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Singularity theory of fitness functions under dimorphism equivalence

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Abstract We apply singularity theory to classify monomorphic singular points as they occur in adaptive dynamics. Our approach is based on a new equivalence relation called dimorphism equivalence, which is the largest equivalence relation on strategy functions that preserves ESS singularities, CvSS singularities, and dimorphisms. Specifically, we classify singularities up to topological codimension two and compute their normal forms and universal unfoldings. These calculations lead to the classification of local mutual invasibility plots that can be seen generically in systems with two parameters.

Keywords Fitness functions · ESS · Singularity theory · Adaptive game theory · Dimorphism

Mathematics Subject Classification 34C23 · 91A22 · 58K40

1 Introduction

The application of game theory to biology has a long history. Evolutionary game theory studies the evolution of phenotypic traits and was originated by Maynard-Smith and Price (1973). Since then, there has been an explosion of interest in evolutionary game theory by mathematicians and other scientists. Adaptive dynamics is a set of techniques and methods that studies the long-term consequences for phenotypes of small mutations in the genotypes. In the past twenty years, adaptive dynamics has been studied by many people, including Dieckmann and Law (1996), Geritz et al. (1997),

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Diekmann (2003), Dieckmann et al. (2004), Dercole and Rinaldi (2008), Hofbauer and Sigmund (2003), Metz et al. (1996), Polechova and Barton (2005), Waxman and Gavrilets (2005). Vutha and Golubitsky (2014) applied singularity theory and adaptive dynamics theory to study ESS and CvSS singularities of strategy functions. This paper expands their research to include the study of dimorphisms.

1.1 Background of adaptive dynamics theory

In this subsection, we recall aspects of evolutionary game theory and adaptive dynamics.

Evolutionary game theory

In evolutionary theory, changes in the environment are often reflected by changes in the residents' ability to reproduce. Organisms that can adapt better normally have higher reproductive rates. In biology, the individual's ability to adapt (or reproduce) is called fitness. Mathematical models define fitness in terms of reasonable biological assumptions that are encoded in fitness functions. The simplest game in evolution is a two-player single trait game. In this case, a fitness function is a real-valued function $f(x, y)$ where x and y are the strategies (or phenotypes) of the players (or organisms). A fitness function $f(x, y)$ represents the fitness advantage of a mutant with phenotype y when competing against a resident with phenotype x. In game theory, $f(x, y) > 0$ means that the mutant has a fitness advantage over the resident. In this paper we assume that all functions and mappings are infinitely differentiable. Since any strategy has 0 advantage against itself, we define

Definition 1.1 The $\mathbf{C}^\infty$ real-valued smooth function $f(x, y)$ is a *fitness function* if $f(x, x) = 0$ for all x.

Remark 1.2 Since a fitness function f vanishes along the diagonal (x, x), certain derivatives of f also vanish along the diagonal. For example,

$$f_x + f_y = 0 \qquad f_{xx} + 2f_{xy} + f_{yy} = 0 \qquad \cdots$$

at (x, x) for all x.

Adaptive dynamics theory

Adaptive dynamics applies a game theoretic approach to study the evolution of heritable phenotypes (or strategies), such as beak lengths of birds. There are two fundamental ideas when applying adaptive dynamics:

(i) The resident population is assumed to be in dynamic equilibrium when mutants appear.
(ii) The eventual fate of mutants can be inferred from the mutant's initial growth rate.

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The evolution of strategies is modeled using fitness functions. The idea is that an environment contains organisms playing all possible strategy (phenotypes), and that a given strategy (phenotype) evolves based on the interactions with nearby strategies (mutations). In adaptive dynamics theory, strategies evolve through a series of advantageous interactions against mutant strategies.

In fact, adaptive dynamics assumes that the resident strategy x increases when a mutant strategy $y > x$ has an advantage over x, and decreases when a mutant strategy $y < x$ has an advantage over x. Note that we can apply Taylor's Theorem at (x, x) to obtain:

$$f(x, y) = f(x, x) + (y - x) f_y(x, x) + \mathcal{O}(y - x)^2 = (y - x) f_y(x, x) + \mathcal{O}(y - x)^2$$

Thus, when $f_y(x, x) > 0$, we have for any y near x

$$f(x, y) > 0 \quad \text{if and only if} \quad y > x$$

Therefore, we see that in evolution the rate of change of strategies and the selection gradient $f_y(x, x)$ of the fitness function f have the same sign. Dieckmann and Law (1996) have applied this approach and obtained the canonical equation of adaptive dynamics:

$$\frac{dx}{dt} = \alpha(x) f_y(x, x) \tag{1.1}$$

where $\alpha(x) > 0$ depends on the resident strategy x. We see that (1.1) has an equilibrium at $\bar{x}$ if and only if $f_y(\bar{x}, \bar{x}) = 0$. Therefore, we define

Definition 1.3 A strategy $\bar{x}$ is a singular strategy if $f_y(\bar{x}, \bar{x}) = 0$ and hence $f_x(\bar{x}, \bar{x}) = 0$.

1.2 Important concepts in adaptive dynamics

There are four properties of singular strategies in adaptive dynamics, which we now describe.

Evolutionarily stable strategy An evolutionarily stable strategy is a resident phenotype (i.e. strategy $\bar{x}$) such that no mutant with phenotype y near $\bar{x}$ can invade the resident.

Convergence stable strategy A convergence stable strategy is a phenotype (i.e. strategy $\bar{x}$) that is a linearly stable equilibrium for the canonical equation of adaptive dynamics (1.1). If a phenotype $\bar{x}$ is a convergence stable strategy, it is a local attractor in adaptive dynamics; that is, all nearby resident strategies will evolve toward $\bar{x}$.

Neighborhood invader strategy A neighborhood invader strategy is a mutant phenotype (i.e. strategy $\bar{x}$) such that it can always invade a nearby resident with phenotype x.

Mutual invasibility strategy In evolution, either mutations do or do not die out. If the mutant with phenotype y has no advantage over the resident with phenotype

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x (i.e. $f(x, y) < 0$), the mutant will die out. On the other hand, if the mutant with phenotype y has an advantage over the resident with phenotype x (i.e. $f(x, y) > 0$), then the mutant's population will grow and they will not die out. However, in some situations, the mutant cannot outcompete the resident once the resident is rare and coexistence of two subpopulations with different phenotypes follows. That is, the system becomes a dimorphic population and this happens when $f(x, y) > 0$ and $f(y, x) > 0$. Such a pair of phenotypes is called a dimorphism. A strategy $\bar{x}$ has mutual invasibility if there exists dimorphism pairs (x, y) arbitrarily near $(\bar{x}, \bar{x})$.

Next we interpret these properties as singularities.

Remark 1.4 In singularity theory local singularities of a function F at some base point X_0 are defined by conditions on the derivatives of F at X_0. These conditions divide into two types: defining conditions and nondegeneracy conditions. Defining conditions assume that certain combinations of derivatives equal 0 and nondenegenacy conditions assume that certain combinations of derivatives do not equal 0. For example, in one parameter bifurcation theory Golubitsky and Schaeffer (1985) a pitchfork bifurcation at (0, 0) is the singularity of a map $F : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$, denoted $F(X, \lambda)$, whose defining conditions are $F = F_X = F_{XX} = F_\lambda = 0$ and whose nondegeneracy conditions are $F_{XXX} \neq 0 \neq F_{X\lambda}$. Singularity theory provides general methods to prove that if F has a pitchfork bifurcation at (0, 0), then F is equivalent by (an appropriately defined) change of coordinates to the normal form $H(X, \lambda) = \varepsilon X^3 + \delta X \lambda$ where $\varepsilon = \text{sgn}(F_{XXX}(0, 0))$ and $\delta = \text{sgn}(F_{X\lambda}(0, 0))$. In steady-state bifurcation theory, the appropriate changes of coordinates are parametrized families of contact equivalences Golubitsky and Schaeffer (1985).

Remark 1.5 The application of singularity theory to adaptive dynamics requires that the properties of singularities defined above be recast in terms of defining and nondegeneracy conditions. To do this we use the long form "Evolutionarily stable strategy" to refer to the definition given above and the abbreviated form "ESS" given in Definition 1.6 below to refer to the the defining and nondegeneracy conditions of the given singularity. Lemma 1.7 fills in the details that the abbreviated forms (such as "ESS") satisfy the long forms (such as "Evolutionarily stable strategy"). In this introduction we give sample results, like the pitchfork results of steady-state bifurcation theory, in the context of the singularities of adaptive dynamics.

Definition 1.6 Let f be a fitness function satisfying the defining conditions $f_x = f_y = 0$ at $(\bar{x}, \bar{x})$. Then

(a) $\bar{x}$ is an ESS if $f_{yy} < 0$ at $(\bar{x}, \bar{x})$.
(b) $\bar{x}$ is a CvSS if $f_{yy} - f_{xx} < 0$ at $(\bar{x}, \bar{x})$.
(c) $\bar{x}$ is an NIS if $f_{xx} > 0$ at $(\bar{x}, \bar{x})$.
(d) $\bar{x}$ is an MIS if $f_{yy} + f_{xx} > 0$ at $(\bar{x}, \bar{x})$.

Lemma 1.7 *Let f be a fitness function with singular strategy $\bar{x}$.*

(a) *If $\bar{x}$ is an ESS, then $\bar{x}$ is an evolutionarily stable strategy.*
(b) *If $\bar{x}$ is a CvSS, then $\bar{x}$ is a convergence stable strategy.*

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(c) *If $\bar{x}$ is an NIS, then $\bar{x}$ is a neighborhood invader strategy.*
(d) *If $\bar{x}$ is an MIS, then $\bar{x}$ is a mutual invasibility strategy.*

Proof We prove each statement in turn.

(a) f has an evolutionarily stable strategy if $f(\bar{x}, \cdot)$ has a maximum at $\bar{x}$. Since $f(\bar{x}, \bar{x}) = 0$, it is sufficient to have that $f_y(\bar{x}, \bar{x}) = 0$ and $f_{yy}(\bar{x}, \bar{x}) < 0$. Remark 1.2 implies that $f_x(\bar{x}, \bar{x}) = 0$ if and only if $f_y(\bar{x}, \bar{x}) = 0$.
(b) Recall that $\bar{x}$ is a linearly stable equilibrium of (1.1) if $f_y(\bar{x}, \bar{x}) = 0$ and

$$\frac{d}{dx}(\alpha(x) f_y(x, x))\bigg|_{x=\bar{x}} < 0$$

By Remark 1.2, we have

$$\begin{aligned}\frac{d}{dx}(\alpha(x) f_y(x, x))\bigg|_{x=\bar{x}} &= \alpha'(\bar{x}) f_y(\bar{x}, \bar{x}) + \alpha(\bar{x})(f_{xy}(\bar{x}, \bar{x}) + f_{yy}(\bar{x}, \bar{x})) \\ &= \frac{1}{2}\alpha(\bar{x})(f_{yy}(\bar{x}, \bar{x}) - f_{xx}(\bar{x}, \bar{x}))\end{aligned}$$

Since $\alpha(x) > 0$, we see that if $\bar{x}$ is a CvSS, then $\bar{x}$ is a convergence stable strategy.
(c) $\bar{x}$ is a neighborhood invader strategy if $f(\cdot, \bar{x})$ has a minimum at $\bar{x}$. It is sufficient to have $f_x(\bar{x}, \bar{x}) = 0$ and $f_{xx}(\bar{x}, \bar{x}) \geq 0$; that is, to have $\bar{x}$ be an NIS.
(d) Let $g(x) = f(x, 2\bar{x} - x)$ and $h(x) = f(2\bar{x} - x, x)$. We claim that if $\bar{x}$ is an MIS, then both $g(x) > 0$ and $h(x) > 0$ when x is close to $\bar{x}$. Therefore, $(x, 2\bar{x} - x)$ is a dimorphism pair and $\bar{x}$ is a mutual invasibility strategy when $\bar{x}$ is an MIS. We prove the claim for g; the result is similar for h. Using Remark 1.2 calculate

$$\begin{aligned} g'(\bar{x}) &= f_x(\bar{x}, \bar{x}) - f_y(\bar{x}, \bar{x}) &= 0 \\ g''(\bar{x}) &= 2(f_{xx}(\bar{x}, \bar{x}) + f_{yy}(\bar{x}, \bar{x})) &> 0 \end{aligned}$$

where $'$ is derivative with respect to x. Then, we have

$$g(x) = \frac{1}{2} g''(\bar{x})(x - \bar{x})^2 + h.o.t.$$

and conclude that $g(x) > 0$ when x is close to $\bar{x}$. □

In fact, the singular properties of adaptive dynamics come in pairs. For example, there is a sister singularity to ESS, denoted ESS* whose nondegeneracy condition is $f_{yy} > 0$. As we have discussed each singular strategy has four pairs of sister characteristics defined by an inequality in certain combinations of derivatives of f. Specifically:

ESS ($f_{yy} < 0$) A resident phenotype that is not invaded by a nearby mutant phenotype.
ESS* ($f_{yy} > 0$) A resident phenotype that can be invaded by any nearby mutant phenotype.

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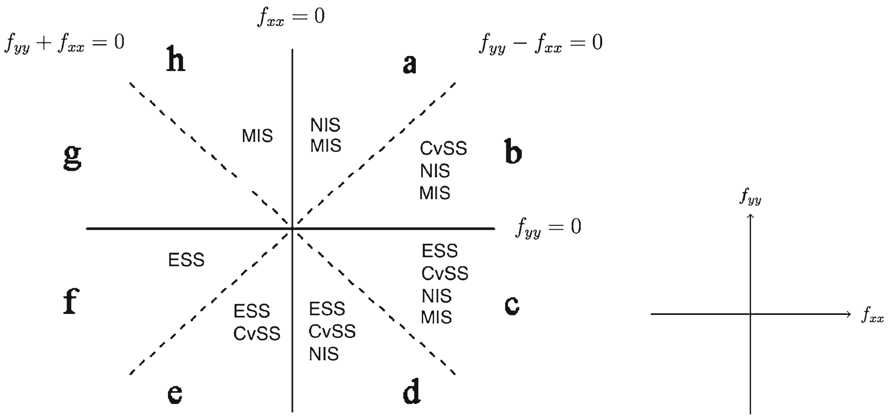

Fig. 1 Classification of singular strategies. Based on the defining conditions of ESS, CvSS, NIS, and MIS, we divide the $f_{xx} f_{yy}$ plane into eight regions. In each region we list the standard properties that the singular strategy has. If the acronym for a pair is missing, then the singularity present is the sister one

CvSS ($f_{yy} - f_{xx} < 0$) A phenotype that is a linearly stable equilibrium for (1.1).
CvSS* ($f_{yy} - f_{xx} > 0$) A phenotype that is a linearly unstable equilibrium for (1.1).
NIS ($f_{xx} > 0$) A mutant phenotype that can invade any nearby resident phenotype.
NIS* ($f_{xx} < 0$) A mutant phenotype that can never invade any nearby resident phenotype.
MIS ($f_{yy} + f_{xx} > 0$) A singular strategy (phenotype) with dimorphism pairs nearby.
MIS* ($f_{yy} + f_{xx} < 0$) A singular strategy (phenotype) with no dimorphism pairs nearby.

Indeed, Geritz et al. (1997) had proposed four different evolutionary scenarios in a fitness function and Dieckmann et al. (2004) had discussed the evolutionary influence of these four properties of singularities. These four types are summarized in Fig. 1, which is a modification of a picture in Geritz et al. (1997). In fact, our figure classifies more, namely, the properties ESS, CvSS, NIS, and MIS, and their sister properties.

Definition 1.8 The region of coexistence for a fitness function f is the set of all dimorphisms of f; that is, the set $\{(x, y) : f(x, y) > 0 \text{ and } f(y, x) > 0\}$.

Remark 1.9 Diekmann (2003), based on Metz et al. (1996), Geritz et al. (1998), discusses consequences of mutual invasibility. Specifically, suppose there is a dimorphism (x, y) near a singular strategy $\bar{x}$ that is a CvSS. Then

(a) If $\bar{x}$ is an ESS, then both strategies x and y will evolve towards $\bar{x}$, and the dimorphism (x, y) is called a converging dimorphism.
(b) If $\bar{x}$ is not an ESS, then an interesting phenomenon called evolutionary branching can occur. When this happens, (x, y) is called a diverging dimorphism. Detailed discussions about evolutionary branching can be found in Geritz et al. (1998), Ito and Dieckmann (2012), Kisdi and Priklopil (2011), Priklopil (2012).

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(c) It follows from items (a,b) that the dimorphism pairs whose existence are guaranteed in Fig. 1 region c are converging and the dimorphism pairs in Fig. 1 region b are diverging.
(d) Note that there are other dimorphism pairs that appear in Fig. 1 regions a and h and they are diverging in the sense that the singularity is CvSS*.

The methods of this paper will enable us to keep track of the existence of dimorphisms, but these methods will not distinguish between converging and diverging dimorphisms.

1.3 Background of singularity theory

In this subsection, we discuss the general ideas of singularity theory and how it helps in studying ESS, CvSS, and dimorphisms in the context of adaptive dynamics. In addition, we introduce mutual invasibility plots that show how ESS, CvSS, and dimorphisms can interact.

Singularity theory studies how certain properties of a class of functions change as parameters are varied. When applying singularity theory, the first step is to determine the most general transformations that preserve the properties that one is trying to study. For example, contact equivalences are the most general transformations that preserve zero sets. Specifically, suppose $f, g : \mathbf{R}^2 \to \mathbf{R}$ are fitness functions. Then f and g are contact equivalent if there exists $S : \mathbf{R}^2 \to \mathbf{R}^+$ and a change of coordinates $\Phi : \mathbf{R}^2 \to \mathbf{R}^2$ such that

$$g(x, y) = S(x, y) f(\Phi(x, y)).$$

It is easy to see that Φ transforms the zero set of g to the zero set of f.

Vutha and Golubitsky (2014) consider a subset of contact equivalences that preserve ESS and CvSS singularities of fitness functions. These equivalences are called strategy equivalences (see Definition 4.1). In this paper we study special forms of strategy equivalences that also preserve dimorphisms, that is, that preserve the property MIS. It then follows that these coordinate changes also preserve the property NIS. We call these equivalences dimorphism equivalences. Specifically:

Definition 1.10 Two fitness functions f and $\hat{f}$ are dimorphism equivalent if

$$\hat{f}(x, y) = S(x, y) f(\Phi(x, y)),$$

where

(a) $S(x, y) > 0$ for all (x, y).
(b) $\Phi(x, y) = (\varphi(x, y), \varphi(y, x))$ where $\varphi\colon \mathbf{R}^2 \to \mathbf{R}$.
(c) $\varphi_x(x, x) > 0$ and $\varphi_y(x, x) = 0$ for all x.

Moreover, we call the dimorphism equivalence strong if $\varphi_x(x, x) \equiv 1$.

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Remark 1.11 Assumption (a) in Definition 1.10 preserves the sign of f and assumption (b) preserves the diagonal $x = y$. In fact, assumption (b) does more—it guarantees that if the fitness functions $f(x, y)$ and $\hat{f}(x, y)$ are dimorphism equivalent, then so are the fitness functions $f(y, x)$ and $\hat{f}(y, x)$. See Lemma 4.3. The technical assumption (c) is part of strategy equivalence and is needed to preserve the properties CvSS and ESS. See Theorem 1.13, specifically identity (4.2), whose proof is given in Sect. 4.

Remark 1.12 We claim that dimorphism equivalence, as defined in Definition 1.10, is the largest class of transformations of fitness functions that preserves the four pairs of sister properties ESS, CvSS, MIS, and NIS. Note that it follows that such transformations also preserve the zero sets $f(x, y) = 0$ and $f(y, x) = 0$. Specifically, Definition 1.10(a) and (c) follow from strategy equivalence (see Definition 4.1), which is the largest set of equivalences that preserve ESS and CvSS. In addition, in order to preserve dimorphism pairs, the same Φ in the dimorphism equivalence that transforms $f(x, y)$ to $\hat{f}(x, y)$ must also transform $f(y, x)$ to $\hat{f}(y, x)$. It follows that Definition 1.10(b) must be satisfied.

Theorem 1.13 *Dimorphism equivalence preserves ESS, CvSS, NIS, MIS and their sister properties for all fitness functions. Moreover, if the fitness functions f and $\hat{f}$ are dimorphism equivalent, then the diffeomorphism Φ maps the regions of coexistence of $\hat{f}$ to those of f.*

Singularity theory provides methods for answering two questions:

1. *Recognition problem* When is a fitness function f dimorphism equivalent to a specific fitness function h on a neighborhood of a singularity?
2. *Universal unfolding problem* Find the smallest family of perturbations $H : \mathbf{R}^2 \times \mathbf{R}^k \to \mathbf{R}$ of h that contains all perturbations of h up to dimorphism equivalence.

Remark 1.14 Note that almost all results about recognition and universal unfolding problems are local results—they are valid only on some (unspecified) neighborhood of the origin. More precisely, singularity theory results are about germs of functions (Golubitsky and Schaeffer 1985, p.54) or (Golubitsky and Guillemin 1974, p.103) and not specifically about functions themselves. The conversion of statements about germs to statements about functions is routine in singularity theory. In this exposition we follow the usual practice of not stating this conversion explicitly.

The function h in the recognition problem is called a normal form and is usually the 'simplest representative' from the equivalence class of h. A major result is that the recognition problem can be solved by examining a finite number of derivatives of f at the singularity. This result relates to Remark 1.5. For example:

Theorem 1.15 *The fitness function f is dimorphism equivalent to*

$$h = (x - y)^4 + (x + y)(x - y) \tag{1.2}$$

on a neighborhood of $(0, 0)$ *if and only if the defining conditions* $f_x = f_{xy} = 0$ *and the nondegeneracy conditions*

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$$f_{xx} > 0 \qquad f_{xx}(f_{x^4} + 6f_{x^2y^2} + f_{y^4}) - 4(f_{x^2y} + f_{xy^2})(3f_{xy^2} + f_{y^3}) > 0$$

are valid at $(0, 0)$.

Singularity theory studies small perturbations of a given fitness function h through its universal unfolding. An unfolding of h is a parametrized family $H(x, y, \alpha)$ such that $H(x, y, 0) = h(x, y)$ where $\alpha \in \mathbf{R}^k$ is a set of parameters. A versal unfolding of h contains all small perturbations of h up to dimorphism equivalence. A universal unfolding of h is a versal unfolding with the smallest number of parameters k. Once a singularity is identified in a normal form h, we can find all possible small perturbations of h in its universal unfolding $H(x, y, \alpha)$ up to dimorphism equivalence. For example, the following theorem is included in Table 3(d).

Theorem 1.16 *A universal unfolding of* (1.2) *is*

$$H = ((x - y)^2 + a)(x - y)^2 + (x + y)(x - y) \tag{1.3}$$

where $a \in \mathbf{R}$ *is near 0.*

Geometry and mutual invasibility plots

Universal unfoldings allow us to study the geometry of small perturbations around a singularity. For a given fitness function we use mutual invasibility plots (MIPs) to illustrate ESS singularities, CvSS singularities, and regions of coexistence. For example, Fig. 2 shows the two possible perturbations of the fitness function (1.2) up to dimorphism equivalence by plotting the data for the universal unfolding (1.3) when $a > 0$ and $a < 0$. In these plots, the blue curve is $H(x, y, a) = 0$, the red curve is $H(y, x, a) = 0$, and the black curve is the line $y = x$. In each region, we can see a pair of signs and they stand for $\mathrm{sgn}(H(x, y, a))$ and $\mathrm{sgn}(H(y, x, a))$. The centered red dot stands for a singularity that is both ESS and CvSS. Plots like these are called mutual invasibility plots (MIPs). Please note that all MIPs drawn in this paper are local in both phase space and parameter space near the degenerate singularity. This is consistent with the theory, which is valid only locally on such a neighborhood.

In Fig. 2, regions of coexistence emerge from the perturbation of a singular fitness function h. The MIP of h is the middle plot of Fig. 2 and we see no region of coexistence.

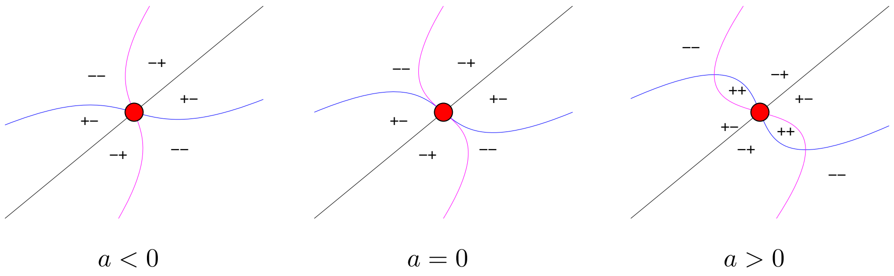

Fig. 2 MIPs of $H = ((x - y)^2 + a)(x - y)^2 + (x + y)(x - y)$

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If we perturb the parameter to $a > 0$ (as in the right plot), the strategy function has two regions of coexistence; whereas, when $a < 0$ (as in the left plot), the strategy function has no region of coexistence. We can think of the perturbation of this singular strategy as creating regions of coexistence.

Remark 1.17 In the adaptive dynamics literature, there is another useful plot called a pairwise invasibility plot (PIP). For any strategy function f, a PIP contains the curve $f(x, y) = 0$ and $\mathrm{sgn}(f)$ in each region bounded by $f(x, y) = 0$.

Codimension

Let $F(\cdot, \alpha)$, where $\alpha \in \mathbf{R}^k$, be a universal unfolding of the fitness function f. Then F contains all perturbations of f up to dimorphism equivalence. The parameters α are often called unfolding parameters. The C^∞ codimension of f is the number of unfolding parameters in the universal unfolding F, that is, k.

The perturbation of f associated with a parameter $\beta \in \mathbf{R}^k$ is $f_\beta(\cdot) = F(\cdot, \beta)$. It is a theorem in singularity theory that the C^∞ codimension of f_β is always less than or equal to the C^∞ codimension of f. We call the parameter $\beta \in \mathbf{R}^k$ a modal parameter if the C^∞ codimension of f_β equals the C^∞ codimension of f. Let $B \subset \mathbf{R}^k$ be the set of modal parameters. The topological codimension of a function f is its C^∞ codimension minus the dimension of B (see Golubitsky and Schaeffer 1985, p. 193). As is standard in singularity theory, we can classify singularities of fitness functions either up to a given C^∞ codimension or up to a given topological codimension. We choose 'topological codimension' because this is the number of parameters that are needed in an application for a particular singularity type to occur generically. Indeed, as can be seen in Table 1, μ is a modal parameter (when $\mu_0 \neq 0, \pm 1$). So the singularity listed in that table has C^∞ codimension 1 and topological codimension 0.

Finally, the motivation behind the term 'topological codimension' is that f_β for $\beta \in B$ is dimorphism equivalent to f if in Definition 1.10 we allow Φ to be a homeomorphism rather than a diffeomoprhism. However, proving such a statement is beyond the scope of this paper.

Classification of singularities

In this paper, we solve the recognition problems, find the universal unfoldings, and plot the MIPs (for all perturbations up to dimorphism equivalence) for singularities

Table 1 The normal form, defining (Def) and non-degeneracy (ND) conditions, and universal unfolding of topological codimension (TC) zero singularities

	Def	ND	TC	Normal Form	Universal Unfolding
(a)		p	0	$h = \epsilon(w + \mu_0 uv)$	
		q_u		$\epsilon = \mathrm{sgn}(p)$	$H = \epsilon(w + \mu uv)$
		$p + q_u$		$\mu_0 = \frac{q_u}{p}$	$\mu \approx \mu_0$
		$p - q_u$		$\mu_0 \neq -1, 0, 1$	

All derivatives are evaluated at the origin

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of topological codimension ≤ 2. In specific, Tables 1, 3, 4 classify all singularities of topological codimension 0, 1, 2 and give a normal form and a universal unfolding for each of these singularities. The MIPs for topological codimensions zero and one are simpler and presented in Sect. 2. The MIPs for topological codimension two are provided in Sect. 3.

1.4 Structure of the paper

In Sect. 2, we summarize the major results of this paper and provide an application of our theory. In Sect. 3, we study the geometry of the unfolding space for each singularity of topological codimension two. For these singularities, we determine the MIPs of all possible perturbations up to dimorphism equivalence. In Sect. 4 we review Vutha and Golubitsky (2014) strategy equivalence and discuss dimorphism equivalence which preserves ESS singularities, CvSS singularities, and regions of coexistence. Theorem 1.13 is proved in this section. In Sect. 5, given a fitness function f, we present a sufficient condition to determine all small perturbations η so that $f+\eta$ is dimorphism equivalent to f. The result is stated in the modified tangent space constant theorem (Theorem 5.5). In Sect. 6, we discuss universal unfoldings of fitness functions up to dimorphism equivalence. Section 7 contains the sketch of the proof of Theorem 2.3 and also sketches a solution to the recognition problem for universal unfoldings by discussing one singularity. See Lemma 7.7. The corresponding results for all singularities of codimension one are given in Wang (2015).

2 Determinacy and unfolding results

In this section, we present our results and explain why these results are important. In addition, we apply our theory to study the Hawk-Dove game Dieckmann and Metz (2006).

2.1 Major results from singularity theory

As discussed in Sect. 1, we apply singularity theory to study certain singular strategies in adaptive dynamics. In this section we present the classification of singularities of topological codimension zero, one, and two. We also present the MIPs associated to the singularities of codimension zero and one.

Note that the classification is done using (u, v) coordinates where

$$u = x + y \qquad \text{and} \qquad v = x - y$$

We also denote $w = v^2$. Using (u, v) coordinates simplifies the statements of our theorems, as well as the calculations in their proofs. We show in Sect. 5 that a general fitness function f can be written as

$$f = p(u, w)w + q(u, w)v. \tag{2.1}$$

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Basically, we can split f into the sum of an even part in v and an odd part in v. Later in the paper we will identify the fitness function f notationally with the pair of functions $[p, q]$.

Topological codimension zero singularities

Table 1 lists all topological codimension zero singularities. We see that ϵ and μ_0 can be obtained from f. The singularity has topological codimension zero only when $\mu_0 \neq -1, 0, 1$. (The singularity has higher topological codimension if $\mu_0 = -1$, or 0, or 1.) In the universal unfolding, the parameter μ near μ_0 is a modal parameter.

Mutual invasibility plots (MIPs) consist of the zero sets ($f(x, y) = 0$ and $f(y, x) = 0$), signs of $f(x, y)$ and $f(y, x)$ in connected components of the complement of the zero sets, and the type of singularities on the diagonal that correspond to the symbols in Table 2.

Figure 3 contains MIPs of the different universal unfoldings of topological codimension zero singularities. We see that the differences originate with the existence of the properties ESS, CvSS, NIS, MIS or their sister properties. The ESS and CvSS

Table 2 Symbols of singular strategies in the MIPs

● (red)	CvSS ESS
■ (red)	CvSS ESS*
● (green)	CvSS* ESS
■ (green)	CvSS* ESS*
● (black)	$CvSS_0$ ESS
■ (black)	$CvSS_0$ ESS*
♦ (red)	CvSS ESS_0
♦ (green)	CvSS* ESS_0

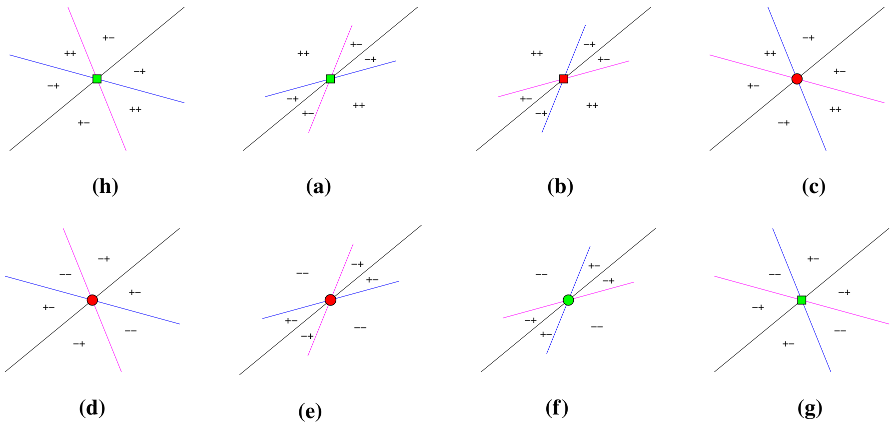

Fig. 3 The MIPs of fitness function $H = \epsilon(w + \mu uv)$ of different $\mu \neq -1, 0, 1$. The figure labels correspond to the regions in Fig. 1

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property of each singular strategy is indicated by a symbol with a certain color and shape as shown in Table 2. The asterisks in Table 2 stands for sister properties and the subscript 0 stands for a transition between a property and its sister property (that is, a degenerate singularity). The MIS property is indicated by the existence of regions of coexistence (++). We do not indicate the NIS property in MIPs because we do not focus on the study of NIS in this paper. We show in Sect. 3 that NIS is preserved under dimorphism equivalence and could be kept track of in MIPs.

Remark 2.1 It helps to note, especially in the higher codimension singularities, that the MIPs of $-f(x, y)$ can be determined directly from the MIPs of $f(x, y)$. The zero sets are identical, $+$ signs are replaced by $-$ signs and $-$ signs by $+$ signs and the properties of the singularities are replaced by their sister properties. Similarly, reflecting MIPs across the diagonal is equivalent to replacing (x, y) by (y, x) (or replacing v by $-v$). This transformation yields the same zero sets, interchanges the signs in each connected component, and changes the singularity type in a precise way. Specifically, ESS $\leftrightarrow$ NIS*; ESS* $\leftrightarrow$ NIS; CvSS $\leftrightarrow$ CvSS*; and MIS and MIS* remain unchanged.

Remark 2.2 As an example of Remark 2.1, we look at Fig. 3a in which the unperturbed fitness function is $h = w + \mu_0 uv$ where $-1 < \mu_0 < 0$. Note that $-h = -w - \mu_0 uv$ falls in Fig. 3e. Swapping x and y yields $h(u, -v) = w - \mu_0 uv$, that is, Fig. 3b. Finally, Fig. 3f is obtained from case (a) by simultaineously swapping x and y and mutiplying h by -1. In fact, the eight MIPs in Fig. 3 can be obtained by applying these transformations to cases (g) and (a).

Topological codimension one singularities

Table 3 contains the detailed information of the four topological codimension one singularities. These singularities are degenerate in ESS, NIS, CvSS, and MIS, respectively. Figure 4 lists the MIPs of the perturbations up to dimorphism equivalence for each singularity and describes a typical transition of their degeneracy. Note that in this figure we have reduced the number of possible triples of pictures from 16 unfoldings to four unfoldings by applying the transformations mentioned in Remark 2.1. In particular, we can take $\varepsilon = +1$ to be representative.

We observe the degenerate singularity in the centered MIPs. In particular, we point out that the degenerate ESS (resp. NIS) singularities are simply special cases when the parameter μ_0 of codimension zero singularity becomes 1 (resp. -1). We also see that the degenerate CvSS singularity contains a modal parameter γ in its universal unfolding and we find that γ has no influence on the type of singular strategy. As for the degenerate MIS, we discuss it in Sect. 1 and already know that this singularity creates new regions of coexistence (cf. Fig. 2). For the complete list of MIPs for these singularities, please refer to Wang (2015).

Topological codimension two singularities

Table 4 lists the three topological codimension two singularities. We will have a detailed discussion of the associated MIPs in Sect. 3.

To summarize:

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Table 3 The normal forms, defining (Def) and non-degeneracy (ND) conditions, and universal unfoldings of topological codimension (TC) one singularities

	Def	ND	TC	Normal form	Universal unfolding
(b) ESS_0	$q_u - p$	p	1	$h = \epsilon(w + uv)$ $\epsilon = \text{sgn}(p)$	$H = \epsilon(w + \mu uv)$
(c) NIS_0	$q_u + p$	p	1	$h = \epsilon(w - uv)$ $\epsilon = \text{sgn}(p)$	$H = \epsilon(w - \mu uv)$
(d) MIS_0	p	q_u $p_w q_u - p_u q_w$	1	$h = \epsilon(\delta w^2 + uv)$ $\epsilon = \text{sgn}(q_u)$ $\delta = \text{sgn}(p_w q_u - p_u q_w)$	$H = \epsilon((a + \delta w)w + uv)$
(e) $CvSS_0$	q_u	p q_{uu}	1	$h = \epsilon(w + (\delta u^2 + \gamma_0 u^3)v)$ $\epsilon = \text{sgn}(p)$ $\delta = \text{sgn}(\frac{q_{uu}}{p})$ $\gamma_0 = \frac{2pq_{uuu} - 6q_{uu}p_u}{3q_{uu}^2}$	$H = \epsilon(w + (a + \delta u^2 + \gamma u^3)v)$

Theorem 2.3 *Suppose $f = p(u, w)w + q(u, w)v$ is a fitness function with a singularity at the origin of topological codimension* (TC) *at most two. Then f is dimorphism equivalent to a normal form in Tables 1, 3, 4. The type of singularity is determined by which defining* (Def) *and non-degeneracy conditions* (ND) *are satisfied.*

Theorem 2.3 is proved in Sect. 7.2.

2.2 Application of the theory

This paper provides the general methodology to simultaneously study ESS singularities, CvSS singularities, and regions of coexistence for fitness functions. Our theorems (specifically the solutions to recognition problems and the determination of universal unfoldings) enable us to find properties of fitness functions that would otherwise be difficult to find. The MIPs in this section and in Sect. 3 show the key properties for universal unfoldings of each singularity up to topological codimension 2. In particular, we can find dimorphisms in fitness functions that depend on parameters by using defining conditions to search for specific degenerate singularities that have dimorphisms in their universal unfoldings.

The Hawk-Dove game

Dieckmann and Metz (2006) considered generalizations of the classical Hawk-Dove game. Vutha and Golubitsky (2014) study this generalization in the context of strategy equivalence and find different types of ESS and CvSS singularities as parameters are varied.

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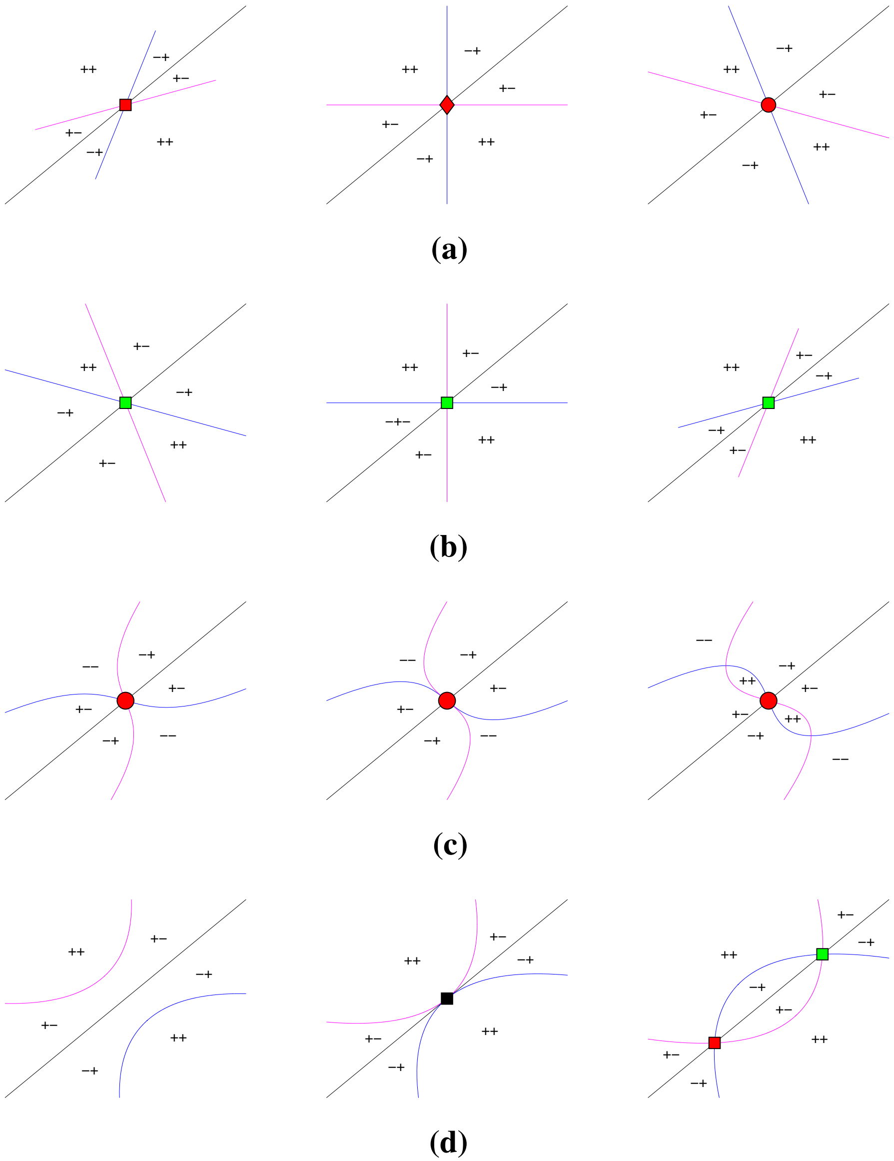

Fig. 4 MIPs of universal unfoldings of the four topological codimension one singularities

The classical Hawk-Dove game has two players A and B who can play either a hawk strategy or a dove strategy with payoffs given in Table 5. Here $V > 0$ is a reward and $C \geq 0$ is a cost.

In fact, Dieckmann and Metz (2006) consider a game where A plays hawk with probability x and B plays hawk with probability y and show that the advantage for B in this game is given by the fitness function

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Table 4 The normal forms, defining (Def) and non-degeneracy (ND) conditions, and universal unfoldings of topological codimension (TC) two singularities

Def	ND	TC	Normal form	Universal unfolding
(f) p	q_u		$h = \epsilon(\delta w^3 + uv)$	$H = \epsilon((a + bw + \delta w^2)w + uv)$
	$p_w q_u - p_u q_w \Delta_1$	2	$\epsilon = \text{sgn}(q_u)$	
			$\delta = \text{sgn}(\Delta_1)$	
			$h = \epsilon(w + (\delta u^3 + \lambda_0 u^5)v)$	
(g) q_u	p	2	$\epsilon = \text{sgn}(p)$	$H = \epsilon(w + (a + bu + \delta u^3 + \lambda u^5)v)$
q_{uu}	q_{u^3}		$\delta = \text{sgn}(\frac{q_{uuu}}{p})$	
			$\lambda_0 = \Delta_2$	
	p_u		$h = \epsilon(uw + (\alpha_0 w + \beta_0 u^2)v)$	
(h) p	q_w	2	$\epsilon = \text{sgn}(p_u)$	$H = \epsilon((a + u)w + (b + \alpha w + \beta u^2)v)$
q_u	q_{uu}		$\alpha_0 = \frac{q_w}{p_u}$	
	$p_u^2 - 2q_w q_{uu}$		$\beta_0 = \frac{q_{uu}}{2p_u}$	

Singularity (f) has double MIS degeneracy; Singularity (g) has double CvSS degeneracy; Singularity (h) has a double degeneracy found by simultaneously having MIS and CvSS degeneracies. Note that when this happens, all properties ESS, CvSS, NIS, and MIS are simultaneously degenerate; one can also think of (h) as a double ESS or a double MIS degeneracy. ($\Delta_1 = q_u^2(q_u p_{ww} - p_u q_{ww}) + q_w^2(q_u p_{uu} - p_u q_{uu}) - 2q_u q_w(q_u p_{uw} - p_u q_{uw})$; $\Delta_2 = -120 p_{uu} q_{uuu}^2 + 60 p_u q_{uuu} q_{uuuu} - 15 p q_{uuuu}^2 + 12 p q_{uuu} q_{uuuuu})/(40 q_{uuu}^3))$

Table 5 The Hawk-Dove game

	Hawk	Dove
Hawk	$\frac{1}{2}(V - C)$	V
Dove	0	$\frac{1}{2}V$

$$f(x, y) = (y - x)(V - Cx) \tag{2.2}$$

Dieckmann and Metz (2006) consider variations of (2.2) that lead to parametrized families of fitness functions, which are based on various ecological assumptions (see Dieckmann and Metz (2006) for details). Their most complicated game has the form

$$f(x, y) = \ln\left(\frac{1 + Q(x, y)}{1 + Q(x, x)}\right) \tag{2.3}$$

where

$$\begin{aligned} P(x) &= r_0 + r_1(x - x_0) + r_2(x - x_0)^2 \\ A(x, y) &= \frac{1}{2}\sqrt{P(x)P(y)} \\ B(x, y) &= V(1 - x + y) - Cxy \\ Q(x, y) &= A(x, y)B(x, y)/R \end{aligned} \tag{2.4}$$

and $R > 0$, $C > 0$, and $V > 0$.

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By applying our singularity theory results we show that the fitness function (2.3) of the generalized Hawk-Dove example has regions of coexistence. Specifically, assume

$$V = \frac{3}{16}C \qquad x_0 = \frac{1}{4} \qquad r_0 = 1 \qquad r_1 = 1 \qquad r_2 = 0 \tag{2.5}$$

We claim that the fitness function $f(x, y)$ has a singularity at $(x, y) = (\frac{1}{4}, \frac{1}{4})$ that is dimorphism equivalent to

$$h(x, y) = (x - y)^2 + \frac{14}{13}(x + y)(x - y) \tag{2.6}$$

This h is one case of the normal form in Table 1 and its MIPs can be found in Fig. 3g where we see the existence of regions of coexistence. Direct calculation verifies the claim:

$$\begin{aligned} f_x &= -\frac{Q_y}{1 + Q} = 0 \\ f_{xy} &= -\frac{39C}{4C + 64R} < 0 \\ f_{xx} - f_{yy} &= \frac{21C}{C + 16R} > 0 \end{aligned}$$

Thus, Table 1 indicates that f is dimorphism equivalent to $w + \mu_0 uv$ where $\mu_0 = \frac{14}{13}$. Therefore, this specific Hawk-Dove game contains a singularity that is dimorphism equivalent to the fitness function (2.6) and has regions of coexistence.

3 Geometry of unfoldings in codimension two

In this section, we list the mutual invasibility plots associated to universal unfoldings of singularities of topological codimension two.

3.1 Transition varieties

Suppose $F(x, y, \alpha)$, where $\alpha \in \mathbf{R}^k$, is a universal unfolding of $f(x, y)$. In $F(x, y, \alpha)$, the classification of small perturbations proceeds by determining parameter values where singularity types change. In the parameter space $\alpha \in \mathbf{R}^k$ of $F(x, y, \alpha)$, there are six varieties where such changes occur. These varieties are based on degeneracies of ESS, CvSS, MIS, NIS, bifurcation, and tangency. Bifurcation points occur at parameter values where the off-diagonal zero set $F(x, y, \alpha) = 0$ is singular (that is, at points where $F = F_x = F_y = 0$). Tangency points in parameter space occur at certain parameter values when $F(x, y, \alpha) = 0$ and $F(y, x, \alpha) = 0$ become tangent at an off-diagonal point. Note that we are studying dimorphism equivalence, so we consider a pair of universal unfolding fitness functions $(F(x, y, \alpha), F(y, x, \alpha))$. Denote

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$$F(x, y, \alpha) = (x - y)G(x, y, \alpha)$$

Then we define

$$\begin{aligned}
\mathscr{E} &= \{\alpha \in \mathbf{R}^k : \exists\, x \text{ such that } F_y = F_{yy} = 0 \text{ at } (x, x, \alpha)\} \quad \textit{(ESS variety)} \\
\mathscr{C} &= \{\alpha \in \mathbf{R}^k : \exists\, x \text{ such that } F_y = F_{yy} - F_{xx} = 0 \text{ at } (x, x, \alpha)\} \quad \textit{(CvSS variety)} \\
\mathscr{M} &= \{\alpha \in \mathbf{R}^k : \exists\, x \text{ such that } F_y = F_{yy} + F_{xx} = 0 \text{ at } (x, x, \alpha)\} \quad \textit{(MIS variety)} \\
\mathscr{N} &= \{\alpha \in \mathbf{R}^k : \exists\, x \text{ such that } F_y = F_{xx} = 0 \text{ at } (x, x, \alpha)\} \quad \textit{(NIS variety)} \\
\mathscr{B} &= \{\alpha \in \mathbf{R}^k : \exists\, x, y \text{ such that } F = F_x = F_y = 0 \text{ at } (x, y, \alpha) \text{ where } x \neq y\} \\
&\quad (Bifurcation\ \ variety) \\
\mathscr{T} &= \{\alpha \in \mathbf{R}^k : \exists\, x, y \text{ such that } G_x(x, y)G_y(y, x) - G_y(x, y)G_x(y, x) = \\
&\quad G(x, y) = G(y, x) = 0 \text{ at } (x, y, \alpha) \text{ where } x \neq y\} \quad \textit{(Tangency variety)}
\end{aligned} \tag{3.1}$$

Note that the first four varieties in (3.1) detect parameter values in the universal unfolding where the properties ESS, CvSS, MIS, and NIS are degenerate. The bifurcation variety detects parameter values where for fixed $\alpha \neq 0$ the set $F(x, y, \alpha) = 0$ is not a simple curve near the diagonal (for example, the curve crosses itself). The tangency variety detects parameter values α where the curves $F(x, y, \alpha) = 0$ and $F(y, x, \alpha) = 0$ are tangent away from the diagonal.

Definition 3.1 The transition variety is the union of the varieties listed in (3.1). That is

$$\mathscr{T}\mathscr{V} = \mathscr{E} \cup \mathscr{C} \cup \mathscr{M} \cup \mathscr{N} \cup \mathscr{B} \cup \mathscr{T}$$

We can simplify the calculation of transition variety. Since the universal unfoldings of fitness functions vanish on the diagonal, we can define

$$F(x, y, \alpha) = P(u, w, \alpha)w + Q(u, w, \alpha)v.$$

Therefore, with direct calculation, we obtain

$$\begin{aligned}
\mathscr{E} &= \{\alpha \in \mathbf{R}^k : \exists\, u \text{ such that } Q = P - Q_u = 0 \text{ at } (u, 0, \alpha)\} \\
\mathscr{C} &= \{\alpha \in \mathbf{R}^k : \exists\, u \text{ such that } Q = Q_u = 0 \text{ at } (u, 0, \alpha)\} \\
\mathscr{M} &= \{\alpha \in \mathbf{R}^k : \exists\, u \text{ such that } Q = P = 0 \text{ at } (u, 0, \alpha)\} \\
\mathscr{N} &= \{\alpha \in \mathbf{R}^k : \exists\, u \text{ such that } Q = P + Q_u = 0 \text{ at } (u, 0, \alpha)\} \\
\mathscr{B} &= \{\alpha \in \mathbf{R}^k : \exists\, u, w \text{ such that} \\
&\quad Pv + Q = P_u v + Q_u = 2P_w w + P + 2Q_w v = 0 \text{ at } (u, v, \alpha) \text{ where } v \neq 0\} \\
\mathscr{T} &= \{\alpha \in \mathbf{R}^k : \exists\, u, w \text{ such that} \\
&\quad P = Q = 2w(P_u Q_w - P_w Q_u) - P Q_u = 0 \text{ at } (u, v, \alpha) \text{ where } v \neq 0\}
\end{aligned} \tag{3.2}$$

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Remark 3.2 This paper began with the study of ESS singularities, CvSS singularities, and dimorphisms. We showed, however, that in addition to these singular properties, dimorphism equivalence also preserves the NIS property. Thus, in the MIPs, we identify NIS and NIS* as different singularities.

3.2 MIPs of topological codimension two singularities

In this subsection, we classify perturbations of topological codimension two singularities. Each type of perturbation is described by a mutual invasibility plot.

(f) Table 4 shows that a universal unfolding of $h = \epsilon(\delta w^3 + uv)$ is

$$H = \epsilon((a + bw + \delta w^2)w + uv)$$

where a, b are unfolding parameters near 0. Note that

$$P(u, w) = \epsilon(a + bw + \delta w^2) \qquad Q(u, w) = \epsilon u$$

Using (3.2) we see that the transition variety of H is:

$$\mathscr{M} = \{a = 0\} \qquad \mathscr{T} = \{b^2 - 4\delta a = 0,\ \delta b < 0\} \qquad \mathscr{E} = \mathscr{C} = \mathscr{N} = \mathscr{B} = \emptyset$$

In principle, there are four sets of MIPs corresponding to the cases $\varepsilon = \pm 1$ and $\delta = \pm 1$. However, using the transformations described in Remark 2.1 we can assume $\varepsilon = +1 = \delta$. (Note: to transform δ from -1 to 1 we also need to transform (a, b) to $(-a, -b)$.) Figure 5 contains the transition variety and the MIPs of H up to dimorphism equivalence for a, b close to 0. We can see the emergence of regions of coexistence as the unfolding parameters are varied.

(g) Table 4 shows that a universal unfolding of $h = \epsilon(w + (\delta u^3 + \lambda_0 u^5)v)$ is

$$H = \epsilon(w + (a + bu + \delta u^3 + \lambda u^5)v)$$

where a, b are unfolding parameters near 0 and λ is a modal parameter near λ_0. Note that

$$P = \epsilon \qquad Q = \epsilon(a + bu + \delta u^3 + \lambda u^5)$$

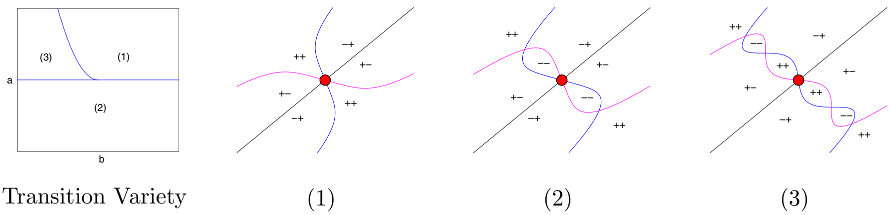

Fig. 5 Transition variety $\{a = 0\} \cup \{a = \frac{b^2}{4}; b < 0\}$. MIPs of $H = (a + bw + w^2)w + uv$

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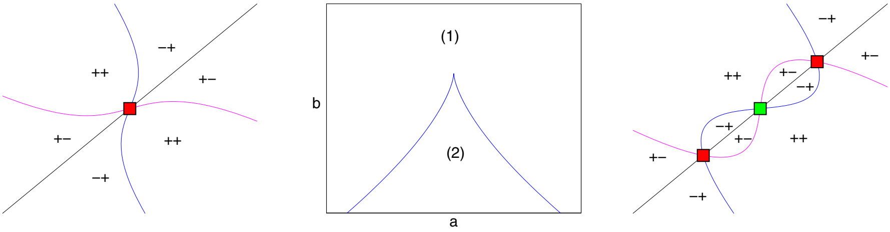

(1) $27a^2 + 4\delta b^3 > 0$ Transition Variety (2) $27a^2 + 4\delta b^3 < 0$

Fig. 6 Transition variety and MIPs of $H = w + (a + bu + u^3)v$

The transition variety of H is given by:

$$\mathscr{C} = \{27a^2 + 4\delta b^3 + o(\lambda b^3) = 0\} \qquad \mathscr{E} = \mathscr{M} = \mathscr{N} = \mathscr{B} = \mathscr{T} = \emptyset$$

In a similar way to the normal form (f), we can derive the MIPs for $\epsilon = -1$ or $\delta = -1$ from those of $\epsilon = +1 = \delta$. Figure 6 contains transition variety and MIPs of H up to dimorphism equivalence when $\lambda = 0$, $\epsilon = 1$, $\delta = 1$. The role of λ is discussed in Remark 3.3.

Remark 3.3 When $\lambda \neq 0$, the transition variety is tilted to the left or the right and becomes a modified cusp. But the MIPs will be similar in the sense that dimorphisms and properties of singularities within each region of parameter space stay the same as those of the case $\lambda = 0$.

(h) Table 4 shows that a universal unfolding of $h = \epsilon(uw + (\alpha_0 w + \beta_0 u^2)v)$ is

$$H = \epsilon((a + u)w + (b + \alpha w + \beta u^2)v)$$

where a, b are unfolding parameters near 0 and α, β are modal parameter near α_0, β_0. Note that

$$P = \epsilon(a + u) \qquad Q = \epsilon(b + \alpha w + \beta u^2)$$

The non-degeneracy conditions in Table 4 shows $\alpha \neq 0$, $\beta \neq 0$, $4\alpha\beta - 1 \neq 0$. Thus, the transition variety of H is the union of four parabolas and a line given by

$$\begin{aligned} \mathscr{E} &= \{\epsilon a^2\beta + b(1 - 2\beta)^2 = 0\} & \mathscr{C} &= \{b = 0\} \\ \mathscr{M} &= \{\epsilon a^2\beta + b = 0\} & \mathscr{N} &= \{\epsilon a^2\beta + b(1 + 2\beta)^2 = 0\} \\ \mathscr{B} &= \{\epsilon a^2\beta - b(4\alpha\beta - 1) = 0\} & \mathscr{T} &= \emptyset \end{aligned}$$

Degeneracies in the transition variety occur when either two parabolas coincide or when a parabola degenerates into a line as the modal parameters α, β are varied. Specifically:

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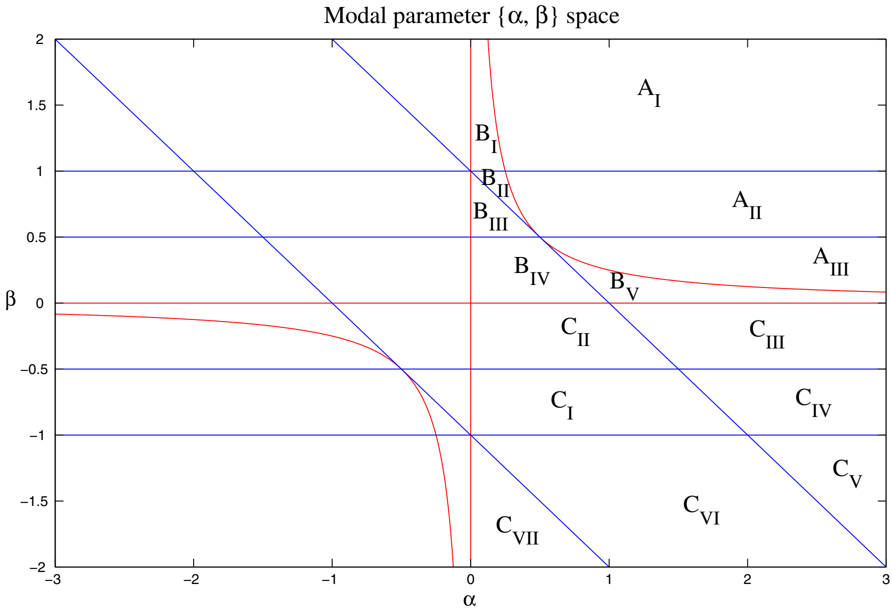

Fig. 7 The modal parameter space (α, β) for $H = (a+u)w + (b+\alpha w+\beta u^2)v$. *Red curves* correspond to the non-degeneracy conditions in Table 4; *blue curves* correspond to degeneracies of the transition variety (see (3.3)). (Note that only the regions in the half plane $\alpha > 0$ are enumerated. The corresponding transition varieties for the enumerated regions are displayed in Fig. 8) (color figure online)

$$\begin{aligned} &\text{(i) If } \beta = \pm\tfrac{1}{2}, \text{ the parabola} \mathscr{E} \text{ or } \mathscr{N} \text{ degenerates to the line } a = 0; \\ &\text{(ii) If } \beta = \pm 1, \text{ the parabola} \mathscr{E} \text{ or } \mathscr{N} \text{ coincides with the parabola} \mathscr{M}; \\ &\text{(iii) If } \alpha + \beta = \pm 1, \text{ the parabola} \mathscr{E} \text{ or } \mathscr{N} \text{ coincides with the parabola} \mathscr{B}. \end{aligned} \tag{3.3}$$

Remark 3.4 When degeneracies in transition variety occur, the singularity's topological codimension is higher than two and we do not consider these more degenerate cases in this paper.

In a similar way to the normal form (f), we can derive the MIPs for $\epsilon = -1$ or $\alpha_0 < 0$ from those of $\epsilon = +1$ and $\alpha_0 > 0$. (Note: to transform α_0 from positive to negative we need to simultaneously transform all parameters $(a, b, \alpha_0, \beta_0)$ to $(-a, -b, -\alpha_0, -\beta_0)$.)

We now consider the transition variety and MIPs of the universal unfolding $H = (a+u)w + (b+\alpha w+\beta u^2)v$ for different parameter values. First, we divide the modal parameter space (α, β) into regions with the same type of transition variety. See Fig. 7. Second, we graph the transition variety of H for different a, b close to 0. Figure 8 plots the transition variety (in the unfolding parameters a, b) for each numbered region in Fig. 7. Last, we draw the MIPs of all small perturbations for each scenario of Fig. 8. The MIPs can be found in Figs. 9, 10, 11.

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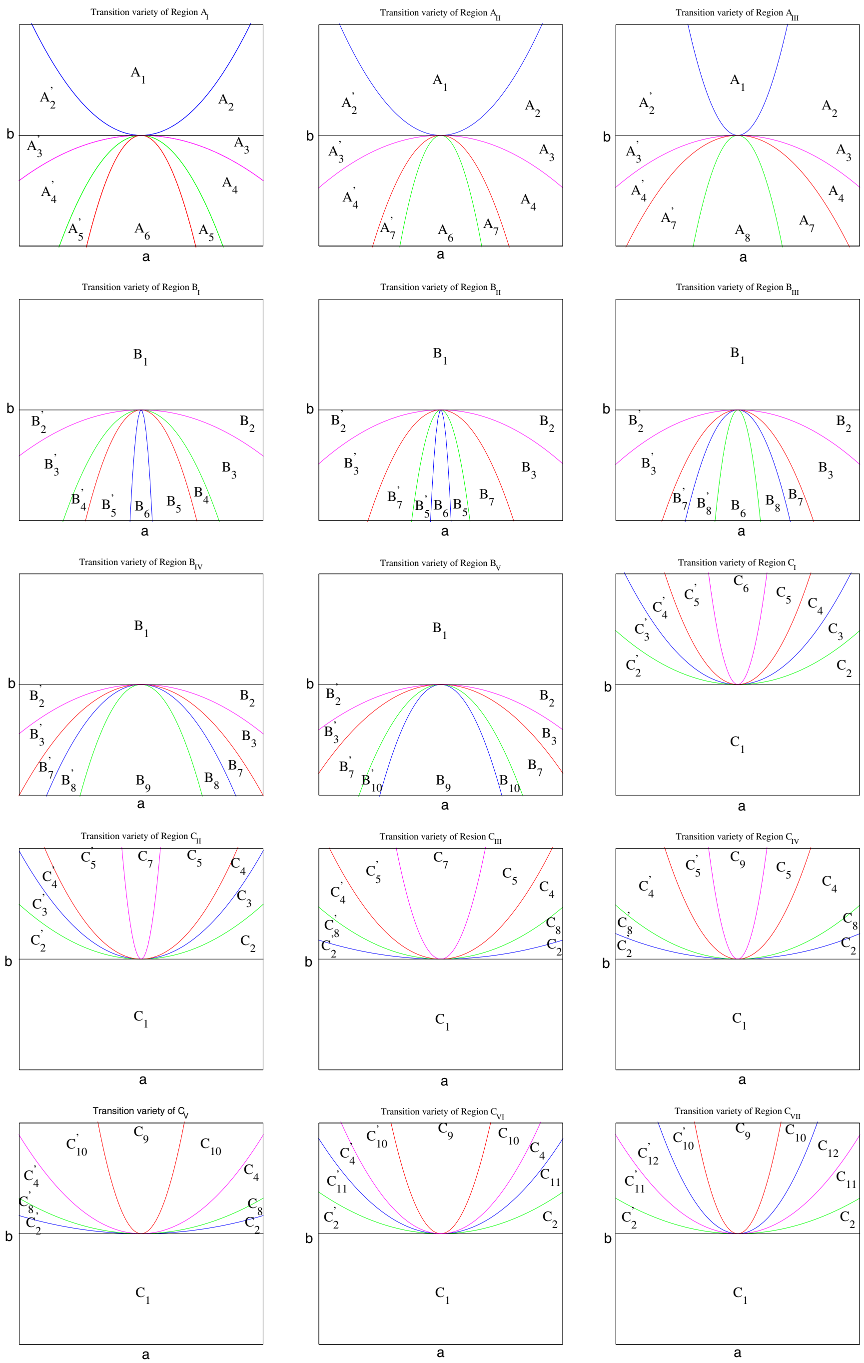

Fig. 8 Transition varieties of $H = (a + u)w + (b + \alpha w + \beta u^2)v$ for different regions in the modal parameters space (α, β) when $\epsilon = 1$, $\alpha > 0$. *Blue* is variety $\mathscr{B}$; *red* is variety $\mathscr{M}$; *green* is variety $\mathscr{E}$; *magenta* is variety $\mathscr{N}$; *black* is variety $\mathscr{C}$ (color figure online)

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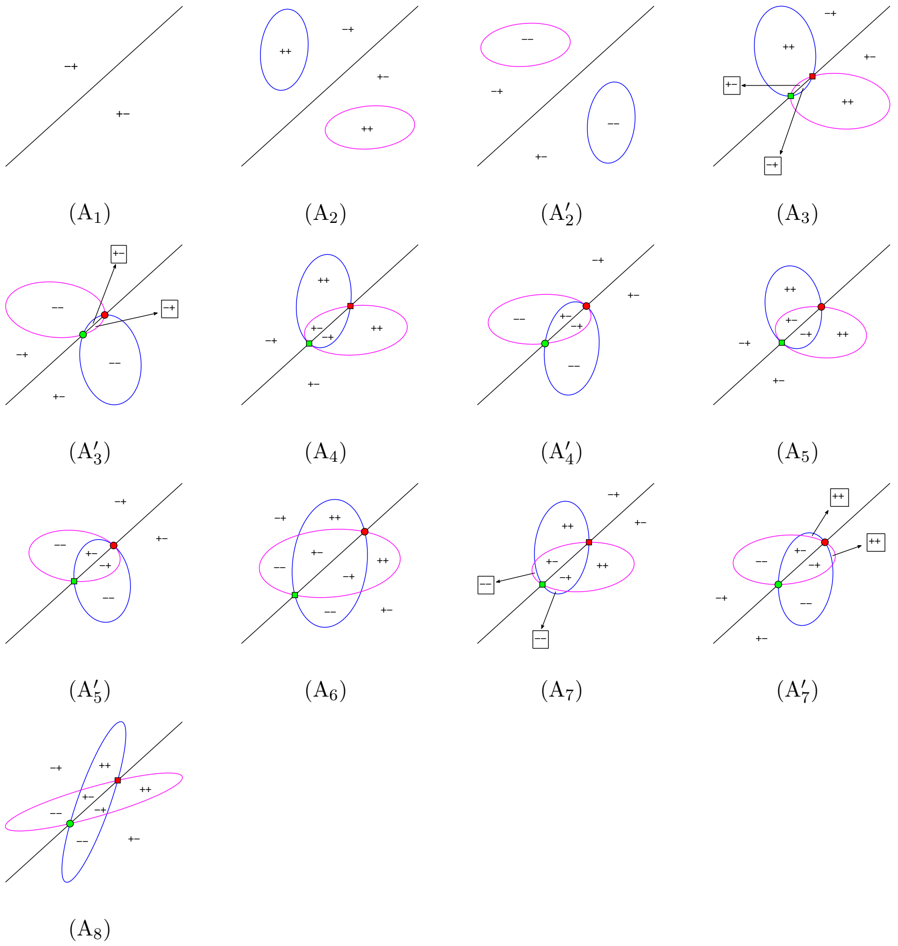

Fig. 9 MIPs for all the non-degenerate perturbation of $H = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions A_i and A'_i

3.3 An example of topological codimension two singularity

Geritz et al. (1999) use adaptive dynamics to study competition between seeds with different sizes when there is a trade-off between seed size and seed number. In particular, they propose a fitness function

$$W_m(m') = f(m')\frac{R}{m'}\left(\sum_{k=0}^{\infty} \frac{c(m')}{c(m') + kc(m)} \times \frac{(N(m))^k}{k} e^{-N(m)}\right)$$

as the fitness of a mutant plant with seed size m' in a monomorphic resident population with seed size m. Here $f(m) = \max\{0, 1 - 2e^{-\beta m}\}$ is the expected reproductive yield per seed as a function of seed size, R is the total amount of resources, $c(m) = e^{\alpha m}$ is a

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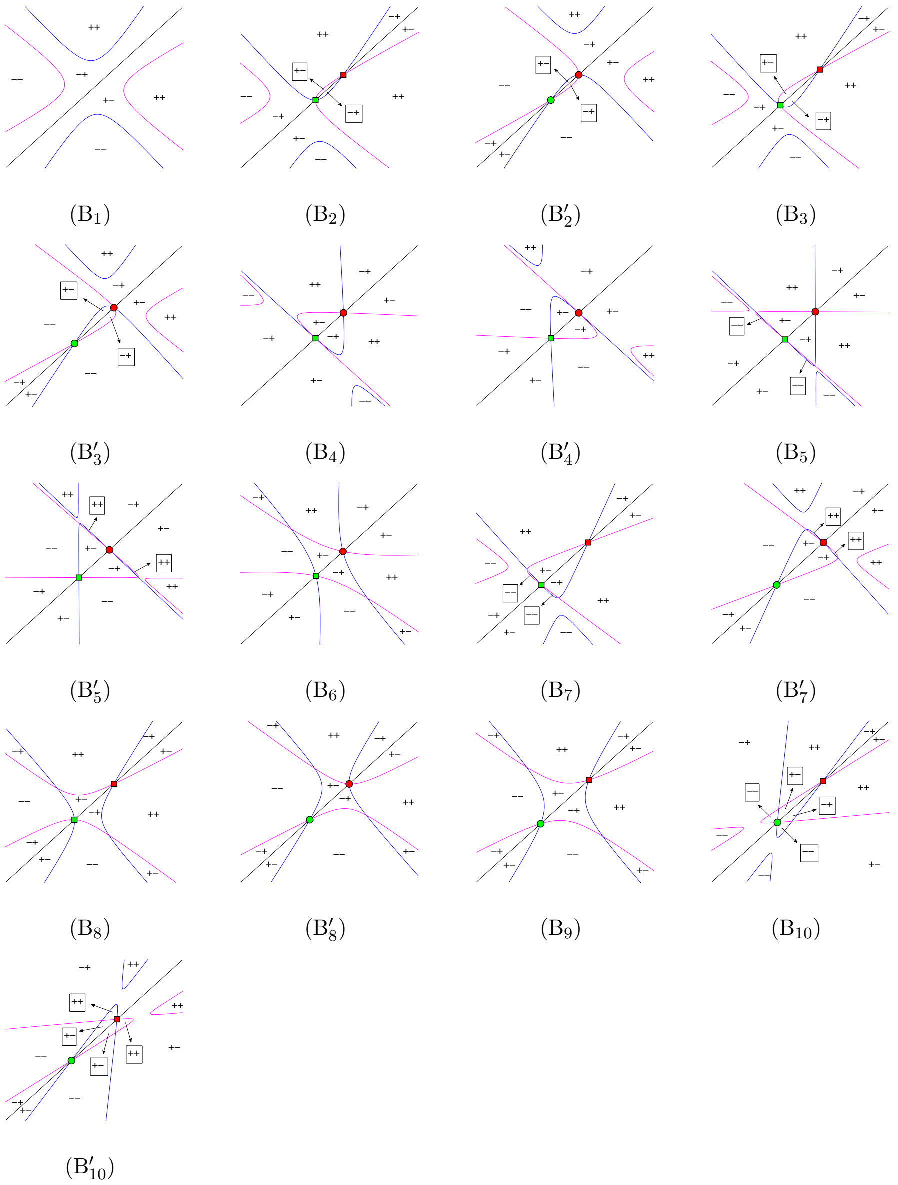

Fig. 10 MIPs for all the non-degenerate perturbation of $H = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions B_i and B'_i

measure of the competitive ability of a seeding with seed size m, and $N(m)$ is the single, positive, and asymptotically stable equilibrium density in a monomorphic resident population with only seeds of size m. Geritz et al. (1999) shows that the fitness function depends, after scaling, on only the two products αR and βR. Specifically, α represents the evolutionary consequences of intermediate levels of competitive asymmetry and β represents the precompetitive environments.

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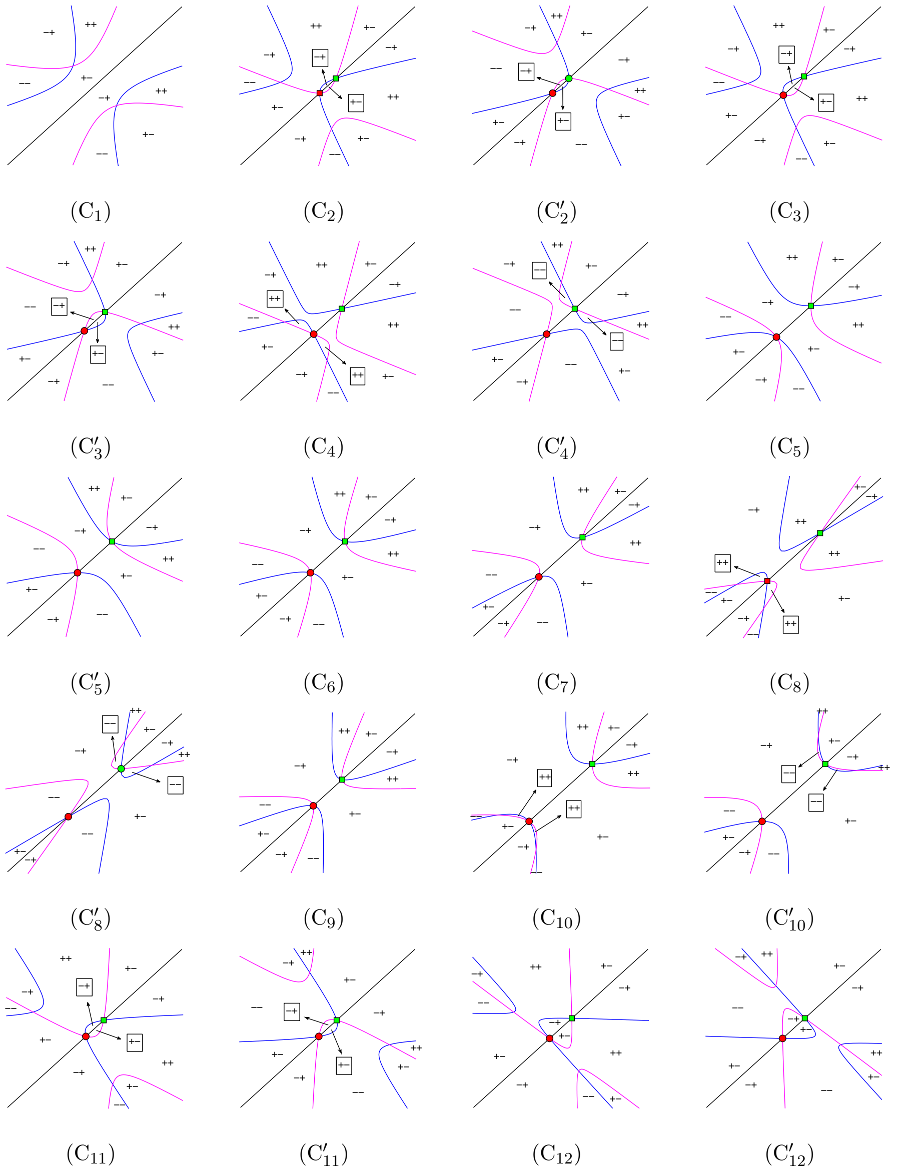

Fig. 11 MIPs for all the non-degenerate perturbation of $H = (a + \epsilon u)w + (b + \alpha w + \beta u^2)v$ when $\{\alpha, \beta\}$ are in regions C_i and C'_i

Figure 12 contains the results of numerical simulations conducted in Geritz et al. (1999) to study the number and properties of singular strategies depending on the values of αR and βR. We claim that the existence of two topological codimension two singularities (namely, (g) and (h)) can be inferred from these plots.

For example, on row $\beta R = 45$ we see the creation of two singular points (at $\alpha R \approx 5.4$) when the closed curve moves across the diagonal. A reasonable inference

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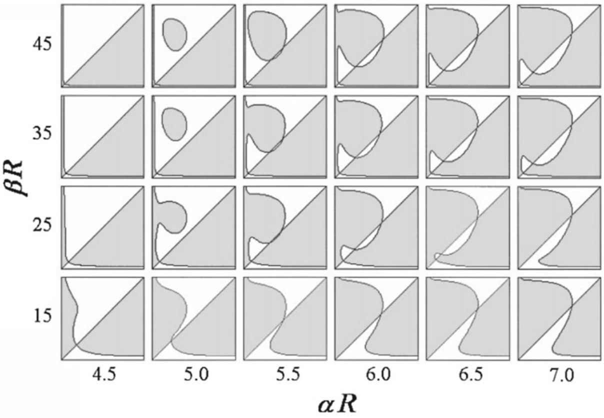

Fig. 12 This figure consists of pairwise invasibility plots of fitness function $W_m(m')$ for different values of αR and βR and is taken from Geritz et al. (1999)

is that these plots are contained in the unfolding of singularity (h). Specifically, we see the MIPS A_1 in Fig. 9 when $\alpha R = 4.5$, A_2 when $\alpha R = 5.0$ and A_3 when $\alpha R = 5.5$. Moreover we also see A_4 when $\alpha R = 7.0$. This suggests that we are near a codimension two singularity of type A_I, A_{II}, or A_{III} in Fig. 8. Figure 9 indicates that as αR continues to change, one of these two newly generated singularities will be ESS* and then change to ESS, which is not shown in Fig. 12 or the analysis in Geritz, etc Geritz et al. (1999).

In addition, in row $\beta R = 25$, Geritz et al. (1999) observe the creation of two singular strategies at $\alpha R \approx 5.5$ and the merger and disappearance of two singular strategies at $\alpha R \approx 6.6$. This is what is seen when varying a with $b < 0$ in the unfolding of singularity (g) in Fig. 6. We therefore infer that varying αR and βR can lead to the existence of a singularity (g) in this fitness function.

4 Dimorphism equivalence

In this section, we review the notion of strategy equivalence (see Definition 4.1) developed in Vutha and Golubitsky (2014). Strategy equivalence preserves ESS and CvSS singularities of fitness functions. However, we show in Example 4.2 that strategy equivalence does not always preserve regions of coexistence. We then introduce a special type of strategy equivalence that does preserve regions of coexistence (Theorem 1.13), which we call dimorphism equivalence (Definition 1.10). To develop dimorphism equivalence, we combine strategy equivalence with concepts motivated by singularity theory with symmetry (see Wang (2015)); the symmetry is just the transposition $(x, y) \mapsto (y, x)$.

Definition 4.1 Two fitness functions f and $\hat{f}$ are strategy equivalent if

$$\hat{f}(x, y) = S(x, y) f(\Phi(x, y))$$

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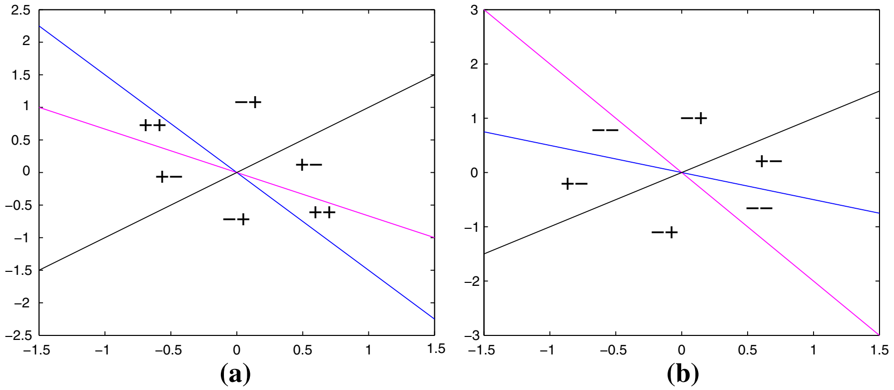

Fig. 13 Fitness functions f and g are strategy equivalent, but have different dimorphism properties because f has regions of coexistence and g does not

where

1. $S(x, y) > 0$ for all x, y.
2. $\Phi \equiv (\Phi_1, \Phi_2)$ where $\Phi_i\colon \mathbf{R}^2 \to \mathbf{R}$, $\det(d\Phi)_{x,y} > 0$ for all x, y.
3. $\Phi(x, x) = (\phi(x), \phi(x))$ for every x where $\phi : \mathbf{R} \to \mathbf{R}$.
4. $\frac{\partial \Phi_1}{\partial y}(x, x) = 0$ for every x.

The following is an example of two fitness functions that are strategy equivalent, but one has regions of coexistence, whereas the other does not. This example implies the need for a strengthened equivalence relation that also preserves regions of coexistence.

Example 4.2 Consider the fitness functions

$$f(x, y) = -(y - x)(2x + y) \quad \text{and} \quad g(x, y) = -(y - x)(x + 2y). \tag{4.1}$$

Vutha and Golubitsky (2014, Theorem 5.5) show that $f(x, y)$ and $g(x, y)$ are strategy equivalent to the normal form $h(x, y) = -(y - x)y$. However, f has regions of coexistence, whereas g does not. See the mutual invasibility plots in Fig. 13.

We note that any two fitness functions that are dimorphism equivalent are also strategy equivalent.

Lemma 4.3 *Let $g(x, y) = f(y, x)$ and $\hat{g}(x, y) = \hat{f}(y, x)$. Suppose that $f(x, y)$ is dimorphism equivalent to $\hat{f}(x, y)$ under (S, Φ). Then g and $\hat{g}$ are also dimorphism equivalent (and hence strategy equivalent) using the same Φ.*

Proof To establish this claim, compute

$$\begin{aligned} \hat{g}(x, y) = \hat{f}(y, x) &= S(y, x) f(\Phi(y, x)) = S(y, x) f(\varphi(y, x), \varphi(x, y)) \\ &= S(y, x) g(\Phi(x, y)) \end{aligned}$$

So g and $\hat{g}$ are dimorphism equivalent using the dimorphism equivalence $(S(y, x), \Phi(x, y))$. □

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Vutha and Golubitsky (2014) note that the coordinate changes defined in strategy equivalence preserve the zero sets of fitness functions. We strengthen that observation as follows.

Lemma 4.4 *If two fitness functions $f(x, y)$ and $\hat{f}(x, y)$ are dimorphism equivalent, that is $\hat{f}(x, y) = S(x, y) f(\Phi(x, y))$, where (S, Φ) satisfies Definition 1.10, then*

1. *$\Phi(x, y)$ maps the zero set of $\hat{f}(x, y)$ to the zero set of $f(x, y)$;*
2. *$\Phi(x, y)$ maps the zero set of $\hat{f}(y, x)$ to the zero set of $f(y, x)$.*

Proof Vutha and Golubitsky (2014) observe that if $f(x, y)$ and $\hat{f}(x, y)$ are strategy equivalent, then the diffeomorphism Φ preserves the zero set of $\hat{f}(x, y)$. That is, Φ maps the zero set of $\hat{f}(x, y)$ to the zero set of $f(x, y)$. Lemma 4.3 implies that $f(y, x)$ and $\hat{f}(y, x)$ are also strategy equivalent using the same Φ, so the diffeomorphism Φ preserves the zero set of $\hat{f}(y, x)$. That is, Φ maps the zero set of $\hat{f}(y, x)$ to the zero set of $f(y, x)$.

Proof of Theorem 1.13 In his thesis Vutha (2013, Proposition 6.6) proves that the properties ESS, CvSS, NIS, and MIS are preserved under strong strategy equivalence. In fact, every dimorphism equivalence is also a strong strategy equivalence, and hence these properties are preserved under dimorphism equivalence. Rather than define strong strategy equivalence, we just include here a direct proof of property preservation under dimorphism equivalence.

Suppose f has a singularity at $(\bar{x}, \bar{x})$ and that $\hat{f} = Sf$. At $(\bar{x}, \bar{x})$ we have

$$\begin{aligned} \hat{f}_{xx} &= S f_{xx} \\ \hat{f}_{yy} &= S f_{yy} \\ \hat{f}_{yy} - \hat{f}_{xx} &= S(f_{yy} - f_{xx}) \\ \hat{f}_{yy} + \hat{f}_{xx} &= S(f_{yy} + f_{xx}) \end{aligned}$$

Hence, all four properties are preserved between f and $\hat{f}$. Next, suppose $\hat{f} = f(\Phi)$ where $\Phi(x, y) = (\varphi(x, y), \varphi(y, x))$. Recall from Definition 1.10 (c) that $\varphi_y(\bar{x}, \bar{x}) \equiv 0$ and that $\varphi_x(\bar{x}, \bar{x}) > 0$. Since $f_x(\bar{x}, \bar{x}) = f_y(\bar{x}, \bar{x}) = 0$, direct calculations show that

$$\begin{aligned} f_{xx}(\varphi(\bar{x}, \bar{x}), \varphi(\bar{x}, \bar{x})) &= \varphi_x(\bar{x}, \bar{x})^2 f_{xx}(\bar{x}, \bar{x}) \\ f_{yy}(\varphi(\bar{x}, \bar{x}), \varphi(\bar{x}, \bar{x})) &= \varphi_x(\bar{x}, \bar{x})^2 f_{yy}(\bar{x}, \bar{x}) \end{aligned} \tag{4.2}$$

Hence

$$\begin{aligned} \hat{f}_{xx} &= \varphi_x^2 f_{xx} \\ \hat{f}_{yy} &= \varphi_x^2 f_{yy} \\ \hat{f}_{yy} - \hat{f}_{xx} &= \varphi_x^2 (f_{yy} - f_{xx}) \\ \hat{f}_{yy} + \hat{f}_{xx} &= \varphi_x^2 (f_{yy} + f_{xx}) \end{aligned}$$

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Therefore, all four properties are also preserved between f and $\hat{f}$. By combining these two cases, we see that ESS, CvSS, NIS, and MIS are all preserved under dimorphism equivalence.

Since $S(x, y) > 0$ and $S(y, x) > 0$, it follows from Lemma 4.4 that Φ maps

$$\{(x, y) : \hat{f}(x, y) > 0 \text{ and } \hat{f}(y, x) > 0\}$$

to

$$\{(x, y) : f(\varphi(x, y), \varphi(y, x)) > 0 \text{ and } f(\varphi(y, x), \varphi(x, y)) > 0\}.$$

That is, dimorphism equivalence preserves regions of coexistence for fitness functions. □

Remark 4.5 In fact, we have shown that dimorphism equivalences simultaneously preserve $\mathrm{sgn}(f(x, y))$ and $\mathrm{sgn}(g(x, y))$, where $g(x, y) \equiv f(y, x)$. Thus, there are four types of region based on the signs of $f(x, y)$ and $g(x, y)$ and dimorphism equivalence preserves these types.

5 Tangent spaces

In this section, we discuss the question: when is $f + \eta$ strongly dimorphism equivalent to f for a small perturbation η? (Recall that strong dimorphism equivalence is defined in Definition 1.10). We assume that f and η are fitness functions and $(0, 0)$ is a singularity for both. To do so, we introduce the restricted tangent space: the subspace of all fitness functions η that satisfy $f + t\eta$ is strongly dimorphism equivalent to f for all small t. We find the general form for the fitness function η in Proposition 5.4. Using the form of η we state the tangent space constant theorem (Theorem 5.5) in the context of strong dimorphism equivalence.

We find that the calculations are more easily done in the coordinate system

$$u = x + y \qquad v = x - y \qquad w = v^2 \tag{5.1}$$

5.1 The form of fitness functions

Lemma 5.1 *Every smooth fitness function $f(u, v)$ has the form*

$$f = p(u, w)w + q(u, w)v,$$

where $p, q : \mathbf{R}^2 \to \mathbf{R}$ are smooth functions.

Proof Every real-valued smooth function $a(u, v)$ can be written in the form

$$a(u, v) = b(u, w) + c(u, w)v \tag{5.2}$$

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where b and c are smooth functions. Equation (5.2) just states that every function in u, v is a sum of an even function in v and an odd function in v. A proof for smooth functions is given in (Golubitsky and Guillemin 1974, p.108). Let f be a fitness function. Since $f(x, x) = 0$ or in (u, v) coordinates $f(u, 0) = 0$, it follows from (5.2) that f can be written in the form

$$f = b(u, w) + q(u, w)v$$

where $b(u, 0) = 0$. By Taylor's Theorem we can write $b(u, w) = p(u, w)w$. □

As noted after (2.1) we identify fitness functions $f = pw + qv$ with

$$[p, q] \in \mathcal{E}^2,$$

where $\mathcal{E}$ is the space of (germs of) smooth real-valued functions in (u, w) coordinates. Later we use the notation $\mathcal{M}$ to indicate the maximal ideal in $\mathcal{E}$ consisting of functions that vanish at the origin.

5.2 Tangent space and restricted tangent Space

Let f be a fitness function with a singularity at the origin. Let Φ_t be a one-parameter family of dimorphism equivalences such that Φ_0 is the identity equivalence. The tangent space $\mathrm{T}(f)$ is the set of functions (more precisely, germs at the origin) of the form

$$\left.\frac{d}{dt}\Phi_t(f)\right|_{t=0}$$

The restricted tangent space $\mathrm{RT}(f)$ is obtained by assuming that Φ_t is an origin preserving strong dimorphism equivalence. Recall that if the diffeomorphism Φ_t induces a dimorphism equivalence then $\Phi_t = (\varphi(x, y, t), \varphi(y, x, t))$ where $\varphi_y(x, x, t) = 0$. Also, if $\Phi_0 = I$, then $\varphi(x, y, 0) = x$. Moreover, if Φ_t is a strong dimorphism equivalence, then $\varphi_x(x, x, t) = 1$ and if Φ is origin preserving, then $\varphi(0, 0, t) = 0$.

Lemma 5.2 *Let Φ_t be a one-parameter family of origin preserving strong dimorphism equivalence diffeomorphisms with $\Phi_0 = I$. In (u, v) coordinates*

$$\varphi(u, v, t) = \varphi^e(u, w, t) + \varphi^o(u, w, t)v$$

where

$$\varphi_y(x, x, t) = 0 \quad \varphi_x(x, x, t) = 1 \quad \varphi(x, y, 0) = x \quad \varphi(0, 0, t) = 0. \tag{5.3}$$

Then φ^o and φ^e have the form

$$\varphi^e(u, w, t) = \frac{1}{2}u + twh^e(u, w, t) \quad \text{and} \quad \varphi^o(u, w, t) = \frac{1}{2} + twh^o(u, w, t) \tag{5.4}$$

where h^e and h^o are arbitrary functions.

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Proof The first two restrictions in (5.3) imply that

$$\varphi^o(u,0,t) = (\varphi^e)_u(u,0,t) = \frac{1}{2}.$$

Hence

$$\varphi^e(u,w,t) = a(t) + \frac{1}{2}u + g^e(u,w,t)w \qquad \varphi^o(u,w,t) = \frac{1}{2} + g^o(u,w,t)w.$$

Restricting to $x = y = 0$ (equivalently $u = v = 0$) yields

$$\varphi(0,0,t) = a(t) = 0.$$

Thus

$$\varphi^e(u,w,t) = \frac{1}{2}u + g^e(u,w,t)w \qquad \varphi^o(u,w,t) = \frac{1}{2} + g^o(u,w,t)w$$

where g^e and g^o are arbitrary. Finally, restricting to $t = 0$ yields

$$x = \varphi(u,v,0) = \frac{1}{2}u + g^e(u,w,0)w + \frac{1}{2}v + g^o(u,w,0)wv$$

Since $u + v = 2x$, it follows that

$$g^e(u,w,0) + g^o(u,w,0)v = 0$$

But since the first term is even in v and the second term is odd in v, it follows that

$$g^e(u,w,0) = g^o(u,w,0) = 0.$$

Hence, by Taylor's Theorem we can factor out a t from g^e and g^o, and (5.4) is verified where h^e and h^o are arbitrary. □

Lemma 5.3 *Consider the map $Z(x,y) = (x^j, y^j)$. Then, in (u,v) coordinates*

$$z(u,v) = \left(\frac{u+v}{2}\right)^j \qquad z^e(u,v) = \left(\frac{u}{2}\right)^j + a(u,w)w$$
$$z^o(u,v) = \frac{j}{2}\left(\frac{u}{2}\right)^{j-1} + b(u,w)w$$

for some polynomials a and b.

Proof Write $Z(x,y) = (z(u,v), z(u,-v))$ and $z(u,v) = z^e(u,w) + z_0(u,w)v$. Since $x = (u+v)/2$ we see that $z(u,v) = (\frac{u+v}{2})^j$. We decompose z into even and odd parts by

$$z(x,y) = \frac{1}{2}(x^j + y^j) + \frac{1}{2}(x^j - y^j).$$

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Converting to uv-coordinates yields

$$z(u, v) = \left(\frac{1}{2}\right)^{j+1} ((u+v)^j + (u-v)^j) + \left(\frac{1}{2}\right)^{j+1} ((u+v)^j - (u-v)^j).$$

Use the binomial expansion on the even part, note that terms with odd powers in v cancel, and observe that all terms after the first have a $v^2 = w$ factor. Hence

$$z^e(u, w) = \left(\frac{1}{2}\right)^j u^j + a(u, w)w.$$

Similarly, use the binomial expansion on the odd part, note that terms with even powers in v cancel, and observe that all nonzero terms after the first have a $v^3 = wv$ factor. Hence

$$z^o(u, w)v = \left(\frac{1}{2}\right)^j ju^{j-1}v + b(u, w)wv.$$

Therefore, z^e and z^o have the forms as claimed. □

Define the submodule $\mathcal{I}(f)$ of $\mathcal{E}^2$ by

$$\mathcal{I}(f) = \langle [p, q], [q, wp], [wp_u, wq_u], [wp + 2w^2 p_w, 2w^2 q_w] \rangle \tag{5.5}$$

and the vector subspace of $\mathcal{E}^2$ by

$$V(f) = \mathbf{R}\{u^{j-1}[jp + up_u + 2jwp_w, uq_u + 2jwq_w], [p_u, q_u]\} \tag{5.6}$$

for $j = 1, 2, \ldots$.

Proposition 5.4 *Let* $f = pw + qv$*, then* $\mathrm{RT}(f) = \mathcal{I}(f)$ *and* $\mathrm{T}(f) = \mathcal{I}(f) + V(f)$.

Proof We calculate the two tangent spaces in order.

Computation of $\mathrm{RT}(f)$ The restricted tangent space is obtained using two kinds of strong dimorphism equivalences

$$S(x, y, t) f(x, y) \quad \text{and} \quad f(\varphi(x, y, t), \varphi(y, x, t)) \tag{5.7}$$

where $S(x, y, 0) = 1$ and φ satisfies (5.3) and (5.4).

Consider the first type of equivalences in (u, v) coordinates, we can assume that

$$S = S^e(u, w) + S^o(u, w)v$$

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where $S^e(0,0) > 0$. Then the tangent vectors given by this type of equivalences can be computed by

$$\begin{aligned}\frac{d}{dt}S(x,y,t)f(x,y)\Big|_{t=0} &= \dot{S}(x,y,0)f(x,y)\\ &= (\dot{S}^e + \dot{S}^o v)(pw+qv)\\ &= \dot{S}^e(pw+qv) + \dot{S}^o(qw+wpv)\end{aligned}$$

where $\cdot$ is the derivative with respect to t. Since $\dot{S}^e$ and $\dot{S}^o$ are arbitrary functions we see that the submodule

$$\langle [p,q],[q,wp]\rangle \subset RT(f)$$

Differentiating the second strong dimorphism equivalence in (5.7) with respect to t and evaluating at $t=0$ gives

$$\dot{\varphi}(x,y,0)f_x(x,y) + \dot{\varphi}(y,x,0)f_y(x,y) \tag{5.8}$$

where $\cdot$ is the derivative with respect to t and φ satisfies the requirements in (5.3). Since $\varphi(y,x,t) = \varphi^e(u,w,t) - \varphi^o(u,w,t)v$, we compute

$$u(\Phi(x,y,t)) = 2\varphi^e \qquad v(\Phi(x,y,t)) = 2\varphi^o v \qquad w(\Phi(x,y,t) = 4(\varphi^o)^2 w \tag{5.9}$$

Then the tangent vectors given by this type of equivalences can be computed using (5.9) as

$$\begin{aligned}&\frac{d}{dt}f(\varphi(x,y,t),\varphi(y,x,t))\\ &\quad= \frac{d}{dt}(p(u(\Phi),v(\Phi)^2)v(\Phi)^2 + q(u(\Phi),v(\Phi)^2)v(\Phi))\\ &\quad= \frac{d}{dt}(4p(2\varphi^e,4(\varphi^o)^2w)(\varphi^o)^2w + 2q(2\varphi^e,4(\varphi^o)^2w)\varphi^o v)\\ &\quad= 8p_1(\varphi^o)^2w\dot{\varphi^e} + 32p_2w^2(\varphi^o)^3\dot{\varphi^o} + 8pw\varphi^o\dot{\varphi^o} + 4q_1\varphi^o v\dot{\varphi^e}\\ &\qquad + 16q_2wv(\varphi^o)^2\dot{\varphi^o} + 2qv\dot{\varphi^o}\\ &\quad= \dot{\varphi^e}(8p_1(\varphi^o)^2w + 4q_1\varphi^o v) + \dot{\varphi^o}(32p_2w^2(\varphi^o)^3\\ &\qquad + 8pw\varphi^o + 16q_2wv(\varphi^o)^2 + 2qv)\end{aligned} \tag{5.10}$$

where p_i, q_i are the derivative of p, q respect to the i^{th} component ($i=1,2$) and $\cdot$ is the derivative with respect to t. It follows from (5.4) that $\varphi^o = \frac{1}{2}$ at $t=0$. Therefore,

$$\frac{d}{dt}f(\varphi(x,y,t),\varphi(y,x,t))\Big|_{t=0} = 2\dot{\varphi^e}(p_u w + q_u v) + 4\dot{\varphi^o}(p_w w^2 + pw + q_w wv + \frac{1}{2}qv) \tag{5.11}$$

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In addition, it also follows from (5.4) that

$$\dot{\varphi^e} = \frac{d}{dt}\varphi^e(u, w, t)\bigg|_{t=0} = h^e(u, w, 0)w$$
$$\dot{\varphi^o} = \frac{d}{dt}\varphi^o(u, w, t)\bigg|_{t=0} = h^o(u, w, 0)w$$

where h^e and h^o are arbitrary. Hence we can take $[wp_u, wq_u]$ and $[2w^2 p_w + 2wp, 2w^2 q_w + wq] - [wp, wq]$ to be the remaining two generators of $\mathcal{I}(f)$ and $\mathrm{RT}(f)$.

Computation of $\mathrm{T}(f)$ We claim that there are two types of dimorphism equivalence diffeomorphisms that were not used in the computation of $\mathrm{RT}(f)$ above. The diffeomorphisms are defined by:

$$\begin{aligned}\Phi(x, y) &= (x + a, y + a)\\ \Phi(x, y) &= (C(x), C(y)).\end{aligned} \tag{5.12}$$

To see this, first let Φ be a diffeomorphisn inducing a dimorphism equivalence and let $\Phi(0, 0) = (b, b)$. Let $\Psi(x, y) = (x - b, y - b)$. Then $\Psi\Phi$ is an origin preserving diffeomorphism that induces a dimorphism equivalence.

Second, let $\Phi(x, y) = (\varphi(x, y), \varphi(y, x))$ be an origin preserving diffeomorphism and let $C(x)$ solve the ODE

$$C'(x) = \frac{1}{\varphi_x(C(x), C(x))}.$$

Let

$$\Psi(x, y) = (C(x), C(y)) \tag{5.13}$$

Then $\Phi\Psi$ is a diffeomorphism that induces a strong dimorphism equivalence.

To determine the tangent vector that derives from translations, we compute

$$\begin{aligned}\frac{d}{dt} f(u + a(t), v, t)\bigg|_{t=0} &= \frac{d}{dt}(p(u + a(t), w)w + q(u + a(t), w)v)\bigg|_{t=0}\\ &= a'(0)(p_u w + q_u v).\end{aligned}$$

It follows that $\mathbf{R}\{[p_u, q_u]\} \subset V(f)$.

To determine the tangent vectors that derive from this class of Ψ we write $\psi(u, v, t) = C(\frac{1}{2}(u + v), t)$ as $\psi = \psi^e(u, v, t) + \psi^o(u, v, t)v$, where $\Psi(\cdot, \cdot, 0)$ is the identity map. It follows from (5.13) that $C(x, 0) = x$. Note that we can decompose ψ into even and odd parts by

$$\psi(x, y, t) = \frac{1}{2}(C(x, t) + C(y, t)) + \frac{1}{2}(C(x, t) - C(y, t)).$$

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Hence the odd part at $t = 0$ is $\frac{1}{2}v$ and $\psi^o(u, v, 0) = \frac{1}{2}$. Using (5.11) we see that

$$\frac{d}{dt} f(\psi(x, y, t), \psi(y, x, t))\bigg|_{t=0} = 2\dot{\psi}^e(p_u w + q_u v) + 4\dot{\psi}^o \left(p_w w^2 + pw + q_w wv + \frac{1}{2}qv\right). \tag{5.14}$$

We now need to determine the forms of $\dot{\psi}^e$ and $\dot{\psi}^o$. In fact

$$\psi^e = \frac{1}{2}\left(C\left(\frac{u+v}{2}, t\right) + C\left(\frac{u-v}{2}, t\right)\right) \quad \text{and}$$
$$\psi^o v = \frac{1}{2}\left(C\left(\frac{u+v}{2}, t\right) - C\left(\frac{u-v}{2}, t\right)\right)$$

Hence

$$\dot{\psi}^e(u, v, 0) = \frac{1}{2}\left(\dot{C}\left(\frac{u+v}{2}, 0\right) + \dot{C}\left(\frac{u-v}{2}, 0\right)\right) \text{ and}$$
$$\dot{\psi}^o(u, v, 0)v = \frac{1}{2}\left(\dot{C}\left(\frac{u+v}{2}, 0\right) - \dot{C}\left(\frac{u-v}{2}, 0\right)\right)$$

There is one generator of $V(f)$ for each j and that generator is given by $\dot{C}(x) = 2^j x^j$ for $j = 1, 2, \ldots$. It follows from Lemma 5.3 (where $Z = \dot{C}$) that there is one generator of $V(f)$ given by

$$\dot{\psi}^e = u^j + a(u, w)w \qquad \text{and} \qquad \dot{\psi}^o = ju^{j-1} + b(u, w)w$$

for some polymials a and b. We are allowed to alter these generators by adding elements from $\mathcal{I}(f)$.

From (5.14) the new generators of $\mathrm{T}(f)$ can be taken to be

$$\dot{\psi}^e[p_u, q_u] + \dot{\psi}^o[2p_w w + 2p, 2q_w w + q]$$

Since $w[p_u, q_u]$ is in the submodule $\mathcal{I}(f)$ we can eliminate a from $\dot{\psi}^e$. Since

$$w[2wp_w + 2p, 2wq_w + q] = [wp + 2w^2 p_w, 2w^2 q_w] + [wp, wq]$$

is in $\mathcal{I}(f)$, we can eliminate the b from $\dot{\psi}^o$. Therefore, the generators of $\mathrm{T}(f)$ have the form

$$u^j[p_u, q_u] + ju^{j-1}[2wp_w + 2p, 2wq_w + q].$$

Finally, since $[p, q] \in \mathcal{I}(f)$, we can write the generators of $\mathrm{T}(f)$ modulo $\mathcal{I}(f)$ as

$$u^{j-1}([up_u, uq_u] + j[2wp_w + p, 2wq_w]) \qquad j = 1, 2, \ldots$$

as claimed in (5.6). □

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5.3 Modified tangent space constant theorem

The definition of $RT(f)$ implies that if $f + t\eta$ is dimorphism equivalent to f for all small t, then $\eta \in RT(f)$. In the converse direction, we have Theorem 5.5.

Theorem 5.5 (Modified tangent space constant theorem) *Let f be a fitness function. If*

$$\mathrm{RT}(f + t\eta) = \mathrm{RT}(f) \ \ for\ all\ t \in [0, 1] \tag{5.15}$$

Then $f + t\eta$ is strongly dimorphism equivalent to f for all $t \in [0, 1]$.

The proof is in Wang (2015). The proof is standard in singularity theory and, for example, is a small modification of an analogous theorem in bifurcation theory (see Golubitsky and Schaeffer 1985, Chapter 2, Theorem 2.2).

6 Universal unfoldings under dimorphism equivalence

In this section we sketch the universal unfolding theory of dimorphism equivalence.

Definition 6.1 Let f be a C^∞ fitness function $\mathbf{R}^2 \to \mathbf{R}$ defined on a neighborhood of the origin. Then $F : \mathbf{R}^2 \times \mathbf{R}^k \to \mathbf{R}$ is a k-parameter unfolding of f if

$$F(x, y, 0) = f(x, y) \qquad F(x, x, \alpha) = 0$$

where the parameter $\alpha \in \mathbf{R}^k$.

Definition 6.2 Let $F(x, y, \alpha)$ be a k-parameter unfolding of f and let $H(x, y, \beta)$ be an l-parameter unfolding of f. We say that H factors through F if there exists maps $S : \mathbf{R}^2 \times \mathbf{R}^l \to \mathbf{R}$, $\Phi : \mathbf{R}^2 \times \mathbf{R}^l \to \mathbf{R}^2$, and $A : \mathbf{R}^l \to \mathbf{R}^k$ such that

$$H(x, y, \beta) = S(x, y, \beta) F(\Phi(x, y, \beta), A(\beta))$$

where

1. $S(x, y, 0) = 1$
2. $\Phi(x, y, 0) = (x, y)$
3. $\Phi(x, y, \beta) = (\varphi(x, y, \beta), \varphi(y, x, \beta))$ where $\varphi : \mathbf{R}^2 \times \mathbf{R}^l \to \mathbf{R}$
4. $(d\Phi)_{x,x,\beta} = c(x, \beta) I_2$ where $c(x, \beta) > 0$
5. $A(0) = 0$

Remark 6.3 We do not require that $\Phi(0, 0, \beta) = (0, 0)$; that is, when β is nonzero, the equivalence need not preserve the origin.

Definition 6.4 An unfolding F of f is versal if every unfolding of f factors through F. A versal unfolding depending on the minimum number of parameters is called universal. That minimum number is called the C^∞ codimension of f.

The topological codimension of f is the C^∞ codimension of f minus the number of modal parameters in the universal unfolding F.

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One of the most important results in singularity theory gives necessary and sufficient conditions for F to be a versal unfolding.

Theorem 6.5 *Let F be a k-parameter unfolding of f. Then*

- *F is a versal unfolding of f if and only if*

$$\mathcal{E}^2 = T(f) + \mathbf{R}\left\{\frac{\partial F}{\partial \alpha_1}(x, y, 0), \ldots, \frac{\partial F}{\partial \alpha_k}(x, y, 0)\right\} \tag{6.1}$$

- *An unfolding F of f is universal if and only if (6.1) is a direct sum.*
- *The number of parameters in F equals the codimension of $T(f)$.*
- *If f has C^∞ codimension k and $z_1, \ldots, z_k \in \mathcal{E}^2$ are chosen so that*

$$\mathcal{E}^2 = T(f) \oplus \mathbf{R}\{z_1, \ldots, z_k\}$$

then

$$F(x, y, \alpha) = f(x, y) + \alpha_1 z_1(x, y) + \cdots + \alpha_k z_k(x, y)$$

is a universal unfolding of f.

This theorem is a special case of results in (Damon 1984, Sect. 9). By applying Theorem 6.5, we obtain a universal unfolding for each normal form in Theorem 2.3 with direct calculations. The details are given in Table 6.

7 Singularity theory proofs

This section addresses two issues: the solution to the recognition problems of singularities in Theorem 2.3 (Sect. 7.2) and the recognition problem for universal unfoldings (Sect. 7.3). The resolution of both issues uses Nakayama's Lemma (Lemma 7.1) and the notion of intrinsic submodules (Definition 7.2). These techniques are introduced in Sect. 7.1.

7.1 Intrinsic submodules

We apply the modified tangent space constant theorem (Theorem 5.5) to solve the recognition problems for low codimension singularities. The proof requires the calculation of the finitely generated submodule $\mathcal{I}(f) \subset \mathcal{E}^2$, where $\mathcal{E}$ is the ring of (germs of) functions $z(u, w)$ (defined on a neighborhood of the origin). We use Nakayama's Lemma (cf. Golubitsky and Schaeffer 1985, Chapter 2), which we now state, to perform these calculations. Recall that $\mathcal{M} \subset \mathcal{E}$ is the maximal ideal consisting of function germs that vanish at the origin.

Lemma 7.1 (Nakayama's Lemma) *Let $\mathcal{I}, \mathcal{J} \subset \mathcal{E}^2$ be finitely generated $\mathcal{E}$-modules. Then*

$$\mathcal{I} \subset \mathcal{J} \quad \textit{if and only if} \quad \mathcal{I} \subset \mathcal{J} + \mathcal{M}\mathcal{I}$$

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Table 6 Universal unfoldings and tangent space for all singularities up to topological codimension two

	Normal form h	TC	$T(h)$	Universal unfolding H
(a)	$\epsilon(w+\mu_0 uv)$	0	$[\mathcal{M}, \mathcal{M}^2+\langle w\rangle]+\mathbf{R}\{[1,\mu_0 u],[0,1]\}$	$\epsilon(w+\mu uv)$
(b)	$\epsilon(w+uv)$	1	$[\mathcal{M}, \mathcal{M}^2+\langle w\rangle]+\mathbf{R}\{[1,u],[0,1]\}$	$\epsilon(w+\mu uv)$
(c)	$\epsilon(w-uv)$	1	$[\mathcal{M}, \mathcal{M}^2+\langle w\rangle]+\mathbf{R}\{[1,-u],[0,1]\}$	$\epsilon(w+\mu uv)$
(d)	$\epsilon(\delta w^2+uv)$	1	$[\mathcal{M}, \mathcal{M}]+\mathbf{R}\{[0,1]\}$	$\epsilon((a+\delta w)w+uv)$
(e)	$\epsilon(w+(\delta u^2+\gamma_0 u^3)v)$	1	$[\mathcal{M}^2+\langle w\rangle, \mathcal{M}^4+\langle w\rangle]+\mathbf{R}\{[1,\delta u^2+\gamma_0 u^3],$ $[1, 2\delta u^2+3\gamma_0 u^3],[u,\delta u^3],[0,2\delta u+3\gamma_0 u^2]\}$	$\epsilon(w+(a+\delta u^2+\gamma u^3)v)$
(f)	$\epsilon(\delta w^3+uv)$	2	$[\mathcal{M}^2, \mathcal{M}]+\mathbf{R}\{[0,1],[u,0]\}$	$\epsilon((a+bw+\delta w^2)w+uv)$
(g)	$\epsilon(w+(\delta u^3+\lambda_0 u^5)v)$	2	$[\mathcal{M}^3+\langle w\rangle, \mathcal{M}^6+\langle w\rangle]+\mathbf{R}\{[1,\delta u^3+\lambda_0 u^5],$ $[1,3\delta u^3+5\lambda_0 u^5],[u^2,\delta u^5],[u,0],[0,u^4],[0,u^2]\}$	$\epsilon(w+(a+bu+\delta u^3+\lambda u^5)v)$
(h)	$\epsilon(uw+(\alpha_0 w+\beta_0 u^2)v)$	2	$[\mathcal{M}^2+\langle w\rangle, \mathcal{M}^3+\mathcal{M}\langle w\rangle]+\mathbf{R}\{[u,\alpha_0 w+\beta_0 u^2],[1,2\beta_0 u]\}$	$\epsilon((a+u)w+(b+\alpha w+\beta u^2)v)$

Definition 7.2 An ideal $P\subset\mathcal{E}$ is intrinsic if $p(\Phi)\in P$ whenever Φ is a dimorphism equivalence and $p\in P$. A submodule $\mathcal{K}\subset\mathcal{E}^2$ is intrinsic if $\gamma([p,q])\in\mathcal{K}$ whenever γ is a dimorphism equivalence and $[p,q]\in\mathcal{K}$.

Note that sums and products of intrinsic ideals are intrinsic and that $\mathcal{M}$ and $\langle w\rangle$ are intrinsic ideals. Note also that sums and products of intrinsic submodules are intrinsic. Specifically:

Proposition 7.3 *Let $P, Q\subset\mathcal{E}$ be intrinsic ideals. Then the submodule $[P,Q]\subset\mathcal{E}^2$ is intrinsic if $Q\subset P$ and $\langle w\rangle P\subset Q$.*

Proof Let $S=S^e+S^o v$ and $\Phi=(\varphi^e+\varphi^o v, \varphi^e-\varphi^o v)$, where $S^e, S^o, \varphi^e, \varphi^o\in\mathcal{E}$. Assume

$$f=pw+qv=[p,q]\in[P,Q].$$

Then

$$Sf=[S^e p+S^o q, S^e q+S^o pw]\quad\text{and}$$
$$f\circ\Phi=[p(2\varphi^e,4(\varphi^o)^2 w)4(\varphi^o)^2, q(2\varphi^e,4(\varphi^o)^2 w)2\varphi^o]$$

are in $[P,Q]$. □

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Corollary 7.4 *The modules $w^s[\mathcal{M}^k, \mathcal{M}^k]$ and $w^s[\mathcal{M}^k, \mathcal{M}^{k+1}]$ are intrinsic, where k and s are non-negative integers. In addition, all $\mathcal{K}$ listed in Table 7 are intrinsic submodules.*

Lemma 7.5 *Suppose $\mathcal{K} \subset \mathcal{I}(f)$ is intrinsic and $\eta \in \mathcal{K}$. Then* $\mathrm{RT}(f + s\eta) = \mathrm{RT}(f)$ *for all s and $f + \eta$ is strongly dimporphism equivalent to f.*

Proof Let $f = p^f w + q^f v$, $\eta = p^\eta w + q^\eta v$, and

$$z(f) = \begin{pmatrix} [p^f, q^f] \\ [q^f, wp^f] \\ [wp_u^f, wq_u^f] \\ [wp^f + 2w^2 p_w^f, 2w^2 q_w^f] \end{pmatrix}$$

We claim that $\mathrm{RT}(\eta) \subset \mathrm{RT}(f)$. Note that any term $r \in \mathrm{RT}(\eta)$ falls in the form

$$r = \frac{d}{dt}\Phi_t(\eta)\bigg|_{t=0}$$

Since $\eta \in \mathcal{K}$ and $\mathcal{K}$ is intrinsic, we know that $\Phi_t(\eta) \in \mathcal{K}$. Therefore, $r \in \mathcal{K}$. Hence we have proved the claim that $\mathrm{RT}(\eta) \subset \mathrm{RT}(f)$. This means that $z(\eta)$ can be written as a combination of $z(f)$. Assume that

$$z(\eta) = Az(f)$$

where A is a 4×4 matrix. Then we know,

$$z(f + s\eta) = z(f) + sz(\eta) = z(f) + sAz(f) = (I + sA)z(f)$$

where I is identity matrix. Note that when $s = 0$, $I + sA = I$ is invertible. Thus, when s is small enough, say $s \leq s_0$, we know that $I + sA$ is also invertible. Therefore, when s is sufficiently small, $z(f + s\eta)$ and $z(f)$ are two different basis of same submodule. That is,

$$\mathrm{RT}(f + s\eta) = \mathrm{RT}(f)$$

We claim that we can always increase s_0 and hence the lemma is valid for all s. Let $g = f + s_0\eta$. Then $\mathcal{K} \subset \mathcal{I}(g)$ since g is dimorphism equivalent to f and $\mathcal{K}$ is intrinsic. Apply the same argument to g. Finally apply the modified tangent space constant theorem (Theorem 5.5) to conclude that $f + \eta$ is equivalent to f. □

7.2 Proof of Theorem 2.3

The necessity is straightforward, see Wang (2015) for details. We present here the proof of sufficiency. We begin with an overview of the method of proof. Generally, for each fitness function f we follow a two step procedure to determine a normal form h for f.

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Table 7 The restricted tangent space for the singularities up to topological codimension two

	Normal form h	TC	$\mathcal{K}$	$RT(h) = \mathcal{I}(h)$
(a)	$\epsilon(w + \mu_0 uv)$	0	$[\mathcal{M}, \mathcal{M}^2 + \langle w \rangle]$	$\mathcal{K} \oplus \mathbf{R}\{[1, \mu_0 u]\}$
(b)	$\epsilon(w + uv)$	1	$[\mathcal{M}, \mathcal{M}^2 + \langle w \rangle]$	$\mathcal{K} \oplus \mathbf{R}\{[1, u]\}$
(c)	$\epsilon(w - uv)$	1	$[\mathcal{M}, \mathcal{M}^2 + \langle w \rangle]$	$\mathcal{K} \oplus \mathbf{R}\{[1, -u]\}$
(d)	$\epsilon(\delta w^2 + uv)$	1	$[\mathcal{M}^2, \mathcal{M}^2]$	$\mathcal{K} \oplus \mathbf{R}\{[\delta w, u], [u, 0], [0, w]\}$
(e)	$\epsilon(w + (\delta u^2 + \gamma_0 u^3)v)$	1	$[\mathcal{M}^3 + \langle w \rangle, \mathcal{M}^5 + \mathcal{M}\langle w \rangle]$	$\mathcal{K} \oplus \mathbf{R}\{[\delta u^2, w], [u^2, \delta u^4], [1, \delta u^2 + \gamma_0 u^3], [u, \delta u^3 + \gamma_0 u^4]\}$
(f)	$\epsilon(\delta w^3 + uv)$	2	$[\mathcal{M}^3, \mathcal{M}^3]$	$[\mathcal{M}^3 + \langle u \rangle, \mathcal{M}^2 + \langle w \rangle] \oplus \mathbf{R}\{[\delta w^2, u]\}$
(g)	$\epsilon(w + (\delta u^3 + \lambda_0 u^5)v)$	2	$[\mathcal{M}^5 + \langle w \rangle, \mathcal{M}^8 + \mathcal{M}^2\langle w \rangle + \langle w^2 \rangle]$	$\mathcal{K} \oplus \mathbf{R}\{[1, \delta u^3 + \lambda_0 u^5], [u, \delta u^4 + \lambda_0 u^6], [\delta u^4, uw], [u^2, \delta u^5 + \lambda_0 u^7], [u^3, \delta u^6], [u^4, \delta u^7], [\delta u^3, w]\}$
(h)	$\epsilon(uw + (\alpha_0 w + \beta_0 u^2)v)$	2	$[\mathcal{M}^4 + \mathcal{M}^2\langle w \rangle + \langle w^2 \rangle, \mathcal{M}^5 + \mathcal{M}^3\langle w \rangle + \mathcal{M}\langle w^2 \rangle]$	$\mathcal{K} \oplus \mathbf{R}\{[u, \alpha_0 w + \beta_0 u^2], [u^2, \alpha_0 uw + \beta_0 u^3], [u^3, \alpha_0 u^2 w + \beta_0 u^4], [\alpha_0 w + \beta_0 u^2, uw], [uw, 2\alpha_0 w^2], [\alpha_0 uw + \beta_0 u^3, u^2 w], [w, 2\beta_0 uw], [uw, 2\beta_0 u^2 w]\}$

TC topological codimension, *RT(h)* restricted tangent space of h

- Step 1: we find an intrinsic submodule $\mathcal{K} \subset \mathrm{RT}(f)$ such that $f + \eta$ is strongly dimorphism equivalent to f for any $\eta \in \mathcal{K}$. Then we reduce f to a polynomial g that is dimorphism equivalent to f. Table 7 provides the details of the submodule $\mathcal{K}$ and the restricted tangent space $\mathrm{RT}(f)$ for each singularity up to topological codimension two.
- Step 2: we find specific dimorphism equivalences that transform g into a normal form h. These calculations can be performed modulo $\mathcal{K}$ because no $\eta \in \mathcal{K}$ will change the dimorphism equivalence class.

When proving this theorem we use Mathematica for many of these calculations. We set notation before giving the details of the proof of Theorem 2.3. Let $f = pw + qv$ be a fitness function and let

$$p = \sum_{i,j=0}^{\infty} p_{ij} u^i w^j \qquad q = \sum_{i,j=0}^{\infty} q_{ij} u^i w^j$$

where $p_{ij} = \frac{1}{i!j!} p_{u^i w^j}(0, 0)$ and $q_{ij} = \frac{1}{i!j!} q_{u^i w^j}(0, 0)$. Recall that

$$\mathrm{RT}(f) = \mathcal{I}(f) = \langle [p, q], [q, wp], [wp_u, wq_u], [wp + 2w^2 p_w, 2w^2 q_w] \rangle$$

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A general dimorphism transformation (S, Φ) is denoted as

$$S = S^e(u, w) + S^o(u, w)v \qquad \Phi = (\varphi^e(u, w) + \varphi^o(u, w)v, \varphi^e(u, w) - \varphi^o(u, w)v)$$

Now we prove the sufficiency of Theorem 2.3 for singularities of topological codimension ≤ 2.

(a) Assume $q_{00} = 0$, $q_{10} \neq 0$, $p_{00} \neq 0$, and let $\mathcal{K} = [\mathcal{M}, \mathcal{M}^2 + \langle w \rangle]$. We show $\mathcal{K} \subset \mathcal{I}(f) + \mathcal{M}\mathcal{K}$ and apply Nakayama's Lemma to conclude $\mathcal{K} \subset \mathcal{I}(f)$. Calculation modulo $\mathcal{M}\mathcal{K}$ yields

$$\begin{aligned} w[p, q] &\equiv [p_{00}w, 0] &&= \gamma_1 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K} \\ [wp_u, wq_u] &\equiv [p_{10}w, q_{10}w] &&= \gamma_2 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K} \\ [q, wp] &\equiv [q_{10}u + q_{01}w, p_{00}w] &&= \gamma_3 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K} \\ u[p, q] &\equiv [p_{00}u, q_{10}u^2] &&= \gamma_4 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K} \end{aligned}$$

We claim that $\gamma_1, \ldots, \gamma_4$ is a basis of the linear space L generated by

$$\langle [u, 0], [w, 0], [0, w], [0, u^2] \rangle$$

because the transition matrix has determinant $q_{10}^3 p_{00} \neq 0$. Since the γ_j generate $\mathcal{K}$, we can conclude that $\mathcal{K} \subset \mathcal{I}(f) + \mathcal{M}\mathcal{K}$, as desired. Proposition 7.5 implies that $\mathcal{K}$ is an intrinsic submodule and Lemma 7.5 implies that $f + \eta$ is strongly dimorphism equivalent to f for any $\eta \in \mathcal{K}$. So we know that f is dimorphism equivalent to $g = p_{00}w + q_{10}uv$. Let $\epsilon = \text{sgn}(p_{00})$. Using the transformation $(S, \Phi) = (\frac{\epsilon}{p_{00}}, I)$ where I is 2×2 identity matrix, we see that g is dimorphism equivalent to $h = \epsilon(w + \mu_0 uv)$ where $\mu_0 = \frac{q_u}{p}$ and $\epsilon = \text{sgn}(p)$.

(b) and (c) Note that singularity (b) and (c) are special cases of singularity (a) and share the normal form of (a) when $\mu_0 = \pm 1$. The proof is the same if we replace $\mu_0 = \pm 1$.

(d) Assume $q_{00} = p_{00} = 0$, $p_{10} \neq 0$, $p_{01}q_{10} - p_{10}q_{01} \neq 0$, and let $\mathcal{K} = [\mathcal{M}^2, \mathcal{M}^2]$. We show $\mathcal{K} \subset \mathcal{I}(f) + \mathcal{M}\mathcal{K}$ and apply Nakayama's Lemma to conclude $\mathcal{K} \subset I(f)$. Calculation modulo $\mathcal{M}\mathcal{K}$ yields

$$\begin{aligned} u[p, q] &\equiv [p_{10}u^2 + p_{01}uw, q_{10}u^2 + q_{01}uw] &&= \gamma_1 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K} \\ w[p, q] &\equiv [p_{10}uw + p_{01}w^2, q_{10}uw + q_{01}w^2] &&= \gamma_2 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K} \\ u[q, wp] &\equiv [q_{10}u^2 + q_{01}uw, 0] &&= \gamma_3 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K} \\ w[q, wp] &\equiv [q_{10}uw + q_{01}w^2, 0] &&= \gamma_4 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K} \\ u[wp_u, wq_u] &\equiv [p_{10}uw, q_{10}uw] &&= \gamma_5 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K} \\ w[wp_u, wq_u] &\equiv [p_{10}w^2, q_{10}w^2] &&= \gamma_6 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K} \end{aligned}$$

We claim that $\gamma_1, \ldots, \gamma_6$ is a basis of the linear space L generated by

$$\langle [u^2, 0], [uw, 0], [w^2, 0], [0, u^2], [0, uw], [0, w^2] \rangle$$

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because the transition matrix has determinant $q_{10}^4(p_{10}q_{01}-p_{01}q_{10}) \neq 0$. Since the γ_j generate $\mathcal{K}$, we can conclude that $\mathcal{K} \subset \mathcal{I}(f)+\mathcal{M}\mathcal{K}$, as desired. Proposition 7.5 implies that $\mathcal{K}$ is an intrinsic submodule and Lemma 7.5 implies that $f+\eta$ is strongly dimorphism equivalent to f for any $\eta \in \mathcal{K}$. So we know that f is dimorphism equivalent to $g = (p_{10}u + p_{01}w)w + (q_{10}u + q_{01}w)v$. Let $\epsilon = \mathrm{sgn}(q_{10})$, $\delta = \mathrm{sgn}(p_{01}q_{10} - p_{10}q_{01})$. Using the transformation (S, Φ) where

$$S^e = \frac{\delta\,(p_{01}q_{10} - p_{10}q_{01})}{\epsilon q_{10}^3} \qquad S^o = -\frac{p_{10}(\sqrt{\delta\,(p_{01}q_{10} - p_{10}q_{01})})}{\epsilon q_{10}^3}$$

$$\varphi^e = \frac{\epsilon q_{10}(\sqrt{\delta(p_{01}q_{10} - p_{10}q_{01})}u - q_{01}w)}{2\delta(p_{01}q_{10} - p_{10}q_{01})} \qquad \varphi^o = \frac{\epsilon q_{10}}{2\sqrt{\delta(p_{01}q_{10} - p_{10}q_{01})}}$$

we see that g is dimorphism equivalent to $h = \epsilon(\delta w^2 + uv)$, where $\epsilon = \mathrm{sgn}(q_u)$, $\delta = \mathrm{sgn}(p_w q_u - p_u q_w)$.

(e) Assume $q_{00} = q_{10} = 0$, $p_{00} \neq 0$, $q_{20} \neq 0$, and let $\mathcal{K} = [\mathcal{M}^3 + \langle w\rangle, \mathcal{M}^5 + \mathcal{M}\langle w\rangle]$. We show $\mathcal{K} \subset \mathcal{I}(f) + \mathcal{M}\mathcal{K}$ and apply Nakayama's Lemma to conclude $\mathcal{K} \subset I(f)$. Calculation modulo $\mathcal{M}\mathcal{K}$ yields

$$\begin{aligned}
w[q, wp] &\equiv [0, p_{00}w^2] && = \gamma_1 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K}\\
w[wp + 2w^2 p_w, 2w^2 q_w] &\equiv [p_{00}w, 2q_{01}w^2] && = \gamma_2 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K}\\
[wp_u, wq_u] &\equiv [p_{10}w, 2q_{20}uw + q_{11}w^2] && = \gamma_3 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K}\\
u[q, wp] &\equiv [q_{20}u^3, p_{00}uw] && = \gamma_4 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K}\\
u^3[p, q] &\equiv [p_{00}u^3, q_{20}u^5] && = \gamma_5 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K}
\end{aligned}$$

We claim that $\gamma_1, \ldots, \gamma_5$ is a basis of the linear space L generated by

$$\langle [u^3, 0], [w, 0], [0, u^5], [0, uw], [0, w^2] \rangle$$

because the transition matrix has determinant $p_{00}^2 q_{20}^3 \neq 0$. Since the γ_j generate $\mathcal{K}$, we can conclude that $\mathcal{K} \subset \mathcal{I}(f) + \mathcal{M}\mathcal{K}$, as desired. Proposition 7.5 implies that $\mathcal{K}$ is an intrinsic submodule and Lemma 7.5 implies that $f + \eta$ is strongly dimorphism equivalent to f for any $\eta \in \mathcal{K}$. So we know that f is dimorphism equivalent to $g = (p_{00} + p_{10}u + p_{20}u^2)w + (q_{20}u^2 + q_{01}w + q_{30}u^3 + q_{40}u^4)v$. Let $\epsilon = \mathrm{sgn}(p_{00})$, $\delta = \mathrm{sgn}(\frac{q_{20}}{p_{00}})$. Using the transformation (S, Φ) where

$$S^e = \frac{q_{20}^2}{\epsilon p_{00}^3} - \frac{\epsilon q_{01}q_{20}}{\delta p_{00}^3}u + \frac{-6p_{00}^2q_{40} + 5p_{20}p_{00}q_{20} + 6p_{10}p_{00}q_{30} - 5p_{10}^2q_{20} - 5q_{01}q_{20}^2\epsilon}{\epsilon p_{00}^3 q_{20}}u^2$$

$$S^o = -\frac{q_{01}q_{20}}{\epsilon\delta p_{00}^3}$$

$$\varphi^e = \frac{\delta p_{00}}{2q_{20}}u + \frac{\delta p_{00}\left(p_{00}^2q_{40} - p_{20}p_{00}q_{20} - p_{10}p_{00}q_{30} + p_{10}^2q_{20} + q_{01}q_{20}^2\right)}{2q_{20}^4}u^3$$

$$\varphi^o = \frac{\delta p_{00}}{2q_{20}} + \frac{3\delta p_{00}\left(p_{00}^2q_{40} - p_{20}p_{00}q_{20} - p_{10}p_{00}q_{30} + p_{10}^2q_{20} + q_{01}q_{20}^2\right)}{2q_{20}^4}u^2$$

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we see that g is dimorphism equivalent to $h = \epsilon(w + (\delta u^2 + \gamma_0 u^3)v)$, where $\gamma_0 = \frac{2pq_{uuu} - 6q_{uu}p_u}{3q_{uu}^2}$, $\epsilon = \text{sgn}(p)$, $\delta = \text{sgn}(\frac{p_{uu}}{p})$.

(f) Assume $p_{00} = q_{00} = p_{10}q_{01} - p_{01}q_{10} = 0$, $q_{10} \neq 0$, $\Delta_1 \neq 0$ ($\Delta_1 = 2q_{10}^2(q_{10}p_{02} - p_{10}q_{01}) + 2q_{01}^2(q_{10}p_{20} - p_{10}q_{20}) - 2q_{10}q_{01}(q_{10}p_{11} - p_{10}q_{11})$), and let $\mathcal{K} = [\mathcal{M}^3, \mathcal{M}^3]$. We show $\mathcal{K} \subset \mathcal{I}(f) + \mathcal{MK}$ and apply Nakayama's Lemma to conclude $\mathcal{K} \subset I(f)$. Calculation modulo $\mathcal{MK}$ yields

$$\begin{aligned}
u^2[p,q] &\equiv [p_{10}u^3 + p_{01}u^2w, q_{10}u^3 + q_{01}u^2w] &&= \gamma_1 \in \mathcal{I}(f) + \mathcal{MK}\\
u^2[q,wp] &\equiv [q_{10}u^3 + q_{01}u^2w, 0] &&= \gamma_2 \in \mathcal{I}(f) + \mathcal{MK}\\
uw[q,wp] &\equiv [q_{10}u^2w + q_{01}uw^2, 0] &&= \gamma_3 \in \mathcal{I}(f) + \mathcal{MK}\\
w^2[q,wp] &\equiv [q_{10}uw^2 + q_{01}w^3, 0] &&= \gamma_4 \in \mathcal{I}(f) + \mathcal{MK}\\
u^2[wp_u, wq_u] &\equiv [p_{10}u^2w, q_{10}u^2w] &&= \gamma_5 \in \mathcal{I}(f) + \mathcal{MK}\\
uw[wp_u, wq_u] &\equiv [p_{10}uw^2, q_{10}uw^2] &&= \gamma_6 \in \mathcal{I}(f) + \mathcal{MK}\\
w^2[wp_u, wq_u] &\equiv [p_{10}w^3, q_{10}w^3] &&= \gamma_7 \in \mathcal{I}(f) + \mathcal{MK}
\end{aligned}$$

Further we have

$$\begin{aligned}
w[p,q] &\equiv [p_{10}uw + p_{01}w^2 + p_{20}u^2w + p_{11}u^2w + p_{02}w^2,\\
&\qquad q_{10}uw + q_{01}w^2 + q_{20}u^2w + q_{11}u^2w + q_{02}w^2] &&= \xi_1 \in \mathcal{I}(f) + \mathcal{MK}\\
u[wp_u, wq_u] &\equiv [p_{10}uw + 2p_{20}u^2w + p_{11}uw^2,\\
&\qquad q_{10}uw + 2q_{20}u^2w + q_{11}uw^2] &&= \xi_2 \in \mathcal{I}(f) + \mathcal{MK}\\
w[wp_u, wq_u] &\equiv [p_{10}w^2 + 2p_{20}uw^2 + p_{11}uw^3,\\
&\qquad q_{10}w^2 + 2q_{20}uw^2 + q_{11}uw^3] &&= \xi_3 \in \mathcal{I}(f) + \mathcal{MK}
\end{aligned}$$

Let

$$\begin{aligned}
\gamma_8 &= q_{10}\xi_1 - q_{10}\xi_2 - q_{01}\xi_3\\
&= [-q_{10}p_{20}u^2w - 2q_{10}p_{20}uw^2 + (q_{10}p_{02} - q_{01}p_{11})w^3,\\
&\quad -q_{10}q_{20}u^2w - 2q_{10}q_{20}uw^2 + (q_{10}q_{02} - q_{01}q_{11})w^3] \in \mathcal{I}(f) + \mathcal{MK}
\end{aligned}$$

We claim that $\gamma_1, \ldots, \gamma_8$ is a basis of the linear space L generated by

$$\langle [u^3, 0], [u^2w, 0], [uw^2, 0], [w^3, 0], [0, u^3], [0, u^2w], [0, uw^2], [0, w^3] \rangle$$

because the transition matrix has determinant $-\frac{1}{2}q_{10}^5\Delta_1 \neq 0$. Since the γ_j generate $\mathcal{K}$, we can conclude that $\mathcal{K} \subset \mathcal{I}(f) + \mathcal{MK}$, as desired. Proposition 7.5 implies that $\mathcal{K}$ is an intrinsic submodule and Lemma 7.5 implies that $f + \eta$ is strongly dimorphism equivalent to f for any $\eta \in \mathcal{K}$. So we know that f is dimorphism equivalent to $(p_{10}u + p_{01}w + p_{20}u^2 + p_{11}uw + p_{02}w^2)w + (q_{10}u + q_{01}w + q_{20}u^2 + q_{11}uw + q_{02}w^2)v$. Let $\epsilon = \text{sgn}(q_{10})$, $\delta = \text{sgn}(\Delta_1)$. Using the transformation (S, Φ) where

$$\begin{aligned}
S^e &= \frac{\epsilon}{\tilde{\Delta}^2 q_{10}} - \frac{q_{20}\epsilon}{\tilde{\Delta} q_{10}^2}u\\
S^o &= -\frac{p_{10}\epsilon}{\tilde{\Delta} q_{10}^2} - \frac{\epsilon\,(p_{20}q_{10} - 2p_{10}q_{20})}{q_{10}^3}u
\end{aligned}$$

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$$+\frac{\tilde{\Delta}\epsilon\left(2p_{20}q_{01}q_{10}-p_{11}q_{10}^2+p_{10}q_{11}q_{10}-2p_{10}q_{01}q_{20}\right)}{q_{10}^4}w$$

$$\varphi^e=\frac{1}{2}\left(\tilde{\Delta}u-\frac{\tilde{\Delta}^2 q_{01}}{q_{10}}w+\frac{\tilde{\Delta}^3\left(p_{10}^2-q_{10}q_{11}+2q_{01}q_{20}\right)}{q_{10}^2}uw\right.$$
$$\left.-\frac{\tilde{\Delta}^4\left(q_{20}q_{01}^2-q_{10}q_{11}q_{01}+q_{02}q_{10}^2\right)}{q_{10}^3}w^2\right)$$
$$\varphi^o=\frac{1}{2}\tilde{\Delta}$$

we see that g is dimorphism equivalent to $h=\epsilon(\delta w^3+uv)$, where $\epsilon=\operatorname{sgn}(p)$, $\delta=\operatorname{sgn}(\Delta_1)$.

(g) Assume $q=q_{10}=q_{20}=0$, $p\neq 0$, $q_{30}\neq 0$, and let $\mathcal{K}=[\mathcal{M}^5+\langle w\rangle, \mathcal{M}^8+\mathcal{M}^2\langle w\rangle+\langle w^2\rangle]$. We show $\mathcal{K}\subset\mathcal{I}(f)+\mathcal{MK}$ and apply Nakayama's Lemma to conclude $\mathcal{K}\subset I(f)$. Calculation modulo $\mathcal{MK}$ yields

$$\begin{aligned}
w[q,wp]&\equiv[0,p_{00}w^2] &&=\gamma_1\in\mathcal{I}(f)+\mathcal{MK}\\
[wp+2w^2p_w,2w^2q_w]&\equiv[p_{00}w,2q_{01}w^2] &&=\gamma_2\in\mathcal{I}(f)+\mathcal{MK}\\
[wp_u,wq_u]&\equiv[p_{10}w,q_{11}w^2+3q_{30}u^2w] &&=\gamma_3\in\mathcal{I}(f)+\mathcal{MK}\\
u^2[q,wp]&\equiv[q_{30}u^5,p_{00}u^2w] &&=\gamma_4\in\mathcal{I}(f)+\mathcal{MK}\\
u^5[p,q]&\equiv[p_{00}u^5,q_{30}u^8] &&=\gamma_5\in\mathcal{I}(f)+\mathcal{MK}
\end{aligned}$$

We claim that $\gamma_1,\ldots,\gamma_5$ is a basis of the linear space L generated by

$$\langle[u^5,0],[w,0],[0,u^8],[0,u^2w],[0,w^2]\rangle$$

because the transition matrix has determinant $3p_{00}^2q_{30}^3\neq 0$. Since the γ_j generate $\mathcal{K}$, we can conclude that $\mathcal{K}\subset\mathcal{I}(f)+\mathcal{MK}$, as desired. Proposition 7.5 implies that $\mathcal{K}$ is an intrinsic submodule and Lemma 7.5 implies that $f+\eta$ is strongly dimorphism equivalent to f for any $\eta\in\mathcal{K}$. So we know that f is dimorphism equivalent to $g=(p_{00}+p_{10}u+p_{20}u^2+p_{30}u^3+p_{40}u^4)w+(q_{01}w+q_{11}uw+q_{30}u^3+q_{40}u^4+q_{50}u^5+q_{60}u^6+q_{70}u^7)v$. Let $\epsilon=\operatorname{sgn}(p_{00})$, $\delta=\operatorname{sgn}(\frac{q_{30}}{p_{00}})$. Using the transformation (S,Φ) where

$$S^e=\frac{q_{30}\epsilon\delta}{p_{00}^2}+\epsilon\frac{4p_{00}q_{40}-5p_{10}q_{30}}{\sqrt{p_{00}^5q_{30}\delta}}u$$
$$+\epsilon\frac{\left(16p_{10}^2q_{30}^2-p_{00}p_{20}q_{30}^2-27p_{00}p_{10}q_{40}q_{30}+12p_{00}^2q_{40}^2\right)}{p_{00}^3q_{30}^2}u^2$$
$$S^o=-\epsilon q_{01}\sqrt{\frac{q_{30}\delta}{p_{00}^5}}+\epsilon\frac{4p_{10}q_{01}q_{30}-p_{00}q_{11}q_{30}-2p_{00}q_{01}q_{40}}{p_{00}^3q_{30}}u$$

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$$\varphi^e = \frac{1}{2}\left(\sqrt{\frac{p_{00}}{q_{30}\delta}}u + \delta\frac{p_{10}q_{30} - p_{00}q_{40}}{q_{30}^2}u^2\right)$$
$$\varphi^o = \frac{1}{2}\left(\sqrt{\frac{p_{00}}{q_{30}\delta}} + 2\delta\frac{p_{10}q_{30} - p_{00}q_{40}}{q_{30}^2}u\right)$$

we see that g is dimorphism equivalent to $\tilde{g} = (\epsilon + \tilde{p}_{30}u^3 + \tilde{p}_{40}u^4)w + (\epsilon\delta u^3 + \tilde{q}_{50}u^5 + \tilde{q}_{60}u^6 + \tilde{q}_{70}u^7)v$, where $\tilde{q}_{50} = \epsilon\frac{-p_{20}q_{30}^2 + p_{10}q_{30}q_{40} - p_{00}q_{40}^2 + p_{00}q_{30}q_{50}}{q_{30}^3}$. Next we apply a second transformation (S, Φ) where

$$S^e = 1 + (7\epsilon\tilde{p}_{30} - 8\epsilon\delta\tilde{q}_{60})u^3 + (4\epsilon\tilde{p}_{40} - 5\epsilon\delta\tilde{q}_{70})u^4$$
$$S^o = 0$$
$$\varphi^e = \frac{1}{2}(u + (\epsilon\tilde{p}_{30} - \epsilon\delta\tilde{q}_{60})u^4 + (\epsilon\tilde{p}_{40} + \epsilon\delta\tilde{q}_{70})u^5)$$
$$\varphi^o = \frac{1}{2}(1 + 4(\epsilon\tilde{p}_{30} - \epsilon\delta\tilde{q}_{60})u^3 + 5(\epsilon\tilde{p}_{40} + \epsilon\delta\tilde{q}_{70})u^4)$$

we show that $\tilde{g}$ is dimorphism equivalent to $h = \epsilon(w + (\delta u^3 + \lambda_0 u^5)v)$ where $\epsilon = \operatorname{sgn}(p)$, $\delta = \operatorname{sgn}(\frac{q_{uuu}}{p})$ and $\lambda_0 = \frac{-120p_{uu}q_{uuu}^2 + 60p_u q_{uuu}q_{uuuu} - 15pq_{uuuu}^2 + 12pq_{uuu}q_{uuuuu}}{40q_{uuu}^3}$.

(h) Assume $p_{00} = q_{00} = q_{10} = 0$, $q_{01} \neq 0$, $p_{10} \neq 0$, $q_{20} \neq 0$, $p_{10}^2 - 4q_{20}q_{01} \neq 0$, and let $\mathcal{K} = [\mathcal{M}^4 + \mathcal{M}^2\langle w\rangle + \langle w^2\rangle, \mathcal{M}^5 + \mathcal{M}^3\langle w\rangle + \mathcal{M}\langle w^2\rangle]$. We show $\mathcal{K} \subset \mathcal{I}(f) + \mathcal{M}\mathcal{K}$ and apply Nakayama's Lemma to conclude $\mathcal{K} \subset I(f)$. Calculation modulo $\mathcal{M}\mathcal{K}$ yields

$$\begin{aligned}
u^3[p, q] &\equiv [p_{10}u^4, q_{01}u^3w + q_{20}u^5] &&= \gamma_1 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K}\\
w^2[p, q] &\equiv [0, q_{01}w^3] &&= \gamma_2 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K}\\
u^2[q, wp] &\equiv [q_{01}u^2w + q_{20}u^4, p_{10}u^3w] &&= \gamma_3 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K}\\
w[q, wp] &\equiv [q_{01}w^2 + q_{20}u^2w, p_{10}uw^2 + p_{01}w^3] &&= \gamma_4 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K}\\
u^2[wp_u, wq_u] &\equiv [p_{10}u^2w, 2q_{20}u^3w] &&= \gamma_5 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K}\\
w[wp_u, wq_u] &\equiv [p_{10}w^2, 2q_{20}uw^2 + q_{11}w^3] &&= \gamma_6 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K}\\
u[wp + 2w^2p_w, 2w^2q_w] &\equiv [p_{10}u^2w, 2q_{01}uw^2] &&= \gamma_7 \in \mathcal{I}(f) + \mathcal{M}\mathcal{K}
\end{aligned}$$

We claim that $\gamma_1, \ldots, \gamma_7$ is a basis of the linear space L generated by

$$\langle [u^4, 0], [w^2, 0], [u^2w, 0], [0, uw^2], [0, u^3w], [0, w^3], [0, u^5]\rangle$$

because the transition matrix has determinant $2p_{10}q_{01}q_{20}^3(p_{10}^2 - 4q_{01}q_{20}) \neq 0$. Since the γ_j generate $\mathcal{K}$, we can conclude that $\mathcal{K} \subset \mathcal{I}(f) + \mathcal{M}\mathcal{K}$, as desired. Proposition 7.5 implies that $\mathcal{K}$ is an intrinsic submodule and Lemma 7.5 implies that $f + \eta$ is strongly dimorphism equivalent to f for any $\eta \in \mathcal{K}$. So we know that f is dimorphism equivalent to $g = (p_{10}u + p_{01}w + p_{20}u^2 + p_{11}uw + p_{30}u^3)w + (q_{01}w + q_{20}u^2 + q_{11}uw + q_{02}w^2 + q_{30}u^3 + q_{21}u^2w + q_{40}u^4)v$. Let $\epsilon = \operatorname{sgn}(p_{10})$.

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Using the transformation (S, Φ) where

$$
\begin{aligned}
S^e =& \epsilon\left(\frac{1}{p_{10}} - \frac{4p_{20}p_{10}^2q_{20} - 5p_{10}^3q_{30} - 4p_{10}q_{11}q_{20}^2 + 16p_{10}q_{01}q_{20}q_{30} + 8p_{01}q_{20}^3 - 8p_{20}q_{01}q_{20}^2}{p_{10}^2q_{20}\left(4q_{01}q_{20} - p_{10}^2\right)}u\right)\\
S^o =& \frac{2p_{01}q_{20}^2 + 2p_{20}q_{01}q_{20} - p_{10}q_{11}q_{20} - p_{10}q_{01}q_{30}}{\epsilon p_{10}q_{20}\left(4q_{01}q_{20} - p_{10}^2\right)}\\
\varphi^e =& \frac{1}{2}\left(u - \frac{p_{01}p_{10}^2q_{20} - p_{10}q_{01}q_{11}q_{20} - p_{10}q_{01}^2q_{30} - 2p_{01}q_{01}q_{20}^2 + 2p_{20}q_{01}^2q_{20}}{p_{10}q_{20}\left(p_{10}^2 - 4q_{01}q_{20}\right)}w\right.\\
&\left. - \frac{-q_{30}p_{10}^3 + p_{20}p_{10}^2q_{20} - p_{10}q_{11}q_{20}^2 + 3p_{10}q_{01}q_{20}q_{30} + 2p_{01}q_{20}^3 - 2p_{20}q_{01}q_{20}^2}{p_{10}q_{20}\left(p_{10}^2 - 4q_{01}q_{20}\right)}u^2\right)\\
\varphi^o =& \frac{1}{2}\left(1 - 2\frac{-q_{30}p_{10}^3 + p_{20}p_{10}^2q_{20} - p_{10}q_{11}q_{20}^2 + 3p_{10}q_{01}q_{20}q_{30} + 2p_{01}q_{20}^3 - 2p_{20}q_{01}q_{20}^2}{p_{10}q_{20}\left(p_{10}^2 - 4q_{01}q_{20}\right)}u\right)
\end{aligned}
$$

we see that g is dimorphism equivalent to $\tilde{g} = (\tilde{p}_{10}u + \tilde{p}_{11}uw + \tilde{p}_{30}u^3)w + (\tilde{q}_{01}w + \tilde{q}_{20}u^2 + \tilde{q}_{02}w^2 + \tilde{q}_{21}u^2w + \tilde{q}_{40}u^4)v$, where $\tilde{p}_{10} = \epsilon$, $\tilde{q}_{01} = \epsilon\frac{q_{01}}{p_{10}}$, $\tilde{q}_{20} = \epsilon\frac{q_{20}}{p_{10}}$. Next we apply another transformation (S, Φ) where

$$
\begin{aligned}
S^e &= 1 - \frac{\tilde{q}_{02}}{\tilde{q}_{01}}w \qquad S^o = -\frac{5\tilde{p}_{30}\tilde{q}_{20} - 7\tilde{p}_{10}\tilde{q}_{40}}{5\tilde{q}_{20}^2}u\\
\varphi^e &= \frac{1}{2}\left(u - \frac{\tilde{q}_{40}}{5\tilde{q}_{20}}u^3\right) \quad \varphi^o = \frac{1}{2}\left(1 - \frac{3\tilde{q}_{40}}{5\tilde{q}_{20}}u^2\right)
\end{aligned}
$$

It is seen that $\tilde{g}$ is dimorphism equivalent to $\bar{g} = (\bar{p}_{10}u + \bar{p}_{11}uw)w + (\bar{q}_{01}w + \bar{q}_{20}u^2 + \bar{q}_{21}u^2w)v$ where $\bar{p}_{10} = \tilde{p}_{10}$, $\bar{q}_{01} = \tilde{q}_{01}$, $\bar{q}_{20} = \tilde{q}_{20}$. At last, we use a third transformation (S, Φ) where

$$
\begin{aligned}
S^e &= 1 - \frac{\bar{p}_{10}\bar{q}_{21} - 2\bar{p}_{11}\bar{q}_{20}}{\bar{p}_{10}\bar{q}_{01}}u^2 \quad S^o = 0\\
\varphi^e &= \frac{1}{2}\left(u - \frac{\bar{p}_{11}}{\bar{p}_{10}}uw\right) \qquad \varphi^o = \frac{1}{2}
\end{aligned}
$$

we see that $\bar{g}$ is dimorphism equivalent to $h = \epsilon(uw + (\alpha_0 w + \beta_0 u^2)v)$, where $\epsilon = \mathrm{sgn}(p_u)$, $\alpha_0 = \frac{q_w}{p_u}$, $\beta_0 = \frac{q_{uu}}{2p_u}$.

7.3 Recognition problem for universal unfoldings

Consider the following: Let $F(x, y, \alpha)$ be an unfolding of a fitness function $f(x, y)$, where f is dimorphism equivalent to a normal form h. When is F a universal unfolding of f? The answer to this question is important in applications. We follow Golubitsky and Schaeffer (1985).

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Let $\gamma = (S, \Phi)$ be a dimorphism equivalence. That is, S and Φ satisfy conditions in Definition 1.10. Denote

$$\gamma(h)(x, y) = S(x, y)h(\Phi(x, y))$$

Lemma 7.6 *Suppose $f = \gamma(h)$. Then $T(f) = \gamma(T(h))$.*

Proof Define a smooth curve of dimorphism equivalences δ_t at h as

$$\delta_t(h) = S(x, y, t)h(\Phi(x, y, t))$$

where S, Φ vary smoothly in t. Assume that δ_0 is the identity map; that is, $S(x, y, 0) = 1$ and $\Phi(x, y, 0) = (x, y)$. In other words $g = \frac{d}{dt}\delta_t(h)|_{t=0}$ is a typical member of $T(h)$. By direct calculation,

$$\gamma(g) = \gamma\frac{d}{dt}\delta_t(h)|_{t=0} = \frac{d}{dt}\gamma(\delta_t(h))|_{t=0} = \frac{d}{dt}\gamma\delta_t\gamma^{-1}\gamma(h)|_{t=0} = \frac{d}{dt}\gamma\delta_t\gamma^{-1}(f))|_{t=0}$$

Let $\hat{\delta}_t = \gamma\delta_t\gamma^{-1}$. Then $\hat{\delta}_0$ is the identity and

$$\gamma(g) = \frac{d}{dt}\hat{\delta}_t(f)|_{t=0}.$$

In other words, $\gamma(g) \in T(f)$ and $\gamma(T(h)) \subset T(f)$. Interchanging the roles of f and h shows that $T(f) \subset \gamma(T(h))$ and equality holds, as claimed. □

Let h be a normal form of $f = [p, q]$. We calculate necessary conditions for F to be a universal unfolding of f when $f = \gamma(h)$, as follows:

(a) Write $T(h) = \mathcal{J} \oplus V_h$ where $\mathcal{J}$ is intrinsic.
(b) Using Lemma 7.6 and the fact that $\mathcal{J}$ is intrinsic, write $T(f) = \mathcal{J} \oplus V_f$.
(c) By Theorem 6.5, F is a k-parameter universal unfolding of f if and only if

$$\mathcal{E}^2 = \mathcal{J} \oplus V_f \oplus \mathbf{R}\{F_{\alpha_1}, \ldots, F_{\alpha_k}\}$$

(d) A complementary space to $\mathcal{J}$ always consists of $\dim(V_f) + k$ dimensions. We can choose a basis for V_f in terms of $[p, q]$ and its derivatives. Then we solve the problem by writing the Taylor coefficients of this basis and F_{α_j} in the monomials that are not in $\mathcal{J}$. It follows that F is a universal unfolding of f if and only if this matrix has a nonzero determinant.

Lemma 7.7 *Suppose f is dimorphism equivalent to $h = w^2 + uv$, and $F = P(u, w, \alpha)w + Q(u, w, \alpha)v$ is a 1-parameter unfolding of $f = p(u, w)w + q(u, w)v$. Then F is a universal unfolding of f if and only if*

$$p_u Q_\alpha - q_u P_\alpha \neq 0$$

at $u = v = \alpha = 0$.

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Proof Table 6 shows $T(h) = [\mathcal{M}, \mathcal{M}] + \mathbf{R}\{[0, 1]\}$. Note that $[\mathcal{M}, \mathcal{M}]$ is intrinsic. We want to write

$$T(f) = [\mathcal{M}, \mathcal{M}] \oplus V_f$$

Since f is dimorphism equivalent to h, we know at (0, 0)

$$q = 0, p = 0 \qquad q_u \neq 0 \qquad p_w q_u - p_u q_w \neq 0$$

Therefore,

$$T(f) = [\mathcal{M}, \mathcal{M}] + \mathbf{R}\{[p_u, q_u]\}$$

The universal unfolding Theorem 6.5 implies that F is a one-parameter universal unfolding of f if and only if

$$\mathbf{R}\{[p_u, q_u], [P_\alpha, Q_\alpha]\}$$

spans $\mathbf{R}\{[1, 0], [0, 1]\}$. That is

$$\det \begin{pmatrix} p_u & q_u \\ P_\alpha & Q_\alpha \end{pmatrix} = p_u Q_\alpha - q_u P_\alpha \neq 0$$

The solution of the recognition problems for the other singularities of low codimension are discussed in Wang (2015).

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