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A Classification of Degenerate Hopf Bifurcations with $O(2)$ Symmetry

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1. INTRODUCTION

In this paper we study degenerate Hopf bifurcations with $O(2)$ symmetry in systems of ordinary differential equations

$$\dot{x} + F(x, \lambda) = 0, \tag{1.1}$$

where $F: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$. We assume that F commutes with a (nontrivial) linear action of $O(2)$ on $\mathbb{R}^n$, that is,

$$F(\gamma x, \lambda) = \gamma F(x, \lambda) \qquad \forall \gamma \in O(2); \tag{1.2}$$

that there is an $O(2)$-invariant equilibrium that, without loss of generality, we take to be $x = 0$; and that there is a value of λ, which for convenience we take to be $\lambda = 0$, at which the Jacobian matrix $DF(0, 0)$ has a pair of purely imaginary eigenvalues which after a rescaling of time in (1.1) may be assumed to be $\pm i$. Generically there are no other eigenvalues on the imaginary axis and the representation of $O(2)$ on the eigenspace corresponding to the eigenvalue i is irreducible. This means that the eigenvalues $\pm i$ are simple or double (see [17]). The simple eigenvalue case may be understood using the standard Hopf bifurcation theorem; here we assume that the eigenvalues at $\pm i$ are double.

There are a number of physical situations where circular symmetry seems to be important and where Hopf bifurcation with double eigenvalues appears. We mention four: oscillation of a flexible pipe [1], the Couette-

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Taylor experiment [10, 7], doubly diffusive waves [21], and porous-plug burner flames [19]. Such systems have several parameters and because of this we may expect degeneracies to occur at special paramcter values. Our study is motivated by the fact that the (quasi) global behavior of such systems is organized by these degeneracies.

There are two types of degeneracies that occur in multiparameter systems, mode interactions and higher order singularities. Mode interactions occur when several eigenvalues of $DF(0, 0)$ appear simulaneously on the imaginary axis. Motivated by the Couette Taylor experiment, studies have been made of $O(2)$-symmetric Hopf-steady state mode interactions [18] and $O(2)$-symmetric Hopf–Hopf mode interactions [3, 5]. The general $O(2)$ Hopf–Hopf mode interactions problem is considered in Chossat, Golubitsky, and Keyfitz [6]. Mode interactions with double zero eigenvalues have been considered in Dangelmayr and Armbruster [8] and Dangelmayr and Knobloch [9].

Higher order singularities occur when certain nondegeneracy conditions in the simplest $O(2)$-Hopf theorem fail. In this paper we classify and unfold those singularities that may be expected to appear in systems (1.1) that depend on two parameters in addition to the bifurcation parameter λ; that is, the singularities of codimension less than or equal to two. The codimension zero and one singularities have been studied by several authors, as we now explain. For case of exposition we assume that all eigenvalues of $DF(0, 0)$, other than $\pm i$, have possitive real part. We assume that the eigenvalue of $DF(0, \lambda)$ corresponding to i (when $\lambda = 0$) crosses the imaginary axis with nonzero speed and that this eigenvalue also has a positive real part when $\lambda < 0$. Thus $x = 0$ is an asymptotically stable equilibrium when $\lambda > 0$.

It is now well known that under these assumptions there exist two families of periodic solutions to (1.1), rotating waves and standing waves See Ruelle [23], Schecter [24], van Gils [26], and Golubitsky and Stewart [17]. Moreover, there is a kind of exchange of stability that is valid generically and which may be expressed as follows. Neither family of periodic solutions is asymptotically stable unless both families bifurcate supercritically, and then precisely one family is stable. The super- or subcritically of each branch, as well as their stabilities is determined by two numbers that depend on the Taylor expansion of $F(x, \lambda)$ at $(0, 0)$ up to order 3. Thus the codimension zero singularities are determined by four nondegeneracy conditions:

(a) eigenvalues crossing the imaginary axis with nonzero speed,

(b) super/sub-criticality of rotating waves.

(c) super/sub-criticality of standing waves, and

(d) the competition between stability of rotating and standing waves.

The codimension one singularities are found by having precisely one of these four conditions fail, and then imposing certain nondegeneracy conditions at a higher order. The most interesting codimension one singularity occurs when (d) fails. As discovered by Erneux and Matkowsky [11], perturbation of such a singularity leads to a branch of 2-tori connecting the standing and rotating wave branches, and, under certain circumstances, this 2-torus can be asymptotically stable. More precisely, Erneux and Matkowsky work with the system (1.1) in normal form and it can be shown (we will do so below), that under such circumstances the flow on this 2-torus must be linear. Recently Chossat [4] has shown that this 2-torus and its linear flow persist even when (1.1) is not assumed to be in normal form.

Swift [25], Knobloch [20], and Nagata [22] have each investigated the codimension one singularities corresponding to degeneracies in (b) and (c) above. In addition, Knobloch has studied certain codimension two degeneracies, the most interesting of which leads to the existence of an invariant 3-torus.

Our paper extends the work described above in several ways:

(1) We include the effects of degeneracies in the bifurcation parameter (that is, failure of (a) above). This is analogous to the classification of degenerate Hopf bifurcations, without symmetry, given by Golubitsky and Langford [14] (see also [15]). Some familiarity with those results will be helpful in the understanding of the results we present here.

(2) Our classification is complete up to codimension two and includes all nondegeneracy conditions.

(3) The universal unfolding theorem guarantees that we have found, up to an appropriate notion of equivalence, all possible perturbations of the singularities we classify.

Our main results are summarized in Table II, where the complete classification is given, and in the figures of Section 5, where the quasi-global information obtained in the universal unfoldings of the singularities is pictured. We regret that this information is sufficiently complicated that the figures are necessarily incomplete; this is, however, an accurate reflection of the complexity of the problem. Nevertheless, the main conclusions are illustrated.

The remainder of the paper is divided into eight sections. In Section 2 we follow Swift [25] in reducing the $O(2)$-symmetric Hopf bifurcation to one of finding zeroes of D_4-equivariant mappings on $\mathbb{R}^2$. This reduction uses the center manifold and Birkhoff normal form theories to obtain D_4-equivariant amplitude equations. In Section 3 we define D_4-equivalence and state our classification results. We discuss how to solve these amplitude equations in Section 4. Section 5 is devoted to constructing the bifurcation diagrams for the normal forms of Section 3. In Section 6 we describe how to find Knobloch's [20] invariant 3-torus using our results.

The proofs for our main theorems are given in Sections 7–9. The necessary singularity theory background is described in Section 7. The calculations needed to use singularity theory are described in Section 8. Here we rely on results from Buzano *et al.* [2]. The main ideas in the proofs, and some of the most difficult calculations, are summarized in Section 9. The calculations have been substantially simplified using a recent result of Gaffney [12] which is described in Section 7.

2. Reduction to Amplitude Equations and D_4-Equivalence

Center manifold theory allows us to study small amplitude periodic solutions to (1.1) by analyzing

$$\dot{x} + f(x, \lambda) = 0. \tag{2.1}$$

where $f: \mathbb{R}^4 \times \mathbb{R} \to \mathbb{R}^4$ commutes with the action of $O(2)$ on $\mathbb{R}^4$, identified with the sum of the $\pm i$ eigenspaces.

There is also a natural action of the circle group S^1 on $\mathbb{R}^4$ which stems from (2.1). $\mathbb{R}^4$ can be identified with the space of 2π-periodic solutions of the linearized system

$$\dot{x} + Df(0, 0) \cdot x = 0 \tag{2.2}$$

and S^1 acts on these 2π-periodic solutions by phase shifting. The theory of Birkhoff normal forms [17] allows us to use nonlinear changes of coordinates to transform (2.1) to commute with the action of $O(2) \times S^1$. More precisely, for each integer k there exists a polynomial change of coordinates so that f commutes with $O(2) \times S^1$ modulo terms of degree greater than k. We note, however, that as k increases to infinity the neighborhood of the origin on which this transformation is valid may shrink to nothing.

In this paper we assume that f commutes with $O(2) \times S^1$ to all orders. This may appear to be a strong restriction, but the local dynamics of any system (2.1) is well approximated by an $O(2) \times S^1$ equivariant system, at least regarding the existence and stability of small amplitude periodic solutions with period near 2π. This can be proved using the theory of Golubitsky and Stewart [17] and Chossat [4].

The group $O(2)$ is generated by $\theta \in SO(2)$, where $0 \leqslant \theta < 2\pi$ and an involution κ. The group of phase shifts S^1 has as typical element φ, where $0 \leqslant \varphi < 2\pi$. It is now well known [25, 26, 7] that it is possible to identify $\mathbb{R}^4$ with $\mathbb{C}^2$ and choose coordinates so that the action of $O(2) \times S^1$ is,

$$\begin{aligned}
&\text{(a)} \quad \theta\cdot(z_1, z_2) = (e^{i\theta}z_1, e^{-i\theta}z_2)\\
&\text{(b)} \quad \kappa\cdot(z_1, z_2) = (z_2, z_1) \qquad (2.3)\\
&\text{(c)} \quad \varphi\cdot(z_1, z_2) = (e^{i\varphi}z_1, e^{i\varphi}z_2).
\end{aligned}$$

The $O(2)\times S^1$ equivariance of f imposes strong restrictions on the terms in its Taylor series expansion. The normal form we use is related to those of Swift [25] and van Gils [26].

PROPOSITION 2.1. (i) *Any $O(2)\times S^1$ invariant function $g\colon \mathbb{C}^2\times\mathbb{R}\to\mathbb{R}$ is a function of (N, Δ, λ), where $N = |z_1|^2 + |z_2|^2$, $\Delta = \delta^2$, and $\delta = |z_2|^2 - |z_1|^2$.*

(ii) *Any $O(2)\times S^1$ equivariant mapping $f\colon \mathbb{C}^2\times\mathbb{R}\to\mathbb{C}^2$ has the form*

$$f(z_1, z_2, \lambda) = (p+iq)\begin{pmatrix} z_1\\ z_2\end{pmatrix} + (r+is)\delta\begin{pmatrix} z_1\\ -z_2\end{pmatrix}, \qquad (2.4)$$

where p, q, r, and s are $O(2)\times S^1$ invariant functions on $\mathbb{C}^2\times\mathbb{R}$.

The proof of this is a straightforward invariant theory calculation. See Swift [25] or Golubitsky, Stewart, and Schaeffer [16].

Remark. With this notation the hypothesis that $Df(0, 0)$ has eigenvalues $\pm i$ becomes $p(0, 0) = 0$ and $q(0, 0) = 1$.

One of the nice facts about the form of the vector field given by (2.4) is that it allows us to separate the four-dimensional system of ordinary differential equations into amplitude and phase equations. If we write $z_1 = xe^{i\psi_1}$, $z_2 = ye^{i\psi_2}$, then the equations

$$\begin{aligned}
\dot z_1 + (p + iq + (r+is)\delta)\, z_1 &= 0\\
\dot z_2 + (p + iq - (r+is)\delta)\, z_2 &= 0
\end{aligned} \qquad (2.5)$$

become

$$\left.\begin{aligned}
\dot x + (p + r\delta)x &= 0\\
\dot y + (p - r\delta)y &= 0
\end{aligned}\right\} \quad \text{amplitude equations} \qquad (2.6a)$$

$$\left.\begin{aligned}
\dot\psi_1 + (q + s\delta) &= 0\\
\dot\psi_2 + (q - s\delta) &= 0,
\end{aligned}\right\} \quad \text{phase equations} \qquad (2.6b)$$

where p, q, r, and s are functions of N, Δ, and λ, where $N = x^2 + y^2$, $\delta = y^2 - x^2$, and $\Delta = \delta^2$. This calculation may be done by differentiating the identity $x^2 = z_1\bar z_1$ to obtain

$$x\dot x = \mathrm{Re}(\dot z_1 \bar z_1) = -(p + r\delta)\, z_1\bar z_1$$

and similarly with $y^2 = z_2\bar{z}_2$. In this paper we are chiefly concerned with the amplitude equations which we think of as defining a vector field on $\mathbb{R}^2$. We follow Swift in noting that these equations are equivariant with respect to the group action on $\mathbb{R}^2$ generated by the symmetries:

$$I: (x, y) \to (x, -y) \qquad \text{and} \qquad J: (x, y) \to (y, x). \tag{2.7}$$

This group is the group of symmetries of the square in $\mathbb{R}^2$ with vertices $(\pm 1, \pm 1)$ or, abstractly, the dihedral group D_4. The D_4-equivariance of the amplitude equations is essential to our classification procedure. It is not hard to show that (2.6a) gives the general form for a D_4-equivariant vector-field on $\mathbb{R}^2$. Compare with Buzano *et al.* [2].

We now describe the correspondence between equilibrium solutions of the amplitude equations (2.6a) and solutions of the original equations (2.5). Observe that if (x_0, y_0) is an equilibrium point of (2.6a) then the submanifold of $\mathbb{R}^4$ defined by $z_1 = x_0 e^{i\psi_1}$, $z_2 = y_0 e^{i\psi_2}$, as ψ_1 and ψ_2 vary, is invariant under the flow described by (2.5). These submanifolds are points (if $x_0 = 0 = y_0$), circles (if $x_0 = 0$, $y_0 \neq 0$ or $x_0 \neq 0$, $y_0 = 0$), or tori (if $x_0 \neq 0$, $y_0 \neq 0$). The solution $z_1 = 0 = z_2$ is always an equilibrium point of (2.5). The possible flows on each of the other invariant submanifolds are restricted by the symmetry conditions. Each submanifold is an orbit of the $SO(2) \times S^1$ action on $\mathbb{R}^4$. This means that the vector field on each invariant orbit is determined by its value at any one point; in particular if it is zero at one point it must be zero on the whole submanifold. Thus the invariant circles are either periodic solutions of (2.5) or, exceptionally, circles of equilibrium points, while the invariant tori have either "linear" flows or, again exceptionally, are tori of equilibrium points. This can also be seen by considering the phase equations (2.6b), since for each invariant orbit both $\dot{\psi}_1$ and $\dot{\psi}_2$ are constant.

For certain tori the equivariance with respect to κ in $O(2)$ places even further restrictions on the flow. The involution κ maps invariant tori to invariant tori, taking $\{z_1 = x_0 e^{i\psi_1}, z_2 = y_0 e^{i\psi_2}\}$ to $\{z_1 = y_0 e^{i\psi_2}, z_2 = x_0 e^{i\psi_1}\}$. If $x_0 = y_0$ (or, equivalently, $x_0 = -y_0$) then this takes the corresponding torus to itself, leaving fixed the subset given by $\psi_1 = \psi_2$, a circle in the torus. The equivariance of the vector field with respect to κ implies that this circle must be invariant under the flow and hence must be a periodic solution. The linearity of the flow on the torus now implies that the whole torus must be filled out by periodic solutions. Again this can also be see by considering (2.6b).

We have shown that the equilibrium points of (2.6a) correspond to four different types of invariant submanifolds of (2.5): an equilibrium point at the origin, "isolated" periodic solutions, invariant tori foliated by periodic solutions, and invariant tori with general linear flows. These may be dis-

TABLE I

	$O(2)\times S^1$ Action on $\mathbb{C}^2$		D_4 Action on $\mathbb{R}^4$		
Label	Isotropy subgroup	Fixed point set	Isotropy subgroup	Fixed point set	Description
0	$O(2)\times S^1$	$\{(0,0)\}$	D_4	$\{(0,0)\}$	Equilibrium point
R	$\widetilde{SO(2)}$	$\{(z,0)\}$	$\{1, I\}$ $I(x,y)=(x,-y)$	$\{(x,0)\}$	Periodic solution, "rotating waves"
S	$\mathbb{Z}_2+\mathbb{Z}_2^c$	$\{(z,z)\}$	$\{1, J\}$ $J(x,y)=(y,x)$	$\{(x,x)\}$	Torus foliated by periodic solutions, "standing waves"
T	$\mathbb{Z}_2^c$	$\{(z_1,z_2)\}$	$\{1\}$	$\{(x,y)\}$	Torus with linear flow

tinguished by either the isotropy subgroup of D_4 at (x_0, y_0) (the subgroup of D_4 fixing the point) or by the isotropy subgroup of $O(2)\times S^1$ at a point in the corresponding invariant submanifold of $\mathbb{R}^4$. This information is summarized in Table I. The group $\widetilde{SO(2)}$ is the subgroup of $SO(2)\times S^1 = \{(e^{i\theta}, e^{i\varphi})\}$ given by $\theta=-\varphi$, while $\mathbb{Z}_2$ is the group generated by κ and $\mathbb{Z}_2^c$ is generated by $(e^{i\pi}, e^{i\pi})\in SO(2)\times S^1$. Note that $\mathbb{Z}_2^c$ fixes every point in $\mathbb{C}^2$. In the table we have only included one representative from each conjugacy class of isotropy subgroups and the fixed point set is the fixed point set of that representative. The solution types labelled R and S are those with

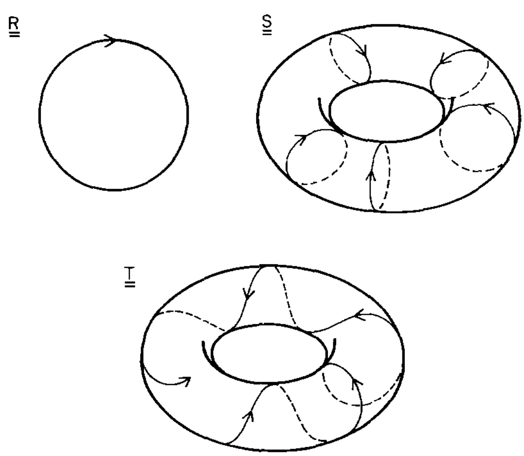

FIG. 2.1. Types of invariant submanifolds given by equilibrium points of the amplitude equations: R: periodic solution; S: torus of periodic solutions; T: torus with linear flow.

"maximal" isotropy subgroups, having two-dimensional fixed point sets $\mathbb{C}^2$ (see Table 1). These are the solutions whose existence near the bifurcation point is proved in Golubitsky and Stewart [17]. These periodic solutions are called "rotating waves" and "standing waves," respectively.

We end this section by observing that the correspondence between equilibrium solutions of the amplitude equations and solutions of the original equations preserves stability.

PROPOSITION 2.2. *An equilibrium solution of the amplitude equations is asymptotically stable if and only if the corresponding equilibrium point, periodic solution, or invariant 2-torus is asymptotically stable in the four-dimensional system.*

Proof. A zero (x_0, y_0) of the amplitude equations is asymptotically stable if, for every trajectory $(x(t), y(t))$ with initial point sufficiently close to (x_0, y_0), the trajectory stays near (x_0, y_0) and $\lim_{t\to\infty}(x(t), y(t)) = (x_0, y_0)$. Let M be the orbit of $SO(2)\times S^1$ given by $|z_1| = x_0$, $|z_2| = y_0$. Then M is asymptotically stable if and only if for every trajectory $(z_1(t), z_2(t))$ of the four-dimensional system, with $(z_1(0), z_2(0))$ sufficiently close to M, $\lim_{t\to\infty}(|z_1(t)|, |z_2(t)|) = (x_0, y_0)$ and so if and only if (x_0, y_0) is asymptotically stable as an equilibrium solutions of the amplitude equations. ∎

Remark. Observe that solutions that tend to a 2-torus of standing waves actually converge to a single periodic orbit on that 2-torus since $\dot{\psi}_t$ tends to a constant in the phase equations (2.6b).

3. NORMAL FORMS FOR THE AMPLITUDE EQUATIONS

In the previous section we reduced the study of equilibrium orbits of $O(2)$-equivariant vector fields near a Hopf bifurcation point to that of D_4-equivariant vector fields on $\mathbb{R}^2$ in a neighborhood of the origin. We also saw that any D_4-equivariant vector field can be written as

$$f(x, y, \lambda) = p(N, \Delta, \lambda)\begin{pmatrix} x \\ y \end{pmatrix} + r(N, \Delta, \lambda)\,\delta\begin{pmatrix} x \\ -y \end{pmatrix}, \tag{3.1}$$

where $\delta = y^2 - x^2$ and p and r are functions of $N = x^2 + y^2$ and $\Delta = \delta^2$.

We are chiefly interested in the equilibrium points of f and so in the solutions of the equation $f(x, y, \lambda) = 0$. We shall study the bifurcations of these solutions as the distinguished parameter λ varies, using singularity theory methods. We refer to the mapping $f(x, y, \lambda)$ as a D_4-bifurcation problem. It is convenient to introduce a notion of codimension for D_4-

bifurcation problems. Loosely speaking the codimension of f is the minimum number of extra parameters needed for a generic family of bifurcation problems to include f. A more precise description is given in the last section.

Our three main theorems give solutions to the following three problems:

(1) Classify all D_4-bifurcation problems of codimension $\leqslant 2$. The classification consists of a list of normal forms such that any vector field with codimension $\leqslant 2$ is equivalent to one of these normal forms. The equivalence relation is defined below.

(2) For each normal form give necessary and sufficient conditions on the partial derivatives of a D_4-bifurcation problem for it to be equivalent to that normal form.

(3) Give a qualitative description of all bifurcation diagrams that can be obtained by perturbing germs of D_4-bifurcation problems of codimension $\leqslant 2$.

Let $u=(x, y)$.

DEFINITION 3.1. Two D_4-bifurcation problems f and g are *D_4-equivalent* if there exists a smooth 2×2 matrix $S(u, \lambda)$ and diffeomorphism $\Phi(u, \lambda)=(Z(u, \lambda), \Lambda(\lambda))$ of $\mathbb{R}^2\times\mathbb{R}$ such that

$$g(u, \lambda)=S(u, \lambda)\, f(Z(u, \lambda), \Lambda(\lambda)) \tag{3.2}$$

and satisfying:

$$\Phi(0, 0)=(0, 0), \tag{3.3a}$$

$$Z(\gamma\cdot u, \lambda)=\gamma Z(u, \lambda) \quad \text{for all } \gamma \text{ in } D_4, \tag{3.3b}$$

$$S(\gamma\cdot u, \lambda)\cdot\gamma=\gamma\cdot S(u, \lambda) \quad \text{for all } \gamma \text{ in } D_4, \tag{3.3c}$$

$$\Lambda'(0)>0, \tag{3.3d}$$

$$S(0, 0)=A\cdot I \text{ and } dZ(0, 0)=a\cdot I, \quad \text{where } A \text{ and } a \text{ are strictly positive real number.} \tag{3.3e}$$

We define $\mathscr{E}_{u,\lambda}(D_4)$ to be the ring of D_4-invariant functions $\mathbb{R}^2\to\mathbb{R}$. Matrices S satisfying (3.3c) are called D_4-equivariant matrices and form a module over $\mathscr{E}_{u,\lambda}(D_4)$. A simple calculation shows that, for any Z satisfying (3.3b), $dZ(u, \lambda)$ is also a D_4-equivariant matrix. Equivariance implies that $S(0, \lambda)=c(\lambda)\cdot I$ for some $c(\lambda)\in\mathscr{E}_\lambda$, and similarly for $dZ(0, \lambda)$. As we discuss next, the extra hypothesis in (3.3e), that $c(0)=A>0$ and $a>0$, is needed to ensure that D_4-equivalence preserves the stability of at least some of the equilibrium points of f.

PROPOSITION 3.2. *Let* f *and* $g = S \cdot f(\Phi(u, \lambda))$ *be* D_4*-equivalent bifurcation problems and let* (u, λ) *be an equilibrium point of* f. *Then the signs of the real parts of the eigenvalues of* $df(u, \lambda)$ *are the same as those of* $dg(\Phi(u, \lambda))$ *if any of the following conditions hold*:

(i) $u = 0$,

(ii) u *is of type* R *or* S,

(iii) u *is of type* T *and* $\det df(u, \lambda) < 0$.

Proof. It is sufficient to prove that for any equivariant matrix S such that $S(0, 0) = A \cdot I$, where $A > 0$, the signs of the real parts of the eigenvalues of $S \cdot df$ are the same as those of df when any of the conditions hold [15, Chap. X, Lemma 3.3 and the following remark]. Observe that if γ is in the isotropy subgroup of u then it follows from (3.3c) that

$$S(u, \lambda) \cdot \gamma = \gamma \cdot S(u, \lambda). \tag{3.4}$$

The isotropy subgroup of $u = 0$ is D_4. The commutativity in (3.4) with D_4 forces $S(0, \lambda)$ to be a multiple of I, say $c(\lambda) \cdot I$. Since we assume $c(0) > 0$ and since $df(0, \lambda)$ is also a multiple of the identity, the eigenvalues of $S(0, \lambda)\, df(0, \lambda)$ have the same signs as those of $df(0, \lambda)$ and (i) is verified. Similarly the matrix $\gamma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is in the isotropy subgroup of solutions of type R. Thus (3.4) implies that S and df are both diagonal and (ii) follows. The argument showing the invariance of the stability of type S solutions is abstractly the same. The nontrivial matrix in the isotropy subgroup of type S solutions is $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Since γ has distinct eigenvalues with eigenvectors $v_1 = (1, 1)$ and $v_2 = (1, -1)$ it follows from (3.4) that S and df also have v_1 and v_2 as eigenvectors. Thus (ii) is verified.

For (iii) we note that, near $(0, 0)$, $\det S(u, \lambda)$ must be positive and so $\det S(u, \lambda)\, df(u, \lambda)$ must have the same sign as $\det df(u, \lambda)$. If this is negative both the eigenvalues of $df(u, \lambda)$ and those of $S(u, \lambda)\, df(u, \lambda)$ must have real parts with opposite signs. ∎

We now come to the statement of our main classification results, for which all the relevant data are contained in Table II. Note that the classification up to codimension one is given in Buzano *et al.* [2] and corresponds to the results of Nagata [22] and Knobloch [20] described in Section 1.

CLASSIFICATION THEOREM. *In a generic two-parameter family all bifurcation problems are* D_4*-equivalent to one of the normal forms listed in Table* II.

RECOGNITION THEOREM. *A* D_4*-equivariant bifurcation problem is* D_4*-*

TABLE II

Normal Forms for D_4—Bifurcation Problems

	Defining conditions	Non-degeneracy conditions	Normal form	Coefficients in normal form	Universal unfolding	Codim	Description	Fig
I	-	$p_X \neq 0,\ r \neq 0$ $p_X \neq r,\ p_\lambda \neq 0$	$(\varepsilon_0\lambda + mX, \varepsilon_1)$ $m \neq 0, \varepsilon_1$	$\varepsilon_0 = \operatorname{sgn} p_\lambda, \varepsilon_1 = \operatorname{sgn} r$ $m = p_X/\lvert r\rvert$	-	0	Bifurcation of R and S branches	5.1
II	$p_X = 0$	$r \neq 0,\ p_\lambda \neq 0$ $p_{XX} \neq 0$	$(\varepsilon_0\lambda + \varepsilon_1 X^2, \varepsilon_2)$	$\varepsilon_0 = \operatorname{sgn} p_\lambda, \varepsilon_1 = \operatorname{sgn} p_{XX}$ $\varepsilon_2 = \operatorname{sgn} r$	$\alpha(X, 0)$	1	Fold in S branch	5.2
III	$p_X = r$	$p_X \neq 0\ \ p_\lambda \neq 0$ $p_{XX} + 2p_\Lambda - 2r_X \neq 0$	$(\varepsilon_0\lambda + \varepsilon_1 X + \varepsilon_2\Lambda, \varepsilon_1)$	$\varepsilon_0 = \operatorname{sgn} p_\lambda, \varepsilon_1 = \operatorname{sgn} p_X$ $\varepsilon_2 = \operatorname{sgn}(p_{XX} + 2p_\Lambda - 2r_X)$	$\alpha(\Lambda, 0)$	1	Fold in R branch	
IV	$r = 0$	$p_X \neq 0\ \ p_\lambda \neq 0$ $p_X r_\Lambda - p_\Lambda r_X \neq 0$ $p_\lambda r_X - p_X r_\lambda \neq 0$	$(\varepsilon_0\lambda + \varepsilon_1 X + m\Lambda, \varepsilon_2 X)$ $m \neq 0$	$\varepsilon_0 = \operatorname{sgn} p_\lambda, \varepsilon_1 = \operatorname{sgn} p_X$ $\varepsilon_2 = \varepsilon_0 \operatorname{sgn}(p_\lambda r_X - p_X r_\lambda)$ $m = \varepsilon_2 p_\lambda^2 (p_X r_\Lambda - p_\Lambda r_X)$ $/(p_\lambda r_X - p_X r_\lambda)$	$\alpha(0, 1)$	1	Bifurcation of T branch	5.3
V	$p_\lambda = 0$	$p_X \neq 0,\ p_X \neq r$ $r \neq 0,\ p_{\lambda\lambda} \neq 0$ $p_{\lambda X} r - p_X r_\lambda \neq 0$	$(\varepsilon_0\lambda^2 + mX + \varepsilon_1\lambda X, \varepsilon_2)$ $m \neq 0, \varepsilon_2$	$\varepsilon_0 = \operatorname{sgn} p_{\lambda\lambda}, \varepsilon_2 = \operatorname{sgn} r$ $\varepsilon_1 = \varepsilon_2 \operatorname{sgn}(p_{\lambda X} r - p_X r_\lambda)$ $m = p_X/r$	$\alpha(1, 0)$	1	Re-entrant R and S branches	5.4
VI	$p_X = 0$ $p_{XX} = 0$	$r \neq 0,\ p_\lambda \neq 0$ $p_{XXX} \neq 0$	$(\varepsilon_0\lambda + \varepsilon_1 X^3, \varepsilon_2)$	$\varepsilon_0 = \operatorname{sgn} p_\lambda, \varepsilon_1 = \operatorname{sgn} p_{XXX}$ $\varepsilon_2 = \operatorname{sgn} r$	$\alpha(X, 0) + \beta(X^2, 0)$	2	Hysteresis in S branch	5.5
VII	$p_X = 0$ $p_{XX} + 2p_\Lambda - 2r_X = 0$	$p_X \neq 0,\ p_\lambda \neq 0$ $p_{XXX} + 6p_{X\Lambda} - 6r_\Lambda - 3r_{XX} \neq 0$	$(\varepsilon_0\lambda + \varepsilon_1 X + \varepsilon_2 X\Lambda, \varepsilon_1)$	$\varepsilon_0 = \operatorname{sgn} p_\lambda, \varepsilon_1 = \operatorname{sgn} p_X$ $\varepsilon_2 = \operatorname{sgn}(p_{XXX} + 6p_{X\Lambda} - 6r_\Lambda - 3r_{XX})$	$\alpha(X, 0) + \beta(\Lambda, 0)$	2	Hysteresis in R branch	
VIII	$r = 0$ $p_X r_\Lambda - p_\Lambda r_X = 0$	$p_X \neq 0,\ p_\lambda \neq 0$ $p_\lambda r_X - p_X r_\lambda \neq 0$ $\xi \neq 0$	$(\varepsilon_0\lambda + \varepsilon_1 X + m\Lambda^2, \varepsilon_2 X)$ $m \neq 0$	$\varepsilon_0 = \operatorname{sgn} p_\lambda, \varepsilon_1 = \operatorname{sgn} p_X$ $\varepsilon_2 = \varepsilon_0 \operatorname{sgn}(p_\lambda r_X - p_X r_\lambda)$ $m = \varepsilon_2 \xi_2$	$\alpha(0, 1) + \beta(\Lambda, 0)$	2	Fold in T branch	5.6

IX	$r = 0$ $p_\lambda r_N - p_N r_\lambda = 0$	$p_N \neq 0,\ p_\lambda \neq 0$ $p_N r_A - p_A r_N \neq 0$ $\xi_2 \neq 0$	$(\varepsilon_0 \lambda + \varepsilon_1 N, \varepsilon_2 A + m\lambda^2)$ $m \neq 0$	$\varepsilon_0 = \operatorname{sgn} p_\lambda, \varepsilon_1 = \operatorname{sgn} p_N$ $\varepsilon_2 = \operatorname{sgn}(p_N r_A - p_A r_N)$ $m = \varepsilon_1 \varepsilon_2 \xi_2$	$\alpha(N, 0) + \beta(0, N)$	2	Two T branches	5.7
X	$p_N = 0$ $r = 0$	$p_\lambda \neq 0, p_{NN} \neq 0, p_A \neq 0$ $r_N \neq 0, p_{NN} + 2p_A - 2r_N \neq 0$ $\xi_3 \neq 0, p_{NN} + 2p_A - r_N \neq 0$	$(\varepsilon_0 \lambda + mN^2 + nA, \varepsilon_1 N + \varepsilon_2 A)$ $m \neq 0, n \neq 0$ $m + n \neq \varepsilon_1, \varepsilon_1/2$	$\varepsilon_0 = \operatorname{sgn} p_\lambda, \varepsilon_1 = \operatorname{sgn} r_N$ $\varepsilon_2 = \varepsilon_0 \operatorname{sgn} \xi_3$ $m = p_{NN}/2\|r_N\|, n = p_A/\|r_N\|$	$\alpha(N, 0) + \beta(0, 1)$	2	Interaction of T bifurcations with R and S folds	5.9
XI	$p_N = 0$ $p_\lambda = 0$	$r \neq 0, p_{NN} \neq 0$ $p_{\lambda\lambda} \neq 0$ $p_{\lambda N}^2 \neq p_{\lambda\lambda} p_{NN}$	$(\varepsilon_0 \lambda^2 + \varepsilon_1 N^2 + m\lambda N, \varepsilon_2)$ $m^2 \neq 4\varepsilon_0 \varepsilon_1$	$\varepsilon_0 = \operatorname{sgn} p_{\lambda\lambda}, \varepsilon_2 = \operatorname{sgn} r$ $\varepsilon_1 = \operatorname{sgn} p_{NN}$ $m = 2p_{\lambda N}/\sqrt{(\|p_{\lambda\lambda} p_{NN}\|)}$	$\alpha(1, 0) + \beta(N, 0)$	2	Creation of isola of type S	5.10
XII	$p_N = r$ $p_\lambda = 0$	$p_N \neq 0, p_{\lambda\lambda} \neq 0$ $p_{NN} + 2p_A - 2r_N \neq 0$ $(p_{\lambda N} - r_\lambda)^2 \neq$ $p_{\lambda\lambda}(p_{NN} + 2p_A - 2r_N)$	$(\varepsilon_0 \lambda^2 + \varepsilon_1 N + \varepsilon_2 A + m\lambda N, \varepsilon_1)$ $m^2 \neq 4\varepsilon_0 \varepsilon_2$	$\varepsilon_0 = \operatorname{sgn} p_{\lambda\lambda}, \varepsilon_1 = \operatorname{sgn} p_N$ $\varepsilon_2 = \operatorname{sgn}(p_{NN} + 2p_A - 2r_N)$ $m = 2\varepsilon_1(p_{\lambda N} - r_\lambda)$ $(\|p_{\lambda\lambda}\|\|p_{NN} + 2p_A - 2r_N\|)^{1/2}$	$\alpha(1, 0) + \beta(N, 0)$	2	Creation of isola of type R	
XIII	$r = 0$ $p_\lambda = 0$	$p_N \neq 0, p_{\lambda\lambda} \neq 0$ $r_\lambda \neq 0, p_N r_A - p_A r_N \neq 0$	$(\varepsilon_0 \lambda^2 + \varepsilon_1 N, \varepsilon_2 \lambda + mA)$ $m \neq 0$	$\varepsilon_0 = \operatorname{sgn} p_{\lambda\lambda}, \varepsilon_1 = \operatorname{sgn} p_N$ $\varepsilon_2 = \operatorname{sgn} r_\lambda$ $m = \varepsilon_1 p_{\lambda\lambda}^2 (p_N r_A - p_A r_N)/r_\lambda^2$	$\alpha(1, 0) + \beta(0, 1)$	2	Re-entrant R & S branches with T branch	5.11
XIV	$p_\lambda = 0$ $p_{\lambda\lambda} = 0$	$p_N \neq 0, p_N \neq r, r \neq 0$ $p_{\lambda\lambda\lambda} \neq 0, p_{\lambda N} r - p_N r_\lambda \neq 0$	$(\varepsilon_0 \lambda^3 + mN + \varepsilon_1 \lambda N, \varepsilon_2)$ $m \neq 0, \varepsilon_2$	$\varepsilon_0 = \operatorname{sgn} p_{\lambda\lambda\lambda}, \varepsilon_2 = \operatorname{sgn} r$ $\varepsilon_1 = \varepsilon_2 \operatorname{sgn}(p_{\lambda N} r - p_N r_\lambda)$ $m = p_N/\|r\|$	$\alpha(1, 0) + \beta(\lambda, 0)$	2	Doubly re-entrant R and S branches	5.12
XV	$p_\lambda = 0$ $p_{\lambda N} r - p_N r_\lambda = 0$	$p_N \neq 0, p_N \neq r$ $r \neq 0, p_{\lambda\lambda} \neq 0$	$(\varepsilon_0 \lambda^2 + mN, \varepsilon_1)$ $m \neq 0, \varepsilon_1$	$\varepsilon_0 = \operatorname{sgn} p_{\lambda\lambda}, \varepsilon_1 = \operatorname{sgn} r$ $m = p_N/r$	$\alpha(1, 0) + \beta(\lambda N, 0)$	2	Symmetric re-entrant R and S branches	

equivalent to a normal form if and only if it satisfies the corresponding sets of "defining" and "non-degeneracy" conditions listed in Table II.

UNFOLDING THEOREM. *Any bifurcation diagram of a perturbation of a germ D_4-equivalent to a given normal form is qualitatively the same as a bifurcation diagram obtained from the universal unfolding of the normal form. The universal unfolding is the family of bifurcation problems obtained by adding to the normal form the terms listed in the table.*

The proofs of these theorems will be given in Section 8, using the singularity theory and calculations outlined in Sections 6 and 7. Examples of the bifurcation diagrams obtained from the unfoldings of the normal forms are given in Section 5.

Notes on Table II. (i) The D_4-vector field (3.1) is written as (p, r), where p and r are elements of $\mathscr{E}_{u,\lambda}(D_4)$. The defining conditions are given in terms of the partial derivatives of p and r, with respect to N, Δ, and λ, at the origin.

(ii) The normal forms are listed under fifteen headings, but under each of these there are, in general, a number of different families of non-equivalent normal forms. Each choice for sign ε_i and distinct values of the moduli, m, n, give different normal forms.

(iii) To obtain the universal unfolding of a normal form we need to add on the terms in the column headed "universal unfolding" and also allow the moduli to vary. However, for most values of the moduli small perturbations do not effect the qualitative properties of the bifurcation diagrams in the unfolding. When we come to discuss the perturbed bifurcation diagrams we will restrict attention to normal forms for which the bifurcation diagrams are persistant under perturbation of the moduli and explicitly consider only perturbations with respect to the parameter α, β.

(iv) The "codimension" given in the table is that referred to above. It is the number of parameters needed for a generic family of vector fields to include a germ equivalent to the normal form (such a family is provided by the universal unfolding). This idea of codimension is the topological (or C^0) codimension of singularity theory and is related, but not identical, to the smooth (C^∞) codimension defined in Section 7.

(v) The following expressions are needed for VIII, IX, and X, respectively.

$$\xi_1 = \frac{p_N p_\lambda^3}{2(p_\lambda r_N - p_N r_\lambda)^4} \{ p_\Delta^2 r_\Delta p_\lambda + p_\lambda p_\Delta (r_\Delta p_{NN} - p_\Delta r_{NN})$$
$$+ 2p_N (p_\Delta r_\lambda r_{N\Delta} - r_\Delta p_\lambda p_{N\Delta}) + p_\lambda p_N (r_N p_{\Delta\Delta} - p_N r_{\Delta\Delta}) \}$$

$$\xi_2 = \frac{1}{2p_N p_\lambda^2 (r_\Delta p_N - p_\Delta r_N)} \{ p_N(p_N r_{\lambda\lambda} - r_N p_{\lambda\lambda})$$
$$+ p_\lambda(p_\lambda r_{NN} - r_\lambda p_{NN}) - 2p_\lambda(p_N r_{\lambda N} - r_N p_{\lambda N}) \}$$

$$\xi_3 = (p_\lambda r_\Delta - p_\Delta r_\lambda) - \frac{r_N}{6p_{NN}(2p_\Delta + p_{NN} - r_N)} \{ 2p_\lambda p_{NNN}(p_\Delta - r_N)$$
$$+ 6p_{NN}(r_\lambda r_N - p_\lambda p_{N\Delta}) + 3p_{NN}(p_\lambda r_{NN} - r_\lambda p_{NN}) \}.$$

4. Solving the Amplitude Equations

In this section we obtain the three sets of equations that are satisfied by the three different types, R, S, and T, of equilibrium points of the amplitude equations (2.6a). We also calculate general expressions that give the stability of these points; for R and S we find the signs of the eigenvalues of df, while for T we give formulas for the signs of the trace and the determinant of df. Recall that if $\det df < 0$ then the eigenvalues are real and have opposite signs and so the equilibrium point is unstable, while if $\det df > 0$ the real parts of the eigenvalues have the same sign and this is positive if trace $df > 0$ and negative if trace $df < 0$.

Proposition 4.1. *The equilibrium points of types R, S, and T of the amplitude equations* (2.6a) *can be found by solving the equations in the second column of the following table. Their stability can be computed from the information given in the third column*

Type	Equations	Stability
R	$y = 0 \quad x > 0$ $p + x^2 r = 0$	Signs of eigenvalues of df: $p_N - r + x^2(2p_\Delta - r_N) - 2x^4 r_\Delta;\ r$
S	$x = y > 0$ $p = 0$	Signs of eigenvalues of df: $p_N;\ -r$
T	$x > y > 0$ $p = 0 \quad r = 0$	Sign of trace df: $Np_N - 2\Delta p_\Delta - \Delta r_N - 2N\Delta r_\Delta$ Sign of $\det df$: $p_\Delta r_N - p_N r_\Delta$

These equations do not give all the equilibrium points. However, the other equilibria can be obtained from these by applying the symmetry operations in D_4.

Proof. The equilibrium points of (2.6a) are the solutions of the equation

$$f(x, y) = p\begin{pmatrix} x \\ y \end{pmatrix} + r\delta\begin{pmatrix} x \\ -y \end{pmatrix} = 0. \tag{4.1}$$

The solutions of type R are those satisfying either $y=0$, $x \neq 0$ or $x=0$, $y \neq 0$. Using the symmetry operation $(x, y) \to (y, x)$ it is sufficient to find the solutions satisfying the first set of conditions and substituting this into (4.1) gives the required equations. Similarly the equations for solutions of type S can be found by substituting $x=y$ into (4.1), noting that then $\delta(x, y)=0$. Solutions where $x=-y$ are found by symmetry.

If $x \neq 0$, $y \neq 0$, $x \neq \pm y$ then $\binom{x}{y}$ and $\binom{x}{-y}$ are independent vectors in $\mathbb{R}^2$ and (4.1) can only be satisfied if $p=0$ and $r=0$ so these are the equations for the solutions of type T.

To obtain the stability information in the table we need to compute df in terms of p and r. Write f as (f_1, f_2), where $f_1 = (p + r\delta)x$ and $f_2 = (p - r\delta)y$. Since df is an equivariant matrix (3.3c) tells us that

$$df_{21}(x, y) = df_{12}(y, x) \qquad \text{and} \qquad df_{22}(x, y) = df_{11}(y, x), \tag{4.2}$$

where the second subscript denotes a derivative with respect to x or y. A straightforward computation gives

$$df_{11}(x, y) = p + 2x^2 p_N - 4x^2\,\delta p_\Delta + (y^2 - 3x^2)r + 2x^2\,\delta r_N - 4x^2\,\Delta r_\Delta \tag{4.3a}$$

$$df_{12}(x, y) = 2xyp_N + 4xy\,\delta p_\Delta + 2xyr + 2xy\,\delta r_N + 4xy\,\Delta r_\Delta \tag{4.3b}$$

$$df_{21}(x, y) = 2xyp_N - 4xy\,\delta p_\Delta + 2xyr - 2xy\,\delta r_N + 4xy\,\Delta r_\Delta \tag{4.3c}$$

$$df_{22}(x, y) = p + 2y^2 p_N + 4y^2\,\delta p_\Delta - (3y^2 - x^2)r - 2y^2\,\delta r_N - 4y^2\,\Delta r_\Delta. \tag{4.3d}$$

At a point of type R for which $y=0$, df is diagonal and so that eigenvalues are equal to $df_{11}(x, 0)$ and $df_{22}(x, 0)$. If, in addition, the point is a solution of (4.1) we have $p = x^2 r$ and the eigenvalues are given by the expressions in the table. At a point of type S, when $x=y$, we have $df_{11}(x, x) = df_{22}(x, x)$ and $df_{12}(x, x) = df_{21}(x, x)$ and so the eigenvalues are $df_{11}(x, x) \pm df_{12}(x, x)$. For a solution of (4.1) we take $p=0$ to obtain the required expressions. Finally the formulas for the trace and determinant of df at a solutions of type T, when $p = 0 = r$, are obtained by routine calculation. ∎

5. Perturbed Bifurcation Diagrams

We now discuss and illustrate the bifurcation diagrams that are obtained by perturbing the normal forms in our classification. We will give essentially all the diagrams for the generic and codimension one normal forms. For the codimension two normal forms we will derive some useful general formulas and illustrate the bifurcation diagrams with some representative examples. In our discussion we will allow time reversal ($f \to -f$) and reversal of the distinguished parameter ($\lambda \to -\lambda$) to reduce the number of normal forms we must explicitly consider.

I. *The Generic Normal Form*

We begin with the generic normal form

$$(\varepsilon_0 \lambda + mN, \varepsilon_1), \qquad m \neq 0, \varepsilon_1.$$

The equilibrium point at the origin is stable if $\varepsilon_0 \lambda > 0$ and unstable if $\varepsilon_0 \lambda < 0$. Since $r(0) \neq 0$ there are no type T equilibrium points. The equations for the types R and S equilibrium points and their stabilities are given in the following table.

Type	Equations	Signs of eigenvalues
R	$y = 0$	$m - \varepsilon_1$
	$\varepsilon_0 \lambda + (m - \varepsilon_1) x^2 = 0$	ε_1
S	$x = y$	m
	$\varepsilon_0 \lambda + 2mx^2 = 0$	$-\varepsilon_1$

There are twelve qualitatively distinct diagrams corresponding to the two possible choices of ε_0 and ε_1 and the three choices of m given by the three regions of $\mathbb{R} \setminus \{0, \varepsilon_1\}$. However, allowing the coordinate changes $f \to -f$ (which interchanges stabilities) and $\lambda \to -\lambda$ (which interchanges left and right in the bifurcation diagrams) means we need only illustrate the diagrams for $\varepsilon_0 = -1$ and $\varepsilon_1 = 1$. This is done in Fig. 5.1.

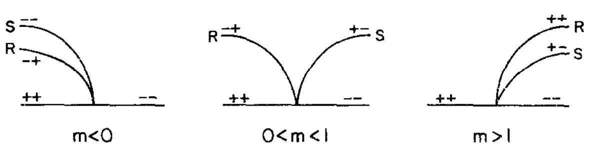

FIG. 5.1. Bifurcation diagrams for I: $\varepsilon_0 = -1$, $\varepsilon_1 = 1$.

For the remaining normal forms we are not so much interested in the bifurcation diagram of the normal form itself as in those of its generic perturbations. All possible perturbations are qualitatively equivalent to those given by the universal unfolding of the normal form and for most values of the moduli they can be obtained by keeping the moduli fixed and varying only the unfolding parameters α and β. When we look at the examples we will exclude from consideration any values of the moduli for which this is not true. This is justified by the fact that the normal forms corresponding to such values of the moduli do not appear in generic two-parameter families of vector fields, they have codimension strictly greater than two.

Codimension One Normal Forms

II. $(\varepsilon_0\lambda + \varepsilon_1 N^2, \varepsilon_2)$

The universal unfolding is given by:

$$p = \varepsilon_0\lambda + \varepsilon_1 N^2 + \alpha N$$

$$r = \varepsilon_2.$$

We will restrict our attention to $\varepsilon_0 = -1$ and $\varepsilon_1 = 1$. As $r \neq 0$ there are not solutions of type T and, with our choice of signs, the origin is stable for $\lambda < 0$ and unstable for $\lambda > 0$. The other solutions are given in the table below and the bifurcation diagrams are shown in Fig. 5.2.

Type	Equations	Signs of eigenvalues
R	$u = 0$	$\alpha - \varepsilon_2 + 2x^2$
	$\lambda - (\alpha - \varepsilon_2)x^2 - x^4 = 0$	ε_2
S	$x = y$	$\alpha + 4x^2$
	$\lambda - 2\alpha x^2 - 4x^4 = 0$	ε_2

III. The bifurcation diagrams for this normal form can be obtained from those of II by interchanging R and S

IV. $(\varepsilon_0\lambda + \varepsilon_1 N + mA, \varepsilon_2 N)$, $m \neq 0$.

The universal unfolding is

$$p = \varepsilon_0\lambda + \varepsilon_1 N + mA$$

$$r = \varepsilon_2 N + \alpha.$$

We can choose $\varepsilon_0 = -1$ and $\varepsilon_2 = 1$ so the origin is stable if $\lambda < 0$ and unstable if $\lambda > 0$. The other equilibrium points are given in the table below. Notice that the T branch can only exist if $\alpha < 0$.

Type	Equations	Stability information
R	$y=0$ $\lambda-(\varepsilon_1-\alpha)x^2-(m-1)x^4=0$	Signs of eigenvalues: $(\varepsilon_1-\alpha)+(2m-1)x^2$, $\alpha-x^2$
S	$x=y$ $\lambda-2\varepsilon_1 x^2=0$	Signs of eigenvalues: ε_1, $-(\alpha+2x^2)$
T	$\lambda-\varepsilon_1 N-m\Delta=0$ $\alpha+N=0$	Sign trace df: $\varepsilon_1 N+(2m-1)\Delta$ Sign det df: m

The values of λ at which the T branch bifurcates from the R and S branches are found to be

$\lambda=-\varepsilon_1\alpha+m\alpha^2$ for the bifurcation from the R branch, and

$\lambda=-\varepsilon_1\alpha$ for the bifurcation from the S branch.

See also the general formula given below. The bifurcation diagrams are given in Fig. 5.3. The only exceptional value of the modulus is $m=0$ and the only differences between the $m>0$ and $m<0$ cases are the direction and stability of the T branch. For $m<0$ the stability of the T branch is an invariant of D_4 equivalence by Proposition 3.3. However we have no such result for $m>0$, though by the exchange of stabilities rule for pitchfork bifurcations we know that near the points of bifurcation from the R and S branches the stability must be as shown in the diagrams.

V. $(\varepsilon_0\lambda^2+mN+\varepsilon_1\lambda N, \varepsilon_2)$, $m\neq 0$, ε_2

The universal unfolding is

$$p=\lambda^2+mN+\varepsilon_1\lambda N+\alpha$$

$$r=\varepsilon_2.$$

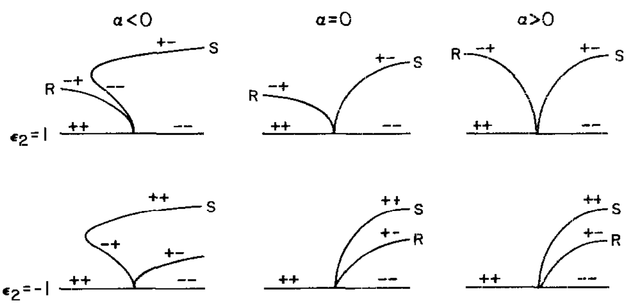

FIG. 5.2. Bifurcation diagrams in the unfolding of II: $\varepsilon_0=-1$, $\varepsilon_1=1$.

We choose $\varepsilon_0 = 1$ and $\varepsilon_1 = 1$. The origin is stable if $\lambda^2 > -\alpha$ and unstable if $\lambda^2 < -\alpha$. There are no solutions of type T. The other solutions are given in the table below.

Type	Equations	Signs of eigenvalues
R	$y = 0$ $\alpha - \lambda^2 + (m - \varepsilon_2 - \lambda)\, x^2 = 0$	$m - \varepsilon_2 + \lambda$ ε_2
S	$x = y$ $\alpha + \lambda^2 + 2(m + \lambda)\, x^2 = 0$	$m + \lambda$ $-\varepsilon_2$

Notice that the R and S solutions can only bifurcate if $\alpha < 0$. The bifurcation diagrams are given for $\varepsilon_2 = -1$, those for $\varepsilon_2 = 1$ can be obtained by interchanging R and S (Fig. 5.4).

Before describing the codimension-two normal forms we shall develop a general idea that was implicit in the discussion above. Most bifurcation diagrams in a universal unfolding are *persistent*, that is, a small change in the unfolding parameters does not change it qualitatively. The bifurcations which may occur in a persistent diagram are limited to those listed in Table III, which also includes the equations for the corresponding bifurcation points.

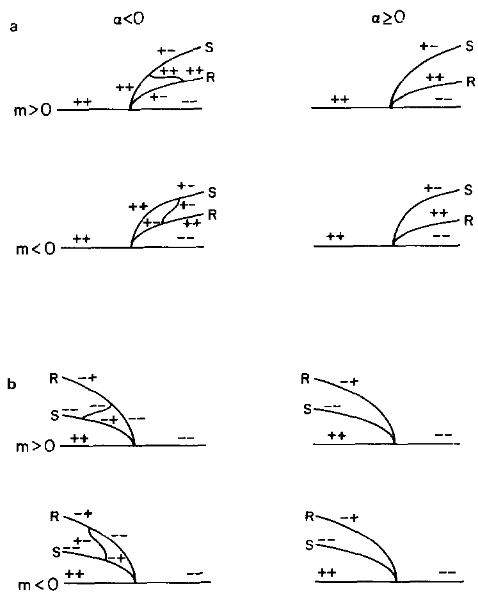

FIG. 5.3. (a) Bifurcation diagrams in the unfolding of IV: $\varepsilon_0 = 1$, $\varepsilon_1 = 1$, $\varepsilon_2 = -1$. (b) Bifurcation diagrams in the unfolding of IV: $\varepsilon_0 = -1$, $\varepsilon_1 = -1$, $\varepsilon_2 = 1$.

TABLE III

Bifurcations in Persistant Diagrams

Label	Description	Equations
$\mathscr{S}_I$	Generic bifurcation from 0	$p = 0;\ x = 0 = y$
$\mathscr{F}_R$	Fold in R branch	$y = 0;\ p - x^2 r = 0;$ $p_N r + x^2(2p_A - r_N) - 2x^4 r_A = 0$
$\mathscr{F}_S$	Fold in S branch	$x = y;\ p = 0;\ p_N = 0$
$\mathscr{F}_T$	Fold in T branch	$p = 0,\ r = 0;\ p_N r_A - p_A r_N = 0$
$\mathscr{P}_R$	Pitchfork bifurcation from R branch to T branch	$y = 0;\ p = 0;\ r = 0$
$\mathscr{P}_S$	Pitchfork bifurcation from S branch to T branch	$x = y,\ p = 0;\ r = 0$

However, on subvarieties of the space of unfolding parameters more degenerate behavior can be seen in the bifurcation diagrams. Roughly speaking, any codimension-one degeneracy defines a hypersurface in an unfolding space such that points on that hypersurface correspond exactly to the bifurcation diagrams containing that degeneracy. Similarly, codimension-two degeneracies define codimension-two subvarieties and so on. The subvariety given by a particular degeneracy is called the *transition variety* of the degeneracy. The transition varieties divide the unfolding space into a finite number of regions in each of which the bifurcation diagrams are all qualitatively the same and it is these diagrams that we illustrate. For codimension-one normal forms the unfolding space is one dimensional and so the transition varieties can only be the origin (since α and β are always considered to be "small"). However, for the codimension-two normal forms the transition varieties of codimension-one degeneracies can be quite complicated. They are of two types, *global* and *local*, corresponding, respectively to (1) and (2) below.

(1) For every pair (X, Y) of generic singularities, listed in Table III, there is a transition variety, denoted $\mathscr{Y}(X, Y)$, consisting of all values of the unfolding parameters for which the corresponding diagrams contain bifurcations of type X and Y at the same value of λ. We do not explicitly calculate the equations for these transition varieties; they can be found by eliminating x and y from the equations for X and similarly for Y, and then eliminating λ from the resulting equations.

(2) There are sixteen possible codimension-one degenerate local bifurcations listed, along with their equations, in Table IV. The first four of

TABLE IV

Codimension One Local Transition Varieties

Label	Description	Equations
$\mathcal{S}_{II}$	Bifurcation from 0; normal form II	$p(0)=0;\ p_u(0)=0$
$\mathcal{S}_{III}$	Bifurcation from 0; normal form III	$p(0)=0;\ p_u(0)=-r(0)$
$\mathcal{S}_{IV}$	Bifurcation from 0, normal form IV	$p(0)=0,\ r(0)=0$
$\mathcal{S}_{V}$	Bifurcation from 0; normal form V	$p(0)=0;\ p_\lambda(0)=0$
$\mathcal{B}_R$	Symmetry preserving bifurcation from R branch	$y=0;\ p-x^2r=0;\ p_\lambda-x^2r_\lambda=0$ $p_u-r+x^2(2p_\Delta-r_u)-2x^4r_\Delta=0$
$\mathcal{B}_S$	Symmetry preserving bifurcation from S branch	$x=y;\ p=0,\ p_u=0;$ $p_\lambda=0$
$\mathcal{C}_R$	Degenerate symmetry breaking bifurcation from R branch	Not needed explicitly
$\mathcal{C}_S$	Degenerate symmetry breaking bifurcation from S branch	Not needed explicitly
$\mathcal{H}_R$	Hysteresis point on R branch	$y=0;\ p-x^2r=0;$ $p_u-r+x^2(2p_\Delta-r_u)-2x^4r_\Delta=0$ $p_{uu}+2p_\Delta-2r_u+x^2(4p_{u\Delta}-6r_\Delta-r_{uu})$ $+4x^4(p_{\Delta\Delta}-r_{u\Delta})-4r_{\Delta\Delta}x^6=0$
$\mathcal{H}_S$	Hysteresis point on S branch	$x=y,\ p=0;\ p_u=0;$ $p_{uu}=0$
$\mathcal{J}_R$	Coalescence of $\mathcal{F}_T$ with $\mathcal{P}_R$	$y=0,\ p=0;\ r=0;$ $p_ur_\Delta-p_\Delta r_u=0$
$\mathcal{J}_S$	Coalescence of $\mathcal{F}_T$ with $\mathcal{P}_S$	$x=y;\ p=0;\ r=0;$ $p_ur_\Delta-p_\Delta r_u=0$
$\mathcal{Q}_R$	Coalescence of $\mathcal{F}_R$ with $\mathcal{P}_R$	$y=0;\ p=0;\ r=0;$ $p_u+x^2(2p_\Delta-r_u)-2x^4r_\Delta=0$
$\mathcal{Q}_S$	Coalescence of $\mathcal{F}_S$ with $\mathcal{P}_S$	$x=y;\ p=0;\ r=0;$ $p_u=0$
$\mathcal{B}_T$	Bifurcation from T branch	$p=0;\ r=0;$ $\operatorname{rank}\begin{pmatrix} p_u & p_\Delta & p_\lambda \\ r_u & r_\Delta & r_\lambda \end{pmatrix}\leqslant 1$
$\mathcal{H}_T$	Hysteresis point on T branch	Not needed

these are those occurring at the origin and listed as II–V in Table II. Most of the others are either the usual codimension-one bifurcations that can occur ($\mathscr{B}_R, \mathscr{B}_S, \mathscr{H}_R, \mathscr{H}_S, \mathscr{B}_T, \mathscr{H}_T$) or the codimension-one bifurcations with one-dimensional critical eigenspace and $\mathbb{Z}_2$ symmetry ($\mathscr{C}_R, \mathscr{C}_S, \mathscr{I}_R, \mathscr{I}_S$), classified in Golubitsky and Schaeffer [15]. The two exceptions, $\mathscr{Q}_R$ and $\mathscr{Q}_S$, are codimension-one bifurcations with two-dimensional critical eigenspace and nontrivial $\mathbb{Z}_2$ symmetry. The equations for these transition varieties can be deduced from the theory developed in Golubitsky and Schaeffer [15].

Codimension Two Normal Forms

VI. $(\varepsilon_0\lambda + \varepsilon_1 N^3, \varepsilon_2)$

The unfolding is

$$p = \varepsilon_0\lambda + \varepsilon_1 N^3 + \beta N^2 + \alpha N$$

$$r = \varepsilon_2.$$

As in the previous examples we can restrict attention to the case $\varepsilon_0 = -1$, $\varepsilon_1 = 1$. The origin is stable if $\lambda < 0$ and unstable if $\lambda > 0$. There are no solutions of type T. The solutions of type R and S are given in the table.

Type	Equations	Signs of eigenvalues
R	$y = 0$	$(\alpha - \varepsilon_2) + 2\beta x^2 + 3x^4$
	$\varepsilon_0\lambda + (\alpha - \varepsilon_2)x^2 + \beta x^4 + x^6 = 0$	ε_2
S	$x = y$	$\alpha + 4\beta x^2 + 12x^4$
	$\varepsilon_0\lambda + 2\alpha x^2 + 4\beta x^4 + 8x^6 = 0$	$-\varepsilon_2$

It is easily checked that the only generic bifurcations occurring are those at the origin and folds in the S branch. The folds occur when

$$x = y \tag{5.1a}$$

$$\varepsilon_0\lambda + 2\alpha x^2 + 4\beta x^4 + 8x^6 = 0 \tag{5.1b}$$

$$\alpha + 4\beta x^2 + 12x^4 = 0. \tag{5.1c}$$

Bifurcation from 0 occurs when $\lambda = 0$ and so $\mathscr{Q}(\mathscr{S}_I; \mathscr{F}_S)$ can be found by eliminating x from Eqs. (5.1) with $\lambda = 0$. Subtracting $2 \times (5.1c)$ from $3 \times (5.1b)$ gives $(\alpha + \beta x^2)\,x^2 = 0$, so α and β must have opposite signs. Substituting $x^2 = -\alpha/\beta$ into (5.1c) gives $\alpha(\alpha - \beta^2/4) = 0$. As $\alpha = 0$ is easily seen to be the equation of the transition variety $\mathscr{S}_{III}$, the relevant part of this

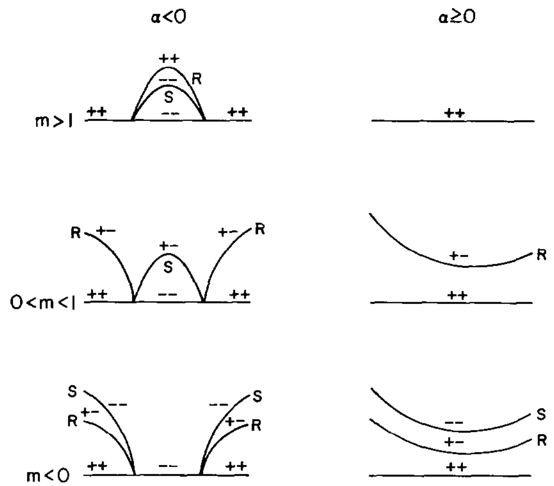

FIG. 5.4. Bifurcation diagrams in the unfolding of V: $\varepsilon_0 = 1$, $\varepsilon_1 = 1$, $\varepsilon_2 = -1$.

equation for $\mathscr{D}(\mathscr{S}_1; \mathscr{F}_S)$ is $\alpha = \beta^2/4$. Thus α is positive and β must be negative and

$$\mathscr{D}(\mathscr{S}_1; \mathscr{F}_S) = \{\alpha = \beta^2/4, \beta < 0\}.$$

The other nontrivial transition varieties are calculated to be

$$\mathscr{S}_{\text{III}} = \{\alpha = 0\},$$

$$\mathscr{H}_S = \{\alpha = \beta^2/3, \beta < 0\}.$$

The bifurcation diagrams are shown in Fig. 5.5.

VII.

The bifurcation diagrams for this normal form are essentially the same as those for VI, but with R and S interchanged.

VIII. $(\varepsilon_0 \lambda + \varepsilon_1 N + m\Delta^2, \varepsilon_2 N)$, $m \neq 0$

The unfolding is

$$p = \varepsilon_0 \lambda + \varepsilon_1 N + m\Delta^2 + \beta\Delta$$

$$r = \varepsilon_2 N + \alpha.$$

We make the choices $\varepsilon_0 = -1$, $\varepsilon_1 = 1$. The origin is stable if $\lambda < 0$ and unstable if $\lambda > 0$. The other solutions are given in the table. Note that the type T solutions can only exist if $\varepsilon_2 \alpha < 0$.

Type	Equations	Stability
R	$y = 0$ $\lambda - (1-\alpha)x^2 - (\beta - \varepsilon_2)x^4 - mx^8 = 0$	Signs of eigenvalues: $(1-\alpha) + (\beta - \varepsilon_2)x^2 + 2mx^6$ $\gamma + \varepsilon_2 x^2$
S	$x = y$ $\lambda - 2x^2 - 4\beta x^4 - 16mx^8 = 0$	Signs of Eigenvalues. 1 $-(\gamma - \varepsilon_2 x^2)$
I	$\lambda - N - \beta A - mA^2 = 0$ $\gamma - \varepsilon_2 N = 0$	Sign trace df: $N + (2\beta - \varepsilon_2)A - 4mA^3$ Sign det df: $\varepsilon_2(\beta + 2mA)$

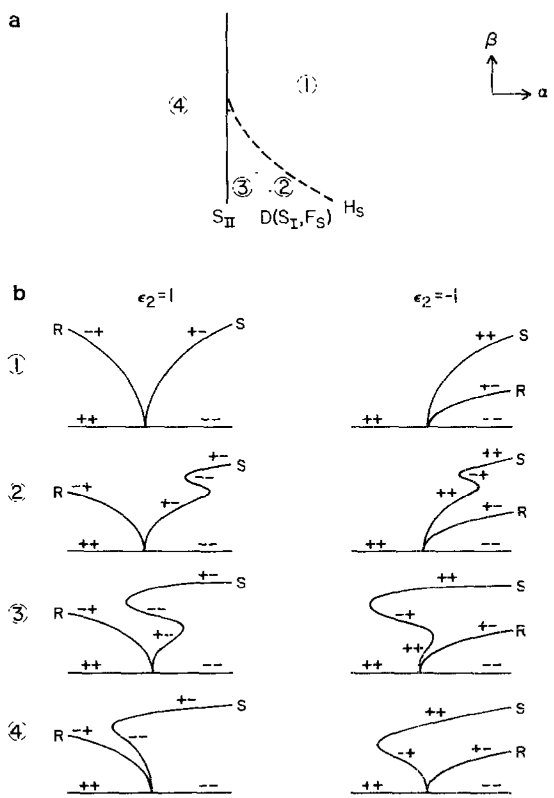

FIG. 5.5. (a) Transition varieties of VI: $\varepsilon_0 = -1$, $\varepsilon_1 = 1$. (b) Bifurcation diagrams in the unfolding of VI: $\varepsilon_0 = -1$, $\varepsilon_1 = 1$

The values of λ at which type T solutions bifurcate from the R and S solutions are:

$$\mathscr{P}_R: \lambda = -[\varepsilon_2 \alpha - \beta\alpha^2 - m\alpha^4]$$

$$\mathscr{P}_S: \lambda = -\varepsilon_2 \alpha.$$

The nontrivial transition varieties are:

$$\mathscr{S}_{IV} = \{\alpha = 0\}$$

$$\mathscr{I}_R = \{\beta + 2m\alpha^2 = 0, \varepsilon_2 \alpha < 0\}$$

$$\mathscr{I}_S = \{\beta = 0, \varepsilon_2 \alpha < 0\}$$

$$\mathscr{D}(\mathscr{P}_R: \mathscr{P}_S) = \{\beta + m\alpha^2 = 0, \varepsilon_2 \alpha < 0\}.$$

In Fig. 5.6 we illustrate only the cases $\varepsilon_2 = 1$ and $m > 0$. The diagrams for $m < 0$ are essentially the same while those for $\varepsilon_2 = -1$ can be obtained by interchanging R and S. The remarks we made in the discussion of IV concerning the stability of the torus branch apply here also.

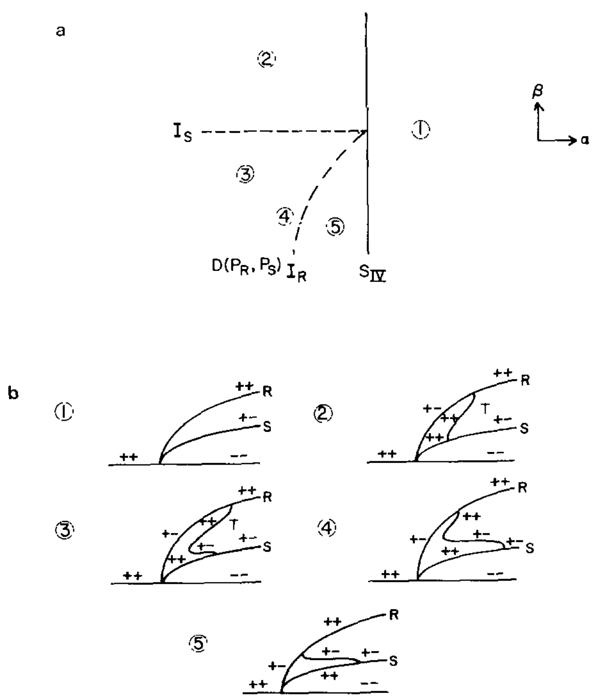

FIG. 5.6. (a) Transition varieties of VIII. $\varepsilon_0 = -1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $m > 0$ (b) Bifurcation diagrams in the unfolding of VIII: $\varepsilon_0 = -1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$, $m > 0$.

IX. $(\varepsilon_0\lambda + \varepsilon_1 N, \varepsilon_2 A + m\lambda^2)$, $m \neq 0$

The unfolding is

$$p = \varepsilon_0\lambda + \varepsilon_1 N$$
$$r = \varepsilon_2 A + m\lambda^2 + \beta N + \alpha.$$

As usual we take $\varepsilon_0 = -1$, $\varepsilon_1 = 1$. The origin is stable if $\lambda < 0$ and unstable if $\lambda > 0$. The other solutions are given by the table:

Type	Equations	Stability
R	$y = 0$ $\lambda - (1-\alpha)x^2 + m\lambda^2 x^2 + \beta x^4 + \varepsilon_2 x^6 = 0$	Signs of eigenvalues: $(1-\alpha)\quad \beta x^2 - 2\varepsilon_2 x^4$ $\alpha - m\lambda^2 - \beta x^2 + \varepsilon_2 x^4$
S	$x = y$ $\lambda - x^2 = 0$	Signs of eigenvalues: ε_1 $(\alpha + m\lambda^2 - 2\beta x^2)$
I	$\lambda - N = 0$ $\alpha + m\lambda^2 + \beta N - \varepsilon_2 A = 0$	Sign trace dI: $N + \beta A - 2\varepsilon_2 NA$ Sign det dI $-\varepsilon_2$

The generic bifurcations in the diagrams are:

$$\mathscr{S}_I: \lambda = 0$$
$$\mathscr{P}_R: \lambda = (m+\varepsilon_2)[-\beta \pm \sqrt{\beta^2 - 4\alpha(m+\varepsilon_2)}]/2$$
$$\mathscr{P}_S: \lambda = m[-\beta \pm \sqrt{\beta^2 - 4\alpha m}]/2.$$

The R and S branches bifurcate supercritically, so only positive values of λ for $\mathscr{P}_R$ and $\mathscr{P}_S$ are relevant. The nontrivial transition varieties are:

$$\mathscr{S}_{IV} = \{\alpha = 0\}$$
$$\mathscr{C}_R = \{\beta^2 - 4\alpha(m+\varepsilon_2) = 0, \text{ sign } \beta = -\text{sign}(m+\varepsilon_2)\}$$
$$\mathscr{C}_S = \{\beta^2 - 4\alpha m = 0, \text{ sign } \beta = -\text{sign } m\}.$$

Note that

$$\mathscr{D}(\mathscr{S}_I; \mathscr{P}_R) = \mathscr{D}(\mathscr{S}_I; \mathscr{P}_S) = \mathscr{D}(\mathscr{P}_R; \mathscr{P}_S) = \mathscr{S}_{IV} = \{\alpha = 0\}.$$

To obtain persistant bifurcation diagrams in the unfolding we need to exclude $m = -\varepsilon_2$ as well as $m = 0$.

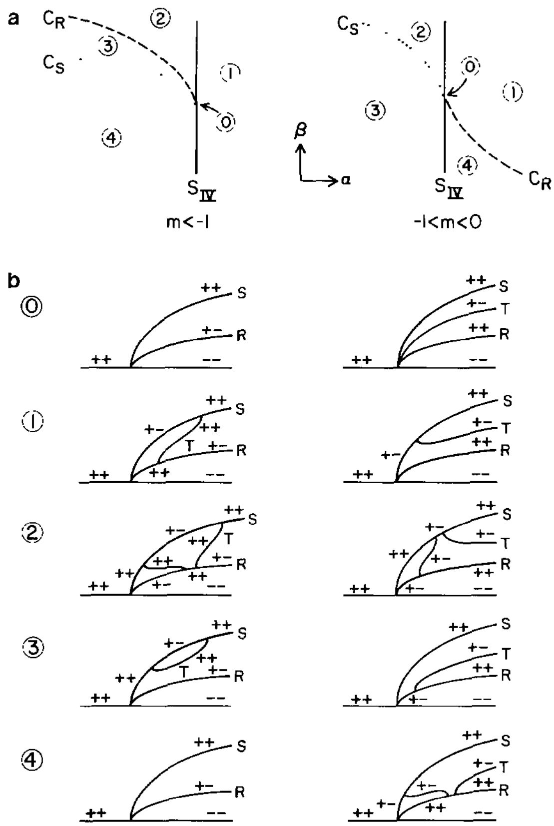

FIG. 5.7 (a) Transition varieties for IX: $\varepsilon_0 = -1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$. (b) Bifurcation diagrams in the unfolding of IX: $\varepsilon_0 = -1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$.

The illustrations in Fig. 5.7 are for $\varepsilon_0 = -1$, $\varepsilon_1 = 1$, $\varepsilon_2 = 1$ and $m < -1$ and $-1 < m < 0$. The diagrams for these choices of ε_i and $m > 0$ are essentially the same as for $m < -1$, though with changed stability assignments. Changing the sign of ε_2 reverses the direction of the T branch. Notice that in this case we have also shown the unperturbed bifurcation diagram at $\alpha = 0 = \beta$.

X. $(\varepsilon_0 \lambda + mN^2 + n\Delta, \varepsilon_1 N + \varepsilon_2 \Delta)$, $m \neq 0$, $n \neq 0$, $m + n \neq \varepsilon_1$, $\varepsilon_1/2$

The unfolding is

$$p = \varepsilon_0 \lambda + mN^2 + n\Delta + \alpha N$$

$$r = \varepsilon_1 N + \varepsilon_2 \Delta + \beta.$$

Choosing $\varepsilon_0 = -1$, $\varepsilon_1 = 1$, the origin is stable if $\lambda < 0$ and unstable if $\lambda > 0$. The other solutions are given in the table:

Type	Equations	Stability
R	$y = 0$ $\lambda - (\alpha - \beta)x^2 - (m+n-1)x^4 + \varepsilon_2 x^6 = 0$	Signs of Eigenvalues: $(\alpha-\beta) + 2(m+n-1) - 3\varepsilon_2 x^4$ $\beta + x^2 - \varepsilon_2 x^4$
S	$x = 1$ $\lambda - 2\alpha x^2 - 4mx^4 = 0$	Signs of eigenvalues: $\alpha + 4mx^2$ $-(\beta + 2x^2)$
I	$\lambda - \alpha N - n\Delta - mN^2 = 0$ $\beta - N + \varepsilon_2 \Delta = 0$	Sign trace df: $\alpha N + (2n-1)\Delta + 2mN^2 - 2\varepsilon_2 N\Delta$ Sign det df: $n - \alpha\varepsilon_2 - 2m\varepsilon_2 N$

The equations for the λ values of the generic bifurcations in the bifurcation diagrams and those for the transition varieties are given in Tables V and VI, respectively. Because of the two moduli, this example is considerably more complicated than the others. The nondegeneracy conditions that appear in the classification ($m \neq 0$, $n \neq 0$, $m+n \neq 1$, $m+n \neq \frac{1}{2}$) divide the moduli space into ten regions and so we have ten normal forms to consider (not counting the different cases $\varepsilon_2 = \pm 1$). However, the situation is even worse than this as the moduli space will need further subdivision to distinguish between germs whose versal unfoldings are not qualitatively the same. We therefore content ourselves with a discussion of just one case, which, nevertheless, we believe encompasses all the important phenomena associated with the normal form. The case we illustrate is given by $m+n-1 > 0$ and $m < 0$. The choice of $\varepsilon_2 = \pm 1$ does not affect the bifur-

TABLE V

Generic Bifurcations in the Unfoldings of X

$\mathscr{S}_I$	$\lambda = 0$
$\mathscr{F}_R$	$\lambda = -(\alpha - \beta)^2/4(m+n-1) + \cdots$
$\mathscr{F}_S$	$\lambda = -\alpha^2/4m$
$\mathscr{P}_R$	$\lambda = (m+n)\beta^2 - \alpha\beta + \cdots$
$\mathscr{P}_S$	$\lambda = m\beta^2 - \alpha\beta$

Note. Here "$+ \cdots$" denotes higher order terms.

TABLE VI

Transition Varieties for X

$\mathcal{S}_{II}$	$\alpha = 0$
$\mathcal{S}_{III}$	$\alpha = \beta$
$\mathcal{S}_{IV}$	$\beta = 0$
$\mathcal{Q}_R$	$\alpha = 2(m+n-1/2)\beta$
$\mathcal{Q}_S$	$\alpha = 2m\beta$
$\mathcal{D}(\mathcal{S}_I; \mathcal{P}_R)$	$\alpha = (m+n)\beta$
$\mathcal{D}(\mathcal{S}_I; \mathcal{P}_S)$	$\alpha = m\beta$
$\mathcal{D}(\mathcal{F}_R; \mathcal{F}_S)$	$\alpha^2/4m = (m+n-1)x^2 - 2\varepsilon_2 x^6$, $\beta - \alpha - 2(m+n-1)x^2 \quad 3\varepsilon_2 x^4$
$\mathcal{D}(\mathcal{F}_R; \mathcal{P}_S)$	$\alpha = 2[(m+n-1/2) \pm \sqrt{n(m+n-1)}]\beta$
$\mathcal{D}(\mathcal{F}_S; \mathcal{P}_R)$	$\alpha = 2[m \pm \sqrt{-mn}]\beta$

cation diagrams. The transition varieties are shown in Fig. 5.8 and the bifurcation diagrams themselves in Fig. 5.9. Note that $\mathcal{D}(\mathcal{F}_R; \mathcal{F}_S) = \varnothing$ in this case.

Notice that in the bifurcation diagrams numbered (3)–(7) the exchange of stabilities rule shows that the T branch must be stable when it bifurcates from the R branch, but unstable when it bifurcates from the S branch. This has important consequences which are described in the next section.

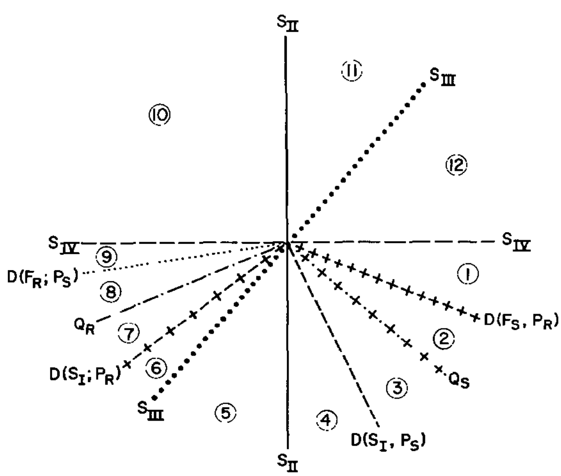

FIG. 5.8. Transition varieties for X: $\varepsilon_0 = 1$, $\varepsilon_1 = 1$, $m+n-1 > 0$, $m < 0$.

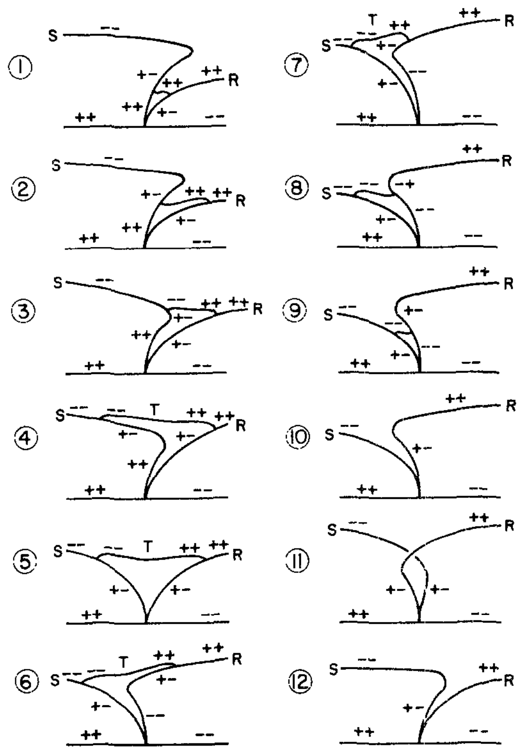

FIG. 5.9 Bifurcattion diagrams in the unfolding of X: $\varepsilon_0 = -1$, $\varepsilon_1 = 1$, $m + n - 1 > 0$, $m < 0$.

XI. $(\varepsilon_0 \lambda^2 + \varepsilon_1 N^2 + m\lambda N, \varepsilon_2)$, $m^2 \neq 4\varepsilon_0\varepsilon_1$

The unfolding is

$$p = \varepsilon_0 \lambda^2 + \varepsilon_1 N^2 + m\lambda N + \alpha + \beta N$$

$$r = \varepsilon_2.$$

Choose $\varepsilon_0 = 1$. The R and S branches are given in the table:

Type	Equations	Signs of eigenvalues
R	$y = 0$	$-\varepsilon_1 + \beta + m\lambda + 2\varepsilon_1 x^2$
	$\lambda^2 + \alpha + (\beta + m\lambda - 1)x^2 + \varepsilon_1 x^4 = 0$	ε_1
S	$x = y$	$\beta + m\lambda + 3\varepsilon_1 x^4$
	$\lambda^2 + \alpha + 2(\beta + m\lambda)x^2 + 4\varepsilon_1 x^4 = 0$	$-\varepsilon_1$

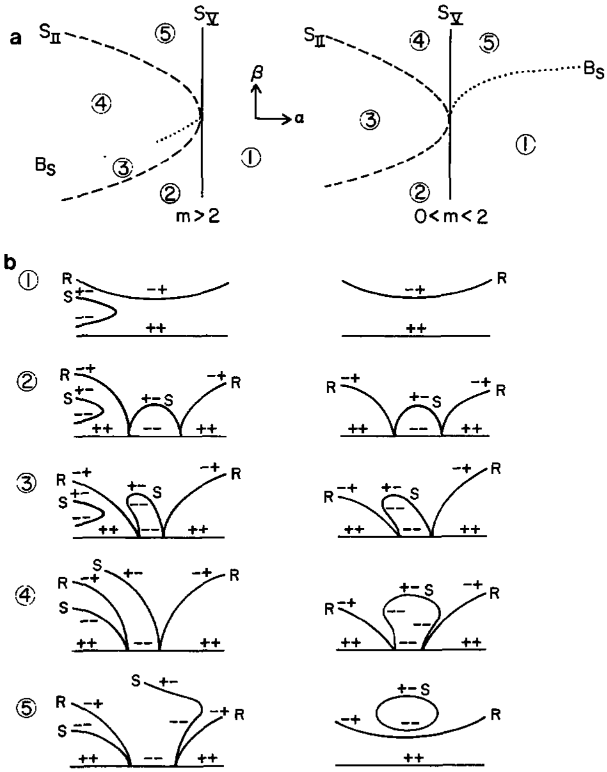

FIG. 5.10. (a) Transition varieties for XI: $\varepsilon_0 = 1$, $\varepsilon_1 = 1$. (b) Bifurcation diagrams in the unfolding of XI: $\varepsilon_0 = 1$, $\varepsilon_1 = 1$.

The behavior of the R branch is essentially as in V, while that of the S branch is given by the $\mathbb{Z}_2$ normal form (8) on page 263 of Golubitsky and Schaeffer [15], to which we refer the reader for a full discussion. In Fig. 5.10 we illustrate the cases $\varepsilon_1 = 1$, $m > 2$ and $\varepsilon_1 = 1$, $0 < m < 2$. Changing the sign of m is equivalent to reversing λ while $\varepsilon_1 = -1$ gives a rather different set of diagrams. Note that $m = 0$ has to be excluded if all persistant bifurcation diagrams are to be obtained by varying α and β.

XII

This is similar to XI with R and S interchanged.

XIII. $(\varepsilon_0\lambda^2 + \varepsilon_1 N, \varepsilon_2\lambda + m\Delta)$, $m \neq 0$

The unfolding is

$$p = \varepsilon_0\lambda^2 + \varepsilon_1 + \alpha$$

$$r = \varepsilon_2\lambda + m\Delta + \beta.$$

We fix $\varepsilon_0 = 1$, $\varepsilon_2 = 1$:

Type	Equations	Stability
R	$y = 0$ $\lambda^2 + \alpha + (\varepsilon_1 \beta - \lambda)x^2 - mx^6 = 0$	Signs of eigenvalues: $\varepsilon_1 \beta - \lambda - 3mx^4$ $\lambda + \beta - mx^4$
S	$x = y$ $\lambda^2 + \alpha + 2\varepsilon_1 x^2 = 0$	Signs of eigenvalues. ε_1 $-(\lambda - \beta)$
T	$\lambda^2 + \alpha + \varepsilon_1 N = 0$ $\lambda + \beta + m\Delta = 0$	Sign trace df: $\varepsilon_1 - 2m\Delta$ Sign det df. $-\varepsilon_1 m$

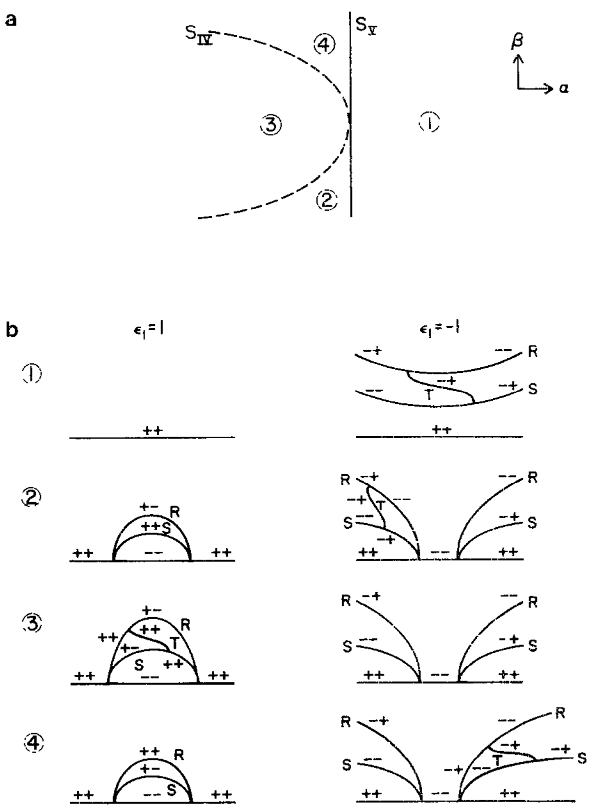

FIG. 5.11. (a) Transition varieties of XIII: $\varepsilon_0 = 1$, $\varepsilon_2 = 1$. (b) Bifurcation diagrams in the unfolding of XIII: $\varepsilon_0 = 1$, $\varepsilon_2 = 1$.

The case $m < 0$ is illustrated in Fig. 5.11. Changing the sign of m reverses the direction of the T branch.

XIV. $(\varepsilon_0 \lambda^3 + mN + \varepsilon_1 \lambda N, \varepsilon_2)$, $m \neq 0$, ε_2

The unfolding is

$$p = \varepsilon_0 \lambda^3 + mN + \varepsilon_1 \lambda N + \alpha + \beta\lambda$$

$$r = \varepsilon_2.$$

Fix $\varepsilon_0 = 1$, $\varepsilon_2 = 1$:

Type	Equations	Signs of Eigenvalues
R	$y = 0$ $\lambda^3 + (m-1)x^2 + \varepsilon_1 \lambda x^2 + \alpha + \beta\lambda = 0$	$m - \varepsilon_2 + \varepsilon_1 \lambda$ ε_2
S	$x = y$ $\lambda^3 + 2mx^2 + 2\varepsilon_1 \lambda x^2 + \alpha + \beta\lambda = 0$	$m - \varepsilon_1 \lambda$ $-\varepsilon_2$

The bifurcation diagrams are illustrated in Fig. 5.12. The choice of ε_1 makes little difference to these diagrams, it simply changes the relative amplitudes of the various R and S branches. A similar phenomenon occurs in XV.

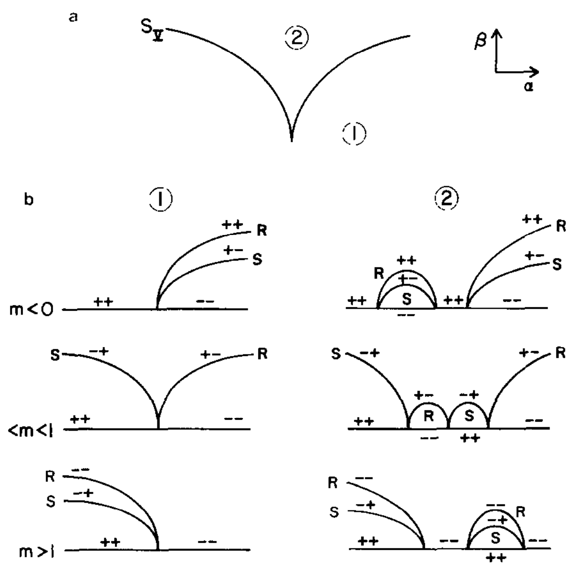

FIG. 5.12. (a) Transition variety of XIV: $\varepsilon_0 = 1$, $\varepsilon_2 = 1$. (b) Bifurcation diagrams in the unfolding of XIV: $\varepsilon_0 = 1$, $\varepsilon_2 = 1$.

XV. $(\varepsilon_0\lambda^2+mN, \varepsilon_1)$, $m\neq 0$, ε_1

The bifurcation diagrams here are essentially the same as those of the codimension-one V. The extra degeneracy that makes it codimension two is simply symmetry with respect to $\lambda\to-\lambda$.

6. Existence of Invariant 3-Tori

In the bifurcation diagrams (3) (7) of normal form X (with $\varepsilon_0=-1$, $\varepsilon_1=1$, $m+n-1>0$, $m<0$) the exchange of stabilities rule shows that the T branch must have eigenvalues with positive real parts when it bifurcates from the R branch, but negative real parts when it bifurcates from the S branch. These eigenvalues vary continuously along the branch and so at some point must pass through the imaginary axis. In the region of moduli space we are considering $n>0$ and so det $df>0$ on the T branch (using the formula given in the discussion of the normal form and noting that we are considering α and N small compared with m and n). Thus the eigenvalues of the T branch cannot pass through 0. Hence they must cross the imaginary axis at a pair of nonzero conjugate points, giving a Hopf bifurcation, i.e., a bifurcation of a periodic solution of the amplitude equations from the T branch of equilibrium solutions.

Lifting this bifurcation back to the full equations (1.3) we obtain the existence of a bifurcation of an invariant 3-torus from a 2-torus with linear flow. This is the 3-torus found by Knobloch [20]. The original periodic solution of the amplitude equations is, of course topologically conjugate to rotation of a circle; this conjugacy can be lifted back to the $SO(2)\times S^1$ equivariant flow on the 3-torus, showing that it must also be conjugate to a linear flow.

We now claim that the existence of this bifurcation is preserved under D_4-equivalence. If g is any bifurcation problem that is D_4 equivalent to the normal form X, with $\varepsilon_0=-1$, $\varepsilon_1=1$, $m+n-1>0$, and $m<0$, then the perturbed bifurcation diagrams of g are the same as those of the normal form. This equality extends to the stabilities of the R and S branches (by Proposition 3.2) and hence those of the T branch near its bifurcation points. In the unfoldings we also still have det $dg>0$, by essentially the same argument as that used in Proposition 3.2. Thus the Hopf bifurcation from the T branch must continue to occur.

Any periodic solution of the amplitude equation created by a Hopf bifurcation from the T branch can only exist for the bounded range of λ values for which the T branch itself exists. Of course there may be more than one Hopf bifurcation from the T branch, but since there is a net change in stability during its existence there must also be a net production of periodic orbits. Thus there must be some other means by which such a periodic orbit is destroyed. The only possibility for a planar system is some form of infinite period bifurcation involving the collision of the periodic orbit with one or more separatrices of the amplitude equations. Note that the existence of this infinite period bifurcation is again preserved under D_4 equivalence. A further study of the normal form X would reveal more details of its dynamics, but it seems probable that most of these will not be invariant under D_4 equivalence.

7. Singularity Theory

The Recognition, Classification of Unfolding Theorems are proved using "singularity theory" techniques, as adapted to bifurcation theory. In this section we briefly review these, referring to Golubitsky and Schaeffer [15], Golubitsky, Stewart, and Schaeffer [16], and Gaffney [12] for proofs and further details. The discussion is given for bifurcation problems which are equivariant with respect to any absolutely irreducible representation of a compact group Γ on $\mathbb{R}^n$. The definition of D_4 equivalence given in Section 3 extends easily to the general case and we use $\sim_\Gamma$ to denote "is Γ-equivalent to."

The Recognition Problem

The recognition problem is concerned with determining when a bifurcation problem is equivalent to a given one. We are first of all interested in knowing when a germ is equivalent to an initial segment of its own Taylor series and so in criteria for deciding whether $f+p$ is equivalent to f for germs f and p in $\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$, the $\mathscr{E}_{\mu,\lambda}(\Gamma)$ module of all D_4-equivariant bifurcation problems.

Definition 7.1. The set of higher order terms of a germ $f\in\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$, denoted $\mathscr{P}(f)$, is defined by

$$\mathscr{P}(f)=\{p\in\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)\colon g+p\sim_\Gamma f\ \forall g\sim_\Gamma f\}.$$

In [16] it is shown that $\mathscr{P}(f)$ is a submodule of $\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$ which depends only on the Γ-equivalence class of f and has the closely related property of being "intrinsic."

Definition 7.2. A submodule $M\subset\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$ is said to be *intrinsic* if for every g and h in $\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$:

$$g\in M \qquad \text{and} \qquad h\sim_\Gamma g\Rightarrow h\in M.$$

For any linear subspace L of $\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$ we define the intrinsic part of L, denoted Itr L, to be the maximal intrinsic submodule contained in L. The usefulness of $\mathscr{P}(f)$ is greatly enhanced by a result of Gaffney [12] which enables us to calculate it. Let $\{\varphi_j(u)\}_{j=1}^{k}$ denote a minimal set of homogeneous generators of $\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$ as an $\mathscr{E}_{u,\lambda}(\Gamma)$ module, with $\varphi_1(u)$ the identity map and degree $\varphi_j \geqslant 2$ for $j=2,..., k$. Similarly let $\{S_1(u),..., S_l(u)\}$ denote a minimal set of homogeneous generators of the $\mathscr{E}_{u,\lambda}(\Gamma)$ module of equivariant matrices, with S_1 the constant identity matrix and degree $S_j \geqslant 1$ for $j=2,..., l$. Both sets of generators can always be chosen to depend on u only. Let $m_{u,\lambda}(\Gamma)$ denote the maximal ideal in $\mathscr{E}_{u,\lambda}(\Gamma)$. For $f \in \vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$ define $\mathscr{K}_1(f)$ to be the $\mathscr{E}_{u,\lambda}(\Gamma)$ module generated by

$$\{m_{u,\lambda}(\Gamma)\cdot df(u,\lambda)\,\varphi_1(u),\ df(u,\lambda)\,\varphi_2(u),..., df(u,\lambda)\,\varphi_k(u),$$
$$m_{u,\lambda}(\Gamma)\cdot S_1(u)\,f(u,\lambda),\ S_2(u)f(u,\lambda),..., S_l(u,\lambda)\,f(u,\lambda)\},$$

and $\mathscr{K}_2(f)$ to be $\mathscr{E}_\lambda$ module generated by $\lambda^2 f_\lambda(u,\lambda)$, where $f_\lambda(u,\lambda)$ denotes the derivative of f with respect to λ.

THEOREM 7.3 [12]. $\mathscr{P}(f) = \mathrm{Itr}(\mathscr{K}_1(f) + \mathscr{K}_2(f))$.

For our representation of $\Gamma = D_4$ on $\mathbb{R}^2$ this description is made more explicit by the calculations in the next section.

The proofs of the recognition and classification theorems given in the last section use this result to calculate the higher order terms that can be discarded in a Taylor series and then uses explicit changes of coordinates to bring the low order terms into the required normal form. General formulas for these coordinate changes are given in the next section.

Unfolding Theory

A k-parameter unfolding of $f \in \vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$ is a germ $F \in \vec{\mathscr{E}}_{\mu,\lambda,\alpha}(\Gamma)$, where $\alpha = (\alpha_1,..., \alpha_k) \in \mathbb{R}^k$ and Γ acts trivially on $\mathbb{R}^k$, with $F(u,\lambda,0) = f(u,\lambda)$. If $F(u,\lambda,\alpha)$ and $G(u,\lambda,\beta)$ are two unfoldings of f we say G factors through F is there exist smooth mappings S, X, Λ, and A such that

$$G(u,\lambda,\beta) = S(u,\lambda,\beta)\,F(X(u,\lambda,\beta), \Lambda(\lambda,\beta), A(\beta)),$$

and for $\beta = 0$ we have $S(u,\lambda,0) = I$, $X(u,\lambda,0) = u$, $\Lambda(\lambda,0) = \lambda$, and $A(0) = 0$. An unfolding F of f is *universal* if every other unfolding of f factors through F. We will also require that a universal unfolding has the minimum number of parameters among unfoldings with this property. Universal unfoldings are unique up to equivalence.

The universal unfolding if f is calculated (if it exists) by means of the "tangent space" of f, denoted $T(f)$.

DEFINITION 7.4. The *tangent space*, $T(f)$, of a germ $f \in \vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$ is the subspace of $\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$ obtained by taking the vector space sum of the $\mathscr{E}_{u,\lambda}(\Gamma)$ submodule generated by

$$\{df(u, \lambda)\, \varphi_1(u), ..., df(u, \lambda)\, \varphi_k(u), S_1(u)\, f(u, \lambda), ..., S_l(u)\, f(u, \lambda)\}$$

and the $\mathscr{E}_\lambda$ submodule generated by $f_\lambda(u, \lambda)$.

Notice that $T(f)$ contains $\mathscr{P}(f)$.

The fundamental theorem of unfolding theory is

THEOREM 7.5 [16]. *Let F be a k-parameter unfolding of $f \in \vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$. Then F is a universal unfolding of f if and only if*

$$\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma) = T(f) + \mathbb{R} \cdot \{\partial F/\partial\alpha_1(u, \lambda, 0), ..., \partial F/\partial\alpha_k(u, \lambda, 0)\}.$$

It clearly follows from this that a bifurcation problem has a universal unfolding if and only if the dimension of $\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)/T(f)$ as a real vector space is finite, and that the number of parameters in the universal unfolding is equal to this number. This is the "C^∞-codimension" of the bifurcation problem and is finite if and only if the dimension of $\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)/\mathscr{P}(f)$ is finite, which in turn is equivalent to the dimension of $\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)/\mathrm{Itr}\, \mathscr{P}(f)$ being finite.

Another easy corollary of the theorem gives a recipe for constructing the universal unfolding of a germ with finite C^∞-codimension.

COROLLARY 7.6 [16]. *Let $f \in \vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$ have C^∞-codimension k and suppose $\{p_1, ..., p_k\} \subset \vec{\mathscr{E}}_{\mu,\lambda}(\Gamma)$ is a set of germs such that*

$$\vec{\mathscr{E}}_{\mu,\lambda}(\Gamma) = T(f) \oplus \mathbb{R} \cdot \{p_1, ..., p_k\}.$$

Then

$$F(u, \lambda, \alpha) = f(u, \lambda) + \sum_{i=1}^{k} \alpha_i p_i(u, \lambda)$$

is a universal unfolding of f. ∎

This reduces the calculation of the universal unfolding of a bifurcation problem to the calculation of $T(f)$. As we show in the last section, this is conveniently done alongside the calculation of $\mathscr{P}(f)$.

8. PRELIMINARY CALCULATIONS

This section consists of calculations of:

(a) a generating set for the module of D_4-equivariant matrices,

(b) generators of $\mathscr{P}(f)$ and $T(f)$ when $f \in \vec{\mathscr{E}}_{\mu,\lambda}(D_4)$.

(c) explicit formulae for the effect of changes of coordinates on low order terms in f, and

(d) intrinsic submodules of $\vec{\mathscr{E}}_{\mu,\lambda}(D_4)$.

(a) *D_4-Equivariant Matrices*

PROPOSITION 8.1. *The module of D_4-equivariant matrices is generated over $\mathscr{E}_{\mu,\lambda}(D_4)$ by*

$$S_1=\begin{pmatrix}1&0\\0&1\end{pmatrix}\quad S_2=\begin{pmatrix}x^2&xy\\xy&y^2\end{pmatrix}\quad S_3=\begin{pmatrix}-x^2&xy\\xy&-y^2\end{pmatrix}\quad S_4=4\begin{pmatrix}0&x^3y\\xy^3&0\end{pmatrix}.$$

This is proved, in a different coordinate system, by Buzano *et al.* [2].

(b) *Generators of $P(f)$ and $T(f)$*

Recall that any $f\in\vec{\mathscr{E}}_{\mu,\lambda}(D_4)$ has the form

$$f(x,y,\lambda)=\begin{pmatrix}f_1(x,y,\lambda)\\f_2(x,y,\lambda)\end{pmatrix}=p(N,\Delta,\lambda)\begin{pmatrix}x\\y\end{pmatrix}+r(N,\Delta,\lambda)\,\delta\begin{pmatrix}x\\-y\end{pmatrix},$$

where $p, r\in\mathscr{E}_{\mu,\lambda}(D_4)$.

As before we identify $\vec{\mathscr{E}}_{\mu,\lambda}(D_4)$ with $\mathscr{E}_{\mu,\lambda}(D_4)\oplus\mathscr{E}_{\mu,\lambda}(D_4)$ and write f as (p,r). Then easy calculations give

$$\begin{aligned}S_1\cdot f&=(p,r)\\S_2\cdot f&=(Np-\Delta r,0)\\S_3\cdot f&=(0,p-Nr)\\S_4\cdot f&=((N^2-\Delta)p,-(N^2-\Delta)r).\end{aligned}\tag{8.1}$$

We also need $df\cdot\varphi_1$ and $df\cdot\varphi_2$, where $\varphi_1=\binom{x}{y}$ and $\varphi_2=\delta\binom{x}{-y}$ are the generators of $\vec{\mathscr{E}}_{\mu,\lambda}(D_4)$. A straightforward calculation using (4.3) gives

$$\begin{aligned}df\cdot\varphi_1&=(p+2Np_N+4\Delta p_\Delta,\ 3r+2Nr_N+4\Delta r_\Delta)\\df\cdot\varphi_2&=(-2\Delta p_N-4N\Delta p_\Delta+\Delta r,\ p-2Nr-2\Delta r_N-4N\Delta r_\Delta).\end{aligned}\tag{8.2}$$

The final ingredient for both $\mathscr{P}(f)$ and $T(f)$ is simply

$$f_\lambda=(p_\lambda,r_\lambda).\tag{8.3}$$

Using the expressions of (8.3), (8.4), and (8.5) in the definitions preceding Theorem 7.3 and Definition 7.4 gives explicit formulae for the generators of $\mathscr{K}_1(f)+\mathscr{K}_2(f)$ and $T(f)$.

(c) *Low Order Terms*

From (3.1) a general D_4-equivalence consists of:

(i) a mapping $Z(u, \lambda)$: $\mathbb{R}^2 \times \mathbb{R} \to \mathbb{R}^2$ which is equivariant with respect to the D_4 action on $\mathbb{R}^2$,

(ii) a D_4-equivariant matrix $S(u, \lambda)$, and

(iii) a mapping $\Lambda(\lambda)$: $\mathbb{R} \to \mathbb{R}$.

These satisfy $dZ(0, 0) = aI$ and $S(0, 0) = AI$, where a and A are strictly positive real numbers and $\Lambda'(0) > 0$. The equivariance of Z and S imply

$$\begin{aligned} Z &= a\begin{pmatrix} x \\ y \end{pmatrix} + b\delta\begin{pmatrix} x \\ -y \end{pmatrix} = (a, b) \\ S &= AS_1 + BS_2 + CS_3 + DS_4, \end{aligned} \tag{8.4}$$

where $a, b, A, B, C, D \in \mathscr{E}_{u,\lambda}(D_4)$.

An easy calculation shows that composing N, δ, and Δ with Z gives

$$\begin{aligned} \tilde{N} &= N \circ Z = a^2N - 2ab\Delta + b^2N\Delta \\ \tilde{\delta} &= \delta \circ Z = (a^2 - 2abN + b^2\Delta)\delta \\ \tilde{\Delta} &= \Delta \circ Z = (a^2 - 2abN + b^2\Delta)^2\Delta. \end{aligned} \tag{8.5}$$

A further calculation shows that the result of applying the coordinate changes Z and Λ to $f = (p, r)$ is

$$(a\tilde{p} + b(a^2 - 2abN + b^2\Delta)\,\Delta\tilde{r},\ b\tilde{p} + a(a^2 - 2abN + b^2\Delta)\tilde{r}), \tag{8.6}$$

where $\tilde{p} = p(\tilde{N}, \tilde{\Delta}, \Lambda)$ and $r = r(\tilde{N}, \tilde{\Delta}, \Lambda)$.

Finally, applying S to (8.6), using (8.1), gives $(\hat{p}, \hat{r})$, where

$$\begin{aligned} \hat{p} &= \{Aa + BaN - (Bb + Da)\Delta + DaN^2\}\,\tilde{p} \\ &\quad + \{[Ab - Ba + BbN + Db(N^2 - \Delta)][a^2 - 2abN + b^2\Delta]\}\,\Delta\tilde{r} \\ \hat{r} &= \{Ab + Ca - CbN - Db(N^2 - \Delta)\}\,\tilde{p} \\ &\quad + \{[a - CaN + (Da + Cb)\Delta - DaN^2][a^2 - 2abN + b^2\Delta]\}\,\tilde{r}. \end{aligned} \tag{8.7}$$

That is, any germ D_4-equivalent to (p, r) can be written as $(\hat{p}, \hat{r})$ for some $a, b, A, B, C, D, \Lambda$.

By taking the Taylor series expansions of $\hat{p}$ and $\hat{r}$ we can extract from (8.7) the coefficients of low order terms of all bifurcation problems D_4-equivalent to (p, r). Those we need for the examples in the next section are given in Table VII. The expressions p_λ, $r_{N\Delta}$, $(Aa)_N$, etc., are partial

TABLE VII

Low Order Terms of Bifurcation Problems D_4-Equivalent to (p, r)

$(\lambda, 0)$	$Aa\Lambda_\lambda p_\lambda$
$(N, 0)$	Aa^3p_N
$(\Delta, 0)$	$2Aa^2bp_N - Aa^5p_\Delta + (Aa^2b - Ba^3)r$
$(\lambda^2, 0)$	$(Aa)_\lambda \Lambda_\lambda p_\lambda + Aa(\Lambda_\lambda)^2p_{\lambda\lambda}/2$
$(\lambda N, 0)$	$((Aa)_N + Ba)\Lambda_\lambda p_\lambda + (Aa)_\lambda a^2p_N + Aa^3\Lambda_\lambda p_{\lambda N}$
$(N^2, 0)$	$(Ba^3 + (Aa)_N a^2)p_N + Aa^5p_{NN}/2$
$(\lambda\Delta, 0)$	$((Aa)_\Delta - (Bb + Da))\Lambda_\lambda p_\lambda - 2(Aa)_\lambda abp_N + (Aa)_\lambda a^4p_\Delta - 2Aa^2b\Lambda_\lambda p_{\lambda N}$ $+ Aa^5\Lambda_\lambda p_{\lambda\Delta} + (Aa^2b - Ba^3)_\lambda r + (Aa^2b - Ba^3)\Lambda_\lambda r_\lambda$
$(N\Delta, 0)$	$(Aab^2 - 3Ba^2b - Da^3 - 2(Aa)_N ab + (Aa)_\Delta a^2)p_N$ $+ (-4Aa^4b + Ba^5 + (Aa)_N a^4)p_\Delta + Aa^7p_{N\Delta}$ $-(-2Aab^2 - 3Ba^2b + (Aa^2)_N)r + (Aa^4b - Ba^5)r_N$
$(0, 1)$	Aa^3r
$(0, \lambda)$	$(Ab + Ca)\Lambda_\lambda p_\lambda + (Aa^3)_\lambda r + Aa^3\Lambda_\lambda r_\lambda$
$(0, N)$	$(Aa^2b + Ca^3)p_N - (2Aa^2b - Ca^3)r + Aa^5r_N$
$(0, \Delta)$	$-2(Aab^2 + Ca^2b)p_N + (Aa^4b + Ca^5)p_\Delta$ $+ (Aab^2 + Ca^2b + Da^3)r - 2Aa^4br_N + Aa^7r_\Delta$

derivatives with respect to the subscripts. All terms are evaluated at 0 and we have assumed throughout that $p(0) = 0$.

(d) *Intrinsic Submodules of* $\vec{\mathscr{E}}_{u,\lambda}(D_4)$

Recall that an intrinsic submodule of $\vec{\mathscr{E}}_{u,\lambda}(D_4)$ is a submodule that is invariant under the action of the group of D_4-equivalences. An ideal in $\mathscr{E}_{u,\lambda}(D_4)$ is also said to be *intrinsic* if it is invariant under the group of coordinate changes

$$(u, \lambda) \to (Z(u, \lambda), \Lambda(\lambda)),$$

where Z and Λ satisfy the conditions in (3.1).

We write submodules of $\vec{\mathscr{E}}_{u,\lambda}(D_4) \cong \mathscr{E}_{u,\lambda}(D_4) \oplus \mathscr{E}_{u,\lambda}(D_4)$ as $I \oplus J$, where I and J are ideals in $\mathscr{E}_{u,\lambda}(D_4)$. The following result is proved using the formula in (8.7) (for (i)) and (8.9) (for (ii)).

PROPOSITION 8.2. (i) *If I is an ideal in $\mathscr{E}_{u,\lambda}(D_4)$ which is a sum of products of the ideals $\langle\lambda\rangle$, $\langle\Lambda\rangle$ and $\langle N, \Delta\rangle$ then I is intrinsic.*

(ii) *A submodule (I, J) of $\vec{\mathscr{E}}_{u,\lambda}(D_4)$ is intrinsic if and only if both I and J are intrinsic ideals in $\mathscr{E}_{u,\lambda}(D_4)$, $I \subset J$, and $\langle\Lambda\rangle J \subset I$.*

Remark. It follows from (ii) that $I \oplus I$ and $\langle \Delta \rangle I \oplus I$ are intrinsic submodules whenever I is an intrinsic ideal.

9. Proof of the Theorems

Most of this section is devoted to outlining the calculations necessary to verify the Recognition Theorem. However, we begin by discussing the Classification Theorem and the Unfolding Theorem. By the general results in Section 7, the latter is reduced to the calculation of $T(f)$ for each normal form f. This is a straightforward exercise and is conveniently carried out alongside the calculations that are necessary for the Recognition Theorem. Examples are given below. For the Classification Theorem we refer to the accompanying flow chart (Table VIII), which describes a partition of the space of k-jets (at 0) of germs of D_4-equivariant maps with $p(0)=0$, denoted J^k, into (semi-algebraic) subvarieties. Specifically we associate to every terminal point of the flow chart the subvariety defined by the set of conditions on the partial derivatives which distinguish that terminal point. It is easily seen that those subvarieties associated with the terminal points labelled by normal forms have codimension in J^k (for k sufficiently large) equal to the number given under "C^0 codimension" in Table II, while all the remaining varieties have codimension greater than or equal to three. Denote the union of the varieties of codimension $\geqslant 3$ by Σ^k. A standard transversality argument implies that the image of the jet extension of a generic two-parameter family of D_4 equivariant germs will not intersect Σ^k and the Classification Theorem therefore follows from the Recognition Theorem.

We now turn to the proof of the Recognition Theorem. The theory and formulae given in the previous two sections have reduced this to routine, though extensive, calculations. The details of these are left to the diligent reader. Here we shall give a procedure that may be followed and illustrate it by a number of examples.

For each normal form f there are two calculations that have to be made:

(1) Check that $\mathscr{P}(f)$ contains the submodule, M, of $\vec{\mathscr{E}}_{\mu,\lambda}(D_4)$ listed in the third column of Table IX.

(2) Check that any germ g satisfying the defining and nondegeneracy conditions for f, given in Table II, is D_4-equivalent to f *modulo* M.

It then follows from the general theory that g is D_4-equivalent to f "to all orders."

Because of the algebraic difficulties with working directly with $\mathscr{K}_1(f)+\mathscr{K}_2(f)$ it is best, in (1), to begin by showing that $\mathscr{K}_1(f)$ (an $\mathscr{E}_{u,\lambda}(D_4)$ submodule of $\vec{\mathscr{E}}_{u,\lambda}(D_4)$) contains the submodule listed in column 2

TABLE VIII

Flow Chart of the Classification

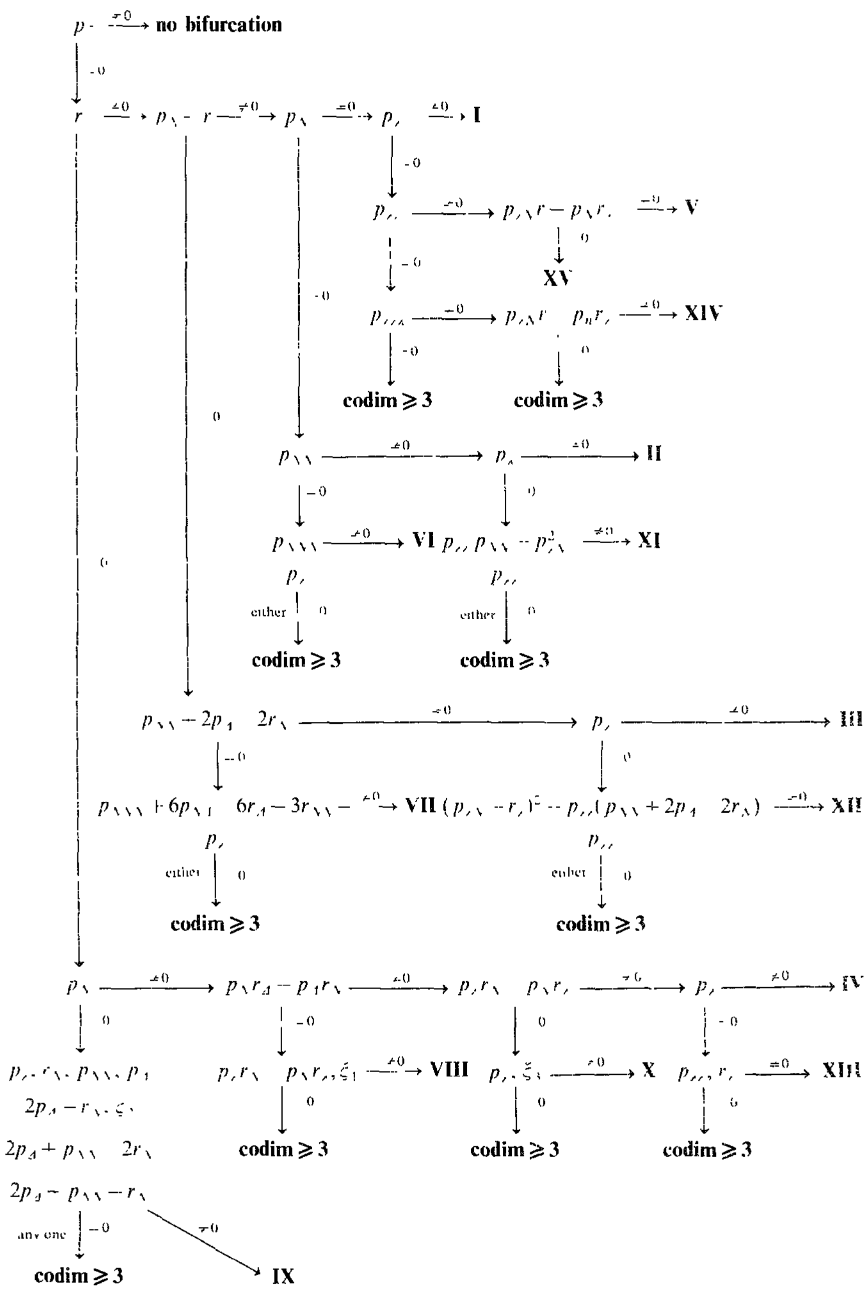

TABLE IX

Algebraic Data for Normal Forms

	$\operatorname{Itr}\cdot\mathscr{K}_1(f)$ contains	$\mathscr{P}(f)$ contains
I	$(\mathcal{M}^2+\langle\Delta\rangle, \mathcal{M})$	As 2nd column
II	$(\mathcal{M}^3+\mathcal{M}\langle\lambda\rangle+\langle\Delta\rangle, \mathcal{M})$	As 2nd column
III	$(\mathcal{M}^3+\mathcal{M}\langle\lambda, \Delta\rangle, \mathcal{M}^2+\langle\lambda, \Delta\rangle)$	As 2nd column
IV	$(\mathcal{M}^2, \mathcal{M}^2)$	As 2nd column
V	$(\mathcal{M}^3+\langle N, \Delta\rangle^2+\langle\Delta\rangle, \mathcal{M}^2+\langle N, \Delta\rangle)$	As 2nd column
VI	$(\mathcal{M}^4+\mathcal{M}(\lambda)+\langle\Delta\rangle, \mathcal{M})$	As 2nd column
VII	$(\mathcal{M}^4+\mathcal{M}^2\langle\Delta\rangle+\langle\Delta\rangle^2+\mathcal{M}\langle\lambda\rangle,$ $\mathcal{M}^3+\mathcal{M}\langle\Delta\rangle+\langle\lambda\rangle)$	As 2nd column
VIII	$(\mathcal{M}^3+\mathcal{M}(\lambda), \mathcal{M}^3+\mathcal{M}\langle\lambda\rangle)$	As 2nd column
IX	$(\mathcal{M}^3+\mathcal{M}\langle\Delta\rangle, \mathcal{M}^3+\mathcal{M}\langle\Delta\rangle)$	As 2nd column
X	$(\mathcal{M}^4+\mathcal{M}^2\langle\Delta\rangle+\langle\Delta\rangle^2+\mathcal{M}\langle\lambda\rangle,$ $\mathcal{M}^3+\mathcal{M}\langle\lambda, \Delta\rangle)$	As 2nd column
XI	$(\mathcal{M}^4+\langle\Delta\rangle, \mathcal{M}^2+\langle N, \Delta\rangle)$	$(\mathcal{M}^3+\langle\Delta\rangle, \mathcal{M})$
XII	$(\mathcal{M}^4+\mathcal{M}^2\langle\Delta\rangle+\langle\Delta\rangle^2, \mathcal{M}^3+\mathcal{M}\langle\Delta\rangle)$	$(\mathcal{M}^3+\mathcal{M}\langle\Delta\rangle, \mathcal{M}^2+\langle\Delta\rangle)$
XIII	$(\mathcal{M}^3+\langle N, \Delta\rangle^2, \mathcal{M}^2+\mathcal{M}\langle N, \Delta\rangle)$	$(\mathcal{M}^3+\mathcal{M}\langle N, \Delta\rangle, \mathcal{M}^2)$
XIV	$(\mathcal{M}^4-\langle N, \Delta\rangle^2+\langle\Delta\rangle, \mathcal{M}^3+\langle N, \Delta\rangle)$	$(\mathcal{M}^4+\mathcal{M}^2\langle N, \Delta\rangle+\langle N, \Delta\rangle^2+\langle\Delta\rangle,$ $\mathcal{M}^2+\langle N, \Delta\rangle)$
XV	$(\mathcal{M}^3+\langle N, \Delta\rangle^2+\langle\Delta\rangle, \mathcal{M}^2+\langle N, \Delta\rangle)$	As 2nd column

of Table IX. This is illustrated in the examples. In the table and the examples we use $\mathcal{M}$ to denote $m_{u,\lambda}(D_4)$.

EXAMPLE 1: NORMAL FORM IV. We have to show that, for $f \sim (\varepsilon_0\lambda+\varepsilon_1 N+m\Delta, \varepsilon_2 N)$ with $m \neq 0$,

(1) $\operatorname{Itr}(\mathscr{K}_1(f)+\mathscr{K}_2(f)) \supset (M^2, M^2)$.

(2) If $g \sim (p, r)$ satisfies

$$r(0)=0, \qquad p_N(0)\neq 0, \qquad p_\lambda(0)\neq 0$$

$$p_N(0)\, r_\Delta(0)-p_\Delta(0)\, r_N(0)\neq 0, \qquad p_\lambda(0)\, r_N(0)-p_N(0)\, r_\lambda(0)\neq 0$$

then it is D_4-equivalent to f, modulo $(\mathcal{M}^2, \mathcal{M}^2)$, with

$$\varepsilon_0=\operatorname{sgn} p_\lambda(0), \qquad \varepsilon_1=\operatorname{sgn} p_N(0), \qquad \varepsilon_2=\varepsilon_0 \operatorname{sgn}(p_\lambda(0)\, r_N(0)-p_N(0)\, r_\lambda(0))$$

$$m=\varepsilon_2\, p_\lambda(0)^2(p_N(0)\, r_\Delta(0)-p_\Delta(0)\, r_N(0))/(p_\lambda(0)\, r_N(0)-p_N(0)\, r_\lambda(0))^2.$$

TABLE X

$(\lambda^2, 0)$	$(\lambda N, 0)$	$(\lambda \Delta, 0)$	$(N^2, 0)$	$(N\Delta, 0)$	$(\Delta^2, 0)$	$(0, \lambda^2)$	$(0, \lambda N)$	$(0, \lambda\Delta)$	$(0, N^2)$	$(0, N\Delta)$	$(0, \Delta^2)$
ε_0	ε_1	m					ε_2				
	ε_0		ε_1	m					ε_2		
		ε_0		ε_1	m					ε_2	
ε_0	$3\varepsilon_1$	$5m$					$5\varepsilon_2$				
	ε_0		3ε	$5m$					$5\varepsilon_2$		
		ε_0		$3\varepsilon_1$	$5m$					$5\varepsilon_2$	
	ε_0		ε_1	$m - \varepsilon_2$							
						ε_0	ε_1	m			
							ε_0		ε_1	m	
								ε_0		ε_1	m
		ε_0		$-\varepsilon_1$	$-m$					ε_2	
		$2\varepsilon_1$				ε_0	ε_1	$m - 2\varepsilon_2$			
				$-2\varepsilon_1$			ε_0		ε_1	$m \;\; 2\varepsilon_2$	
					$-2\varepsilon_1$			ε_0		ε_1	$m - \varepsilon_2$

We will also show that

(3) $T(f) = (\mathcal{M}^2 + \langle \lambda \rangle, \mathcal{M}) + 5$ further elements, and a universal unfolding of f is given by adding the term $\alpha(0, 1)$ to the normal form.

Using the generators of $\mathcal{K}_1(f)$ given in Section 8(b) we see that

$$\begin{aligned}
\mathcal{K}_1(f) = \langle \lambda, N, \Delta \rangle \cdot \{&(\varepsilon_0\lambda + \varepsilon_1 N + m\Delta, \varepsilon_2 N), (\varepsilon_0\lambda + 3\varepsilon_1 N + 5m\Delta, 5\varepsilon_2 N)\} \\
+ \mathcal{E}_{u,\lambda}(D_4) \cdot \{&(\varepsilon_0 \lambda N + \varepsilon_1 N^2 + (m - \varepsilon_2) N\Delta, 0), \\
&(0, \varepsilon_0\lambda + \varepsilon_1 N + m\Delta - \varepsilon_2 N^2), \\
&((N^2 - \Delta)(\varepsilon_0\lambda + \varepsilon_1 N + m\Delta), -\varepsilon_2 N(N^2 - \Delta)), \\
&(-2\varepsilon_1\Delta + (\varepsilon_2 - 4m) N\Delta, \varepsilon_0\lambda + \varepsilon_1 N + (m - 2\varepsilon_2)\Delta - 2\varepsilon_2 N^2)\}.
\end{aligned}$$

We prove that $\mathcal{K}_1(f) \supset (\mathcal{M}^2, \mathcal{M}^2)$ by showing that the inclusion holds modulo $(\mathcal{M}^3, \mathcal{M}^3)$ and then applying Nakayama's lemma [16]. Showing the inclusion mod$(\mathcal{M}^3, \mathcal{M}^3)$ is an exercise in linear algebra; it amounts to showing that the matrix in Table X has rank 12 (providing $m \neq 0$). This is left to the reader. Thus $\mathcal{K}_1(f) \supset (\mathcal{M}^2, \mathcal{M}^2)$ and since this in intrinsic, $P(f) \supset (\mathcal{M}^2, \mathcal{M}^2)$.

Next we calculate $T(f)$. Since $T(f) \supset \mathcal{K}_1(f) \supset (\mathcal{M}^2, \mathcal{M}^2)$ we only need to carry out the calculation mod$(\mathcal{M}^2, \mathcal{M}^2)$. As a vector space over $\mathbb{R}$, $T(f)/(\mathcal{M}^2, \mathcal{M}^2)$ is generated by the following matrix:

$(1,0)$	$(\lambda,0)$	$(N,0)$	$(\Delta,0)$	$(0,1)$	$(0,\lambda)$	$(0,N)$	$(0,\Delta)$
					ε_0	ε_1	m
	ε_0		$-m$			$-\varepsilon_2$	
		ε_1	$2m$			$2\varepsilon_2$	
			ε_1				ε_2
ε_0							
	ε_0						

These relations are independent and so $T(f)=(\mathcal{M}^2+\langle\lambda\rangle, \mathcal{M}^2)$ together with 5 further generators over $\mathbb{R}$. A complement to $T(f)$ is given by $\mathbb{R}\cdot\{(0,1),(\Delta,0)\}$ and so, since the coefficient of $(\Delta,0)$ in f is a modulus, a universal unfolding is obtained by adding the term $\alpha(0,1)$.

Finally, we use the formulae given in Table VII to show that any $g\sim(p,r)$ satisfying the defining and nondegeneracy conditions of f is D_4-equivalent to $f \bmod(\mathcal{M}^2, \mathcal{M}^2)$. Since $p(0)=0$ and $r(0)=0$ the coefficients of $(1,0)$ and $(0,1)$ must always be 0. We need to choose values of A, C, a, b, and A_λ so that, evaluating all terms at 0:

(a) coefficient of $(\lambda,0)=AaA_\lambda p_\lambda=\operatorname{sgn} p_\lambda$

(b) coefficient of $(N,0)=Aa^3p_N=\operatorname{sgn} p_N$

(c) coefficient of $(\Delta,0)=-2Aa^2bp_N+Aa^5p_\Delta$
$=\operatorname{sgn}(p_\lambda/(p_\lambda r_N-p_Nr_\lambda))(p_\lambda/(p_\lambda r_N-p_Nr_\lambda))^2(p_Nr_\Delta-p_\Delta r_N)$

(d) coefficient of $(0,\lambda)=(Ab+Ca)A_\lambda p_\lambda+Aa^3A_\lambda r_\lambda=0$

(e) coefficient of $(0,N)=a^2(Ab+Ca)p_N+Aa^5r_N$
$=\operatorname{sgn}((p_\lambda r_N-p_Nr_\lambda)/p_\lambda)$

(f) coefficient of $(0,\Delta)=-2ab(Ab+Ca)+a^4(Ab+Ca)p_\Delta-2Aa^4br_N+Aa^7r_\Delta=0$.

Thus we need, from (a), (b), and (d),

$$AaA_\lambda=1/|p_\lambda|,\qquad Aa^3=1/|p_N|,\qquad Ab+Ca=-Aa^3r_\lambda/p_\lambda=-r_\lambda/|p_N|\,p_\lambda.$$

Substituting for $Ab+Ca$ in (c) gives

$$Aa^5=|p_\lambda|/|p_\lambda r_N-p_Nr_\lambda|$$

and substituting into (f) gives

$$Aa^4/p_\lambda\{a^3(p_\lambda r_\Delta-r_\lambda p_\Delta)-2b(p_\lambda r_N-r_\lambda p_N)\}=0$$

and so we must have

$$b=a^3(p_\lambda r_\Delta-r_\lambda p_\Delta)/(p_\lambda r_N-r_\lambda p_N)/2.$$

These conditions completely determine A, C, a, b, Λ, and it is easily checked that the values obtained for these also satisfy (c).

EXAMPLE 2: NORMAL FORM XII. Our second example is a normal form for which $\mathscr{P}(f)$ is strictly larger than Itr $\mathscr{K}_1(f)$. Using the formulae of Section 8(b) we obtain

$$\begin{aligned}\mathscr{K}_1(f) = \langle \lambda, N, \Delta \rangle \cdot \{ &(\varepsilon_0\lambda^2 + \varepsilon_1 N + m\lambda N + \varepsilon_2\Delta, \varepsilon_1), \\ &(\varepsilon_0\lambda^2 + 3\varepsilon_1 N + 3m\lambda N + 5\varepsilon_2\Delta, 3\varepsilon_1)\} \\ + \mathscr{E}_{u,\lambda}(D_4) \cdot \{ &(\varepsilon_0\lambda^2 N + \varepsilon_1 N^2 + m\lambda N^2 + \varepsilon_2 N\Delta, 0), \\ &(0, \varepsilon_0\lambda^2 + m\lambda N + \varepsilon_2\Delta), \\ &((N^2 - \Delta)(\varepsilon_0\lambda^2 + \varepsilon_1 N + m\lambda N + \varepsilon_2\Delta), \varepsilon_1(N^2 - \Delta))\}, \\ &(-\varepsilon_1\Delta - 2m\lambda\Delta - 4\varepsilon_2 N\Delta, \varepsilon_0\lambda^2 - \varepsilon_1 N + m\lambda N + \varepsilon_2\Delta)\}, \\ \mathscr{K}_2(f) = \mathscr{E}_\lambda \cdot \{ &(2\varepsilon_0\lambda + mN, 0)\}.\end{aligned}$$

We leave the reader to verify that

$$\mathscr{K}_1(f) \supset M = (\mathscr{M}^4 + \mathscr{M}^2\langle \Delta \rangle + \langle \Delta^2 \rangle, \mathscr{M}^3 + \mathscr{M}\langle \Delta \rangle)$$

using the same method as in the previous example.

Modulo M, $\mathscr{K}_1(f) + \mathscr{K}_2(f)$ is generated over $\mathbb{R}$ by the elements shown in Table XI (omitting some obvious redundancies). The combination of rows (1) − (2) − (3) + (4) gives $-2(m(\lambda\Delta, 0) + 2\varepsilon_2(N\Delta, 0))$ as an element of $\mathscr{K}_1(f) + \mathscr{K}_2(f)$. Combining this with (5) shows that if $m^2 \neq 4\varepsilon_0\varepsilon_2$ then $\mathscr{K}_1(f) + \mathscr{K}_2(f)$ contains $(\lambda\Delta, 0)$ and $(N\Delta, 0)$. It now follows easily that in fact

$$\mathscr{K}_1(f) + \mathscr{K}_2(f) \supset (\mathscr{M}^3 + \mathscr{M}\langle \Delta \rangle, \mathscr{M}^2 + \langle \Delta \rangle).$$

This is intrinsic and so

$$\mathscr{P}(f) \supset (\mathscr{M}^3 + \mathscr{M}\langle \Delta \rangle, \mathscr{M}^2 + \langle \Delta \rangle).$$

It remains to show that any $g \sim (p, r)$ satisfying the defining and nondegeneracy conditions of f is D_4-equivalent to f mod$(\mathscr{M}^3 + \mathscr{M}\langle \Delta \rangle, \mathscr{M}^2 + \langle \Delta \rangle)$ and to calculate $T(f)$. These can be done as in the previous example and are left to the reader.

TABLE XI

$(\lambda,0)$	$(N,0)$	$(\Delta,0)$	$(\lambda^2,0)$	$(\lambda N,0)$	$(\lambda\Delta,0)$	$(N^2,0)$	$(N\Delta,0)$	$(\lambda^3,0)$	$(\lambda^2 N,0)$	$(\lambda N^2,0)$	$(N^3,0)$	$(0,\lambda)$	$(0,N)$	$(0,\Delta)$	$(0,\lambda^2)$	$(0,\lambda N)$	$(0,N^2)$	
				ε_1	ε_2			ε_0	m			ε_1						
						ε_1	ε_2		ε_0	m			ε_1					(1)
							ε_1							ε_1				
									ε_1						ε_1			
										ε_1						ε_1		
											ε_1						ε_1	
				$3\varepsilon_1$	$5\varepsilon_2$			ε_3	$3m$			$3\varepsilon_1$						
						$3\varepsilon_1$	$5\varepsilon_2$		ε_0	$3m$			$3\varepsilon_1$					
		$-\varepsilon_1$				ε_1	ε_2		ε_0	m								(2)
					$-\varepsilon_1$					ε_1								
							$-\varepsilon_1$				ε_1							
														ε_2	ε_0	m		(3)
		$-\varepsilon_1$			$-2m$		$-4\varepsilon_2$						$-\varepsilon_1$	ε_2	ε_0	m		(4)
$2\varepsilon_0$	m																	
			$2\varepsilon_0$	m														
				$2\varepsilon_0$		m												
					$2\varepsilon_0$		m											(5)
								$2\varepsilon_0$	m									
									$2\varepsilon_0$	m								
										$2\varepsilon_0$	m							

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