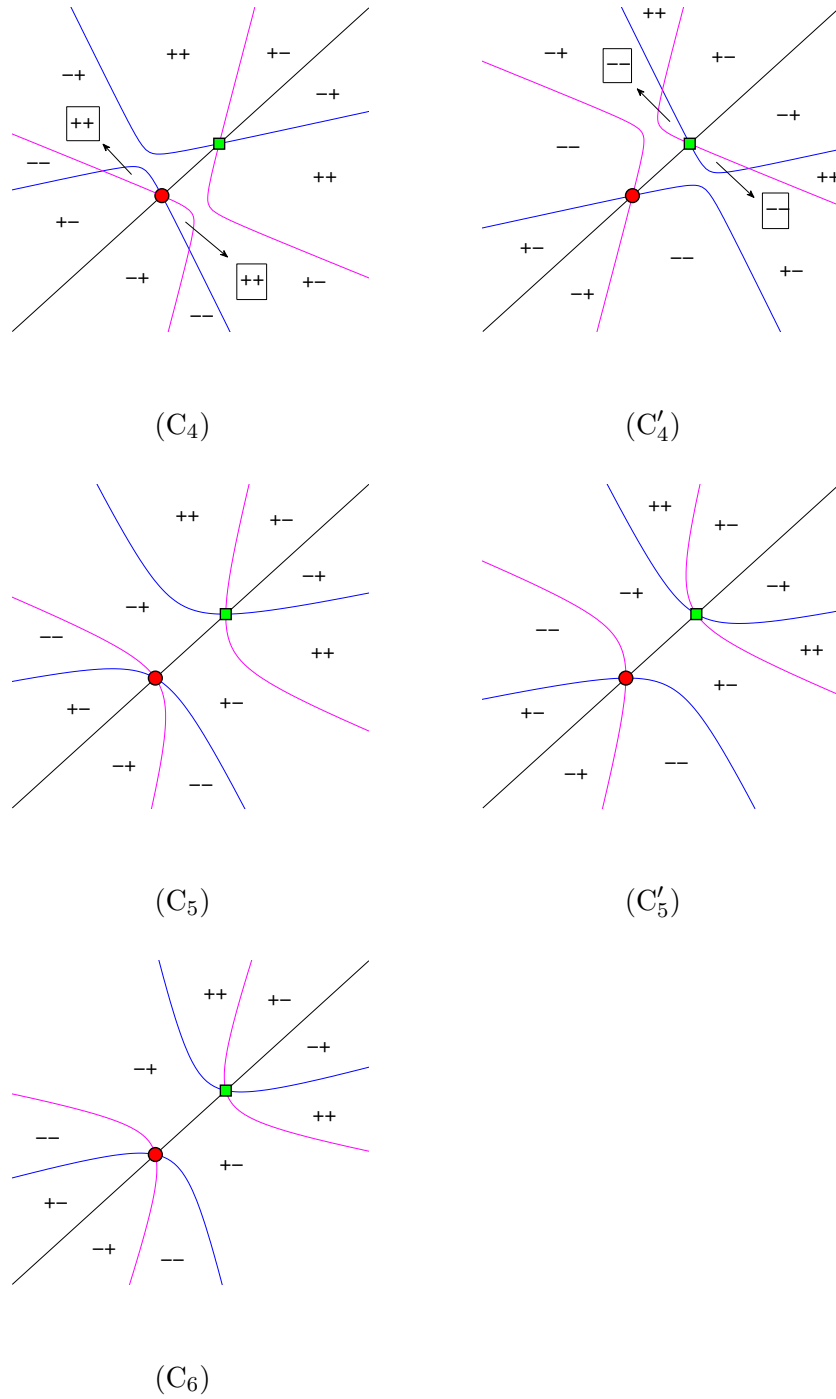
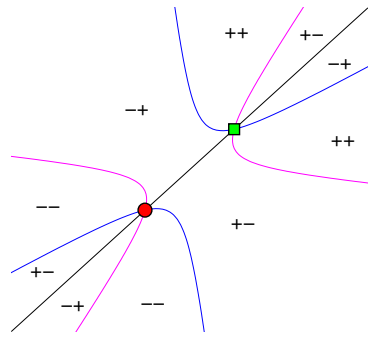
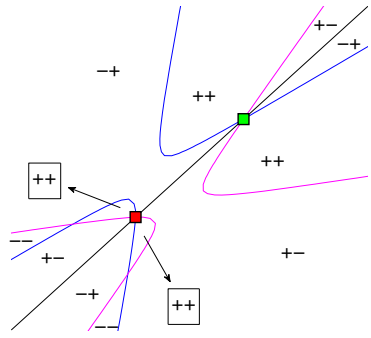


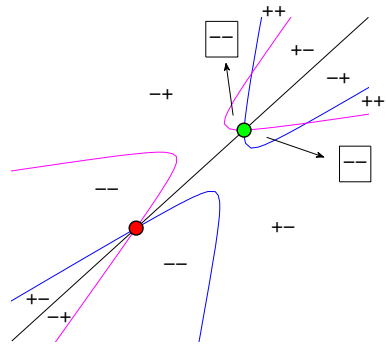
Figure 8.17: MIPs for all the non-degenerate perturbation of $F = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions corresponding to C_4 to C_6 .



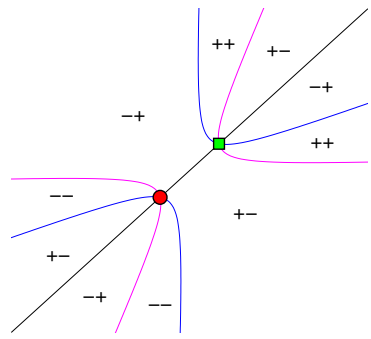
(C₇)



(C₈)



(C'₈)



(C₉)

Figure 8.18: MIPs for all the non-degenerate perturbation of $F = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions corresponding to C₇ to C₉.

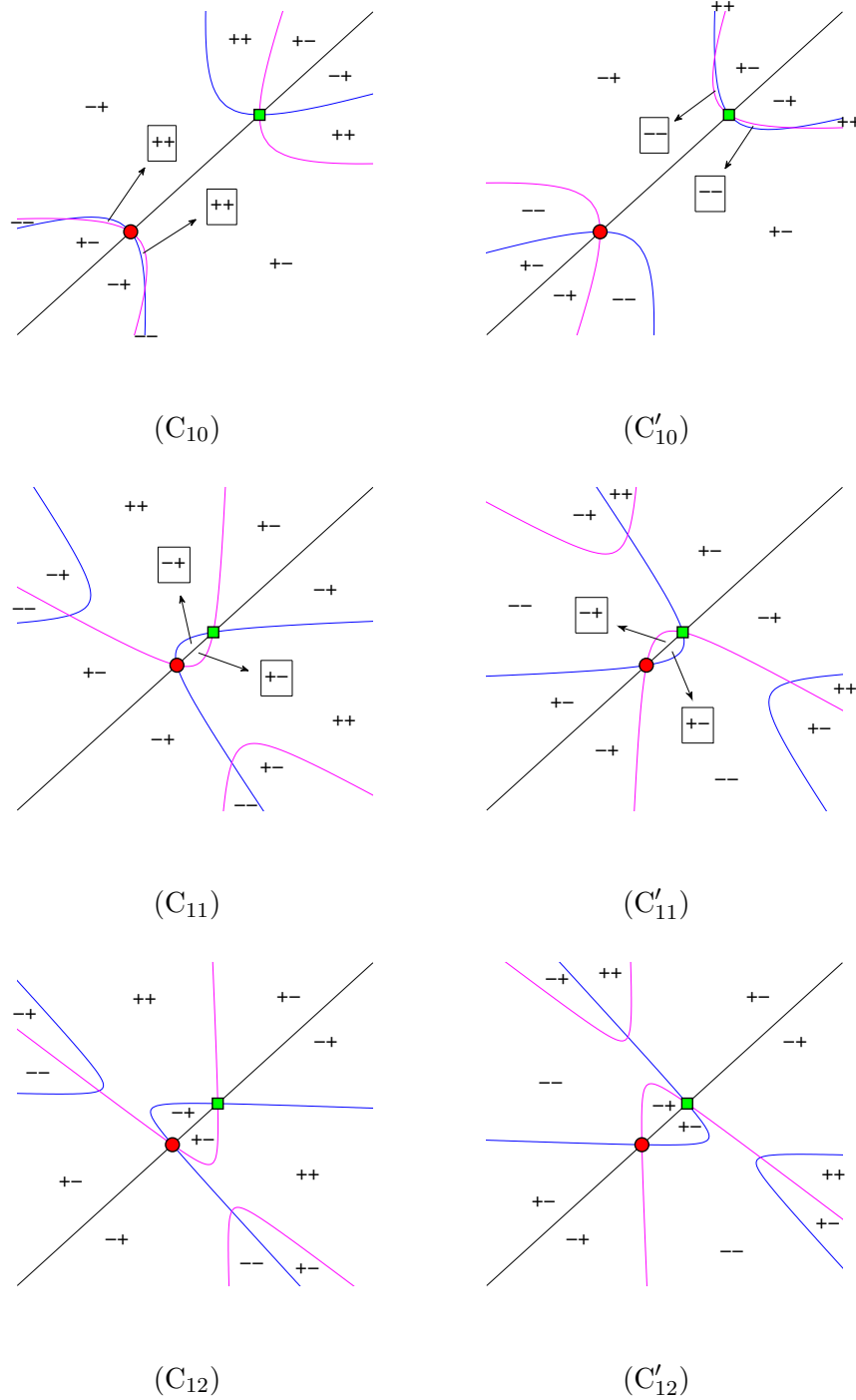


Figure 8.19: MIPs for all the non-degenerate perturbation of $F = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions corresponding to C_{10} to C_{12} .

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