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Coordinate changes for network dynamics

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ABSTRACT

Consider networks in which all arrows are distinct and all cells are distinct. In this context, we obtain complete descriptions of the groups of diffeomorphisms that preserve network dynamics in the following sense: changing coordinates via the diffeomorphism transforms the space of admissible maps to itself. Five distinct actions are considered: left, right, contact, conjugacy, and vector field. Key features are the left core, right core, and core of the network. The core is a subnetwork with special combinatorial features, and it represents a partition of the cells into all-to-all connected subnetworks that couple to each other in a feed-forward manner. For the left/right actions, the group consists of all diffeomorphisms that are admissible for the left/right core, respectively. The contact action is a pair (B, Φ) where B is determined by the left core and Φ by the right core. For the conjugacy and vector field actions, the group is generated by diffeomorphisms that are admissible for the core together with graph automorphisms of the network; that is, permutations of the cells that map arrows to arrows but need not preserve arrow type. The proofs are combinatorial for the left and right actions, but require a mixture of Lie theory and the structure theory of associative algebras in the other cases, together with a nonlinear-to-linear reduction theorem.

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1. Introduction

Dynamical systems associated with networks form a special class whose structure is determined by the network connections. The topology of the connections restricts the variables that can appear in components of the ordinary differential equation (ODE), and a suitable notion of 'equivalent' connections also requires the same coupling function to occur in corresponding components. These restrictions can be formalized in terms of *admissible vector fields*, which informally are those vector fields that encode the network topology [1–3]. This is achieved by associating a variable with each cell (or node) of the network and associating coupling terms between cell variables with the arrows (directed edges). The corresponding class of ODEs, which we call admissible, determines the general dynamical properties of the network.

This paper was motivated by a question arising in biochemistry: the notion of homeostasis, where at an equilibrium of a system of ODEs, some variable remains approximately constant when a parameter varies. The dynamical system viewpoint suggests that a formal

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definition of homeostasis should be invariant under appropriate coordinate changes. The implications of this approach are discussed in [4,5]. Here we discuss, from a purely mathematical viewpoint, a more general question: which coordinate changes are appropriate for network dynamics? By analogy with topological dynamics, they should preserve those qualitative features of the dynamics that are relevant in an appropriate context. But they should also preserve the key feature that distinguishes network dynamics from a general dynamical system: the existence of distinguished variables corresponding to the cells of the network. It is, therefore, natural to require the coordinate changes to preserve the space of admissible vector fields [2,3].

Coordinate changes can act in several distinct ways. The main results of this paper characterize the appropriate changes of coordinates (diffeomorphisms) for five important cases: right composition, left composition, contact equivalence, conjugacy for maps, and vector field changes. See Definition 2.4. Each type of diffeomorphism is useful in an appropriate context. The first two are the simplest and play a key role in the analysis of the other three. Contact equivalences are the most general equivalences that preserve zeros of mappings. Conjugacy is natural for discrete dynamics (iterated maps). Vector field equivalence preserves the entire phase portrait (and its projections to cell coordinates in the network case) for continuous dynamics.

These characterizations solve a basic issue in network dynamics. The structure is richer than might be expected, and depends on the network topology.

Rink and Sanders [6,7] have studied a related issue in connection with normal forms for bifurcations in coupled cell networks. In particular, they provide examples of networks in which strongly admissible diffeomorphisms (2.3) are not the only ones that transform admissible maps to admissible maps. Their results explain many otherwise puzzling features of network dynamics and bifurcations in terms of 'hidden symmetries' and semigroup equivariance (see also [8]). The strongest implications of their techniques occur when the network is homogeneous (all cells have isomorphic input sets), whereas our results require the network to be fully inhomogeneous (distinct cells have non-isomorphic input sets). In this sense, our results are complementary to theirs, addressing different but related questions.

The definition of network-preserving diffeomorphisms can easily be extended to general networks, where distinct cells may be input-isomorphic or when there are non-trivial vertex symmetries (some cell receives several arrow-equivalent inputs). Self-loops and multiple arrows can also be allowed. Characterizing network-preserving diffeomorphisms for general networks seems to require new methods, and we do not discuss this problem here.

This paper uses a large quantity of notation. A list of the main symbols, with references to their definitions, is provided as an appendix at the end of the paper.

1.1. Outline of paper

Throughout we work in the class of fully inhomogeneous networks, in which all arrow types are different, and there are no multiple arrows or self-loops.

Section 2 recalls standard network formalism from [2,3], in particular, the notion of an admissible map, specialized to fully inhomogeneous networks, where the formalism is simpler. Five types of coordinate change are defined: right, left, contact, conjugacy, and vector field. The groups of network-preserving diffeomorphisms for these changes are defined. The first main result is stated as Theorem 2.5, which characterizes left, right, and contact network-preserving diffeomorphisms in terms of combinatorial properties of input and output sets of the network. The second main result, Theorem 2.10, characterizes conjugacy and vector field network-preserving diffeomorphisms. This theorem involves the finite group of graph automorphisms of the network, which is defined.

Section 3 contains the proof of Theorem 2.5, which is relatively straightforward. The combinatorial statement is then reinterpreted in terms of the network structure by defining the left and right cores of the network, and their intersection, the core. Theorem 3.4 proves that the left (respectively right) network-preserving diffeomorphism of a network are precisely the admissible diffeomorphism for its left (respectively right) core.

Section 4 deals with a technical issue: proving that the five groups of network-preserving diffeomorphisms actually are groups. The difficulty is that the inverse of a network-preserving diffeomorphism is not obviously a network-preserving diffeomorphism. This fact is not required in any subsequent proofs, but we discuss it now because the question is a natural one. We answer it using G-structures. This concept reduces the result to the linear case, where it follows by finiteness of the dimension (see Remark 4.3). The proof for the left, right, and contact actions is straightforward. For the conjugacy and vector field actions we present a proof that assumes two results from the linear case: Theorem 6.2 and Lemma 9.1.

This section also introduces a combinatorial structure that we call a shape. Shapes are the basic objects employed in this paper; they characterize a fully inhomogeneous network (via its adjacency matrix), its admissible maps (in terms of which cell variables occur in which components), its linear admissible maps (their form as matrices), and the three cores of the network. The proof of Theorem 2.10 is based on shapes of linear admissible maps, and of network-preserving diffeomorphisms for various actions.

Further key properties of shapes and cores, needed later, are developed in Section 5. Associated with any shape is a square matrix in which an asterisk * denotes an arbitrary entry and a zero 0 denotes a zero entry. Closed shapes, defined by a transitivity property, correspond to cores. We prove that the set of matrices with a given closed shape is an associative algebra, and the subset of invertible matrices of that shape is therefore a group; indeed, a Lie group. We also describe how graph automorphisms act on shapes.

Section 6 outlines the algebraic strategy that we use to prove the linear version of the problem, Theorem 6.2. This result is central to the paper, because the analogous nonlinear Theorem 2.10 is a fairly straightforward consequence.

Sections 7–11 pass from the Lie group of network-preserving matrices to its Lie algebra, analyze its structure and its relation to the core of the network, and introduce the graph automorphisms $\mathrm{gaut}(\mathcal{G})$ and their properties. This part of the paper is technical and algebraic, so we provide extra detail for readers less familiar with these techniques. The Wedderburn–Malcev theorem [9–11] from the theory of associative algebras completes the proof by decomposing any network-preserving diffeomorphism for the conjugacy action into one that lies in the core, multiplied by a graph automorphism.

Finally, Section 12 completes the proof of Theorem 2.10 by reducing the problem from nonlinear network-preserving diffeomorphisms acting on nonlinear admissible maps to linear network-preserving diffeomorphisms (invertible matrices) acting on linear admissible maps.

2. Network-preserving coordinate changes

We describe some basic concepts of the network formalism, specialized to the types of networks we consider in this paper. In [2,3] we defined a *coupled cell network* to be a directed graph, whose arrows and cells may be classified into distinct types. Here we simplify the discussion by assuming that all cells have distinct types and all arrows have distinct types. Formally:

Definition 2.1: A network $\mathcal{G}$ is *fully inhomogeneous* if distinct arrows are inequivalent.

Throughout this paper we assume that $\mathcal{G}$ is fully inhomogeneous. All we actually need is that distinct cells are not input-isomorphic. However, such a network is ODE-equivalent to one in which all arrows differ (see [12]); that is, it defines the same space of admissible vector fields (see below).

For simplicity we assume all cells are one-dimensional, but it is straightforward to rewrite the theorems and their proofs for the general case in which cell c has phase space $\mathbb{R}^{n_c}$, $n_c \geq 1$, at the expense of complicating the notation. We omit the details.

To each coupled cell network there is associated a specific class of vector fields (or equivalently ODEs), which are said to be 'admissible'. Informally, the cells of the graph are identified with systems of ODEs, and the arrows of the graph are interpreted as couplings between those systems. In the general theory arrows of the same type determine identical couplings, and the tail cells of the arrows determine which variables occur in the corresponding component of the vector field. It is especially straightforward to describe admissible systems for fully inhomogeneous networks, as follows.

Given a fully inhomogeneous n-cell network, we associate to each cell a state variable $x_i \in \mathbb{R}$ for $i = 1, \dots, n$. If $I = \{i_1, \dots, i_s\}$ is a set of distinct indices between 1 and n, let

$$x_I = (x_{i_1}, \dots, x_{i_s})$$

Without loss of generality we can assume $i_1 < \cdots < i_s$. Next we define input sets, output sets, and admissible vector fields.

Definition 2.2: The *extended input set* $J(i)$ of cell i is the set of all j such that either $j = i$ or there exists an arrow connecting cell j to cell i. The *extended output set* $O(i)$ is the set of all j such that either $j = i$ or there exists an arrow connecting cell i to cell j.

For fully inhomogeneous networks, admissible systems have the form

$$\begin{aligned} \dot{x}_1 &= f_1(x_{J(1)}) \\ &\vdots \\ \dot{x}_n &= f_n(x_{J(n)}) \end{aligned} \tag{2.1}$$

where, in general, the f_j are distinct.

Example 2.3: Consider the four-cell network $\mathcal{G}$ in Figure 1. Let $x = (x_1, x_2, x_3, x_4) \in \mathbb{R}^4$. The admissible maps $F : \mathbb{R}^4 \to \mathbb{R}^4$, (that is, admissible vector fields) are those of the

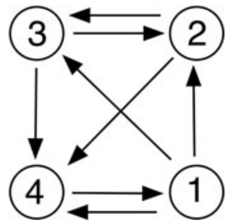

Figure 1. Example of a four-cell network. All arrows are different, but for simplicity this is not made explicit in the figure.

form

$$F(x) = \begin{bmatrix} f_1(x_1, x_4) \\ f_2(x_1, x_2, x_3) \\ f_3(x_1, x_2, x_3) \\ f_4(x_1, x_2, x_3, x_4) \end{bmatrix} \tag{2.2}$$

where the f_j are distinct functions.

In a general dynamical system, the most natural coordinate changes are vector field changes

$$F \mapsto (D\Phi)_x^{-1} F\Phi(x)$$

where $x = (x_1, \ldots, x_n)$ is a coordinate system on phase space, Φ is a diffeomorphism on phase space, and $(D\Phi)_x$ is the Jacobian matrix of Φ evaluated at x. These changes preserve all qualitative features of the dynamics, such as the topological type of attractors. There are no special constraints on Φ other than smoothness and invertibility.

In equivariant dynamical systems, a context that is now well understood, a group of symmetries acts on the vector field and preserves its structure [13]. The diffeomorphism Φ is required to be equivariant. Such a coordinate change not only preserves the qualitative dynamics, it also preserves the symmetries of states.

Coupled cell systems are loosely analogous to equivariant ones, but now the constraints are imposed by the architecture of the network. One major difference is that the composition of two equivariant maps is also equivariant, but the composition of two admissible maps need not be admissible. However, there is a subclass of 'strongly admissible' maps, whose composition with any admissible map (in either order) remains admissible (see [2]). For fully inhomogeneous networks, $\Phi = (\phi_1, \ldots, \phi_n)$ is *strongly admissible* if $\phi_j(x) = \phi_j(x_j)$; that is,

$$\Phi(x_1, \ldots, x_n) = (\phi_1(x_1), \ldots, \phi_n(x_n)) \tag{2.3}$$

This property has applications to several basic questions in network dynamics. The role of strongly admissible maps is to provide coordinate changes that preserve particular features of admissible maps.

It is reasonable to ask whether the strongly admissible maps are the only coordinate changes with such properties. We will show that for some networks, strongly admissible maps are *not* the only such coordinate changes.

2.1. Five types of coordinate change

The precise characterization of network-preserving maps depends on the network, and also on the type of coordinate change concerned. We consider five types of coordinate change, as follows.

Definition 2.4: Let $F : \mathbb{R}^n \to \mathbb{R}^n$ be an admissible smooth map, and let $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism.

(1) The *right action* of Φ transforms F into

$$G(x) = F\Phi(x)$$

(2) The *left action* of Φ transforms F into

$$G(x) = \Phi F(x)$$

(3) The *contact action* transforms F into

$$G(x) = B(x)F\Phi(x)$$

where $B(x)$ be an invertible $n \times n$ matrix.

(4) The *conjugacy* action transforms F into

$$G(x) = \Phi^{-1}F\Phi(x)$$

(5) The *vector field* action transforms F into

$$G(x) = (D\Phi)_x^{-1}F\Phi(x)$$

Contact equivalence is used when studying the zeros of a map, or equilibria of systems of differential equations. Vector field equivalence is a special form of contact equivalence, and it preserves the dynamics.

2.2. Statements of the principal results

In each case in Definition 2.4, we ask for conditions on Φ that ensure that G is admissible for all admissible F. Such a Φ is said to be *network-preserving* in the relevant context.

Specifically, we let $\mathcal{G}$ be a fully inhomogeneous network and define}

$$\begin{aligned}
\mathcal{D}^L_{\mathcal{G}} &= \{\text{left network-preserving diffeomorphisms}\Phi\}\\
\mathcal{D}^R_{\mathcal{G}} &= \{\text{right network-preserving diffeomorphisms}\Phi\}\\
\mathcal{D}^{\square}_{\mathcal{G}} &= \{\text{diffeomorphisms } \Phi \text{ that are simultaneously left and right network-preserving}\}\\
\mathcal{D}^C_{\mathcal{G}} &= \{\text{conjugacy network-preserving diffeomorphisms}\Phi\}\\
\mathcal{D}^V_{\mathcal{G}} &= \{\text{vector field network-preserving diffeomorphisms}\Phi\}
\end{aligned}$$

We often omit the subscript $\mathcal{G}$ when the network is clear. It is straightforward to show that each of the above sets contains the identity diffeomorphism and is closed under composition; that is, it is a semigroup with identity. In fact, all of these semigroups are groups, and $\mathcal{D}^C_{\mathcal{G}} = \mathcal{D}^V_{\mathcal{G}}$. Proofs of these statements rely on relations between the linear and nonlinear settings. The contact action is closely related to $\mathcal{D}^{\square}_{\mathcal{G}}$ and we do not define an explicit group for the contact action.

The main results of this paper characterize these groups of network-preserving diffeomorphisms explicitly.

For right, left, and contact actions, this is straightforward. Define

$$R(i) \equiv \{j \in J(i) : O(j) \supseteq O(i)\} \tag{2.4}$$

$$L(i) \equiv \{j \in J(i) : J(j) \subseteq J(i)\} \tag{2.5}$$

$$\square(i) \equiv L(i) \cap R(i). \tag{2.6}$$

Theorem 2.5:

(1) *A diffeomorphism $\Phi = (\phi_1, \ldots, \phi_n)$ is left network-preserving if and only if*

$$\phi_i(x) = \phi_i(x_{L(i)}) \quad \forall i \tag{2.7}$$

(2) *A diffeomorphism $\Phi = (\phi_1, \ldots, \phi_n)$ is right network-preserving if and only if*

$$\phi_i(x) = \phi_i(x_{R(i)}) \quad \forall i \tag{2.8}$$

(3) *A diffeomorphism $\Phi = (\phi_1, \ldots, \phi_n)$, and an invertible $n \times n$ matrix-valued function B with $B(x) = [b_{ij}(x)]$, are contact network-preserving if and only if Φ satisfies (2.8) and B satisfies*

$$b_{ij}(x) = \begin{cases} b_{ij}(x_{J(i)}) & j \in L(i) \\ 0 & j \notin L(i) \end{cases} \tag{2.9}$$

Theorem 2.6: *The semigroups $\mathcal{D}^L_{\mathcal{G}}$, $\mathcal{D}^R_{\mathcal{G}}$, $\mathcal{D}^{\square}_{\mathcal{G}}$ are groups.*

Theorem 2.5 is proved in Section 3 and Theorem 2.6 is proved in Section 4 (see Proposition 4.6). Theorems 2.5 (1, 2) and 2.6 give a simple sufficient condition for Φ to be a conjugacy or vector field network-preserving diffeomorphism, namely:

Lemma 2.7: *Let Φ be a diffeomorphism.*

(1) *$\Phi = (\phi_1, \dots, \phi_n)$ is left and right network-preserving if and only if*

$$\phi_i(x) = \phi_i(x_{\square(i)}) \quad \forall i \tag{2.10}$$

(2) *If Φ is both left and right network-preserving, then Φ is conjugacy network-preserving.*
(3) *If Φ is conjugacy network-preserving, then Φ is vector field network-preserving.*

In particular, (2,3) *imply $\mathcal{D}_{\mathcal{G}}^{\square} \subset \mathcal{D}_{\mathcal{G}}^{C} \subset \mathcal{D}_{\mathcal{G}}^{V}$.*

Proof: (1) follows directly from Theorem 2.5 (1,2) and (2) is straightforward. To prove (3) let $F(x)$ be admissible. Observe that

$$\frac{d}{dt}\Phi^{-1}(I + tF)\Phi(x)|_{t=0} = (D\Phi)_x^{-1}F(\Phi(x))$$

Since the left-hand side is admissible, so is the right. Therefore vector field changes of coordinates by Φ preserve admissibility. ∎

2.3. Graph automorphisms

Conditions (2, 3) of Lemma 2.7 state sufficient conditions for Φ to be network-preserving for the conjugacy and vector field actions, but it turns out that these conditions are not always necessary. Specifically, some networks have conjugacy (and vector field) network-preserving diffeomorphisms that are not of the form (2.10), namely graph automorphisms. We introduce this notion briefly here, and establish its properties in Section 9.

In the theory of coupled cell networks, a permutation of the cells that preserves all arrows *and their arrow types* is called a *symmetry* of the network. Network symmetries preserve solutions of any admissible ODE. Graph automorphisms are more general:

Definition 2.8: A *graph automorphism* of $\mathcal{G}$ is a permutation of the cells that preserves all arrows but not necessarily their arrow types. The graph automorphisms form a group, which we denote by $\mathrm{gaut}(\mathcal{G})$.

We can interpret graph automorphisms as permutation matrices acting on $\mathbb{R}^n$. Graph automorphisms transform solutions of any admissible ODE to solutions of an admissible ODE; however, it need not be the same ODE.

Example 2.9: Let $\mathcal{G}$ be a three-cell unidirectional cycle. Admissible maps have the form

$$F(x) = \begin{bmatrix} f_1(x_1, x_2) \\ f_2(x_2, x_3) \\ f_3(x_3, x_1) \end{bmatrix}$$

Observe that

$$\mathcal{T} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{bmatrix}$$

is a graph automorphism of $\mathcal{G}$. It can be verified directly that $\mathcal{T}$ is a conjugacy network-preserving diffeomorphism. To see this, compute

$$\begin{aligned} \mathcal{T}^{-1}F(\mathcal{T}(x)) &= \begin{bmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{bmatrix} \begin{bmatrix} f_1(x_2, x_3) \\ f_2(x_3, x_1) \\ f_3(x_2, x_1) \end{bmatrix} \\ &= \begin{bmatrix} f_3(x_2, x_1) \\ f_1(x_2, x_3) \\ f_2(x_3, x_1) \end{bmatrix} \equiv \begin{bmatrix} g_1(x_1, x_2) \\ g_2(x_2, x_3) \\ g_3(x_3, x_1) \end{bmatrix} \end{aligned}$$

for suitable g_j, which is admissible.

Our main result proves that the conjugacy and vector field network-preserving diffeomorphisms are generated by (2.10) and the graph automorphisms.

Theorem 2.10: *The groups $\mathcal{D}^C_{\mathcal{G}}$ and $\mathcal{D}^V_{\mathcal{G}}$ of all network-preserving diffeomorphisms for the conjugacy and vector field actions are both generated by $\mathcal{D}^{\square}_{\mathcal{G}}$ together with* $\mathrm{gaut}(\mathcal{G})$ *(so are equal). Moreover, $\mathcal{D}^{\square}_{\mathcal{G}} \lhd \mathcal{D}^C_{\mathcal{G}}$, so*

$$\mathcal{D}^C_{\mathcal{G}} = \mathcal{D}^{\square}_{\mathcal{G}}.\mathrm{gaut}(\mathcal{G})$$

We prove Theorem 2.5 in the next section. The remainder of the paper is needed to prove Theorem 2.10.

3. Left and right actions

The classification of network-preserving for right actions was given in [4]. For completeness, we include that proof here. We begin with a technical lemma. Let

$$\bar{R}(i) = \bigcap_{m \in O(i)} J(m) \tag{3.1}$$

Lemma 3.1: $\bar{R}(i) = R(i)$.

Proof: If $j \in \bar{R}(i)$, then $j \in J(m)$ for every $m \in O(i)$, so in particular $j \in J(i)$. That is, $m \in O(j)$ for every $m \in O(i)$, which implies $O(i) \subseteq O(j)$ and $j \in R(i)$. Conversely, suppose $j \in R(i)$. Then $j \in J(i)$ and $O(i) \subseteq O(j)$. It follows that if $m \in O(i)$, then $m \in O(j)$. Or, if $m \in O(i)$, then $j \in J(m)$. So $j \in \bar{R}(i)$. ■

For the first three types of coordinate change, where the characterization is fairly easy to prove, we have:

Proof of Theorem 2.5 (1): Let Φ be a diffeomorphism satisfying (2.7) and let F be an admissible map. We need to show that $G = \Phi F$ is also admissible. Observe that

$$G(x) = (\phi_1(F(x)), \ldots, \phi_n(F(x)))$$

The variables appearing in the ith coordinate are those in $g_i(x) = \phi_i(F_{L(i)}(x))$. In particular, if $j \in L(i)$, then the variables in $f_j(x) = f_j(x_{J(j)})$ are in the ith coordinate and these are the variables in $J(j)$. It follows from the definition of $L(i)$ that $J(j) \subset J(i)$; hence G is admissible.

Conversely, suppose $G = \Phi F$ is admissible whenever F is admissible. It follows that the variables in $g_i(x) = \phi_i(F(x))$ must be in $J(i)$. Note that $F(x) = x$ is admissible; hence Φ is also admissible. So the variables in ϕ_i must be in $J(i)$. Suppose $j \in J(i)$ is a variable that actually appears in ϕ_i; that is, ϕ_i is not independent of x_j. Then the variables in $\phi_i(F(x))$ can include variables only in $J(j)$. But these variables must be in $J(i)$; so $J(j) \subset J(i)$ and $j \in L(i)$. Therefore, (2.7) is valid.

The condition 'ϕ_i is not independent of x_j' can be rephrased as '$\partial\phi_i/\partial x_j$ is not identically zero.' The above proof can be restated more formally in terms of partial derivatives that do or do not vanish identically. ■

Proof of Theorem 2.5 (2): Suppose Φ is right network-preserving. We capture the restrictions on $\Phi = (\phi_1, \ldots, \phi_n)$ as follows. First, let $V(k)$ be the indices that the function ϕ_k depends on. Fix j and let $i \in O(j)$. We claim that $V(j) \subset J(i)$. It follows that

$$V(j) \subset \bigcap_{i \in O(j)} J(i) = \bar{R}(j)$$

Let $F = [f_1, \ldots, f_n]$ where $f_k(x) = 0$ when $k \neq i$ and $f_i(x) = x_j$. Since $j \in J(i)$, F is admissible; so $G = F\Phi$ is also admissible. Writing $G = [g_1, \ldots, g_n]$, it follows that $g_i(x) = \phi_j(x) = \phi_j(x_{V(j)})$ and $g_k(x) = 0$ when $k \neq i$. In order for G to be admissible, we must have $V(j) \subset J(i)$.

Conversely, let Φ be a diffeomorphism satisfying (2.8) and let $F = [f_1, \ldots, f_n]$ be admissible. Then let

$$G(x) \equiv F\Phi(x) = (f_1(\phi_{J(1)}(x)), \ldots, f_n(\phi_{J(n)}(x)))$$

We need to show that G is admissible; that is, we need to show that $g_i(x)$ depends only on variables in $J(i)$.

We see that $g_i(x) = f_i(\phi_{J(i)}(x))$. This function can depend on a variable x_k only if $k \in \bar{R}(j)$ for some $j \in J(i)$ (or $i \in O(j)$). Since $i \in O(j)$, it follows that $\bar{R}(j) \subset J(i)$. Therefore, $j \in J(i)$, and $G = F\Phi$ is admissible. ■

Proof of Theorem 2.5 (3): In contact equivalence, B and Φ are independent actions. Since Φ acts by right action, the form of Φ follows from (1). The matrix B acts by left equivalence; so the form of each row follows from (2.7). ■

3.1. Left and right cores

There are several alternative ways to state the conditions of Theorem 2.5. The following version is useful for examples suited to hand calculation, and is also used in some later proofs.

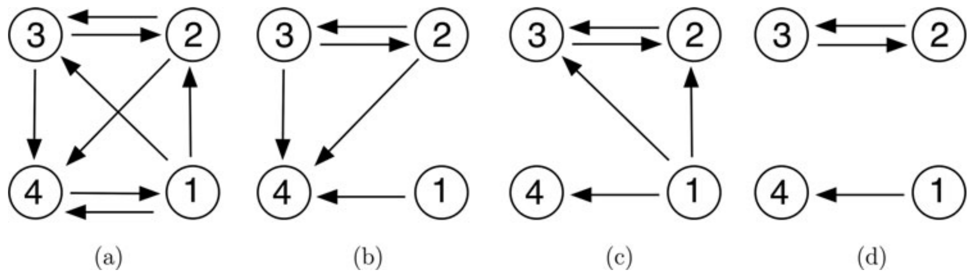

Figure 2. A four-cell network and its cores: (a) network; (b) left core; (c) right core; (d) core. All arrows are different, but for simplicity, this is not made explicit in the figure.

The main point of the theorem is that a network $\mathcal{G}$ has three distinguished subnetworks, which determine certain classes of admissible maps.

Definition 3.2: Let $\mathcal{G}$ be a fully inhomogeneous network.

(1) The *left core* $\mathcal{G}^{\mathrm{L}}$ is the network whose cells are the cells of $\mathcal{G}$ and whose arrows are the arrows $j \Longrightarrow i$ in $\mathcal{G}$ that satisfy: for every diagram in $\mathcal{G}$ of the form

$$\begin{array}{ccc} k & & \\ \downarrow & & \\ j & \Longrightarrow & i \end{array} \quad \text{there exists an arrow such that} \quad \begin{array}{ccc} k & & \\ \downarrow & \searrow & \\ j & \Longrightarrow & i \end{array}$$

(2) The *right core* $\mathcal{G}^{\mathrm{R}}$ is the network whose cells are the cells of $\mathcal{G}$ and whose arrows are the arrows $j \Longrightarrow i$ in $\mathcal{G}$ that satisfy: for every diagram in $\mathcal{G}$ of the form

$$\begin{array}{ccc} j & \Longrightarrow & i \\ & & \downarrow \\ & & k \end{array} \quad \text{there exists an arrow such that} \quad \begin{array}{ccc} j & \Longrightarrow & i \\ & \searrow & \downarrow \\ & & k \end{array}$$

(3) The *core* of $\mathcal{G}$ is $\mathcal{G}^{\square} = \mathcal{G}^{\mathrm{L}} \cap \mathcal{G}^{\mathrm{R}}$.

Example 3.3: The network in Figure 1 shows that the left core, right core, and core can all be different. The network and its three types of core are shown in Figure 2.

It is easy to prove that the core of the core is the same as the core; that is, $(\mathcal{G}^{\square})^{\square} = \mathcal{G}^{\square}$. The same goes for left and right cores, and there are other relations of this general kind. We do not require such properties in this paper.

In the language of cores, we can restate Theorem 2.5 as:

Theorem 3.4: *The left network-preserving diffeomorphisms are precisely the admissible diffeomorphisms for $\mathcal{G}^{\mathrm{L}}$. The right network-preserving diffeomorphisms are precisely the admissible diffeomorphisms for $\mathcal{G}^{\mathrm{R}}$.*

Proof: For right network-preserving admissible diffeomorphisms Φ, Theorem 2.5 implies that ϕ_i depends on x_j if and only if $j \in R(i)$. But $j \in R(i)$ if and only if $k \in J(j)$ implies

$J(j)\subseteq J(i)$. So $k \in J(i)$, which means there is an arrow from k to i. That is, $j \Longrightarrow i$ is in the right core of $\mathcal{G}$.

For left network-preserving admissible diffeomorphisms Φ, Theorem 2.5 implies that ϕ_i depends on x_j if and only if $j \in L(i)$. But $j \in L(i)$ if and only if $k \in O(i)$ implies $O(i)\subseteq O(j)$. So $k \in O(j)$, which means there is an arrow from j to k. That is, $j \Longrightarrow i$ is in the left core of $\mathcal{G}$. ■

Intuitively, in the right core case, the arrow from j to i corresponds to a condition about which variables appear in a suitable component of Φ; similarly, the arrow from k to j corresponds to a condition about which variables appear in a suitable component of F (that is, admissibility). The third arrow imposes admissibility on the composition ΦF, which is the left action. The left core case is similar, but now the arrows compose in the reverse order.

4. Group property for network-preserving diffeomorphisms

4.1. G-structures

We now address a technical issue about diffeomorphisms. For each type of coordinate change, we want the set of network-preserving diffeomorphisms to form a group under composition. In fact, it does, but closure under inverses is not immediately obvious because the space of admissible maps is infinite-dimensional. One standard way to deal with this issue is to use a basic idea from the theory of G-structures (see for example [14]).

Definition 4.1: Let G be a Lie group with a fixed representation on $\mathbb{R}^n$. The *G-structure* $\mathcal{A}_G$ is the set of diffeomorphisms $\Phi : \mathbb{R}^n \to \mathbb{R}^n$ for which $(D\Phi)_x \in G$ for every $x \in \mathbb{R}^n$.

Proposition 4.2: *The G-structure $\mathcal{A}_G$ is a group.*

Proof: This is a simple consequence of the chain rule for derivatives and the fact that $(D\Phi)_x^{-1}$ is also in G. ■

Remark 4.3: The linear analog is even simpler. Let V be a subspace of $n \times n$ matrices. Let G_V be the set of invertible $n \times n$ matrices g such that $gV\subset V$. Then G_V is a Lie group. Clearly G_V is closed under composition. Moreover, if $gv = 0$ for some $v \in V$, then $v = 0$ since g is invertible. It follows that $gV = V$ and hence that $g^{-1}V = V$. So $g^{-1} \in G_V$ and G_V is a group.

4.2. Shapes

We use G-structures to prove that the sets of left, right, contact, conjugacy, and vector field network-preserving diffeomorphisms are groups. To do so, we introduce the notion of 'shape', a combinatorial object that arises in three closely related contexts: a network, its admissible maps, and its linear admissible maps. Then we use the shape to define suitable Lie groups G_V.

Definition 4.4: Let $\mathcal{G}$ be an n-cell network and let $\mathcal{C} = \{1, \dots , n\}$ index the cells.

(1) A *shape* is a subset $\mathcal{S} \subseteq \mathcal{C} \times \mathcal{C}$ that contains all pairs (i, i) for $i \in \mathcal{C}$. Its *size* is $n = |\mathcal{C}|$.

(2) The *shape* of a fully inhomogeneous network $\mathcal{G}$ is the set of all pairs of cells (i, j) such that $i = j$ or there is an arrow from cell j to cell i. We denote this by $\mathcal{S}_\mathcal{G}$, omitting the subscript when it is obvious.
(3) If $\mathcal{S}$ is a shape, the space $\mathcal{M}_\mathcal{S}$ consists of all maps $F : \mathbb{R}^n \to \mathbb{R}^n$ such that each component f_i depends only on variables x_j for which $(i, j) \in \mathcal{S}$.
(4) If $\mathcal{S}$ is a shape, the space $\mathcal{L}_\mathcal{S}$ consists of all linear maps in $\mathcal{M}_\mathcal{S}$.

Informally, we say that the admissible maps (admissible linear maps) are those of shape $\mathcal{S} = \mathcal{S}_\mathcal{G}$.

A key result, which we use repeatedly, is part (3) of the following theorem:

Theorem 4.5:

(1) *Suppose that $\mathcal{G}$ has shape $\mathcal{S}$. Then every admissible map has shape $\mathcal{S}$.*
(2) *A linear map has shape $\mathcal{S}$ if and only if the corresponding matrix has shape $\mathcal{S}$.*
(3) *A map F has shape $\mathcal{S}$ if and only if $(DF)_x$ has shape $\mathcal{S}$ for every $x \in \mathbb{R}^n$.*

Proof: (1) is a restatement of the definition of an admissible map. (2) Let L be a linear map with matrix M. The matrix entry m_{ij} is zero if and only if $L_i(x)$ is independent of x_j. (3) The map $F = (f_1, \dots, f_n)$ has shape $\mathcal{S}$ if and only if $f_i(x)$ is independent of x_j whenever $(i, j) \notin \mathcal{S}$. This happens if and only if $\partial f_i / \partial x_j = 0$ at all points x. But this is the (i, j) entry of $(DF)_x$, so F has shape $\mathcal{S}$ if and only if $(DF)_x$ has shape $\mathcal{S}$ for every $x \in \mathbb{R}^n$. ■

Theorem 2.6 follows directly from the next proposition.

Proposition 4.6: *The sets of left ($\mathcal{D}^{\mathrm{L}}_\mathcal{G}$), right ($\mathcal{D}^{\mathrm{R}}_\mathcal{G}$), and contact network-preserving diffeomorphisms are groups.*

Proof: Let F be an admissible map of a fixed network $\mathcal{G}$ whose shape is $\mathcal{S}$. Let $G^L_\mathcal{S}$ be the set of invertible $n \times n$ matrices that preserve $\mathcal{L}_\mathcal{S}$ under left multiplication. Because $\mathcal{L}_\mathcal{S}$ is finite-dimensional, this set is a group. It is clearly a Lie group.

By Theorem 4.5 (3), a diffeomorphism Φ is left network-preserving, if and only if

$$(D\Phi)_x(\mathcal{S}) \subset \mathcal{S} \tag{4.1}$$

That is, Φ is a left network-preserving diffeomorphism if and only if $(D\Phi)_x \in G^L_\mathcal{S}$ for all $x \in \mathbb{R}^n$, where $G^L_\mathcal{S}$ is the group of all invertible linear maps that are left network-preserving on the space of linear $\mathcal{G}$-admissible maps. Therefore, $\mathcal{D}^{\mathrm{L}}_\mathcal{G}$ is a $G^L_\mathcal{S}$-structure. By Proposition 4.2 it is a group.

The argument for the right and contact actions is similar, replacing left multiplication by the appropriate actions of linear maps (or, for contact equivalence, pairs of linear maps).

By assuming some results from Sections 6–12 on network-preserving linear maps, we can use a similar method to deal with the conjugacy and vector field actions. We give the proof here as extra motivation for those sections. We emphasize that nothing in Sections 6–12 depends on $\mathcal{D}^C_\mathcal{G}$ or $\mathcal{D}^V_\mathcal{G}$ being closed under inverses.

Anticipating Definition 6.1, write $\Gamma^{\Box}$ for the group of all invertible matrices of shape $\mathcal{S}^{\Box}$. Define $\Gamma = \Gamma^{\Box}.\mathrm{gaut}(\mathcal{G})$. In Theorem 6.2 we prove this is a (Lie) group, equal to either of the linear analogs of $\mathcal{D}^C_{\mathcal{G}}$ and $\mathcal{D}^V_{\mathcal{G}}$. ■

Proposition 4.7:

(1) *Both* $\mathcal{D}^C_{\mathcal{G}}$ *and* $\mathcal{D}^V_{\mathcal{G}}$ *equal the* Γ*-structure* $\mathcal{A}_\Gamma$.
(2) *The sets of conjugacy* ($\mathcal{D}^C_{\mathcal{G}}$) *and vector field* ($\mathcal{D}^V_{\mathcal{G}}$) *network-preserving diffeomorphisms are groups.*
(3) $\mathcal{D}^C_{\mathcal{G}} = \mathcal{D}^V_{\mathcal{G}}$.

Proof: We prove (3). Then (1) and (2) are direct consequences. That is, we prove that $\mathcal{D}^C_{\mathcal{G}}$ and $\mathcal{D}^V_{\mathcal{G}}$ both comprise all diffeomorphisms Φ such that

$$(D\Phi)_{x_0} \in \Gamma \quad \forall x_0 \in \mathbb{R}^n \tag{4.2}$$

First, consider $\mathcal{D}^C_{\mathcal{G}}$. Let $\Phi \in \mathcal{D}^C_{\mathcal{G}}$ and let $x_0 \in \mathbb{R}^n$. Let $\mathcal{L}$ be the space of linear admissible maps for $\mathcal{G}$. If $L \in \mathcal{L}$, the map

$$F(x) = Lx - L\Phi(x_0) + \Phi(x_0)$$

is admissible (because $L\Phi(x_0) + \Phi(x_0)$ is a fixed vector in $\mathbb{R}^n$) and $F(\Phi(x_0)) = \Phi(x_0)$.

Since $\Phi \in \mathcal{D}^C_{\mathcal{G}}$, the map $\Phi^{-1}F\Phi$ is admissible. By Theorem 4.5 (3), this is equivalent to

$$D(\Phi^{-1}F\Phi)_{x_0} \in \mathcal{L} \quad \forall x_0 \in \mathbb{R}^n \tag{4.3}$$

Differentiate and use the chain rule to get

$$\begin{aligned} D(\Phi^{-1}F\Phi)_{x_0} &= ((D\Phi^{-1})_{F\Phi(x_0)})(DF)_{\Phi(x_0)}(D\Phi)_{x_0} \\ &= ((D\Phi)_{\Phi^{-1}F\Phi(x_0)})^{-1}(DF)_{\Phi(x_0)}(D\Phi)_{x_0} \end{aligned} \tag{4.4}$$

Since $DF = L$ and $F(\Phi(x_0)) = \Phi(x_0)$, this becomes

$$D(\Phi^{-1}F\Phi)_{x_0} = (D\Phi)^{-1}_{x_0}L(D\Phi)_{x_0}$$

By assumption, this is in $\mathcal{L}$, so $(D\Phi)_{x_0} \in \Gamma$.

Conversely, assume (4.2). We must prove that $\Phi \in \mathcal{D}^C_{\mathcal{G}}$; that is, $\Phi^{1-}F\Phi$ is $\mathcal{G}$-admissible for all $\mathcal{G}$-admissible maps F. By Theorem 4.5 (3), this is equivalent to (4.3), and again we deduce (4.4).

Since $D\Phi_{x_0} \in \Gamma = \Gamma^{\Box}\mathrm{gaut}(\mathcal{G})$, we can write

$$D\Phi_{x_0} = M(x_0)\alpha(x_0) \qquad M(x_0) \in \Gamma^{\Box}, \alpha(x_0) \in \mathrm{gaut}(\mathcal{G}) \tag{4.5}$$

To complete the proof, we need one further result, Lemma 12.4. This implies (by continuity relative to x_0) that in (4.5) we can assume $\alpha(x_0)$ is independent of x_0. Therefore

$$D\Phi_{x_0} = M(x_0)\alpha \qquad M(x_0) \in \Gamma^{\Box}, \alpha \in \mathrm{gaut}(\mathcal{G}) \tag{4.6}$$

This time we do not have $F(\Phi(x_0)) = \Phi(x_0)$, but the proof does not need this condition. Observe that

$$((D\Phi)_{\Phi^{-1}F\Phi(x_0)})^{-1}(DF)_{\Phi(x_0)}(D\Phi)_{x_0} = (M(\Phi^{-1}F\Phi(x_0))\alpha)^{-1}(DF)_{\Phi(x_0)}M(x_0)\alpha$$
$$= \alpha^{-1}[M(\Phi^{-1}F\Phi(x_0))^{-1}(DF)_{\Phi(x_0)}M(x_0)]\alpha$$

Now $A = M(\Phi^{-1}F\Phi(x_0))^{-1}(DF)_{\Phi(x_0)}M(x_0) \in \mathcal{L}$ because $M(y)$ is both left and right network-preserving for all $y \in \mathbb{R}^n$ and $(DF)_{\Phi(x_0)} \in \mathcal{L}$ by Theorem 4.5(3). Moreover,

$$\alpha^{-1}A\alpha \in \mathcal{L}$$

by Lemma 9.1. This proves (4.3) and hence that $\Phi \in \mathcal{D}_{\mathcal{G}}^C$.

The proof for $\mathcal{D}_{\mathcal{G}}^V$ is almost identical, since the linearized equation (4.3) also leads to (4.4) because $D(\Phi)_y$ is linear, hence equal to its derivative. ■

5. Further properties of shapes and cores

The results in this paper require more concepts related to shapes (Definition 4.4). We require these concepts for abstract shapes; they are analogous to and motivated by specific instances of shapes that have previously been discussed. Some features are therefore very similar to earlier ones, but now they occur in a more general and more abstract context. For completeness and clarity, we state the definitions and properties in detail.

Definition 5.1:

(1) The *diagram* of a shape $\mathcal{S} \subseteq \mathcal{C} \times \mathcal{C}$ is the $n \times n$ symbolic matrix $D = (d_{ij})$ with entries $d_{ij} = {}^*$ if $(i, j) \in \mathcal{S}$ and $d_{ij} = 0$ if $(i, j) \notin \mathcal{S}$.
(2) A shape $\mathcal{T} \subseteq \mathcal{C} \times \mathcal{C}$ is a *subshape* of a shape $\mathcal{S} \subseteq \mathcal{C} \times \mathcal{C}$ if and only if $\mathcal{T} \subseteq \mathcal{S}$.
(3) If $\mathcal{S}$ is a shape and $i \in \mathcal{C}$, the *(extended) input set* of i is

$$J_{\mathcal{S}}(i) = \{j \in \mathcal{C} : (i, j) \in \mathcal{S}\}$$

and the *(extended) output set* of i is

$$O_{\mathcal{S}}(i) = \{j \in \mathcal{C} : (j, i) \in \mathcal{S}\}$$

We omit the subscript $\mathcal{S}$ when the shape concerned is clear.

5.1. Core shapes

The main results of this paper depend on the notion of the core of a shape, which appears as three variants: the *right core*, *left core*, and *core*. We have already defined cores for networks (see Section 3.1); the definition for core shapes is essentially equivalent but focuses on the key combinatorial features of the problem.

Definition 5.2: Let $\mathcal{S}$ be a shape. Define

(1) The *right core* of $\mathcal{S}$ is the subshape

$$\mathcal{S}^{\mathrm{R}} = \{(i, j) \in \mathcal{S} : O_{\mathcal{S}}(i) \subseteq O_{\mathcal{S}}(j)\} \tag{5.1}$$

(2) The *left core* of $\mathcal{S}$ is the subshape

$$\mathcal{S}^{\mathrm{L}} = \{(i, j) \in \mathcal{S} : J_{\mathcal{S}}(i) \supseteq J_{\mathcal{S}}(j)\} \tag{5.2}$$

(3) The *core* of $\mathcal{S}$ is the subshape

$$\mathcal{S}^{\square} = \mathcal{S}^{\mathrm{R}} \cap \mathcal{S}^{\mathrm{L}} \tag{5.3}$$

Note that all three types of core shapes contain the diagonal $\{(i, i) : i \in \mathcal{C}\}$ because shapes contain the diagonal. In effect, we are using extended input and output sets. Recall that core networks (see Definition 3.2) are subnetworks and do not contain self-couplings.

Example 5.3: Let

$$\mathcal{S} = \begin{bmatrix} * & 0 & 0 & * \\ * & * & * & 0 \\ * & * & * & 0 \\ * & * & * & * \end{bmatrix}$$

This shape corresponds to the adjacency matrix of the network in Figure 2. Here

$$\mathcal{S}^{\mathrm{L}} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & * & * & * \end{bmatrix} \quad \mathcal{S}^{\mathrm{R}} = \begin{bmatrix} * & 0 & 0 & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ * & 0 & 0 & * \end{bmatrix} \quad \mathcal{S}^{\square} = \begin{bmatrix} * & 0 & 0 & 0 \\ 0 & * & * & 0 \\ 0 & * & * & 0 \\ * & 0 & 0 & * \end{bmatrix} \tag{5.4}$$

5.2. *Closed shapes*

In this subsection we define an important class of shapes related to cores, the closed shapes, and prove some basic properties. Throughout, $\mathcal{S} \subseteq \mathcal{C} \times \mathcal{C}$ is a shape and $\mathcal{C} = \{1, 2, \ldots, n\}$. Denote the set of all $n \times n$ matrices by $\mathbf{M}(n)$, and the group of non-singular $n \times n$ matrices by $\mathbf{GL}(n)$.

Definition 5.4: Let $\mathcal{S}$ be a shape. Then,

$$\mathcal{M}(\mathcal{S}) = \{M \in \mathbf{M}(n) : M \text{ has shape } \mathcal{S}\}$$
$$\mathcal{M}^*(\mathcal{S}) = \{M \in \mathbf{GL}(n) : M \text{ has shape } \mathcal{S}\} = \mathbf{GL}(n) \cap \mathcal{M}(\mathcal{S})$$

The elementary matrices E_{ij} (whose only non-zero entry is a 1 in position (i, j)) form a basis for $\mathbf{M}(n)$. They satisfy

$$E_{ij}E_{kl} = \delta_{jk}E_{il} \tag{5.5}$$

where δ_{jk} is the Kronecker delta. The set $\mathcal{M}(\mathcal{S})$ of all matrices of shape $\mathcal{S}$ is a vector space. It has a natural basis: those elementary matrices E_{ij} for which $(i, j) \in \mathcal{S}$.

Next, we focus on a special class of shapes:

Definition 5.5: A shape $\mathcal{S}$ is *closed* if (i, j) and $(j, k) \in \mathcal{S}$ implies that $(i, k) \in \mathcal{S}$.

Equivalently, the following *transitivity property* holds:

$$\mathcal{S} \text{ is transitive interpreted as a relation} \tag{5.6}$$

Closed shapes arise in the following context:

Theorem 5.6: *For any shape $\mathcal{S}$, the cores $\mathcal{S}^L$, $\mathcal{S}^R$, and $\mathcal{S}^\square$ are closed.*

Proof: Since $\mathcal{S}^\square \subseteq \mathcal{S}$, transitivity follows from the diagrams in Definition 3.2 when all arrows are in the appropriate core. It is also an easy consequence of the equivalent Definition 5.2. ■

The key property of a closed shape is:

Lemma 5.7: *A shape $\mathcal{S}$ is closed if and only if the product of any two matrices of shape $\mathcal{S}$ also has shape $\mathcal{S}$.*

Proof: Suppose $\mathcal{S}$ is closed. Let M, N be matrices of shape $\mathcal{S}$. Then

$$(MN)_{ij} = \sum_k m_{ik} n_{kj}$$

Suppose further that $(i, j) \notin \mathcal{S}$. The transitivity property (5.6) implies that for every k, either $(i, k) \notin \mathcal{S}$ or $(k, j) \notin \mathcal{S}$. Therefore, either $m_{ik} = 0$ or $n_{kj} = 0$, so $(MN)_{ij} = 0$.

For the converse, observe that the matrices M of shape $\mathcal{S}$ include all elementary matrices E_{ij} for $(i, j) \in \mathcal{S}$. Suppose that $(i, j), (j, k) \in \mathcal{S}$. By (5.5) and closure under matrix products, E_{ik} has shape $\mathcal{S}$. That is, $(i, k) \in \mathcal{S}$. ■

Example 5.8: The shape

$$\begin{bmatrix} * & * & * & * \\ * & * & * & 0 \\ * & * & * & 0 \\ * & 0 & 0 & * \end{bmatrix}$$

consists of all pairs

$$(1, 1)\ (1, 2)\ (1, 3)\ (1, 4)\ (2, 1)\ (2, 2)\ (2, 3)\ (3, 1)\ (3, 2)\ (3, 3)\ (4, 1)\ (4, 4)$$

This shape is not closed, because (for example) $(2, 1), (1, 4) \in \mathcal{S}$ but $(2, 4) \notin \mathcal{S}$.

Example 5.9: The subshape

$$\begin{bmatrix} * & 0 & 0 & 0 \\ * & * & * & 0 \\ * & * & * & 0 \\ 0 & 0 & 0 & * \end{bmatrix}$$

of Example 5.8 is closed.

Theorem 5.10: *For any closed shape $\mathcal{S}$, and in particular any core $\mathcal{S}^{\square}$:*

(1) *$\mathcal{M}(\mathcal{S})$ is an associative algebra over $\mathbb{R}$.*
(2) *$\mathcal{M}^*(\mathcal{S})$ is a group under matrix multiplication.*

Proof: $\mathcal{M}(\mathcal{S})$ is a real vector space. Since $\mathcal{S}$ is closed, by Lemma 5.7 it is an algebra.

The set $\mathcal{M}^*(\mathcal{S})$ is closed under matrix multiplication and contains the identity. We have to show that it is closed under inverses. The inverse of a non-singular matrix M is a polynomial in M. Indeed, the Cayley–Hamilton theorem tells us that M satisfies its characteristic equation, so there is a polynomial

$$p(M) = M^n + a_{n-1}M^{n-1} + \cdots + a_1 M + a_0 \mathrm{I} = 0$$

with $a_0 = \pm \det M \neq 0$. Here I is the identity matrix. Now

$$\mathrm{I} = M(a_0^{-1}(-M^{n-1} - a_{n-1}M^{n-2} - \cdots - a_1))$$

so M has inverse $a_0^{-1}(-a_{n-1}M^{n-2} - \cdots - a_1)$. By Lemma 5.7, any polynomial in M has the same shape as M, so M^{-1} has shape $\mathcal{S}$. ■

6. The linear case

As stated in the introduction, we first prove Theorem 2.10 in the linear case (made explicit as Theorem 6.2). Then we reduce the nonlinear case to the linear one. To state it we need:

Definition 6.1: Let $\mathcal{S}$ be a shape. Then $\Gamma_{\mathcal{G}}^{\square}$ is the group of all invertible matrices of shape $\mathcal{S}^{\square}$.

By 'the linear case', we mean the case where both the diffeomorphisms and the admissible maps are linear maps on $\mathbb{R}^n$. 'Network-preserving' is now restricted to this context. By Remark 4.3, the network-preserving linear diffeomorphisms form a group for each of the five actions. In the linear case, both vector field and conjugacy actions reduce to conjugacy, where a linear diffeomorphism M acts on a linear admissible map L to yield $M^{-1}LM$. Henceforth we refer to this as the conjugacy action. The linear case of Theorem 2.10 is therefore:

Theorem 6.2: *The group $\Gamma_{\mathcal{G}}$ of all network-preserving invertible linear maps for the conjugacy action is generated by $\Gamma_{\mathcal{G}}^{\square}$ and $\mathrm{gaut}(\mathcal{G})$. Moreover, $\Gamma_{\mathcal{G}}^{\square} \lhd \Gamma_{\mathcal{G}}$ so*

$$\Gamma_{\mathcal{G}} = \Gamma_{\mathcal{G}}^{\square}.\mathrm{gaut}(\mathcal{G})$$

6.1. Strategy of proof for Theorem 6.2

The remainder of the paper, except for Section 12, characterizes linear network-preserving conjugacies. It is a technical exercise in two areas of algebra: Lie theory and associative algebras. We sketch the main ideas here, and provide details in the following sections.

It is straightforward to prove that $\Gamma^{\square}$ is a subgroup of Γ; indeed, a normal subgroup. We will show that $\Gamma^{\square}$ comprises 'almost all' of Γ, in the sense that $\Gamma/\Gamma^{\square}$ is finite. Moreover, we will prove that this quotient group is induced by $\mathrm{gaut}(\mathcal{G})$, in the sense that $\Gamma = \Gamma^{\square}.\mathrm{gaut}(\mathcal{G})$. This is the most technical part of the paper.

Since Γ is a (real) Lie group, we can consider its Lie algebra $\mathrm{Lie}(\Gamma)$. This preserves admissibility under the commutator action

$$\mathrm{ad}_M(L) = [L, M] = LM - ML$$

By considering the shape $\mathcal{S}$ of the network, hence of L, a simple calculation proves that $\mathrm{ad}_M(L)$ is admissible for all admissible L if and only if both LM and ML are admissible (see Theorem 7.6). So on the Lie algebra level the characterization reduces to the combination of the left and right actions. By Theorem 2.5 (1,2), a matrix M is network-preserving under the commutator action if and only if the shape of M is the core $\mathcal{S}^{\square}$. (This can also be proved by a calculation with elementary matrices.)

This fact has several key consequences. First, M is itself-admissible. Second, a simple property of the core shows that M can be put in lower block-triangular form by suitably re-ordering the cells. Third, the matrices of shape $\mathcal{S}^{\square}$ form an *associative* algebra $\mathcal{L}^{\square}$ under matrix multiplication.

When passing from a Lie group Γ to its Lie algebra $\mathrm{Lie}(\Gamma)$, a standard issue arises. Namely, the Lie algebra captures only the local structure of the group near the identity. Results can often be parlayed to the subgroup generated by a neighborhood of the identity, which is the connected component Γ° of the identity. However, if Γ is not connected, as is often the case, the other connected components are not addressed.

The group of graph automorphisms $\mathrm{gaut}(\mathcal{G})$, which is finite, deals with the other components. It is easily proved to be a subgroup of Γ. To prove that no further type of network-preserving matrix is required beyond graph automorphisms, we have to extract from the definition of Γ enough permutation matrices to give $\mathrm{gaut}(\mathcal{G})$, and show that these together with $\Gamma^{\square}$ generate Γ. The main technical problem is what do these group elements permute? To be graph automorphisms, they should permute the cells $\mathcal{C}$ of $\mathcal{G}$, but we need to define them in terms of the algebra $\mathcal{L}^{\square}$. It is obvious that elements of Γ act as automorphisms of this algebra. Now, $\mathcal{L}^{\square}$ has block-triangular structure, so the natural objects to permute are the diagonal blocks.

The subalgebra of diagonal blocks is not invariant under the conjugation action of network-preserving matrices, but this problem can be circumvented by appealing to a classical result, the Malcev–Wedderburn theorem (Theorem 11.1). This states that every finite-dimensional associative algebra A over $\mathbb{R}$ has a semi-simple–nilpotent decomposition

$$A = N \dot{+} S$$

where N is a nilpotent ideal and S is a semi-simple subalgebra (direct sum of simple algebras). For $\mathcal{L}^{\square}$ the nilpotent part is the subalgebra of zero-triangular block matrices, and a natural choice for S is the algebra of diagonal blocks. However, in the semi-simple–nilpotent decomposition, N is unique but S needs not be. The Malcev–Wedderburn theorem controls the lack of uniqueness, stating that S is unique up to conjugation by an element $\exp(\nu)$ (or equivalently $I + \nu$) where $\nu \in N$.

Appealing to this result lets us decompose $M \in \Gamma$ as a product of an element of $\Gamma^{\square}$ and a permutation matrix π acting on the diagonal blocks. It is then straightforward to replace π by an element of $\mathrm{gaut}(\mathcal{G})$, proving that

$$\Gamma = \Gamma^{\circ}.\mathrm{gaut}(\mathcal{G})$$

This is the desired result.

We now spell out the details, which require some formal definitions and routine verifications, as well as the ideas sketched above.

7. Conjugacy action: linear case

In this section, we transform the linear version of the problem for conjugacy coordinate changes into the corresponding question for the commutator action of the associated Lie algebra, and solve the Lie algebra version.

7.1. Basic set-up and notation

Definition 7.1:

(1) Let $\mathcal{L}_{\mathcal{G}}$ denote the vector space of linear admissible maps for $\mathcal{G}$.
(2) The group $\Gamma_{\mathcal{G}}$ consists of all invertible matrices M such that for all $L \in \mathcal{L}_{\mathcal{G}}$ the conjugate

$$\mathrm{Ad}_M(L) = M^{-1}LM \tag{7.1}$$

belongs to $\mathcal{L}_{\mathcal{G}}$.

Remark 7.2: To avoid notational clutter, we consider a fixed but arbitrary fully inhomogeneous network $\mathcal{G}$ and omit the subscript $\mathcal{G}$ from now on. In particular, $\mathcal{L} = \mathcal{L}_{\mathcal{G}}$ and $\Gamma = \Gamma_{\mathcal{G}}$.

Our goal is to characterize the Lie group Γ. The first step is completed in Corollary 7.8 where we show that $\Gamma^{\square} \subset \Gamma$. We begin by enumerating a basis for $\mathcal{L}$ in Lemma 7.3.

Standard notation for coupled cell networks can be found in [1–3]. Write the set of cells $\mathcal{C}$ of the network $\mathcal{G}$ as

$$\mathcal{C} = \{1, 2, \ldots, n\}$$

and let $x = (x_1, \ldots, x_n)$ be cell coordinates on network phase space $\mathbb{R}^n$. Recall that for simplicity we take cells to be one-dimensional throughout this paper.

Lemma 7.3: *Let $\mathcal{G}$ be a fully inhomogeneous graph. Then the set of elementary matrices E_{ij} for which $j \in J(i)$ is a basis for $\mathcal{L}$.*

Proof: The linear map $F_{ij} = (f_1, \ldots, f_n)$ corresponding to E_{ij} satisfies $f_j(x) = x_i$ and $f_k = 0$ for all $k \neq j$. Since all arrows in $\mathcal{G}$ are inequivalent, $F_{ij} \in \mathcal{L}$. These maps are linearly independent and span the linear (admissible) maps in $\mathcal{L}$. ■

Definition 7.4: To avoid confusion between Lie and associative algebra structures, we use $\mathbf{gl}(n)$ for the Lie algebra of all $n \times n$ matrices under the commutator operation

$$[M, N] = MN - NM.$$

The notation $\mathbf{M}(n)$ now refers to the space of all $n \times n$ matrices considered as an associative algebra.

We first describe the connected component Γ° of the identity in Γ. To do this, we require the following facts from Lie theory.

For any real Lie group G, there is a one-to-one correspondence between one-parameter subgroups of G and tangent vectors in $T_I G$, the underlying vector space of Lie(G). The exponential map

$$\exp : \mathrm{Lie}(G) \to G$$

is a local diffeomorphism near 0 and I. See Adams [15, Theorem 2.6] or Bröcker and tom Dieck [16]. The image of exp generates the connected component Γ° of I, see Adams [15, Proposition 2.16] or Bröcker and tom Dieck [16]. So

$$\Gamma^\circ = \langle \exp(\mathrm{Lie}(\Gamma)) \rangle \tag{7.2}$$

Note that in general exp (Lie(Γ)) need not equal Γ°. The standard example is $\mathbf{SL}_2(\mathbb{R})$, see Adams [15] and Bröcker and tom Dieck [16].

7.2. Adjoint action

The next proposition is well known.

Proposition 7.5: *The action of the Lie algebra Lie(Γ) on $\mathbf{gl}(n)$ corresponding to the conjugation action (7.1) of Γ is the commutator or adjoint*

$$\mathrm{ad}_M(F) = [F, M] = FM - MF \qquad F \in \mathrm{Lie}(\Gamma) \qquad M \in \mathbf{gl}(n)$$

Proof: See Fulton and Harris [17]. ■

The key theorem about the adjoint action is:

Theorem 7.6: *Let $\mathcal{G}$ be a fully inhomogeneous network. Then the following are equivalent:*

(1) *A matrix M satisfies $[L, M] \in \mathcal{L}$ for all $L \in \mathcal{L}$; that is, $\mathcal{L}$ is ad_M invariant.*
(2) *A matrix M satisfies $ML \in \mathcal{L}$ and $LM \in \mathcal{L}$ for all $L \in \mathcal{L}$.*

Proof: As noted in Lemma 7.3 the space $\mathcal{L}$ has natural basis of elementary matrices E_{ij} where $j \in J(i)$. If the statement $[L, M] \in \mathcal{L}$ holds for each $L = E_{ij}$, it holds for any admissible L by linearity.

Condition (1) is valid if and only if

$$[E_{ij}, M] \in \mathcal{L} \qquad 1 \le i \le n \text{ and } j \in J(i) \tag{7.3}$$

Fix (i, j) so that $j \in J(i)$. By Lemma 7.3 E_{ij} is admissible, so $E_{ij} \in \mathcal{L}$. Let

$$M = \sum_{k,l} m_{kl} E_{kl},$$

then

$$ME_{ij} = \sum_{k,l} m_{kl} E_{kl} E_{ij} = \sum_{k,l} \delta_{li} m_{kl} E_{kj} = \sum_{k} m_{ki} E_{kj} \tag{7.4}$$

and

$$E_{ij}M = \sum_{k,l} m_{kl} E_{ij} E_{kl} = \sum_{k,l} \delta_{jk} m_{kl} E_{il} = \sum_{l} m_{jl} E_{il} \tag{7.5}$$

The commutator $[E_{ij}, M]$ is the difference between these expressions.

Observe that, except for E_{ij}, the E_{pq} that occur in the sums (7.4) and (7.5) are all distinct, hence linearly independent. However, since $E_{ij} \in \mathcal{L}$, this term is irrelevant to (7.3).

It follows that $[E_{ij}, M] \in \mathcal{L}$ for a given choice of (i, j) if and only if $ME_{ij} \in \mathcal{L}$ and $E_{ij}M \in \mathcal{L}$. So (1) implies (2). It is obvious that (2) implies (1), so they are equivalent. ■

Definition 7.7: Let $\mathcal{G}$ be a network of shape $\mathcal{S}$. Define

$$\mathcal{L}^{\square} = \mathcal{M}_{\mathcal{S}^{\square}} \subseteq \mathcal{L}$$

to be the subspace of linear maps of shape $\mathcal{S}^{\square}$, as defined in (5.3).

Corollary 7.8: *Let $\mathcal{G}$ be a fully inhomogeneous network of shape $\mathcal{S}$. Then*

(1) *the set of all matrices M satisfying Theorem 7.6 (1) or (2) is $\mathcal{L}^{\square}$;*
(2) *the group of all invertible matrices M satisfying Theorem 7.6 (1) or (2) is $\Gamma^{\square}$.*

Proof: These statements follow from Theorem 7.6 and the linear version of Theorem 2.5, rewritten in 'core' terminology. Indeed,

$$\mathcal{S}^{\square} = \mathcal{S}^{\mathrm{R}} \cap \mathcal{S}^{\mathrm{L}}$$

The corollary can also be proved directly from (7.4) and (7.5). ■

Remark 7.9: A core network $\mathcal{G}^{\square}$, or equivalently any network corresponding to a closed shape, has a highly constrained structure. Every network decomposes into transitive components, connected together in a feed-forward manner (see [18]). By Definition 3.2, each transitive component of $\mathcal{G}^{\square}$ is clearly all-to-all connected. Moreover, if there exists an arrow connecting a cell in one transitive component $\mathcal{G}_1^{\square}$ to a cell in a distinct transitive component $\mathcal{G}_2^{\square}$, then every cell in $\mathcal{G}_1^{\square}$ is connected to every cell in $\mathcal{G}_2^{\square}$ (in the same direction). Figure 2(d) is an example.

8. Lie-theoretic details

Recall from Definition 7.1 and Remark 7.2 that $\mathcal{L}$ denotes the space of linear admissible maps for $\mathcal{G}$ and Γ denotes the group of invertible linear maps preserving $\mathcal{L}$ under the conjugacy action. Further, recall from Definition 6.1 that $\Gamma^{\square}$ is the group of all invertible matrices of shape $\mathcal{S}^{\square}$. That is,

$$\Gamma^{\square} = \mathbf{GL}(n) \cap \mathcal{M}_{\mathcal{S}^{\square}} \subseteq \Gamma$$

In this section we establish key properties of $\Gamma^{\square}$.

Matrices with a given closed shape have some strong properties:

Theorem 8.1: *The space $\mathcal{L}^{\square}$ is an associative algebra under matrix multiplication.*

Proof: Closure under vector space operations is clear. Let M, N be such that $[M, L] \in \mathcal{L}$ for all $L \in \mathcal{L}$ and $[N, L] \in \mathcal{L}$ for all $L \in \mathcal{L}$. By Theorem 7.6 (1, 2), $ML, LM, NL, LN \in \mathcal{L}$ for all $L \in \mathcal{L}$. Now

$$[L, MN] = (MN)L - L(MN) = M(NL) - (LM)N \in \mathcal{L}$$

and we are finished. ■

Remark 8.2: For similar reasons, both $\mathcal{L}^{\mathrm{R}}$ and $\mathcal{L}^{\mathrm{L}}$ are also associative algebras under matrix multiplication. Their intersection is $\mathcal{L}^{\square}$.

Theorem 8.3: *By suitably permuting cells, the matrices of $\mathcal{L}^{\square}$ can simultaneously be put in the form*

$$\begin{bmatrix} \mathbf{M}(n_1) & 0 & 0 \dots & 0 \\ * & \mathbf{M}(n_2) & 0 \dots & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots \\ * & * & * \dots & \mathbf{M}(n_p) \end{bmatrix} \tag{8.1}$$

*Although we show *s below the diagonal, actually subdiagonal blocks are either completely general, or completely* 0*, as specified by the closed shape.*

Proof: As in Remark 7.9, decompose the set of cells $\mathcal{C}$ of $\mathcal{G}$ into transitive components with a feed-forward structure induced by the arrows of $\mathcal{G}$. The set of transitive components is partially ordered by the existence of connecting arrows. Reorder $\mathcal{C}$ to be compatible with this partial order. Each transitive component is all-to-all connected, by definition, so each diagonal block is a full matrix algebra $\mathbf{M}(n_j)$.

The off-diagonal blocks are arbitrary matrices of the correct size, for the following reason. If there is an arrow from any cell in transitive component i to a cell in transitive component $j \neq i$, the transitivity condition implies that there is an arrow from *every* cell in transitive component i to every cell in transitive component j. ■

Example 8.4: This structure is visible in Figure 2(d). The transitive components are $\{1\}$, $\{2, 3\}$and$\{4\}$. The corresponding shape $\mathcal{S}^\square$ is stated in (5.4), and this has block-triangular structure (with four zeros below the diagonal). Note that it is not triangular: the block in positions $\{2, 3\}$ has an entry above the diagonal.

Definition 8.5: The group Δ is the subgroup of $\mathcal{L}^\square$ consisting of block-diagonal matrices:

$$\Delta = \mathbf{GL}(n_1) \oplus \cdots \oplus \mathbf{GL}(n_p)$$

Theorem 8.6: *The group* $\Gamma^\square$ *is generated by exp* $(Lie(\Gamma))$ *and* Δ*.*

Proof: Order cells as in Theorem 8.3 and group entries according to transitive components. Now $\Gamma^\square$ consists of all block matrices of shape

$$\begin{bmatrix} \mathbf{GL}(n_1) & 0 & 0 \ldots & 0 \\ * & \mathbf{GL}(n_2) & 0 \ldots & 0 \\ \vdots & \vdots & \vdots \ddots & \vdots \\ * & * & * \ldots & \mathbf{GL}(n_p) \end{bmatrix} \tag{8.2}$$

where some of the * blocks may be identically zero, according to the shape $\mathcal{S}^\square$. Otherwise every block can be any possible matrix (non-singular for the diagonal blocks).

First, we deal with subdiagonal blocks. The space of all subdiagonal blocks is a nilpotent associative algebra N. If $\nu \in N$, then $I + \nu \in \Gamma^\square$ is in the image of exp; that is, $I + \nu = \exp(\mu)$ where $\mu = \log(I + \nu)$, which converges since ν is nilpotent.

The diagonal blocks are provided by Δ. Since $\mathrm{I} + N$ and Δ generate $\Gamma^\square$, the result is proved. ■

Corollary 8.7: *Let* Γ *be the group of conjugacy action network-preserving invertible matrices for* $\mathcal{G}$*, and let* Γ° *be the connected component of the identity in* Γ*. Then*

$$\Gamma^\circ \subseteq \exp(\mathrm{Lie}(\Gamma))$$

In particular, the matrices in Γ° *have the same shape as those in exp* $(Lie(\Gamma))$*.*

Proof: The matrix $\exp(M)$ is a convergent power series in M, so it is in the associative algebra $\mathrm{Lie}(\Gamma)$. By (7.2) the image of exp generates the connected component of the identity in Γ. The rest is supplied by the strongly admissible (that is, diagonal) network-preserving (linear) diffeomorphisms, as explained in Theorem 8.6. Note that inverses are taken care of since $(\exp(M))^{-1} = \exp(-M)$ has the same shape as M. ■

Corollary 8.8: *The matrices in $\langle \Gamma^\circ, \Delta \rangle$ are precisely the non-singular matrices with the same shape as those in $\exp(Lie(\Gamma))$.*

Another direct consequence of Theorem 8.6 is:

Corollary 8.9:

$$\Gamma^\square = \langle \Gamma^\circ, \Delta \rangle$$

9. Graph automorphisms

Now we establish some basic facts about network symmetries, needed for the proof of Theorem 6.2. Recall that a graph automorphism of $\mathcal{G}$ is a permutation of the cells that preserves arrows, but not necessarily arrow types. The graph automorphisms form a finite group $\mathrm{gaut}(\mathcal{G})$, which we interpret as $n \times n$ permutation matrices acting by conjugacy as in Lemma 9.1.

Recall from Definition 6.1 that the group of all non-singular matrices of shape $\mathcal{S}^\square$ is denoted by $\Gamma^\square$. By Corollary 8.9, $\Gamma^\square = \langle \Gamma^\circ, \Delta \rangle$ where Δ is the group of non-singular block-diagonal matrices.

Example 2.9 shows that the discrete group Γ/Γ° need not be trivial. This network has $\mathbb{Z}_3$ symmetry *provided we ignore differences in the arrow types*. The new shapes are translations of the diagonal shape by the two non-trivial elements of $\mathbb{Z}_3$.

This disjoint decomposition is too neat to be general, but we will prove that the only missing generators for the network-preserving matrices are the graph automorphisms of $\mathcal{G}$. More precisely, $\Gamma = \Gamma^\circ.\mathrm{gaut}(\mathcal{G})$.

Lemma 9.1: *Let π be a permutation of $\mathcal{C}$ with permutation matrix P. Then π is a graph automorphism of $\mathcal{G}$ if and only if*

$$P^{-1}\mathcal{L}P = \mathcal{L} \tag{9.1}$$

Proof: Let $\mathcal{S}$ be the shape of $\mathcal{L}$. Define a matrix A by

$$a_{ij} = \begin{cases} 1 \text{ if } (i,j) \in \mathcal{S} \\ 0 \text{ if } (i,j) \notin \mathcal{S} \end{cases}$$

Then A is the adjacency matrix of $\mathcal{G}$ if we ignore arrow types.

If (9.1) holds, then $P^{-1}AP \in \mathcal{L}$. Since the entries of $P^{-1}AP$ are all 0 or 1, and the number of 1's is conserved, we must have $P^{-1}AP = A$. Therefore, $\pi \in \mathrm{gaut}(\mathcal{G})$.

The converse is similar. The condition $\pi \in \mathrm{gaut}(\mathcal{G})$ is equivalent to

$$j \in J(i) \iff \pi(j) \in J(\pi(i)) \quad i \in \mathcal{C} \tag{9.2}$$

Let $(i, j) \in \mathcal{S}$. Then

$$P^{-1}E_{ij}P = E_{\pi(i),\pi(j)}$$

which has shape in $\mathcal{S}$ by (9.2). Since the E_{ij} span $\mathcal{L}$, we have $P^{-1}\mathcal{L}P \subseteq \mathcal{L}$. But dim $\mathcal{L}$ is finite and the map $M \mapsto P^{-1}MP$ is linear and one-to-one, so $P^{-1}\mathcal{L}P = \mathcal{L}$. ∎

Lemma 9.2: *Interpreting permutations as the corresponding permutation matrices,*

$$\mathrm{gaut}(\mathcal{G}) \subseteq \Gamma$$

Proof: Let π be a graph automorphism of $\mathcal{G}$ with associated permutation matrix P. The automorphism condition can be phrased as (9.2). So the permuted shape is $\mathcal{S}$ with both rows and columns permuted by π. That is, if M has shape $\mathcal{S}$ then so does $P^{-1}MP$. Therefore $P \in \Gamma$. ∎

We now proceed towards the proof of Theorem 6.2. By definition, $\gamma \in \Gamma$ if it preserves $\mathcal{L}$ under conjugation, that is, $\gamma^{-1}\mathcal{L}\gamma = \mathcal{L}$. We now show that the conjugation action also preserves $\mathcal{L}^{\square}$, the set of all matrices of shape $\mathcal{S}^{\square}$.

Lemma 9.3: *If $\gamma \in \Gamma$ then $\gamma^{-1}\mathcal{L}^{\square}\gamma = \mathcal{L}^{\square}$.*

Proof: By Theorem 5.6 the shape $\mathcal{S}^{\square}$ is closed. Theorem 5.10 then implies that $\mathcal{L}^{\square}$ is an associative algebra.

We claim that $\gamma^{-1}\mathcal{L}^{\square}\gamma = \mathcal{L}^{\square}$ for all $\gamma \in \Gamma$. To prove this, let $L \in \mathcal{L}^{\square}$. By Theorem 7.6, this is equivalent to both $LM, ML \in \mathcal{L}$ for all $M \in \mathcal{L}$. Therefore,

$$\gamma^{-1}L\gamma.\gamma^{-1}M\gamma = \gamma^{-1}LM\gamma \in \mathcal{L}$$
$$\gamma^{-1}M\gamma.\gamma^{-1}L\gamma = \gamma^{-1}ML\gamma \in \mathcal{L}$$

The map $M \mapsto \gamma^{-1}M\gamma$ is linear and one-to-one from $\mathcal{L}$ into $\mathcal{L}$. By finiteness of dimension it is also onto, hence bijective. So for fixed γ the elements $\gamma^{-1}M\gamma$ run through $\mathcal{L}$ if M runs through $\mathcal{L}$. Therefore,$\gamma^{-1}L\gamma \in \mathcal{L}^{\square}$, so $\gamma^{-1}\mathcal{L}^{\square}\gamma \subseteq \mathcal{L}^{\square}$. A similar dimension argument now proves that $\gamma^{-1}\mathcal{L}^{\square}\gamma = \mathcal{L}^{\square}$. ∎

Next, we summarize some key results, two of which have already been proved.

Theorem 9.4:

(1) $\langle \Gamma^{\square}, \mathrm{gaut}(\mathcal{G})\rangle \subseteq \Gamma$
(2) $\Gamma^{\circ} \subseteq \Gamma^{\square} \subseteq \Gamma$
(3) $\Gamma^{\square} \lhd \Gamma$

Proof:

(1) By Theorem 7.6, $\Gamma^{\square} \subseteq \Gamma$. Lemma 9.2 shows that $\mathrm{gaut}(\mathcal{G}) \subseteq \Gamma$.
(2) Γ° is the image of exp, so it lies inside $\Gamma^{\square}$. We already know that $\Gamma^{\square} \subseteq \Gamma$ by part (1).

(3) Lie theory implies that in general, $\Gamma^\circ \lhd \Gamma$, but in the non-compact case the quotient need not be abelian, so we argue differently. Let $L \in \Gamma^\square$, so equivalently $L \in \mathcal{L}^\square$ and L is non-singular. By Lemma 9.3 $\gamma^{-1}L\gamma \in \mathcal{L}^\square$. It is clearly non-singular, so $\gamma^{-1}L\gamma \in \Gamma^\square$. Therefore $\Gamma^\square \lhd \Gamma$.

■

The main step remaining to prove Theorem 6.2 is to show that Γ induces a permutation of the cells. Then it is routine to prove that this permutation is a graph automorphism, by using the structure of $\mathcal{L}^\square$.

10. Algebra structure of $\mathcal{L}^\square$

Theorem 8.1 shows that $\mathcal{L}^\square$ is an associative algebra over $\mathbb{R}$. We now establish some detailed structural features. We need these in Section 11 to apply the Malcev–Wedderburn theorem, which is the key step in proving that $\Gamma/\Gamma^\square$ is determined by $\mathrm{gaut}(\mathcal{G})$. This in turn is crucial for the nonlinear case. The basic ideas are routine, but the entire proof rests on them, so we define the structures involved precisely. We sketch some simple proofs to make the description self-contained.

We need some notation:

Definition 10.1: A *block* is a subset of $\mathcal{C} \times \mathcal{C}$ of the form $\mathcal{C}_i \times \mathcal{C}_j$ where $\mathcal{C}_i$ and $\mathcal{C}_j$ are subsets of $\mathcal{C}$. Partition $\mathcal{C}$ into transitive components $\mathcal{C}_1, \ldots, \mathcal{C}_p$ for a closed shape $\mathcal{S}$. Define the *block subspace*

$$\mathcal{B}_{ij} = \{M \in \mathcal{M}_\mathcal{S} : m_{kl} = 0 \; \forall (k,l) \notin \mathcal{C}_i \times \mathcal{C}_j\}$$

That is, the entries of $M \in \mathcal{B}_{ij}$ are arbitrary in the block formed by $\mathcal{C}_i \times \mathcal{C}_j$, and zero everywhere else.

Clearly,

$$\mathcal{M}_\mathcal{S} = \bigoplus_{i,j} \mathcal{B}_{ij}$$

Definition 10.2: Let $\mathcal{A}$ be a finite-dimensional associative algebra over $\mathbb{R}$.

(1) The *nil radical* N of $\mathcal{A}$ is the unique maximal nilpotent ideal of $\mathcal{A}$.
(2) A subalgebra $S \subseteq \mathcal{A}$ is *semisimple* if its radical is zero.
(3) A *Wedderburn decomposition* of $\mathcal{A}$ is a semidirect product

$$\mathcal{A} = N \ltimes S \tag{10.1}$$

where S is a semi-simple subalgebra.
(4) The *unipotent subgroup* of $\mathcal{A}$ is $U = \mathrm{I} + N$.

Remark 10.3: The Wedderburn decomposition of an associative algebra is valid over any field of characteristic zero. Indeed, there always exists a semi-simple subalgebra S satisfying (10.1) [9–11], but in general it is not unique (see Theorem 11.1). The unipotent subgroup is a group because N being an ideal implies closure under products, and for all $\nu \in N$ we have

$$(\mathrm{I}+\nu)^{-1} = \mathrm{I} - \nu + \nu^2 - \nu^3 + \cdots \tag{10.2}$$

which terminates because N is nilpotent.

Using (5.5), we see that

$$\mathcal{B}_{ij}\mathcal{B}_{kl} = \delta_{jk}\mathcal{B}_{il} \tag{10.3}$$

Definition 10.4: In $\mathcal{M}_{\mathcal{S}^\square}$ let

$$N = \bigoplus_{j<i} \mathcal{B}_{ij} \qquad S = \bigoplus_{i} \mathcal{B}_{ii} \tag{10.4}$$

The following result is well known for block-triangular matrices and it is easy to prove by the same reasoning for matrices of given closed shape. We state and prove it for completeness.

Theorem 10.5: *Use the above notation and let n_i be the size of transitive component i. Then*

(1) *S is semi-simple, isomorphic to*

$$\bigoplus_i \mathbf{M}_{n_i}$$

(2) *N in (10.4) is the nil radical of $\mathcal{L}^\square$.*
(3) *$\mathcal{L}^\square$ has a Wedderburn decomposition $\mathcal{L}^\square = N \ltimes S$.*
(4) *The invertible elements of $\mathcal{L}^\square$ are precisely those in*

$$N \oplus \bigoplus_i \mathbf{GL}(n_i) = U \ltimes \bigoplus_i \mathbf{GL}(n_i)$$

Proof:

(1) is obvious.
(2) For $k = 1, 2, \ldots,$ let

$$T_k = \bigoplus_{j \le i-k} \mathcal{B}_{ij}$$

Then, using (10.3),

$$T_1 = N \qquad T_p = \{0\} \qquad T_k T_l \subseteq T_{k+l}$$

Therefore, $N^p \subseteq T_p = \{0\}$, and N is nilpotent.
Further, N is an ideal, because

$$NS \subseteq \bigoplus_{j<i} \mathcal{B}_{ij} \bigoplus_{k} \mathcal{B}_{kk} \subseteq \bigoplus_{j<i} \mathcal{B}_{ij}\mathcal{B}_{jj} \subseteq \bigoplus_{j<i} \mathcal{B}_{ij} = N$$

again using (10.3).
To show that N is maximal, we prove the equivalent fact that $\mathcal{L}^{\square}/N$ is semi-simple. This follows since $N \cap S = \{0\}$ so $\mathcal{L}^{\square}/N \cong S$.

(3) Clearly $\mathcal{L}^{\square} = N \oplus S$ (as vector space). Since $N \cap S = \{0\}$ and N is an ideal, this is a semidirect sum.
(4) This is a consequence of the more general result in Lemma 10.6.

■

Lemma 10.6: *If $\mathcal{A} = N \ltimes S$ is a Wedderburn decomposition, then*

(1) *$\nu + \sigma$ (where $\nu \in N$, $\sigma \in S$) is invertible if and only if $\sigma \in S^*$, where S^* is the group of invertible elements of S;*
(2) *the invertible elements of $\mathcal{A}$ are precisely those in $U \ltimes S^*$.*

Proof: (1) Let $\alpha \in U$. Then α is invertible by (10.2). If $\sigma \in S^*$ then σ is invertible by definition. Therefore $\alpha\sigma$ is invertible.

Conversely, suppose that $\nu + \sigma$ is invertible, where $\nu \in N$, $\sigma \in S$. Let the inverse be $\mu + \tau$ with $\mu \in N$, $\tau \in S$. Then

$$\begin{aligned} \mathrm{I} &= (\nu + \sigma)(\mu + \tau) \\ &= (\nu\mu + \sigma\mu + \nu\tau) + (\sigma\tau) \end{aligned}$$

Here $\nu\mu + \sigma\mu + \nu\tau \in N$ and $\sigma\tau \in S$. But $\mathrm{I} \in S$ so $\sigma\tau = \mathrm{I}$. Therefore $\sigma \in S^*$ and $\sigma^{-1} = \tau$ exists in S^*. Now

$$\nu + \sigma = (\mathrm{I} + \nu\sigma^{-1})\sigma \in US^*$$

The subgroup U is normal, because

$$\sigma^{-1}(\mathrm{I} + \nu)\sigma = \sigma^{-1}\sigma + \sigma^{-1}\nu\sigma \in \mathrm{I} + N = U$$

for all $\sigma \in S^*$.

Finally, we claim that $U \cap S^* = \{\mathrm{I}\}$. Suppose that $\nu \in N$ and $\mathrm{I} + \nu \in S^*$. Since $\mathrm{I} \in S^*$, we have $\nu \in S^*$. So $\nu \in N \cap S = \{0\}$, and

$$U \cap S^* = \{\mathrm{I}\} + 0 = \{\mathrm{I}\}$$

■

Recall that we write $\mathbf{M}(n)$ for the associative algebra of all $n \times n$ matrices. Theorem 10.5(3) has established a Wedderburn decomposition for $\mathcal{L}^{\square}$, and this is what we now exploit.

By Lemma 9.3, each $\gamma \in \Gamma$ acts on $\mathcal{L}^{\square}$ by conjugation

$$\mathrm{Ad}_{\gamma} L = \gamma^{-1} L \gamma$$

Clearly Ad_{γ} is an algebra automorphism of $\mathcal{L}^{\square}$. Since automorphisms preserve the nil radical,

$$\gamma^{-1} N \gamma = N$$

for any $\gamma \in \Gamma$. It follows that γ also acts via Ad_{γ} on $\mathcal{L}^{\square}/N \equiv \mathbf{M}(n_1) \oplus \cdots \oplus \mathbf{M}(n_p)$.

Each direct summand $\mathbf{M}(n_k)$ is a semi-simple associative algebra, so these are precisely the minimal ideals of $\mathbf{M}(n_1) \oplus \cdots \oplus \mathbf{M}(n_p)$. Therefore, this action of γ permutes these summands. That is, Γ induces a permutation of the transitive components: for each $\gamma \in \Gamma$ we have

$$\mathrm{Ad}_{\gamma} \mathbf{M}(n_i) \equiv \mathbf{M}(n_{\pi_{\gamma}(i)}) \pmod N \tag{10.5}$$

where π_{γ} is a permutation of the transitive components $\{1, \ldots, p\}$.

Because the elements of $U = \mathrm{I} + N$ are unipotent, they lie in the image of $\exp$, hence in Γ°. Conjugation by elements of U acts *trivially* on the diagonal blocks $\mathbf{M}(n_1) \oplus \cdots \oplus \mathbf{M}(n_p) \pmod N$, in the sense that the unipotent elements map each diagonal block to itself setwise (the identity permutation on blocks).

11. Wedderburn–Malcev theorem

We now complete the proof of Theorem 6.2 for the linear case. The final idea needed is the Wedderburn-Malcev theorem [9–11], which states:

Theorem 11.1 (Wedderburn-Malcev): *Let A be a finite-dimensional associative algebra (with identity) over a field of characteristic zero (in our case, $\mathbb{R}$). Let N be its nil radical, which is an ideal. Then*

(1) *there exists a semi-simple subalgebra S such that $A = N \ltimes S$;*
(2) *if S_1, S_2 both satisfy (1), then there is an invertible element $\alpha \in U$ such that*

$$S_2 = \alpha^{-1} S_1 \alpha$$

Remark 11.2: The result is often stated with $\alpha \in A$ rather than U, but we can (and will) take $\alpha \in U$, the subgroup of unipotent elements. A proof is given in Curtis and Reiner [10, Section 72].

Now we can remove the $\pmod N$ in Equation (10.5). For technical reasons we need the following standard result.

Lemma 11.3: *An invertible matrix M satisfies*

$$M^{-1} \mathcal{B}_{ii} M = \mathcal{B}_{ii} \quad \forall i \tag{11.6}$$

if and only if $M \in \mathcal{B}_{11} \oplus \cdots \oplus \mathcal{B}_{pp}$ *(that is,* $M \in S$*).*

Proof: Equation (11.6) is equivalent to

$$\mathcal{B}_{ii}M \subseteq M\mathcal{B}_{ii} \quad \forall i$$

Write M as a block matrix with respect to the transitive components:

$$M = [M_{ij}] \qquad i,\ j \text{ transitive components}$$

Fix i and let E be the identity matrix in block $\mathcal{B}_{ii}$. Then

$$EM = \begin{bmatrix} 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \\ M_{i1} & M_{i2} & \cdots & M_{ip} \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

with non-zero entries only in row i. On the other hand, $M\mathcal{B}_{ii}$ is the set of all MQ where $Q \in \mathcal{B}_{ii}$, which are of the form

$$MQ = \begin{bmatrix} 0 & \cdots & 0 & M_{1i}Q & 0 & \cdots & 0 \\ 0 & \cdots & 0 & M_{2i}Q & 0 & \cdots & 0 \\ \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 0 & M_{pi}Q & 0 & \cdots & 0 \end{bmatrix}$$

with non-zero entries only in column i. Since $EM = MQ$, it follows that $M_{ij} = 0$ if $j \neq i$. Since this holds for all i, $M \in S$ as claimed. ■

Proposition 11.4: *Let* $\gamma \in \Gamma$*. Then there exists* $\mu = (I + \nu) \in U$ *such that*

$$(\gamma\mu^{-1})^{-1}S(\gamma\mu^{-1}) = S$$

Proof: Consider the blocks $\mathcal{B}_{ii}$, isomorphic to $\mathbf{M}(n_i)$, down the diagonal. In Definition 10.4, we defined

$$S = \mathcal{B}_{11} \oplus \cdots \oplus \mathcal{B}_{kk}$$

By 10.5, the subalgebra S is a semi-simple complement to N. If $\gamma \in \Gamma$, we know that

$$S_1 = \gamma^{-1}S\gamma$$

is also a complement. The Wedderburn–Malcev theorem implies that there exists $\mu = 1 + \nu \in U$ such that

$$S_1 = \mu^{-1}S\mu$$

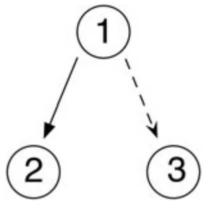

Figure 3. Example of a three-cell network with graph automorphism (2 3).

Therefore

$$\begin{aligned}(\gamma\mu^{-1})^{-1}S(\mu\gamma) &= \gamma^{-1}\mu^{-1}S\gamma\mu^{-1}\\ &= \mu S_1\mu^{-1}\\ &= S\end{aligned}$$

■

Proposition 11.5: *The group* Γ *is generated by* $\Gamma^{\square}$ *together with* $\mathrm{gaut}(\mathcal{G})$.

Proof: Let $\gamma \in \Gamma$. The blocks $\mathcal{B}_{ii}$ are distinguished subalgebras of S. Namely, they are its unique minimal ideals. Therefore, any automorphism of S permutes the diagonal blocks *setwise*.

Lift π to any permutation ρ of $\mathcal{C}$ that induces the same permutation on the blocks. Then $\rho^{-1}\gamma = \rho^{-1}\pi\mu$ fixes each diagonal block, individually, as a set. By Lemma 11.3 it therefore lies in the corresponding $\mathbf{GL}(n_j)$, and this lies in $\Gamma^{\square}$ by definition. Therefore

$$\rho^{-1}\gamma \in \Gamma^{\square}$$

so

$$\gamma \in \rho^{-1}\Gamma^{\square} \subseteq \Gamma^{\square}.\mathbb{S}_n$$

where $\mathbb{S}_n$ is the group of all permutation matrices.

It remains to replace $\mathbb{S}_n$ by $\mathrm{gaut}(\mathcal{G})$. Since $\Gamma^{\square} \subseteq \Gamma$, a product $\delta\theta$, with $\delta \in \Gamma^{\square}$ and $\theta \in \mathbb{S}_n$ conjugates $\mathcal{L}$ to itself if and only if θ does. By Lemma 9.1, this is the case if and only if $\theta \in \mathrm{gaut}(\mathcal{G})$. ■

This completes the proof of Theorem 6.2.

12. Proof of the nonlinear case

Finally we prove the nonlinear case, Theorem 2.10, by reducing it to Theorem 6.2. This requires some tactical maneuvers.

12.1. An example

As motivation, consider the three-cell network $\mathcal{G}$ of Figure 3.

The core shape $\mathcal{S}^\square$ for $\mathcal{G}$ is the same as the shape for $\mathcal{G}$, because the transitivity property 5.6 holds. This shape is

$$\mathcal{S}^\square = \begin{bmatrix} * & 0 & 0 \\ * & * & 0 \\ * & 0 & * \end{bmatrix}$$

There is one non-trivial graph automorphism $\pi = (12)$ with permutation matrix P and corresponding shape $\mathcal{T}$ given by

$$P = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{bmatrix} \quad \mathcal{T} = \begin{bmatrix} * & 0 & 0 \\ 0 & 0 & * \\ 0 & * & 0 \end{bmatrix}$$

The shape $\mathcal{T}$ does not lie in $\mathcal{S}^\square$. The group $\mathrm{gaut}(\mathcal{G})$ is cyclic of order 2, and

$$\Gamma^\square \cap \mathrm{gaut}(\mathcal{G}) = \mathbf{1}$$

The group $\Gamma = \Gamma^\square.\mathrm{gaut}(\mathcal{G})$ is the union of two shapes: $\mathcal{S}^\square$ and (using notation defined more generally below) $\mathcal{S}^\square\pi$, where

$$\mathcal{S}^\square\pi = \begin{bmatrix} * & 0 & 0 \\ * & 0 & * \\ * & * & 0 \end{bmatrix}$$

These shapes have non-zero intersection, namely

$$\mathcal{S} \cap \mathcal{S}^\square\pi = \begin{bmatrix} * & 0 & 0 \\ * & 0 & 0 \\ * & 0 & 0 \end{bmatrix}$$

However, this intersection contains no non-singular matrices.

12.2. Permutations of shapes

An analogous result holds in general, and it is needed to complete the reduction from the nonlinear case to the linear case. The example motivates:

Definition 12.1: Let $\mathcal{S}$ be a shape and let $\pi \in \mathbb{S}_n$ be a permutation of $\mathcal{C}$. Define the shape

$$\mathcal{S}\pi = \{(i, \pi(j)) : (i, j) \in \mathcal{S}\}$$

The next result is clear.

Lemma 12.2: *Let $\mathcal{S}$ be a shape and let $\pi \in \mathbb{S}_n$ be a permutation of $\mathcal{C}$. Let P be the permutation matrix corresponding to π. If M has shape $\mathcal{S}$ then MP has shape $\mathcal{S}\pi$.*

It is easy to construct networks $\mathcal{G}$ where $\Gamma^{\square} \cap \text{gaut}(\mathcal{G}) > \mathbf{1}$. To deal with this possibility it is convenient to choose a *transversal* (that is, a set of coset representatives) $\mathbb{T}$ to $\Gamma^{\square} \cap \text{gaut}(\mathcal{G})$ in $\text{gaut}(\mathcal{G})$.

Recall that $\mathcal{M}^*(\mathcal{S})$ is the set of non-singular matrices of shape $\mathcal{S}$. Translating the definition of $\Gamma = \Gamma^{\square}.\text{gaut}(\mathcal{G})$ into shape notation, we clearly have:

Lemma 12.3: *With the above notation,*

$$\Gamma = \bigcup_{\pi \in \mathbb{T}} \mathcal{M}^*(\mathcal{S}^{\square}\pi) \tag{12.7}$$

Proof: We use $\mathbb{T}$ here because it is clear that $\pi \in \Gamma^{\square} \cap \text{gaut}(\mathcal{G})$ implies $\mathcal{S}^{\square}\pi = \mathcal{S}^{\square}$. The rest is routine. □

The crucial result in the reduction from the nonlinear case to the linear case, which we need for a continuity argument, is: ■

Lemma 12.4: *If $\pi \neq \rho \in \mathbb{T}$ then*

$$\mathcal{M}^*(\mathcal{S}^{\square}\pi) \cap \mathcal{M}^*(\mathcal{S}^{\square}\rho) = \emptyset$$

Proof: Multiplying on the right by ρ^{-1} and setting $\alpha = \pi\rho^{-1}$ this becomes:

$$\text{If } \alpha \notin \Gamma^{\square} \cap \text{gaut}(\mathcal{G}) \text{ then every matrix in } (\mathcal{S}^{\square}) \cap (\mathcal{S}^{\square}\alpha) \text{ is singular.} \tag{12.8}$$

To prove (12.8), assume for a contradiction that M is a non-singular matrix in $(\mathcal{S}^{\square}) \cap (\mathcal{S}^{\square}\alpha)$. Order the cells $\mathcal{C}$ so that M is block-triangular, as in (8.1). The shape $\mathcal{S}$ has the same block-triangular form. Partition the cells into subsets $K_1, \ldots, K_p$ corresponding to this block structure. Let B_{ij} denote the block matrix occurring in rows K_i and columns K_j.

It is clear that every element of $\text{gaut}(\mathcal{G})$ also belongs to $\text{gaut}(\mathcal{G}^{\square})$, because the definition of the core $\mathcal{G}^{\square}$ is preserved by graph automorphisms. Therefore the (right) action of α maps each K_i to some unique K_j, where $1 \leq j \leq p$. That is, α preserves this partition. We prove inductively that α fixes every block setwise.

The structure of $\mathcal{G}^{\square}$ is described in Remark 7.9. In particular every block, including those off the diagonal, either consists entirely of *s or consists entirely of 0s.

The right action of α permutes the columns of the matrix. Since α preserves the partition, it permutes columns of blocks, and preserves rows. If α does not fix the top left block B_{11} then it must move some other block B_{1j} to that position. However, all such blocks are zero. Now M is triangular and also lies in $\mathcal{S}\alpha$. So $B_{11} = 0$ and M is singular.

The only alternative is that α fixes K_1 setwise. Now α permutes all the other sets $K_2, \ldots, K_p$. Delete the rows and columns corresponding to K_1; inductively the same argument proves that α fixes all K_j setwise. The permutation matrix corresponding to α is therefore block-diagonal, so $\alpha \in \Gamma^{\square}$, contradiction. ■

12.3. *Proof of Theorem 2.10*

We deduce Theorem 2.10 from Theorem 6.2 by passing to the Jacobian.

Proposition 4.7 states that $\mathcal{D}^C_{\mathcal{G}} = \mathcal{D}^V_{\mathcal{G}}$ are equal to the Γ-structure $\mathcal{A}_\Gamma$, where $\Gamma = \Gamma^{\square}.\mathrm{gaut}(\mathcal{G})$. It remains to prove that

$$\mathcal{D}^C_{\mathcal{G}} = \mathcal{D}^{\square}_{\mathcal{G}}.\mathrm{gaut}(\mathcal{G})$$

The chain rule shows that if $\Phi \in \mathcal{D}^{\square}_{\mathcal{G}}.\mathrm{gaut}(\mathcal{G})$ then $\Phi \in \mathcal{A}_\Gamma$, which we know equals $\mathcal{D}^C_{\mathcal{G}}$. That is,

$$\mathcal{D}^{\square}_{\mathcal{G}}.\mathrm{gaut}(\mathcal{G}) \subseteq \mathcal{D}^C_{\mathcal{G}}$$

We must prove the reverse inclusion.

If $\Phi \in \mathcal{D}^C_{\mathcal{G}} = \mathcal{A}_\Gamma$ then $D\Phi_{x_0} \in \Gamma$ for all $x_0 \in \mathbb{R}^n$, so

$$D\Phi_{x_0} = M(x_0)\alpha(x_0)$$

for $M(x_0) \in \Gamma^{\square}$, $\alpha(x_0) \in \mathrm{gaut}(\mathcal{G})$.

Lemma 12.4 implies that, by continuity, we may choose $\alpha \in \mathrm{gaut}(\mathcal{G})$ so that $\alpha(x_0) = \alpha$ for all $x_0 \in \mathbb{R}^n$. Therefore

$$D\Phi_{x_0} = M(x_0)\alpha$$

so

$$D(\Phi\alpha^{-1})_{x_0} = D\Phi_{x_0}\alpha^{-1} = M(x_0) \in \Gamma^{\square} \quad \forall x_0 \in \mathbb{R}^n$$

which implies that $\Phi\alpha^{-1} \in \mathcal{D}^{\square}_{\mathcal{G}}$ by Theorem 4.5(3). Therefore

$$\Phi \in \mathcal{D}^{\square}_{\mathcal{G}}\alpha \subseteq \mathcal{D}^{\square}_{\mathcal{G}}.\mathrm{gaut}(\mathcal{G})$$

as required. □

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Appendix: Notation

We list the main notation used in this paper in order of appearance, with a reference to the section or subsection containing a definition.

Symbol	Meaning	Definition
$\mathcal{G}$	fully inhomogeneous network	§ 2
$\mathcal{C}$	set of cells of network $\mathcal{G}$	§ 2
$\mathcal{D}^L_{\mathcal{G}}$	left network-preserving diffeomorphisms	§ 2.2
$\mathcal{D}^R_{\mathcal{G}}$	right network-preserving diffeomorphisms	§ 2.2
$\mathcal{D}^{\Box}_{\mathcal{G}}$	left and right network-preserving diffeomorphisms	§ 2.2
$\mathcal{D}^C_{\mathcal{G}}$	conjugacy network-preserving diffeomorphisms	§ 2.2
$\mathcal{D}^V_{\mathcal{G}}$	vector field network-preserving diffeomorphisms	§ 2.2
$J(i)$	extended input set of cell i	§ 2.2
$O(i)$	extended output set of cell i	§ 2.2
$R(i)$	$\{j \in J(i) : O(j) \supseteq O(i)\}$	§ 2.2
$L(i)$	$\{j \in J(i) : J(j) \subseteq J(i)\}$	§ 2.2
$\Box(i)$	$L(i) \cap R(i)$	§ 2.2
$\mathrm{gaut}(\mathcal{G})$	group of graph automorphisms of $\mathcal{G}$	§ 2.2
$\bar{R}(i)$	$\bigcap_{m \in O(i)} J(m)$	§ 3
$\mathcal{G}^{\mathrm{L}}$	left core of $\mathcal{G}$	§ 3.1
$\mathcal{G}^{\mathrm{R}}$	right core of $\mathcal{G}$	§ 3.1
$\mathcal{G}^{\Box}$	core $\mathcal{G}^{\mathrm{L}} \cap \mathcal{G}^{\mathrm{R}}$ of $\mathcal{G}$	§ 3.1
$\mathcal{A}_G$	G-structure of Lie group G	§ 4
$\mathcal{S}$	shape, subset of $\mathcal{C} \times \mathcal{C}$	§ 4.2
$\mathcal{S}_{\mathcal{G}}$	shape of $\mathcal{G}$	§ 4.2
$\mathcal{M}_{\mathcal{S}}$	space of all maps of shape $\mathcal{S}$	§ 4.2
$\mathcal{L}_{\mathcal{S}}$	space of all linear maps of shape $\mathcal{S}$	§ 4.2
$J_{\mathcal{S}}(i)$	extended input set of i for shape $\mathcal{S}$	§ 5
$O_{\mathcal{S}}(i)$	extended output set of i for shape $\mathcal{S}$	§ 5
$\mathcal{S}^{\mathrm{R}}$	right core of shape $\mathcal{S}$	§ 5.1
$\mathcal{S}^{\mathrm{L}}$	left core of shape $\mathcal{S}$	§ 5.1
$\mathcal{S}^{\Box}$	core of shape $\mathcal{S}$	§ 5.1
$\mathbf{M}(n)$	space of $n \times n$ matrices over $\mathbb{R}$	§ 5.2
$\mathbf{GL}(n)$	group of nonsingular $n \times n$ matrices over $\mathbb{R}$	§ 5.2
E_{ij}	elementary matrix	§ 5.2
$\mathcal{M}(\mathcal{S})$	set of matrices of shape $\mathcal{S}$	§ 5.2
$\mathcal{M}^*(\mathcal{S})$	set of invertible matrices of shape $\mathcal{S}$	§ 5.2
$\Gamma^{\Box}_{\mathcal{G}}$	group of invertible matrices of shape $\mathcal{S}^{\Box}$	§ 6
ad_M	$\mathrm{ad}_M(L) = [L, M] = LM - ML$	§ 6
$\mathrm{Lie}(\Gamma)$	Lie algebra of Lie group Γ	§ 6
Γ°	connected component of the identity of Lie group Γ	§ 6
Ad_M	conjugacy action $\mathrm{Ad}_M(L) = M^{-1}LM$	§ 7.1
$\mathcal{L}_{\mathcal{G}}$	vector space of linear admissible maps for $\mathcal{G}$	§ 7.1
$\Gamma_{\mathcal{G}}$	group of all invertible matrices leaving $\mathcal{L}_{\mathcal{G}}$ invariant (conjugacy)	§ 7.1
$\mathcal{L}$	abbreviated notation for $\mathcal{L}_{\mathcal{G}}$	§ 7.1
Γ	abbreviated notation for $\Gamma_{\mathcal{G}}$	§ 7.1
$\Gamma^{\Box}$	abbreviated notation for $\Gamma^{\Box}_{\mathcal{G}}$	§ 7.1
$\mathbf{gl}(n)$	Lie algebra of all $n \times n$ matrices over $\mathbb{R}$ under commutator	§ 7.1
$\mathbf{M}(n)$	associative algebra of all $n \times n$ matrices over $\mathbb{R}$	§ 7.1
$\mathcal{L}^{\Box}$	space of linear maps of shape $\mathcal{S}^{\Box}$	§ 7.2
$\mathcal{L}^{\mathrm{R}}$	space of linear maps of shape $\mathcal{S}^{\mathrm{R}}$	§ 8
$\mathcal{L}^{\mathrm{L}}$	space of linear maps of shape $\mathcal{S}^{\mathrm{L}}$	§ 8
Δ	subgroup of $\mathcal{L}^{\Box}$ consisting of block-diagonal matrices	§ 8
$\mathcal{B}_{ij}$	block subspace	§ 10
N	nil radical of associative algebra $\mathcal{A}$	§ 10
U	unipotent subgroup of associative algebra $\mathcal{A}$	§ 10
$\mathcal{G}^{\Box}$	core subnetwork of network $\mathcal{G}$	§ 12
$\mathcal{S}\pi$	$\{(i, \pi(j)) : (i, j) \in \mathcal{S}\}$ for permutation π	§ 12.2