An algebraic criterion for symmetric Hopf bifurcation†

By Martin Golubitsky[1] and Ian Stewart[2]

[1] *Department of Mathematics, University of Houston, Houston, Texas 77204-3476, U.S.A*

[2] *Mathematics Institute, University of Warwick, Coventry CV4 7AL, U.K.*

The equivariant Hopf bifurcation theorem states that bifurcating branches of periodic solutions with certain symmetries exist when the fixed-point subspace of that subgroup of symmetries is two dimensional. We show that there is a group-theoretic restriction on the subgroup of symmetries in order for that subgroup to have a two-dimensional fixed-point subspace in *any* representation. We illustrate this technique for all irreducible representations of $\boldsymbol{SO}(3)$ on the space V_l of spherical harmonics for l even.

1. Introduction

The theory of Hopf bifurcation in the presence of symmetry leads to algebraic questions concerning a Lie group of the form $\Gamma \times \boldsymbol{S}^1$. Here $\Gamma \subset \boldsymbol{O}(\boldsymbol{n})$ is a Lie subgroup acting absolutely irreducibly on $\boldsymbol{R}^n$ and by the diagonal action on $\boldsymbol{C}^n \cong \boldsymbol{R}^{2n} = \boldsymbol{R}^n \oplus \boldsymbol{R}^n$. We assume that a parametrized family of Γ-equivariant ODEs on $\boldsymbol{R}^{2n} \cong \boldsymbol{C}^n$ undergoes a Hopf bifurcation from an invariant equilibrium as that parameter is varied. By irreducibility it follows that this Hopf bifurcation is non-resonant in the sense that no eigenvalue is a non-trivial integer multiple of any other.

Hopf bifurcation induces an $\boldsymbol{S}^1$ phase shift symmetry on $\boldsymbol{C}^n$ that acts fixed-point freely, by non-resonance, and commutes with the action of Γ. If we identify $\boldsymbol{S}^1$ with the unit complex numbers, then the action of $\boldsymbol{S}^1$ induced by Hopf bifurcation is just the one given by complex scalar multiplication of $\boldsymbol{C}^n$. The Hopf bifurcation version of the equivariant branching lemma (see Golubitsky & Stewart 1985; Golubitsky *et al.* 1988) states that if $\Sigma \subset \Gamma \times \boldsymbol{S}^1$ is an isotropy subgroup of this action and if

$$\dim \operatorname{Fix}(\Sigma) = 2, \tag{1.1}$$

then generically there exists a branch of time-periodic solutions with symmetry exactly equal to Σ branching from the point of Hopf bifurcation.

By Proposition XVI, 7.2 of Golubitsky *et al.* (1988) isotropy subgroups $\Sigma \subset \Gamma \times \boldsymbol{S}^1$ are always of the form

$$G^{\Theta} = \{(g, \Theta(g)) : g \in G\},$$

where $G \subset \Gamma$ and $\Theta : G \to \boldsymbol{S}^1$ is a group homomorphism. One way to classify all Σ such that (1.1) holds is to list all closed subgroups $G \subset \Gamma$ and all possible homomorphisms Θ; then to compute $\dim \operatorname{Fix}(\Sigma)$ in each case, discarding any that do not satisfy (1.1); and finally to eliminate any redundancies. This method was used by Golubitsky *et al.* (1988) for all irreducible representations of $\Gamma = \boldsymbol{O}(3)$.

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In this paper we determine a relatively simple group-theoretic criterion that Σ must satisfy in order for (1.1) to be valid. See Corollary 2.2. This criterion eliminates large numbers of pairs (G, Θ) from consideration. Moreover, it is important to emphasize that this criterion is independent of the representation of Γ; it just depends on the group Γ. Thus the criterion greatly simplifies the classification of all Σ satisfying (1.1).

In the last section this criterion is used to find periodic solutions in $\boldsymbol{SO}(\mathbf{3})$-equivariant ODEs; these solutions were found previously in Golubitsky & Stewart (1985) but our criterion substantially simplifies the calculations. In Dionne *et al.* (1993) this criterion is used to help determine spatially periodic time-periodic solutions in euclidean equivariant dynamical systems.

Finally, we note that (1.1) also guarantees the existence of periodic orbits in hamiltonian systems near equilibria, given that suitable genericity hypotheses are valid (Montaldi 1988). Much more is proved there; namely, there exist $\dim \operatorname{Fix}(\Sigma)/2$ periodic solutions. But the case $\dim \operatorname{Fix}(\Sigma) = 2$ is particularly important and accessible, and Corollary 2.2 is equally applicable in this case.

2. The algebraic restriction

As indicated in the Introduction we assume that
(*a*) $\Sigma \subset \Gamma \times \boldsymbol{S}^1$ is an isotropy subgroup,
(*b*) $\dim \operatorname{Fix}(\Sigma) = 2$.

Since Σ is an isotropy subgroup, it is a twisted subgroup. More precisely, let G be the projection of Σ into Γ. Since $\boldsymbol{S}^1$ acts fixed-point freely on $\boldsymbol{R}^{2n}$, this projection induces an isomorphism of Σ onto a subgroup G. It follows that Σ has the form

$$\{(g, \Theta(g)) : g \in G\} \equiv G^{\Theta}.$$

We call G^{Θ} a *twisted* subgroup.

It is easy to check that $\Theta : G \to \boldsymbol{S}^1$ is a homomorphism. Let $K = \ker \Theta = \Sigma \cap \Gamma$. Then Σ is determined by the pair of subgroups $G \supset K$ where K is normal in G; that is, $G \subset N_{\Gamma}(K)$. Since Θ is a homomorphism G/K is isomorphic to a subgroup of $\boldsymbol{S}^1$ and hence is either cyclic or $\boldsymbol{S}^1$. In particular, G/K is abelian. Note that G, K and Θ are uniquely determined by Σ.

We now consider in more detail the consequences of (*b*): $\dim \operatorname{Fix}(G^{\Theta}) = 2$. We will show that there is an additional algebraic restriction on the pair (G, K) in order that (*b*) be satisfied. Let

$$C(G,K) = \{\gamma \in \Gamma : \gamma g \gamma^{-1} g^{-1} \in K \; \forall g \in G\}.$$

Note that if L is a subgroup satisfying $G \subset L \subset N_{\Gamma}(K)$, where L/K is abelian, then

$$C(G,K) \supset L. \tag{2.1}$$

(Let l be in L and let g be in $G \subset L$. The fact that L/K is abelian implies that $lg \equiv gl \bmod K$ or that $lgl^{-1}g^{-1} \in K$. Hence $l \in C(G,K)$.) In particular, for all pairs G, K that correspond to isotropy subgroups, G/K is abelian. Hence

$$C(G,K) \supset G. \tag{2.2}$$

We will prove:

Theorem 2.1. *If* $\dim \operatorname{Fix}(G^{\Theta}) = 2$, *then* $C(G,K) = G$.

Corollary 2.2. *If* $\dim \operatorname{Fix}(G^{\Theta}) = 2$, *then* G/K *is a maximal abelian subgroup of* $N_{\Gamma}(K)/K$.

Proof. Suppose that there exists a subgroup L such that

$$K \subset G \subset L \subset N_\Gamma(K),$$

where L/K is abelian. To show that G/K is maximal abelian in $N_\Gamma(K)/K$ we must show that $L = G$.

Since L/K is abelian, (2.1) implies that $L \subset C(G,K)$. Under the assumption that $\dim \mathrm{Fix}(\Sigma) = 2$, Theorem 2.1 implies that $L \subset C(G,K) = G \subset L$ and hence that $L = G$. □

The criterion that G/K is maximal abelian substantially simplifies the search for those isotropy subgroups with two-dimensional fixed-point subspaces. Hence, this criterion makes it possible to find such subgroups when more direct methods lead to intractable calculations.

Remark 2.3. In steady-state bifurcations there is an analogous restriction on possible isotropy subgroups having one-dimensional fixed-point subspaces (that is, on those subgroups satisfying the hypotheses of the equivariant branching lemma). Suppose K is such a subgroup, then $N_\Gamma(K)/K$ must act fixed-point freely on the one-dimensional fixed-point subspace of K. Hence the group $N_\Gamma(K)/K$ is either $\mathbf{1}$ or $\boldsymbol{Z}_2$, that is, either K equals $N_\Gamma(K)$ or K is of index two in $N_\Gamma(K)$.

We begin the proof of Theorem 2.1 by stating and proving two lemmas.

Lemma 2.4. *Suppose that* $\dim \mathrm{Fix}(\Sigma) = 2$. *Then* $N_{\Gamma \times \boldsymbol{S}^1}(\Sigma)/\Sigma \cong \boldsymbol{S}^1$.

Proof. Since Σ is an isotropy subgroup and $\dim \mathrm{Fix}(\Sigma) = 2$, Σ is a maximal isotropy subgroup. It follows that $N(\Sigma)/\Sigma$ acts fixed-point freely and orthogonally on the plane $\mathrm{Fix}(\Sigma)$. That the action is fixed-point free follows from Lemma XIII.10.2c of Golubitsky *et al.* (1988). Hence $N(\Sigma)/\Sigma$ is a Lie subgroup of $\boldsymbol{SO}(\mathbf{2})$ and contains no reflections, so it consists entirely of rotations and thus is isomorphic to a subgroup of $\boldsymbol{S}^1$. Conversely, $\boldsymbol{S}^1$ normalizes any subgroup of $\Gamma \times \boldsymbol{S}^1$. For twisted subgroups $\Sigma \cap \boldsymbol{S}^1 = \{e\}$ and $\boldsymbol{S}^1 \subset N(\Sigma)/\Sigma$. Hence $N(\Sigma)/\Sigma \cong \boldsymbol{S}^1$. □

Lemma 2.5.

$$N_{\Gamma \times \boldsymbol{S}^1}(G^\Theta) = C(G,K) \times \boldsymbol{S}^1.$$

Proof. Suppose that (γ, ψ) normalizes G^Θ. Then

$$(\gamma,\psi)\,(g,\Theta(g))\,(\gamma,\psi)^{-1} = (\gamma g \gamma^{-1}, \psi + \Theta(g) - \psi) = (\gamma g \gamma^{-1}, \Theta(g))$$

is in G^Θ. Hence $\gamma g \gamma^{-1} \in G$ and $\Theta(\gamma g \gamma^{-1}) = \Theta(g)$.

Since $K = \ker(\Theta)$, the last equality holds if and only if $\gamma g \gamma^{-1} g^{-1} \in K$. Hence $(\gamma,\psi) \in C(G,K) \times \boldsymbol{S}^1$ and $N_{\Gamma \times \boldsymbol{S}^1}(G^\Theta) \subset C(G,K) \times \boldsymbol{S}^1$.

Conversely, if $(\gamma,\psi) \in C(G,K) \times \boldsymbol{S}^1$, then

$$(\gamma,\psi)\,(g,\Theta(g))\,(\gamma,\psi)^{-1} = (\gamma g \gamma^{-1}, \Theta(g)).$$

Since $\gamma g \gamma^{-1} g^{-1} \in K$, it follows that $\gamma g \gamma^{-1} \in G$ and that $\Theta(\gamma g \gamma^{-1}) = \Theta(g)$. Hence

$$(\gamma g \gamma^{-1}, \Theta(g)) = (\gamma g \gamma^{-1}, \Theta(\gamma g \gamma^{-1})) \in G^\Theta.$$ □

Proof of Theorem 2.1. From equation (2.2) it follows that $C(G,K) \supset G$. We must prove that the reverse conclusion is valid when $\dim \mathrm{Fix}(G^\Theta) = 2$.

Lemma 2.4 states that if $\dim \mathrm{Fix}\,(G^\Theta) = 2$, then $N_{\Gamma\times \boldsymbol{S}^1}(G^\Theta)/G^\Theta \cong \boldsymbol{S}^1$. By Lemma 2.5

$$N_{\Gamma\times \boldsymbol{S}^1}(G^\Theta)/G^\Theta = (C(G,K)\times \boldsymbol{S}^1)/G^\Theta.$$

Thus

$$(C(G,K)\times \boldsymbol{S}^1)/G^\Theta \cong \boldsymbol{S}^1.$$

Since $e \in C(G,K)$ it follows that every coset in $(C(G,K)\times \boldsymbol{S}^1)/G^\Theta$ has a representative of the form (e,θ). Thus for each $c \in C(G,K)$ there exists $\theta \in \boldsymbol{S}^1$ such that

$$(c,0) \equiv (e,\theta) \bmod G^\Theta. \tag{2.3}$$

Equivalences $\bmod G^\Theta$ can be computed as follows: suppose

$$(c_1,s_1) \sim (c_2,s_2) \bmod G^\Theta.$$

Then

$$(c_1 c_2^{-1}, s_1 - s_2) \in G^\Theta,$$

and

$$c_1 c_2^{-1} \in G \quad \text{and} \quad s_1 - s_2 = \Theta(c_1 c_2^{-1}).$$

Thus (2.3) implies that $c \in G$ and $\Theta(c) = \theta$. Hence $C(G,K) \subset G$, as claimed. □

3. Application to $SO(3)$ Hopf bifurcation

In this section we show how to apply Corollary 2.2 to find periodic solutions via Hopf bifurcation in the presence of $\boldsymbol{SO}(3)$ symmetry. These solutions were previously computed in Golubitsky & Stewart (1985), but required more involved calculations. In Dionne *et al.* (1993) we shall also use Corollary 2.2 to calculate spatially periodic time-periodic solutions via Hopf bifurcation to euclidean equivariant planar systems of PDES.

Corollary 2.2 suggests a strategy for determining the pairs (G,K) that can form a G^Θ with $\dim \mathrm{Fix}\,(G^\Theta) = 2$. This strategy splits into three steps.

1. Classify up to conjugacy all closed subgroups $K \subset \Gamma$.
2. Determine those G that can pair with K (i.e. $K \lhd G$, G/K cyclic and maximal abelian in $N_\Gamma(K)/K$).
3. Use formulas to compute $\dim \mathrm{Fix}\,(G^\Theta)$ in terms of $\dim \mathrm{Fix}\,(G)$ and $\dim \mathrm{Fix}\,(K)$ for the representation of Γ on $\boldsymbol{R}^n$.

We comment on point (2). The methods based on the results in this paper work best when G/K is cyclic. When $G/K \cong \boldsymbol{S}^1$, the case of rotating waves, other methods are needed. We will use special trace formula methods to handle the rotating waves for $\boldsymbol{SO}(3)$ Hopf bifurcations at the end of this section.

We elaborate on point (3). In Golubitsky *et al.* (1988) the trace formula is used to compute $\dim \mathrm{Fix}\,(G)$ in terms of $d(K)$ and $d(G)$ where

$$d(T) = \dim \mathrm{Fix}\,(T)$$

for any subgroup $T \subset \Gamma$ (where $\mathrm{Fix}\,(T)$ is the fixed-point subset in $\boldsymbol{R}^n$). Such formulas are valid when G/K is isomorphic either to $\boldsymbol{1}, \boldsymbol{Z}_2, \boldsymbol{Z}_3, \boldsymbol{Z}_4$ or $\boldsymbol{Z}_6$. See table 1 for a list of these formulas for the first three cases. (The last two cases do not occur in our example.)

We now discuss Hopf bifurcation with $\boldsymbol{SO}(3)$ symmetry. The proper subgroups of $\boldsymbol{SO}(3)$ up to conjugacy are:

$$\boldsymbol{Z}_m (m \geqslant 2), \boldsymbol{D}_m (m \geqslant 2), \boldsymbol{T}, \boldsymbol{O}, \boldsymbol{I}, \boldsymbol{SO}(2), \boldsymbol{O}(2).$$

Table 1. *Trace formulas for twisted subgroups*

G/K	$\dim \mathrm{Fix}\,(G^{\Theta})$
$\mathbf{1}$	$2d(G)$
$\mathbf{Z}_2$	$2(d(K)-d(G))$
$\mathbf{Z}_3$	$d(K)-d(G)$

Table 2. *Subgroups of* $\mathbf{SO}(3)$

K	$N_{\mathbf{SO}(3)}(K)$	G	$d(K)$ for l even
$\mathbf{1}$	$\mathbf{SO}(3)$	$\mathbf{SO}(2)$	$2l+1$
$\mathbf{1}$	$\mathbf{SO}(3)$	$\mathbf{D}_2$	—
$\mathbf{Z}_m(m \geqslant 2)$	$\mathbf{O}(2)$	$\mathbf{SO}(2)$	$2[l/m]+1$
$\mathbf{D}_2$	$\mathbf{O}$	$\mathbf{T}$	$[\frac{1}{2}l]+1$
$\mathbf{D}_2$	$\mathbf{O}$	$\mathbf{D}_4$	—
$\mathbf{D}_m(m \geqslant 3)$	$\mathbf{D}_{2m}$	$\mathbf{D}_{2m}$	$[l/m]+1$
$\mathbf{T}$	$\mathbf{O}$	$\mathbf{O}$	$2[\frac{1}{3}l]+[\frac{1}{2}l]-l+1$
$\mathbf{O}$	$\mathbf{O}$	$\mathbf{O}$	$[\frac{1}{4}l]+[\frac{1}{3}l]+[\frac{1}{2}l]-l+1$
$\mathbf{I}$	$\mathbf{I}$	$\mathbf{I}$	$[\frac{1}{5}l]+[\frac{1}{3}l]+[\frac{1}{2}l]-l+1$
$\mathbf{SO}(2)$	$\mathbf{O}(2)$	$\mathbf{O}(2)$	1
$\mathbf{O}(2)$	$\mathbf{O}(2)$	$\mathbf{O}(2)$	1

Table 3. *Pairs* G, K *for which* $\dim \mathrm{Fix}\,(G^{\Theta}) = 2$

K	G	l even
$\mathbf{1}$	$\mathbf{SO}(2)$	all even l
$\mathbf{Z}_m(m \geqslant 2)$	$\mathbf{SO}(2)$	$m \leqslant l$
$\mathbf{D}_2$	$\mathbf{T}$	2–6
$\mathbf{D}_2$	$\mathbf{D}_4$	2–4
$\mathbf{D}_m(m \geqslant 3)$	$\mathbf{D}_{2m}$	$\frac{1}{3}l^* < m \leqslant l$
$\mathbf{T}$	$\mathbf{O}$	2
$\mathbf{O}$	$\mathbf{O}$	4–10, 14
$\mathbf{I}$	$\mathbf{I}$	6, 10–12, 16–28, 32–34, 38, 44
$\mathbf{O}(2)$	$\mathbf{O}(2)$	all even l

The group $\mathbf{Z}_m$ is generated by a rotation of order m; the group $\mathbf{D}_m$ is the associated dihedral group. The groups $\mathbf{T}, \mathbf{O}$ and $\mathbf{I}$ are the (orientation preserving) symmetries of the tetrahedron, cube and icosahedron. In table 2 we list the normalizers of these groups in $\mathbf{SO}(3)$ and the possible Gs that could give twisted groups with two-dimensional fixed-point subspaces.

It is well known that there is up to isomorphism a unique irreducible representation of $\mathbf{SO}(3)$ in each odd dimension $2l+1$. The computation of $d(K)$ for each of these representations is carried out in Golubitsky *et al.* (1988). There are some differences in the formulas depending on the parity of l. The case l even is more important for applications, so we focus on the computation of $d(K)$ in table 2 for that case.

Using the trace formulas listed in table 1 it is now an exercise to find those pairs $G \supset K$ for which G/K is cyclic and $\dim \mathrm{Fix}\,(G^{\Theta}) = 2$. These pairs are listed in table 3. As noted previously, the case where $G/K \cong \mathbf{S}^1$, the case of rotating waves, is special. We treat this case below.

There is an issue that we have not yet addressed. In our enumeration of twisted groups having two-dimensional fixed-point subspaces, we may have listed too many solutions. Some of these subgroups may have the same fixed-point subspaces and be redundant in the sense that they lead to the same solution.

In particular, suppose we have two twisted groups $G_1^{\Theta_1}$ and $G_2^{\Theta_2}$ whose fixed-point subspaces are two-dimensional. Suppose, in addition, that $G_1 \supset G_2$ and the restriction of Θ_1 to G_2 equals Θ_2. Then $G_2^{\Theta_2}$ is redundant. (It follows that $K_1 \supset K_2$; indeed, $K_2 = K_1 \cap G_2$.) With this discussion in mind it is not hard to show that the pairs of subgroups $\boldsymbol{D}_{2m}$, $\boldsymbol{D}_m$ are redundant when $\frac{1}{3}l < m \leqslant \frac{1}{2}l$, which explains the asterisk in table 3.

Finally we discuss the rotating waves. In these cases, $G = \boldsymbol{SO}(2)$ and $K = \boldsymbol{Z}_m$ for $m \geqslant 1$. We can assume that G is a fixed Cartan group of $\boldsymbol{SO}(3)$. We recall the trace formula from Golubitsky *et al.* (1988):

$$\dim \mathrm{Fix}_{V \oplus V}(G^{\Theta}) = 2 \int_G \cos(\Theta(g))\,\mathrm{tr}\,(g).$$

If we parametrize $\boldsymbol{SO}(2)$ by ψ, then $\Theta(\psi) = m\psi$ in order for the kernel K of the twist to be $\boldsymbol{Z}_m$. So the dimension is:

$$2 \int_{-\pi}^{\pi} \cos(m\psi)\,\mathrm{tr}\,(\psi)\,\mathrm{d}\psi.$$

Now since $\boldsymbol{SO}(2)$ is a Cartan subgroup we can use the Cartan decomposition to compute $\mathrm{tr}\,(\psi) = 1 + 2(\cos(\psi) + \ldots + \cos(l\psi))$. It now follows that the dimension is:

$$4 \int_{-\pi}^{\pi} \cos^2(m\psi)\,\mathrm{d}\psi$$

when $1 \leqslant m \leqslant l$. Moreover this integral always equals 2. Thus there is a rotating wave for each $1 \leqslant m \leqslant l$.

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