GENERICITY, BIFURCATION AND SYMMETRY

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GENERICITY, BIFURCATION AND SYMMETRY

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In these lectures I would like to discuss how the existence of symmetries alters the type of bifurcation behavior that one expects to observe. In the first lecture I will concentrate on the structure and *dynamics* of steady-state bifurcation from equilibria. It is here that the influence of symmetries on linearized equations will be discussed and some facts from elementary representation theory introduced. The second lecture will be devoted to effects of symmetry on period-doubling in maps with a short description of an application to large arrays of Josephson junctions. In the final lecture I will describe how certain standard choices of boundary conditions (particularly Neumann) can be thought of as symmetry constraints and how this fact alters notions of genericity. It accord with the style that has developed in the lectures at this workshop, the lectures are of different length.

Much of the background material for these lectures may be found in [GSS]. The descriptions of the more advanced topics will be brief as the results concerning these topics have or will appear elsewhere.

Lecture 1: *Bifurcation From Equilibria*

Consider the system of ODE

$$\frac{dx}{dt} = f(x, \lambda) \qquad x \in \mathbf{R}^n \ , \ \lambda \in \mathbf{R} \tag{1.1}$$

with an equilibrium at (x_0, λ_0), that is,

$$f(x_0, \lambda_0) = 0.$$

Let $A \equiv (df)_{x_0,\lambda_0}$ be the $n \times n$ Jacobian matrix obtained by differentiation with respect to x. Then (1.1) becomes:

$$\frac{dx}{dt} = A(x - x_0) + \cdots$$

Recall that if A is hyperbolic (that is, all eigenvalues have nonzero real part), then all the dynamics of (1.1) are determined by A near x_0.

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DEFINITION 1.1. (1.1) has a *bifurcation point* at (x_0, λ_0) if some eigenvalue of A lies on the imaginary axis.

Without loss of generality we may assume $\lambda_0 = 0$.

Basic Question: What are the typical transitions in the dynamics of (1.1)?

It is well known that generically the typical transitions are controlled by the transitions from hyperbolicity in the matrix A as λ is varied. Indeed, there are two possibilities; A has a

(a) simple zero eigenvalue *Steady-State Bifurcation*

(b) a pair of simple, complex-conjugate, purely imaginary eigenvalues. *Hopf bifurcation*

Nonlinear theory then implies that in case (a) the dynamics can be reduced (using center manifolds) to one dimension and the expected transition is a limit-point or saddle-node bifurcation with a transition from 0 to 2 equilibria. In case (b) the dynamics can be reduced to two dimensions and one expects a single branch of periodic solutions to emanate from this bifurcation. See [GH1].

We now consider how both the linear and the nonlinear transitions change when (1.1) has a nontrivial group of symmetries.

SYMMETRY

Let $\Gamma \subset O(n)$ be a Lie group of orthogonal matrices.

DEFINITION 1.2. (1.1) has *symmetry* Γ if

$$f(\gamma x, \lambda) = \gamma f(x, \lambda) \qquad \text{for all} \quad \gamma \in \Gamma. \tag{1.2}$$

There are two immediate consequences of the commutativity relation (1.2):

(a) The equilibrium x_0 has *symmetry*. Define the *isotropy subgroup* of Γ at x_0 to be

$$\Sigma_{x_0} \equiv \{\gamma \in \Gamma : \gamma x_0 = x_0\}$$

In our discussion we will assume that the equilibrium x_0 is a fully symmetric, i.e. $\Sigma_{x_0} = \Gamma$. Then, without loss of generality, we may assume $x_0 = 0$.

(b) The chain rule implies $(df)_{\gamma x, \lambda}\gamma = \gamma(df)_{x,\lambda}$ and hence

$$A\gamma = \gamma A.$$

that is, the matrix A *commutes* with Γ. Thus to understand the dynamics of (1.1) we must first understand the form of matrices that commute with Γ. This topic in representation theory has been well studied and we briefly review the relevant theory.

Elementary Representation Theory

Definitions 1.3. Let W be a subspace of $V \cong \mathbf{R}^n$.

(a) $W \subset V$ is Γ-*invariant* if $\gamma(W) = W$ for all $\gamma \in \Gamma$

(b) $W \subset V$ if Γ-*irreducible* if the only Γ-invariant subspaces of W are $\{0\}$ and W.

It is well known that any representation may be decomposed into a direct sum of irreducible representations; the simplicity of the proof of this decomposition is, however, not always appreciated.

Theorem 1.4 (*The Decomposition Theorem*). *There exist* Γ-*irreducible subspaces* $V_1, \ldots, V_s$ *such that*

$$V = V_1 \oplus \cdots \oplus V_s \ .$$

Proof. Since $\Gamma \subset O(n)$, the standard inner product is Γ-invariant; that is, $(\gamma v, \gamma w) = (v, w)$ for all $\gamma \in \Gamma$.

Now suppose V has proper Γ-invariant subspace W. Then define

$$W^\perp = \{v \in V : (v, W) = 0\}$$

and observe that $W^\perp$ is Γ-invariant. Since

$$V = W \oplus W^\perp.$$

the theorem is proved by induction on the dimension of V. □

We begin our discussion of commuting matrices by first considering commuting matrices for an irreducible representation U.

Theorem 1.5. *The space of matrices commuting with an irreducible representation is isomorphic to* $\mathbf{R}, \mathbf{C}$ *or* $\mathbf{H}$ *(where* $\mathbf{H}$ *denotes the quaternions).*

Proof. Observe that the vector space

$$\mathcal{D} = \{\text{matrices on } U \text{ commuting with } \Gamma\}$$

is an algebra over $\mathbf{R}$; that is, we can add, multiply and scalar multiply commuting matrices. In addition, $\mathcal{D}$ is a division algebra, that is, every nonzero matrix B in $\mathcal{D}$ is invertible. To verify this point note that for $B \in \mathcal{D}$

$$\ker B \text{ is } \Gamma\text{-invariant}$$

and irreducibility implies

$$\ker B = U \text{ or } \ker B = \{0\}$$

Hence, either $B = 0$ or B is invertible and $B^{-1} \in \mathcal{D}$. The classical Wedderburn Theorem implies that $\mathcal{D}$ is isomorphic either to $\mathbf{R}, \mathbf{C}$ or $\mathbf{H}$.

DEFINITION 1.6. U is *absolutely* irreducible if the only matrices commuting with Γ are multiples of the identity ($\mathcal{D} \cong \mathbf{R}$) and *nonabsolutely* irreducible otherwise.

Examples. (a) $SO(2)$ acts nonabsolutely irreducibly on $\mathbf{R}^2$.

(b) $O(2)$ acts absolutely irreducibly on $\mathbf{R}^2$.

THEOREM 1.7. *Generically, in one-parameter bifurcation, steady-state bifurcation satisfies:*

(a) the algebraic multiplicity of the zero eigenvalue equals the geometric multiplicity, and

(b) Γ acts absolutely irreducibly on $\ker A$.

Sketch of Proof.

(I) At a bifurcation point, do a center manifold reduction (which can be done preserving symmetries - Ruelle [R]). Thus, we can assume all eigenvalues of A are on the imaginary axis.

(II) Suppose A has a zero eigenvalue.

Choose a Γ-irreducible subspace $U \subset \ker A$, and define

$$M : V \to V \quad \text{by} \quad \begin{cases} 0 & \text{on} \quad U \\ I & \text{on} \quad U^{\perp} \end{cases}$$

Perturb (1.1) to:

$$\frac{dx}{dt} = f(x,\lambda) + \varepsilon M x \equiv f_\varepsilon(x,\lambda).$$

For nonzero ε, $A_\varepsilon \equiv (df_\varepsilon)_{0,0} = A + \varepsilon M$ satisfies:

(a) Geometric multiplicity of eigenvalue zero
= algebraic multiplicity of eigenvalue zero.

(b) Γ acts irreducibly on $\ker A_\varepsilon = U$.

Now reduce the bifurcation problem to U (by center manifold).

(III) If $\dim U = 1$, then Γ acts absolutely irreducibly. So assume $\dim U \geq 2$. We claim that $f(0,\lambda) = 0$, that is, $x = 0$ is a 'trivial' equilibrium.

DEFINITION 1.8. Let $\Sigma \subset \Gamma$ be a subgroup. Define the *fixed-point subspace* of Σ to be:

$$\mathrm{Fix}(\Sigma) \equiv \{v \in V : \sigma v = v \text{ for all } \sigma \in \Sigma\}$$

LEMMA 1.9. $f : \mathrm{Fix}(\Sigma) \times \mathbf{R} \to \mathrm{Fix}(\Sigma)$.

Proof. $f(v,\lambda) = f(\sigma v,\lambda) = \sigma f(v,\lambda)$ for all $\sigma \in \Sigma$.

□

To prove the claim observe that $\mathrm{Fix}(\Gamma) = \{0\}$ since Γ acts irreducibly and nontrivially. Thus, $f(0,\lambda) = 0$.

Define $A_\lambda = (df)_{0,\lambda}$ and observe that A_λ commutes with Γ. Hence, A_λ is in $\mathcal{D}$ and corresponds to a curve $d(\lambda) \in \mathbf{R}, \mathbf{C}$ or $\mathbf{H}$ with $d(0) = 0$.

(IV) Suppose that Γ acts nonabsolutely irreducibly on U. Then the curve $d(\lambda)$ is in either $\mathbf{C}$ or $\mathbf{H}$. The hyperplane $\{z \in \mathcal{D} : \mathrm{Re}(z) = 0\}$ corresponds to matrices with purely imaginary eigenvalues. Hence the curve $d(\lambda)$ can be perturbed to $e(\lambda)$ where $e(\lambda)$ crosses $\mathrm{Re}(z) = 0$ at $\lambda = 0$ with nonzero speed, but NOT THROUGH 0. Since the curve $e(\lambda)$ corresponds to a family of matrices B_λ, we can perturb (1.1) to:

$$\frac{dx}{dt} = f(x,\lambda) + (B_\lambda - A_\lambda)x.$$

□

The absolute irreducibility noted in Theorem 1.9 can be used to transfer the analytic problem of existence of branches of equilibria to an algebraic one, as the next theorem shows.

THEOREM 1.10 (*Equivariant Branching Lemma*). (Vanderbauwhede [V], Cicogna [C])

(a) Let $\Gamma \subset O(n)$ be a Lie group acting absolutely irreducibly on $V \equiv \mathbf{R}^n$.

(b) Assume that (1.1) has a bifurcation at $\lambda = 0$ and symmetry Γ.

(c) Let $\Sigma \subset \Gamma$ be a subgroup such that

$$\dim\ \mathrm{Fix}(\Sigma) = 1. \tag{1.3}$$

then there is a unique branch of equilibria having symmetry Σ if

$$c'(0) \neq 0$$

where $(df)_{0,\lambda} = c(\lambda)I$ by (a) and $c(0) = 0$ by (b).

Proof. We know that $f : \mathrm{Fix}(\Sigma) \times \mathbf{R} \to \mathrm{Fix}(\Sigma)$. Let v_0 be a nonzero vector in $\mathrm{Fix}(\Sigma)$, and define $g : \mathbf{R} \times \mathbf{R} \to \mathbf{R}$ by

$$g(s,\lambda)v_0 = f(sv_0,\lambda).$$

We that $g(0,\lambda) = 0$ since irreducibility implies that 0 is a 'trivial' solution. Hence $g(s,\lambda) = h(s,\lambda)s$ by Taylor's Theorem where

$$h(0,0) = c(0) = 0 \quad \text{and} \quad h_s(0,0) = c'(0) \neq 0.$$

Using the Implicity Function Theorem solve

$$h(s,\lambda) = 0 \quad \text{for} \quad \lambda = \Lambda(s).$$

□

Examples. (a) $O(2)$ acts on $\mathbf{R}^2 \cong \mathbf{C}$. Let $\Sigma = \{1, \kappa\} \cong \mathbf{Z}_2$ where $\kappa z = \bar{z}$. Then $\mathrm{Fix}(\Sigma) = \mathbf{R} \subset \mathbf{C}$ has dimension one. Hence, in circularly symmetric bifurcation problems we expect equilibria with a reflectional symmetry.

(b) The irreducible representations of $SO(3)$ are given by the spherical harmonics of order ℓ denoted by V_ℓ. (Note that $\dim V_\ell = 2\ell+1$.) The Cartan decomposition of V_ℓ is:

$$V_\ell = \mathbf{R} \oplus \mathbf{C}^\ell$$

where the action of $SO(2) \subset SO(3)$ on V_ℓ is given by:

$$\theta \cdot (x,\; z_1, \ldots, z_l) = (x, e^{i\theta} z_1, \ldots, e^{li\theta} z_l).$$

It follows that $\mathrm{Fix}(SO(2)) = \{(x, 0, \ldots, 0)\}$ has dimension one. Since solutions with $SO(2)$ symmetry have an axis of symmetry, we have proved:

COROLLARY 1.11. *In steady-state bifurcations involving spherical symmetry, generically (in the sense that eigenvalues go through zero with nonzero speed) there exist a branch of axisymmetric equilibria.*

(c) (Nontrivial *dynamics* in steady-state bifurcation) Let $\Gamma \subset O(3)$ be the 24 element group generated by:

$$\sigma(x,y,z) = (y,z,x)$$
$$\varepsilon(x,y,z) = (\varepsilon_1 x, \varepsilon_2 y, \varepsilon_3 z) \qquad \text{where} \qquad \varepsilon_j = \pm 1.$$

Bifurcation with this group action was studied by May and Leonard [ML] in the context of three competing populations and Busse and Heikes [BH] in the context of convection in a rotating layer. Later, Guckenheimer & Holmes [GH2] studied this group of symmetries abstractly. They showed that it is possible to have a structurally stable (in the world of Γ symmetry), asymptotically stable, primary branch of heteroclinic connections. We outline this construction.

Up to conjugacy the isotropy subgroups of Γ are:

$$\Sigma_2 = \{\varepsilon_1, 1, 1)\} \qquad \mathrm{Fix}(\Sigma_2) = \{(0,y,z)\}$$
$$\Sigma_3 = \{(1, \sigma, \sigma^2\} \qquad \mathrm{Fix}(\Sigma_3) = \{(x,x,x)\}$$
$$\Sigma_4 = \{(\varepsilon_1, \varepsilon_2, 1)\} \qquad \mathrm{Fix}(\Sigma_4) = \{(0,0,z)\}.$$

Hence, generically, there exist two types of equilibria with isotropy Σ_3 and Σ_4, respectively.

We determine the dynamics associated with this Γ-equivariant bifurcation by describing explicitly the general Γ-equivariant mapping f. Write f in coordinates as $f = (X, Y, Z)$. Then

(i) $f(\sigma v, \lambda) = \sigma f(v, \lambda)$ implies :

$$Y(x, y, z, \lambda) = X(y, z, x, \lambda)$$

$$Z(x, y, z, \lambda) = X(z, x, y, \lambda).$$

((ii) $f(\varepsilon v, \lambda) = \varepsilon f(v, \lambda)$ implies

$X(x, y, z, \lambda)$ is odd in x and even in y and z.

Thus, we can write $X(x, y, z, \lambda) = a(x^2, y^2, z^2, \lambda)x$. The genericity condition is $a_\lambda(0) \neq 0$. We assume:

(H1) $$a_\lambda(0) > 0$$

so that the trivial solution losses stability at $\lambda = 0$. Rescale λ so that:

$$a_\lambda(0) = 1.$$

To third order f has the form:

$$\begin{pmatrix} (\lambda + \alpha x^2 + \beta y^2 + \gamma z^2)x \\ (\lambda + \gamma x^2 + \alpha y^2 + \beta z^2)y \\ (\lambda + \beta x^2 + \gamma y^2 + \alpha z^2)z \end{pmatrix} .$$

Hence, computing $f|\operatorname{Fix}(\Sigma_4) \times \mathbf{R} = 0$ yields the equation $\alpha z^2 = \lambda$. Thus, if we assume

(H2) $$\alpha < 0,$$

then the Σ_4 equilibrium A will exist for $\lambda > 0$ and, by exhance of stability, be stable inside the z-coordinate axis $\operatorname{Fix}(\Sigma_4)$.

Since $(df)_A$ commutes with its isotropy subgroup Σ_4. It follows that $(df)_A$ is diagonal and that, to lowest order, the two eigenvalues outside $\operatorname{Fix}(\Sigma_4)$ are:

$(\gamma - \alpha)z^2$ (in the x-direction)

$(\beta - \alpha)z^2$ (in the y-direction)

If we assume:

(H3) $$\beta < \alpha < \gamma$$

then A will be a sink in the flow-invariant yz-plane $\mathrm{Fix}(\Sigma_2)$ and a saddle in the xz-plane. By considering $f|\mathrm{Fix}(\Sigma_2) \times \mathbf{R}$ one can show that there are no equilibria in the yz-plane that lie off the coordinate axes when (H3) is valid.

Thus the unstable manifold leaving A in the xz-plane must either be unbounded or tend to the equilibrium on the x-axis. Indeed, the saddle-sink connection can be shown to exist if:

$$a \ll 0.$$

Hence the heteroclinic cycle exists.

Finally, we note that this connection is structurally stable, since the coordinate planes are always flow invariant (being fixed point subspaces) and planar saddle-sink connections are structurally stable. A calculation shows that this heteroclinic connection can be asymptotically stable.

Thus, intermittancy is an expected phenomena in symmetric systems. Further examples of complicated dynamics emanating from a steady-state bifurcation may be found in Field [F3]. See also [AGH]

We end this lecture by discussing the general form of linear commuting maps when the representation is not irreducible. This result is useful when computing the asymptotic stability of nontrivial equilibria, and when considering mode interactions in multiparameter systems. This material is included mainly for completeness and may be skipped on a first reading.

DEFINITIONS 1.12. Let Γ act on a space V.

(a) Let W_1 and W_2 be Γ-irreducible subspaces of V. Then W_1 and W_2 are Γ-*isomorphic* if there exists a linear map $L : W_1 \to W_2$ that commutes with Γ, that is, $L\gamma = \gamma L$ for all $\gamma \varepsilon \Gamma$.

(b) Let W be a Γ-irreducible subspace of V. The *isotypic component* of V corresponding to W is the sum of all Γ-irreducible subspaces of V are Γ-isomorphic to W.

Examples. (a) For each integer ℓ let $O(2)$ act on $V_\ell \cong \mathbf{C}$ by:

$$\theta \cdot z = e^{\ell i\theta} z \quad \text{and} \quad \kappa \cdot z = \bar{z}.$$

The actions for ℓ_1 and ℓ_2 are $O(2)$-isomorphic iff $\ell_1 = \pm\ell_2$.

(b) Let the permutation group S_3 act on $\mathbf{C}$ as symmetries of a equilateral triangle and on $\mathbf{R}^3$ by permuting axes. The second action has a two dimensional S_3 irreducible subspace consisting of points in $\mathbf{R}^3$ whose coordinates sum to zero. The actions of S_3 on $\mathbf{C}$ and V are isomorphic.

THEOREM 1.13. *Let $U_1, \ldots, U_t$ be the distinct Γ-irreducible representations appearing in a decomposition of V guaranteed by the Decomposition Theorem. Then V is the direct sum of isotypic components:*

$$V = V_{u_1} \oplus \cdots \oplus V_{u_t}. \tag{1.4}$$

COROLLARY 1.14. *Let $A : V \to V$ be linear and commute with Γ. Then A can be block diagonalized by (1.4), that is,*

$$A(V_{u_j}) \subset V_{u_j} \quad \text{for} \quad j = 1, \ldots, t.$$

Lecture 2: *Period-Doubling, Symmetry and Josephson Functions*

In this lecture I want to describe how symmetry affects period-doubling bifurcations and apply these ideas to coupled systems of Josephson junctions. The theory follows closely the discussion of steady-state bifurcations given in the first lecture - but with an important difference. The period-doubling bifurcation itself introduces a reflectional symmetry.

Let $f : V \times \mathbf{R} \to V$ be a smooth Γ-equivariant mapping and let $f(\cdot, \lambda_0)$ have a fixed-point at x_0.

DEFINITION 2.1. f has a *period-doubling* bifurcation at (x_0, λ_0) if -1 is an eigenvalue of the Jacobian matrix $(df)_{x_0,\lambda_0}$.

We assume that x_0 is Γ-invariant and hence, without loss of generality, we may assume that $(x_0, \lambda_0) = (0, 0)$. Our discussion of genericity for steady-state bifurcation applies equally well to period-doubling bifurcation. In particular, genericity implies that the geometric multiplicity of the eigenvalue -1 equals the algebraic multiplicity and that Γ acts absolutely irreducibly on the eigenspace V_{-1} corresponding to the eigenvalue -1.

After a center manifold reduction we may assume that $V = V_{-1}$. Observe that irreducibility implies that f has a trivial fixed point, i.e. that $f(0, \lambda) = 0$. Similarly, absolute irreducibility implies that

$$(df)_{0,\lambda} = c(\lambda) I$$

where $c(0) = -1$. Indeed, genericity implies that $c'(0) \neq 0$.

The question we address is: find all branches of period two points of f in the neighborhood of the period-doubling bifurcation at $(0, 0)$. We prove the following analogue of the Equivariant Branching Lemma.

Define the group

$$\widehat{\Gamma} = \begin{cases} \Gamma & \text{if} \quad -I \in \Gamma \\ \Gamma \oplus \mathbf{Z}_2(-I) & \text{if} \quad -I \notin \Gamma. \end{cases}$$

Note that $\widehat{\Gamma}$ acts naturally on V.

THEOREM 2.2. *Let $\Sigma \subset \widehat{\Gamma}$ be a subgroup satisfying:*

$$\dim\,(\mathrm{Fix}(\Sigma)) = 1.$$

Then there exists a unique branch of period two points for f emanating from the origin.

A proof of this theorem, based on normal hyperbolicity is given in [ChG]. Here, however, we present a very simple proof using Liapunov-Schmidt reduction. This proof was derived independently by Peckham & Kevrikidis, Roberts and Vanderbauwhede (private communications). The idea of the proof is to convert the problem of finding period two points of f to one of finding zeroes of a derived mapping F. The Equivariant Branching Lemma is then used to prove the existence of branches of zeroes of F. The proof is a discrete analogue of the proof of existence of periodic solutions given by Liapunov–Schmidt reduction in Hopf bifurcation.

Proof. Observe that finding a point x such that $f(f(x)) = x$ is equivalent to finding solutions to the system of equations

$$y = f(x) \qquad \text{and} \qquad x = f(y).$$

Given this, define $F : V \times V \to V \times V$ by

$$F(x, y) = (f(x) - y,\ f(y) - x).$$

Then $F(x, y) = (0, 0)$ if and only if x and y are period two points of f.

Next use Liapunov–Schmidt reduction to solve $F = 0$. To do this, compute

$$(dF)_{0,0} = \begin{pmatrix} -I & -I \\ -I & -I \end{pmatrix}$$

and observe that

$$\widetilde{V} \equiv \ker\,(dF)_{0,0} = \{(x, -x) \in V \times V : x \in V\}$$

is isomorphic to V. Now use Liapunov-Schmidt reduction to find implicitly a mapping $g : \widetilde{V} \to \widetilde{V}$ whose zeroes near the origin are in one to one correspondence with the zeroes of F.

Now we consider the equivariance properties of g. Since Liapunov–Schmidt reduction can be performed in such a way as to preserve symmetries, we need only consider the equivariance of F. Note that Γ acts (via the diagonal action) on $V \times V$ and that F commutes with the action of Γ. In addition F commutes with the reflection symmetry $(x, y) \to (y, x)$. Since this symmetry acts as $-I$ on $\widetilde{V}$, it follows that the reduced mapping g commutes with the group $\widehat{\Gamma}$ acting on $\widetilde{V}$.

Finally, we note that the assumption on $\mathrm{Fix}(\Sigma)$ is precisely what is needed to apply the Equivariant Branching Lemma. $\square$

Arrays of Coupled Oscillators

We will apply this theorem to find period two points emanating from period-doubling bifurcations in the presence of S_N symmetry, where S_N is the permutation group on N letters. To motivate this discussion, we begin by considering arrays of coupled oscillators.

An array of *coupled oscillators* is a system of ODE of the form:

$$\begin{aligned} \dot{y}_1 &= g_1(y_1, \ldots, y_N) \\ &\cdots \qquad\qquad\qquad\qquad y_j \in \mathbf{R}^k \\ \dot{y}_N &= g_N(y_1, \ldots, y_N). \end{aligned} \tag{2.1}$$

These oscillators are *identical* if $g_1 = \cdots = g_N \equiv g$ and *identically coupled* if

$$g(y_1, y_2, \ldots, y_N) = g(y_1, y_{\sigma(2)}, \ldots, y_{\sigma(N)})$$

for every permutation on $N-1$ letters σ. Observe that systems of identical, identically coupled, coupled oscillators are precisely those systems (2.1) that have S_N symmetry.

An *in-phase* solution to (2.1) is one lying in the plane

$$y_1 = \cdots = y_N \equiv y. \tag{2.3}$$

The fact that the plane defined by (2.3) is flow-invariant follows from the fact that the plane (2.3) is just $\mathrm{Fix}(S_N)$. In-phase solutions satisfy the differential equation:

$$\dot{y} = g(y, \ldots, y) \tag{2.4}$$

An interesting example of a system of identical coupled oscillators is a large array of Josephson junctions that has been studied by Hadley, Beasley and Wiesenfeld [HBW1, HBW2]. The second order system of ODE for Josephson junctions is

$$\beta\ddot{\phi}_j + \dot{\phi}_j + \sin(\phi_j) + I_L = I_B \qquad (j = 1, \ldots N) \tag{2.5}$$

where

ϕ_j	is the difference in phase of the "quasiclassical superconducting" wave functions on the two sides of the j-th junction
B	is the capacitance of each junction
I_B	is the bias current of the circuit
I_L	is the load current.

To complete the system, assumptions must be made on how the circuit is loaded. for example, if the array is *capacitative* loaded, then

$$I_L = \sum_{j=1}^{N} \ddot{\phi}_j$$

while if the array is *resistive* loaded, then

$$I_L = \sum_{j=1}^{N} \dot{\phi}_j \ .$$

We make several observations about this system of ODE.

(a) In-phase periodic solutions exist for a large range of the parameters $\beta,\ I_B$.

(b) The Poincaré maps for the in-phase periodic solutions can loose stability by either a fixed-point or a period-doubling bifurcation - but not by a Hopf bifurcation. Both of these types of bifurcations have been found in numerical computation.

(c) When the in-phase periodic solutions exists, it is asymptotically stable in the plane (2.3), and hence is unique.

Next we address the question of what types of solutions are expected to emanate from the bifurcations noted in (b). More detail may be found in [AGK].

Fix β, I_B at a point where an in-phase periodic solution $y(t)$ exists. Choose a Poincaré section S as follows:

$$S = L \oplus W \cong \mathbf{R}^n \tag{2.6}$$

where L is the cross-section to $y(t)$ in the plane of in-phase solutions and

$$W = \{(\phi_1, \ldots, \phi_N) : \phi_1 + \cdots + \phi_N = 0\}.$$

Let $P : S \to S$ be the Poincaré map: $P(0) = 0$ since the in-phase solution is periodic.

Observe that S is S_N-invariant and that uniqueness of solutions to systems of ODE forces P to be S_N-equivariant. Indeed, we may write

$$S = L \oplus V \oplus V \tag{2.7}$$

as a direct sum of S_N-irreducible subspaces, where

$$V = \{x_1 + \cdots + x_N = 0\} \cong \mathbf{R}^{N-1} \subset \mathbf{R}^N.$$

Suppose that the in-phase periodic solution is undergoing a bifurcation; that is, either of the generalized eigenspaces E_1 or E_{-1} corresponding to the eigenvalues ± 1 is nonzero. Invoking genericity, we expect the action of S_N on $E_{\pm 1}$ to be absolutely irreducible. It follows from (2.6, 2.7) that when these eigenspaces are nonzero they will generally be isomorphic to either L or V. If they happened to be isomorphic to L, then the bifurcation would produce a new in-phase periodic solution, thus contradicting (c). Hence $E_{\pm 1} \cong V$.

We consider first the possibility of a fixed-point bifurcation for the Poincaré map; that is, $E_1 \cong V$. Field and Richardson [FR] show that generically all fixed points of P have isotropy with one-dimensional fixed point subspaces. Hence the Equivariant Branching Lemma (applied to $Q(s) = P(s) - s$) implies the existence of all the expected fixed points of P. Up to permutation, the fixed points are classified as follows. Divide the oscillators into two blocks: one block having k oscillators and the other having $N-k$ oscillators. The bifurcating fixed points have the first k and the last $N-k$ coordinates equal. Their symmetry group is:

$$\Sigma_k = S_k \times S_{N-k}. \tag{2.8}$$

Unfortunately, [IG] show that if there is a nonzero equivariant quadratic, then generally all solutions found using the Equivariant Branching Lemma are asymptotically unstable. The action of S_N on V has such a nonzero equivariant quadratic mapping.

At a period-doubling bifurcation the local bifurcation results are more interesting. Since $-I \notin S_N$ as it acts on V, we use $\widehat{S}_N$ and Theorem 2.2 to find period two solutions. There is another class of periodic solutions obtained in this way. Divide the oscillators into three blocks, the first two having k elements and the third having $N-2k$ elements. The isotropy of such solutions is the group:

$$T_k \quad \text{generated by} \quad S_k \times S_k \times S_{N-2k} \text{ and } (x,y,z) \to -(y,x,z).$$

Theorem 2.2 implies the existence of period two points having symmetries Σ_k and symmetries T_k. The interpretation of these properties of these solutions for the Josephson function model is most interesting. As noted above, the periodic solutions with isotropy Σ_k divide the oscillators into two blocks, each block consisting of in-phase oscillators with period approximately twice that of the original in-phase solution. The periodic solutions with isotropy T_k divide the junctions into three blocks, the first two blocks consisting of in-phase oscillation but with a half period phase shift between the two blocks. The third block consists of junctions with in-phase oscillation but with a period *comparable* to the period of the in-phase periodic solution.

Certain of these solutions can be asymptotically stable (for more details, see [AGK]) and have been observed in numerical experiments on the resistive load Josephson junction model (but not with the capacitive loaded model). [AGK] also prove that there exists period two solutions to this S_N symmetric period-doubling bifurcation with submaximal isotropy. For these solutions the oscillators divide into three blocks of unequal size.

Lecture 3: *Genericity and Boundary Conditions*

The Faraday experiment provides an example where a general qualitative analysis based on period-doubling bifurcations may connect theory with experiment. This connection highlights the effects that boundary conditions may have on genericity (see [CGGKS] where the issues raised here are discussed more fully).

In the Faraday experiment a fluid layer is subjected to a vertical oscillation at frequency ω and forcing amplitude A. When A is small the fluid remains essentially flat and when A is increased the flat surface bifurcates to a standing wave at frequency $\omega/2$. What is measured in the experiments of Gollub and coworkers [CG, GS] is a *stroboscopic map* S which pictures the surface of the fluid at each period of the forcing. Since, after the bifurcation, the fluid surface returns to its original form each second iterate of S, we have a period-doubling bifurcation.

The experiments of Ciliberto and Gollub [CG] focus on fluid layers with circular cross-section, while the experiments of Gollub and Simonelli [GS] focus on the square cross-section case. A qualitative analysis of the circular cross-section experiment, along the lines that we describe here for the square cross-section, is given by Crawford, Knobloch and Riecke [CKR].

The following points are observed in the experiments [GS].

(a) For most values of the forcing frequency ω the initial bifurcation from stability of the flat surface as the amplitude A is increased is by a period-doubling bifurcation. Spatial modes are detected and described by their wave numbers in both horizontal directions, such as (3,1), (3,2), (4,0).

(b) For isolated values of ω the flat surface loses stability to two modes simultaneously.

Given this information we may make several reasonable assumptions concerning the mathematical analysis of any model purporting to describe the Faraday experiment.

(a) Since The experiment is square symmetric, the loss of stability of the flat surface to say a (3,1) mode would imply loss of stability to the (1,3) mode as well; that is, the eigenspace E_{-1} corresponding to the period-doubling -1 eigenvalue is at least double. Generically, it is precisely double, and hence $E_{-1} \cong \mathbf{C}$.

(b) At the mode interaction point E_{-1} is isomorphic to $\mathbf{C}^2$.

(c) Assuming that a center manifold reduction is possible, the dynamics of the (stroboscopic map of the) Faraday experiment near the mode interaction point is controlled by the dynamics of a D_4-equivariant mapping $f : \mathbf{C}^2 \to \mathbf{C}^2$.

These assumptions, however, lead to a difficulty. The representation of the symmetry group of the square, D_4, on the eigenspace E_{-1} at a generic (non-mode-interaction point) is either an irreducible two-dimensional representation of D_4 or the sum of two one-dimensional irreducibles.

In the latter case we must then question why a nongeneric situation occurs in this experiment (since generically eigenspaces are irreducible). In the former case the representation is irreducible, but a different problem occurs at a point of mode interaction. Up to isomorphism the two-dimensional irreducible representation of D_4 is unique. Hence, at a codimension two point of mode interaction of two two-dimensional, isomorphic, irreducible representations, generically we expect the linearization (of f) to be nilpotent. This nilpotency would imply that there is only one independent set of eigenfunctions (not two), and hence that the two distinct modes would, in fact, have to merge together at the codimension two point (and be physically indistinguishable).

We are faced with a dilemma: either something nongeneric (the reducibility of the eigenspace) occurs in models of the Faraday experiment, or something is wrong with the experimental observation of distinct modes in the square cross-section case.

We present here an alternative explanation based on some subtleties of genericity and boundary conditions. See [CGGKS].

Fujii, Mimura and Nishiura [FMN] and Armbruster and Dangelmayr [AD] observed that the bifurcation of steady solutions in reaction-diffusion equations on the line changed from what might have been expected when Neumann boundary conditions (NBC) were assumed. We abstract part of their reasoning here.

Any solution u to a reaction-diffusion equation on $[0, \pi]$ with NBC can be extended in a solution v to that same equation with periodic boundary conditions (PBC) on $[-\pi, \pi]$ by extending the solution to be even across zero. More precisely, define:

$$v(x,t) = u(-x,t) \quad \text{for all} \quad x < 0. \tag{3.1}$$

Conversely solutions v to the PBC problem that are also even (which, using (3.1) is a fixed point subspace condition for the symmetry $x \to -x$) is a solution to the NBC model.

What is gained by the extension to PBC is the introduction of O(2) symmetry into the problem (translational symmetry of the reaction-diffusion equation modulo the 2π periodicity of the boundary conditions). The idea for determining genericity is to look at the generic PBC case (that is, O(2) symmetric bifurcation) and then restrict (by fixed-point subspace arguments) to the NBC case.

Similar statements about genericity are valid for Dirichlet boundary conditions (DBC), although DBC does require an extra reflectional symmetry on the differential operator to be valid in order to make the extension to PBC. This symmetry is valid, for example, in the Navier–Stokes equations. Indeed, similar statements hold for systems and for higher dimensional domains with various mixtures of boundary conditions.

We now return to the Faraday experiment. In any analytic model of the experiment one must solve for both the *surface deformation* $\zeta(x, y)$ and the *fluid velocity field* $u(x, y, z)$. Typically, in models, no-slip or Dirichlet boundary conditions are valid for u along the lateral boundaries and Neumann boundary conditions are assumed on ζ (that is, the fluid surface is assumed to be perpendicular to the side walls).

As our discussion above indicates these boundary conditions have the effect of introducing T^2 symmetry into the bifurcation problem. The two-torus T^2 is obtained by planar translations modulo the double periodicity of the square. Thus, the full symmetry group Γ of the Faraday experiment with square geometry is generated by D_4 and T^2.

The consequence of having this enlarged symmetry group is that all of the two-dimensional eigenspaces noted above are irreducible representations of Γ, and that, at mode interaction points, distinct modes have distinct irreducible representations. Hence, the linearization is diagonal rather than nilpotent and distinct modes need not merge (thus agreeing with experimental observation).

Period-doubling bifurcations at points of mode interaction in the Faraday experiment is being studied in [CGK] using the group $\widehat{\Gamma}$.

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