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GENERIC BIFURCATION OF HAMILTONIAN SYSTEMS WITH SYMMETRY

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We study generic bifurcations of equilibria in one-parameter Hamiltonian systems with symmetry group Γ where eigenvalues of the linearized system go through zero. Theorem 3.3 classifies expected actions of Γ on the generalized eigenspace of this zero eigenvalue. Generic one degree of freedom symmetric systems are classified in section 4; remarks concerning systems with more degrees of freedom are given in section 5.

1. Introduction

Hamiltonian systems can undergo an enormous variety of bifurcations. However, the "generic" possibilities – not destroyed by small changes in the Hamiltonian – are more restricted. For example it is well known that for a Hamiltonian system with one degree of freedom and no symmetry, the generic bifurcation when eigenvalues pass through zero is the standard "fish" picture, or saddle-node, shown in fig. 1 below. Indeed there is a simple "normal form" for such a bifurcation, see Meyer [1].

Symmetries in Hamiltonian systems are very common (and their study has a lengthy history). It has been observed in contexts other than the Hamiltonian one (e.g. Golubitsky and Stewart [2]) that the presence of symmetry forces other kinds of behaviour, which are non-generic in general but generic in the world of symmetric systems. It is often possible to use these symmetries to obtain detailed descriptions of the possible types of behaviour.

Of course similar remarks are valid for Hamiltonian systems. The object of this paper is to establish a general setting in which to study generic bifurcations of Hamiltonian systems with symmetry (cf. Meyer [3]). That is, we assume that the Hamiltonian H is invariant under a compact Lie group Γ which preserves the symplectic structure, and we seek conditions on the group action for bifurcations that cannot be changed by small Γ-invariant perturbations of H. Our main result in this direction is theorem 3.3, which states that generically the action of Γ on the zero eigenspace is either irreducible (but not absolutely irreducible), or a diagonal sum of two absolutely irreducible actions. (A representation, or group action, is *absolutely irreducible* if it is irreducible, and the only commuting linear mappings are real scalar multiples of the identity.) We derive this from theorem 2.1, which establishes a canonical decomposition of a symplectic vector space, on which Γ acts, into mutually orthogonal symplectic invariant subspaces.

Because this theorem has important implications for the structure of bifurcating Hamiltonian systems with symmetry, we feel that it is worth presenting an elementary proof based on the interplay between representation theory and sym-

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plectic structure. Similar results can be found in the representation theory literature, for example in Serre [4] p. 108.

This study was to a large extent motivated by work of Lewis, Marsden, and Ratiu [5] and Lewis, Marsden, Montgomery and Ratiu [6] on bifurcations occurring in the dynamics of a rotating liquid drop, where a circularly symmetric equilibrium state can lose stability and break symmetry. Their results suggested that it would be fruitful to seek a general setting for such problems. We have found that such a setting exists, and that numerous special examples that have previously been analysed on a case-by-case basis can all be subsumed into the same broad picture. It is that broad picture that we wish to develop here: the examples are deliberately chosen for simplicity and the results in specific cases are not always new.

As an example, in section 4 we briefly survey generic bifurcations (when eigenvalues pass through zero) of Hamiltonian systems with one degree of freedom, for the symmetry groups $\mathbb{1}$, $\mathbb{Z}_n$, and SO(2). In this way we quickly obtain normal forms for the generic bifurcations. Our aim is to present a unified and simple treatment, capable of being generalized to other cases.

In a subsequent paper we hope to develop this approach for bifurcations through purely imaginary eigenvalues, where new and interesting phenomena arise.

2. The equivariant decomposition theorem

Let Z be a symplectic vector space with symplectic form Ω. Assume that Γ is a compact Lie group acting symplectically on Z, that is,

$$\Omega(\gamma v, \gamma w) = \Omega(v, w) \quad \forall \gamma \in \Gamma; \quad v, w \in Z. \tag{2.1}$$

We call a subspace $W \subset Z$ *Γ-symplectic* if W is Γ-invariant and symplectic. By *symplectic*, we mean that $\Omega | W$ is nondegenerate.

Our main theorem is:

Theorem 2.1. Let $V \subset Z$ be a Γ-symplectic subspace. Then there exist Ω-orthogonal Γ-symplectic subspaces $W_1, \ldots, W_t$ such that

$$V = W_1 \oplus \cdots \oplus W_t, \tag{2.2}$$

where for each j either

a) Γ acts irreducibly but not absolutely irreducibly on W_j; or

b) $W_j = V_j \oplus V_j$ where Γ acts absolutely irreducibly on V_j and by the diagonal action on $V_j \oplus V_j$. (2.3)

Remarks 2.2.

a) Recall that a representation of Γ on V is *absolutely irreducible* if the only linear mappings $V \to V$ which commute with Γ are scalar multiples of the identity. For complex representations Schur's Lemma states that irreducibility and absolute irreducibility are equivalent.

b) For real representations (the context of this paper) this is no longer the case. Now the space $\mathscr{D}$ of commuting linear mappings is a real division algebra. By the classical Wedderburn Theorem (see Kirillov [7]) $\mathscr{D}$ is either $\mathbb{R}$, $\mathbb{C}$, or $\mathbb{H}$, where $\mathbb{H}$ denotes the quaternions. Now V may be viewed as a vector space over $\mathscr{D}$. There is a consequence of this observation which we shall need later. Suppose that every commuting linear map in $\mathscr{D}$ has real eigenvalues: then $\mathscr{D} \cong \mathbb{R}$.

c) The canonical form for Ω follows directly from theorem 2.1 in the case $\Gamma = \mathbb{1}$. The trivial group has only one irreducible representation, and this is absolutely irreducible. Hence

$$V = (V_1 \oplus V_1) \oplus \cdots \oplus (V_n \oplus V_n), \tag{2.4}$$

where the 2-planes $V_j \oplus V_j$ are mutually Ω-orthogonal. Choosing a suitable basis $\{p_j, q_j\}$ for V_j and rescaling if necessary we obtain

$$\Omega = \sum \mathrm{d}p_j \wedge \mathrm{d}q_j$$

which is the classical canonical form.

d) Let $V \subset Z$ be a Γ-invariant subspace. It is always possible to decompose

$$V = K \oplus W, \tag{2.5}$$

where

$K = \ker(\Omega | V) = \{v \in V | \Omega(v, V) \equiv 0\}$

and W is Γ-symplectic. It follows trivially from (2.1) that K is Γ-invariant, and since Γ is compact we can find a Γ-invariant complement W to K. It is clear that $\Omega | W$ is nondegenerate, otherwise K would be too small. Thus W is Γ-symplectic.

There are several ways to prove theorem 2.1, at different levels of abstraction. We have chosen to present a relatively concrete one in the hope of making the paper more widely accessible. We prove some preliminary results before presenting the proof of theorem 2.1. The first is a direct consequence of nondegeneracy of the form Ω, but we include a proof since the result is crucial.

Proposition 2.3. Every Γ-symplectic subspace has an Ω-orthogonal Γ-symplectic complement.

Proof. Let $W \subset V$ be Γ-symplectic and define

$$Y = \{y \in V | \Omega(y, W) \equiv 0\}. \tag{2.6}$$

Certainly Y is Γ-invariant and $\Omega(Y, W) \equiv 0$. We claim that $V = W \oplus Y$. By the nondegeneracy of Ω we have $W \cap Y = \{0\}$. Also by nondegeneracy, $\dim V = \dim W + \dim Y$. Hence $V = Y \oplus W$.

To complete the proof we must show that $\Omega | Y$ is nondegenerate. Suppose that $\Omega(y, Y) \equiv 0$ for some $y \in Y$. By (2.6) $\Omega(y, V) \equiv 0$, contrary to nondegeneracy. □

Lemma 2.4.

a) Let W_1 and W_2 be Γ-irreducible subspaces of V. Then either

i) W_1 and W_2 are Ω-orthogonal, or

ii) Ω is a nondegenerate pairing of W_1 and W_2 and hence $W_1 \cong W_2$. (2.7)

b) If W is Γ-irreducible, then W is either isotropic or Γ-symplectic.

Remarks 2.5.

a) *A* subspace W is *isotropic* if $\Omega(W, W) \equiv 0$.

b) A consequence of (2.7ii) is that the representations of Γ on W_1 and W_2 are isomorphic. The nondegeneracy of the pairing is equivalent to the following:

The mapping φ: $W_1 \to W_2^*$ defined by

$$\varphi(w) = \Omega(w, \cdot) \tag{2.8}$$

is an isomorphism, where the action of $\gamma \in \Gamma$ on $w_2^* \in W_2^*$ is defined by

$$\gamma \cdot w_2^*(w_2) = w_2^*(\gamma^{-1} w_2). \tag{2.9}$$

To obtain (2.8) use (2.1), which implies that $\varphi(\gamma w) = \gamma(\varphi w)$. Note that (2.9) implies that $W_1 \cong W_2^*$ as Γ-modules. However, for compact groups, any Γ-module W is always isomorphic to its dual W^*, so $W_1 \cong W_2$.

To prove the last statement choose a metric $\langle\ ,\ \rangle$ on W in which Γ acts orthogonally. This is possible since Γ is compact (see e.g. Adams [8]). It is now easy to check that ψ: $W \to W^*$ defined by $\psi(w) = \langle w, \cdot \rangle$ is a linear isomorphism commuting with Γ. □

Proof of lemma 2.4. Statement (b) follows directly from (a) by setting $W = W_1 = W_2$ and recalling the definition of an isotropic subspace (remark 2.5a).

To prove (a) define ψ: $W_1 \to W_2^*$ as in (2.8). Since ψ commutes with Γ, $\ker \psi$ and $\operatorname{Im} \psi$ are Γ-invariant. Irreducibility of W_1 implies that $\ker \psi = W_1$ (whence W_1 and W_2 are Ω-orthogonal) or $\ker \psi = \{0\}$, which we now assume. Then $\operatorname{Im} \psi \neq \{0\}$ so the irreducibility of W_2 (and W_2^*) implies that $\operatorname{Im} \psi = W_2^*$. Thus the pairing of W_1 with W_2 by Ω is nondegenerate, so ψ: $W_1 \to W_2^*$ is an isomorphism. □

Lemma 2.6. Let W be a Γ-symplectic subspace and let $\langle\ ,\ \rangle$ be a Γ-invariant metric on W. Define J: $W \to W$ by

$$\Omega(v, w) = \langle v, Jw \rangle \quad \forall v, w \in W. \tag{2.10}$$

Then J commutes with the action of Γ.

Proof. We have assumed that Γ acts symplectically on W, so (2.1) holds. Combining this with

(2.10) we find that

$$\langle v, Jw\rangle = \Omega(v,w) = \Omega(\gamma v, \gamma w) = \langle \gamma v, J\gamma w\rangle = \langle v, \gamma^t J\gamma w\rangle. \tag{2.11}$$

Hence

$$J = \gamma^t J\gamma.$$

Since γ acts orthogonally in this metric we have $\gamma^t = \gamma^{-1}$, so $\gamma J = J\gamma$ as claimed. □

Lemma 2.7. Let Γ act on the subspace Y and by the diagonal action on $Y \oplus Y$. Let A: $Y \to Y$ be linear and define

$$V_A = \{(v, Av) \in Y \oplus Y | v \in Y\}. \tag{2.12}$$

Then V_A is a Γ-invariant subspace if and only if A commutes with Γ.

Proof. $\gamma \cdot (v, Av) = (\gamma v, \gamma Av) \in V_A$ if and only if A applied to the first component γv equals the second component γAv. This identity is equivalent to $A\gamma = \gamma A$. □

Lemma 2.8. Let Γ act irreducibly on Y and by the diagonal action on $Y \oplus Y$. Suppose that $Y \oplus Y$ is Γ-symplectic and that every irreducible subspace of $Y \oplus Y$ is isotropic. Then Γ acts absolutely irreducibly on Y.

Proof. Our method is based on remark 2.2b. We show that every computing linear mapping A: $Y \to Y$ has real eigenvalues. Hence the space of Γ-commuting mappings Δ is isomorphic to $\mathbb{R}$ and Γ acts absolutely irreducibly on Y.

We begin by choosing a Γ-invariant metric $\langle\ ,\ \rangle$ on $Y \oplus Y$ in which $Y \times \{0\}$ and $\{0\} \times Y$ are orthogonal, and for which

$$\langle (v,0),(w,0)\rangle = \langle (0,v),(0,w)\rangle \quad \forall v, w \in Y.$$

Define the mapping J by (2.10) and observe that $J(Y \times \{0\})$ is orthogonal to $Y \times \{0\}$. This follows from the fact that $Y \times \{0\}$ is isotropic. Let $v, w \in Y \times \{0\}$. Then $0 = \Omega(v, w) = \langle v, Jw\rangle$. Hence Jw is orthogonal to $Y \times \{0\}$ as claimed.

Next, choose an orthonormal basis $u_1, \ldots, u_n$ for V. Then $(u_1, 0), \ldots, (u_n, 0)$; $(0, u_1), \ldots, (0, u_n)$ is an orthonormal basis for $Y \oplus Y$. The matrix of J in this basis has the form

$$\begin{bmatrix} 0 & \tilde{J} \\ \tilde{K} & 0 \end{bmatrix}.$$

Moreover, the antisymmetry $\Omega(v, w) = -\Omega(w, v)$ implies that $J^t = -J$. Thus J has the form

$$J = \begin{bmatrix} 0 & \tilde{J} \\ -\tilde{J}^t & 0 \end{bmatrix} \tag{2.13}$$

in this basis. Since J commutes with Γ on $Y \oplus Y$ it follows that $\tilde{J}$ commutes with Γ on Y. Moreover $\tilde{J}$ is invertible since Ω is nondegenerate. Now let A: $Y \to Y$ commute with Γ. Then the subspace V_A is also Γ-irreducible since it is Γ-isomorphic to $Y \times \{0\}$. Hence V_A is isotropic, by hypothesis.

Since $Y \times \{0\}$, $\{0\} \times Y$ and V_A are isotropic we compute, for all $v, w \in Y$:

$$\begin{aligned} 0 &= \Omega((v, Av),(w, Aw)) \\ &= \Omega((v,0),(0, Aw)) + \Omega((0, Av),(w,0)) \\ &= \langle (v,0), J(0, Aw)\rangle + \langle (0, Av), J(w,0)\rangle \\ &= \langle v, \tilde{J}Aw\rangle + \langle Av, -\tilde{J}^t w\rangle \\ &= \langle v, (\tilde{J}A - A^t\tilde{J}^t)w\rangle, \end{aligned} \tag{2.14}$$

using (2.13). By (2.14) we have

$$\tilde{J}A = (\tilde{J}A)^t. \tag{2.15}$$

The identity (2.15) implies that all eigenvalues of $\tilde{J}A$ are real for every Γ-commuting matrix A. But $\tilde{J}$ is invertible and commutes with Γ. Therefore the eigenvalues of A are real – just replace A by $\tilde{J}^{-1}A$. By remark 2.2b the result follows. □

Proof of theorem 2.1. Let V be a Γ-symplectic subspace of Z. Suppose there is a Γ-irreducible subspace W_1 of V. By proposition 2.3 we may write

$$V = W_1 \oplus V_1,$$

where V_1 is a Γ-symplectic subspace of V, Ω-orthogonal to W_1. Inductively we may decompose V as

$$V = W_1 \oplus \cdots \oplus W_t \oplus \tilde{V},$$

where each W_j is Γ-irreducible and Γ-symplectic, the subspaces W_j and $\tilde{V}$ are mutually Ω-orthogonal, and no irreducible subspace of $\tilde{V}$ is Γ-symplectic. By lemma 2.4b every Γ-irreducible subspace of $\tilde{V}$ must be isotropic.

The action of Γ on W_j cannot be absolutely irreducible. To verify this claim, apply lemma 2.6 to W_j to obtain a linear mapping J_j commuting with Γ. Since $\Omega | W_j$ is antisymmetric, J cannot be a multiple of the identity, so the action of Γ on W_j is not absolutely irreducible.

We begin the next step by enumerating the distinct irreducible subspaces $Y_1, \ldots, Y_r$ of $\tilde{V}$. Define

$$U_j = \sum \{\text{all irreducible subspaces } Y \cong Y_j\}.$$

By standard representation theory

$$\tilde{V} = U_1 \oplus \cdots \oplus U_r,$$

with each U_j Γ-invariant. Lemma 2.4a implies that the U_j are pairwise Ω-orthogonal. It follows that each U_j is Γ-symplectic.

To complete the proof of the theorem it suffices to show that theorem 2.1 is valid for $V = U_j$. Let Y be a Γ-irreducible subspace of U_j. There is another Γ-irreducible subspace $Y' \subset V_j$ such that $Y \oplus Y'$ is Γ-symplectic. To see this, observe that by lemma 2.4a, for any Γ-irreducible $Y' \subset U_j$, if $Y \oplus Y'$ is not Γ-symplectic, then $\Omega(Y, Y') \equiv 0$. Thus if no such Y' exists, then $\Omega | U_j$ has a nontrivial kernel containing Y, contrary to nondegeneracy.

Having chosen Y' so that $W_1 = Y \oplus Y'$ is Γ-symplectic, choose a Γ-symplectic complement $\tilde{W}$ to W_1 in U_j. Inductively we may decompose U_j as $U_j = W_1 \oplus \cdots \oplus W_t$, where the W_j are Ω-orthogonal, Γ-symplectic, and of the form $W_j = V_j \oplus V_j$ where Γ acts irreducibly on V_j. (All irreducible subspaces of U_j are isomorphic to each other by the definition of U_j.)

Finally, note that every irreducible subspace of W_j is isotropic, since every irreducible subspace of $\tilde{V}$ is isotropic. By lemma 2.8, Γ acts absolutely irreducibly on V_j, as desired. □

3. Bifurcations with zero eigenvalue

This section is divided into three subsections. In the first we give a short proof of the known result (cf. Williamson [9]) that generalized eigenspaces of symplectic maps are symplectic. In the second section we use theorem 2.1 to give a generic description of the kernels of symplectic mappings in the equivariant context. We then restrict to Hamiltonian systems on the 'generic' kernel and show, in the third subsection, how the eigenvalues make the transition through 0. In the absolutely irreducible ($V \oplus V$) case, the change is from purely imaginary through zero to real. In the nonabsolutely irreducible case the eigenvalues are always purely imaginary.

3.1. *Eigenspaces are Γ-symplectic*

In this section we give a quick proof of a result of Williamson [9]. Let $\mathfrak{sp}(V)$ denote the (real) symplectic Lie algebra, consisting of $\mathbb{R}$-linear maps $A\colon V \to V$ such that $\Omega(Av, w) + \Omega(v, Aw) = 0$.

Proposition 3.1. Let V be a symplectic vector space and let $A \in \mathfrak{sp}(V)$. Then E_μ, the sum of the generalized eigenspaces corresponding to the eigenvalues $\mu, \bar{\mu}, -\mu, -\bar{\mu}$, is symplectic. If A commutes with Γ then E_μ is Γ-symplectic.

Proof. Let $B = (A - \mu)(A - \bar{\mu})(A + \mu)(A + \bar{\mu}) = A^4 - 2\operatorname{Re}\mu^2 \cdot A^2 + |\mu|^4$. It is easy to see that $E_\mu = \ker B^l$ for all $l \geq$ some integer k (for example, consider Jordan canonical form). Now

$$E_\mu \oplus \operatorname{Im} B^k = V. \tag{3.1}$$

Next note that since $A \in \mathfrak{sp}(V)$ we have

$$\Omega(Av, w) = -\Omega(v, Aw).$$

Therefore $\Omega(A^2v, w) = \Omega(v, A^2w)$, so that

$$\Omega(Bv, w) = \Omega(v, Bw).$$

Inductively,

$$\Omega(B^k v, w) = \Omega(v, B^k w). \tag{3.2}$$

Eq. (3.2) implies that E_μ and $\operatorname{Im} B^k$ are Ω-orthogonal. Hence both $\Omega|E_\mu$ and $\Omega|\operatorname{Im} B^k$ are nondegenerate.

Finally note that if A commutes with Γ then Γ obviously leaves E_μ invariant. Thus E_μ is Γ-symplectic. □

3.2. *Genericity of zero eigenvalues*

Let $\mathfrak{sp}_\Gamma(V)$ denote the Lie algebra of symplectic maps commuting with Γ, and let $A \in \mathfrak{sp}_\Gamma(V)$. Proposition 3.1 implies that E_0, the generalized eigenspace of A corresponding to the eigenvalue 0, is Γ-symplectic. By theorem 2.1 we may decompose

$$E_0 = W_1 \oplus \cdots \oplus W_s \oplus (V_1 \oplus V_1) \oplus \cdots \oplus (V_t \oplus V_t),$$

where Γ acts nonabsolutely irreducibly on W_j and absolutely irreducibly on $V_k \oplus V_k$. In addition the W_j and $V_k \oplus V_k$ are Γ-symplectic and mutually Ω-orthogonal.

We claim that generically in a 1-parameter family of A's, the generalized eigenspace E_0 is either a single W or a single $V \oplus V$ (though not necessarily those occurring in a given decomposition of V, since this decomposition is not unique). We need two preliminary lemmas.

Lemma 3.1. Let Γ act absolutely irreducibly on V and diagonally on the Γ-symplectic subspace $V \oplus V$. Let $\langle\ ,\ \rangle$ be a Γ-invariant metric on $V \oplus V$. Then we can choose orthonormal bases $\{p_1, \ldots, p_n\}$ on $V \times \{0\}$ and $\{q_1, \ldots, q_n\}$ on $\{0\} \times V$ such that:

a) $\Omega = \sum_{j=1}^{n} \mathrm{d}p_j \wedge \mathrm{d}q_j$;

b) $\mathfrak{sp}_\Gamma(V \oplus V)$ consists of matrices of the form

$$\begin{bmatrix} aI & bI \\ cI & -aI \end{bmatrix}$$

relative to this basis.

Remark. Lemma 3.1a shows that in the $V \oplus V$ case the symplectic form may be put in canonical form by an equivariant linear change of coordinates. Another way to say this is that in a suitable coordinate system Γ acts separately on configuration space and momentum space, with isomorphic (and symplectically related) actions.

Proof. Let $A \in \mathfrak{sp}_\Gamma(V \oplus V)$. Since A commutes with Γ and Γ acts absolutely irreducibly on V, we have

$$A = \begin{bmatrix} aI & bI \\ cI & dI \end{bmatrix} \tag{3.3}$$

in block form, for suitable $a, b, c, d \in \mathbb{R}$. We assume the choice of an orthonormal basis. In particular, if we define J by

$$\Omega(v, w) = \langle v, Jw \rangle$$

as in (2.10), then by lemma 2.6 J commutes with Γ, so J has the form (3.3).
Moreover $J^t = -J$, so that

$$J = \begin{bmatrix} 0 & bI \\ -bI & 0 \end{bmatrix}.$$

Rescaling if necessary we may assume that $b = 1$, proving (a).

Since $\Omega(Av, w) = -\Omega(v, Aw)$ we have

$$\langle Av, Jw \rangle = -\langle v, JAw \rangle,$$

whence

$$A^t J = -JA. \tag{3.4}$$

Substituting (3.3) into (3.4) we find that $d=-a$, verifying (b). □

Lemma 3.2. Let Γ act irreducibly but not absolutely irreducibly on the Γ-symplectic subspace W. Let $A \in \mathfrak{sp}_\Gamma(W)$. Then A is semisimple and the eigenvalues of A lie on the imaginary axis.

Proof. Let $\mathscr{D}$ be the division algebra of linear mappings on W commuting with Γ. Since W is nonabsolutely irreducible, by remark 2.2b we have $\mathscr{D} \cong \mathbb{C}$ or $\mathbb{H}$. We may view W as a (left) vector space over $\mathscr{D}$. More precisely, $W \cong \mathscr{D}^n$ for some n and scalar multiplication by a commuting map d: $W \to W$ is defined to be $d \cdot (d_1, \dots, d_n) = (dd_1, \dots, dd_n)$. This shows that $\mathscr{D}$ acts semisimply on W.

If $A \in \mathfrak{sp}_\Gamma(W)$ then by definition A commutes with Γ and hence must act semisimply. If in addition A is symplectic, then $\operatorname{Tr} A = 0$. Now the trace of d is just $n \cdot \operatorname{Tr}(L_d)$ where L_d is left multiplication of d on $\mathscr{D}$. Identifying $\mathscr{D}$ with $\mathbb{C}$ or $\mathbb{H}$ it follows easily that $\operatorname{Tr}(L_d) = 2\operatorname{Re}(d)$. Thus A has trace zero precisely when A is identified with $b\mathrm{i}$ ($\mathscr{D} \cong \mathbb{C}$) or $b\mathrm{i} + c\mathrm{j} + d\mathrm{k}$ ($\mathscr{D} \cong \mathbb{H}$). In each case, it can be checked that the eigenvalues of L_d, hence of A, are purely imaginary. □

Theorem 3.3. Let A_λ be a 1-parameter family in $\mathfrak{sp}_\Gamma(V)$ such that 0 is an eigenvalue for A_0. Generically, either

a) $E_0 = W$,

or (3.5)

b) $E_0 = V \oplus V$,

where Γ acts nonabsolutely irreducibly on W, and absolutely irreducibly on V.

Proof. Let $V = E_0 \oplus Z$ where $Z = \Sigma_{\mu \neq 0} E_\mu$. Then E_0 and Z are A-invariant and Ω-orthogonal. Suppose that $\tilde{B} \in \mathfrak{sp}_\Gamma(E_0)$. Then we may define $B \in \mathfrak{sp}_\Gamma(V)$ by $B|E_0 = \tilde{B}$ and $B|Z \equiv 0$. Observe that

$$E_0(A + \varepsilon B) \subset E_0,$$

since $(A+\varepsilon B)|Z = A|Z$ whose eigenvalues are all nonzero by construction.

Next we choose $\tilde{B}$ so that $E_0(A+\varepsilon B)$ has the form either of a W (3.5a) or a $V \oplus V$ (3.5b). Begin by rewriting (3.2) as

$$E_0 = U_1 \oplus \cdots \oplus U_{s+t},$$

where the U_j are Γ-symplectic and Γ-orthogonal. (Each U_j is either a W_k or a $V_k \oplus V_k$.) Define $\tilde{B}$ as follows. Assume

i) $\tilde{B}|U_1 = 0$;

and for $j \geq 2$

ii) $\tilde{B}|U_j$ is identical with multiplication by i if U_j is a W_k, cf. lemma 3.2;

iii) $\tilde{B}|U_j$ is $\begin{bmatrix} I & 0 \\ 0 & -I \end{bmatrix}$ if U_j is a $V_k \oplus V_k$, cf. lemma 3.1b.

We claim now that

$$E_0(A_0 + \varepsilon B) = U_1.$$

Certainly $E_0(A_0+\varepsilon B) \supset U_1$, since $(A_0+\varepsilon B)|U_1 = A_0|U_1$. To verify the claim we show that $(A_0 + \varepsilon B)|U_j$ ($j \geq 2$) must have nonzero eigenvalues when $\varepsilon \neq 0$. Suppose U_j is a W_k. Then $A_0|W_k$ is semisimple and since all eigenvalues of $A_0|W_k$ are zero, it follows that $A_0|W_k \equiv 0$. Therefore $(A_0 + \varepsilon B)U_j = \varepsilon\tilde{B}|U_j$ which has all eigenvalues equal to $\pm\varepsilon\mathrm{i}$ by (ii) above. Suppose finally that U_j is a $V_k \oplus V_k$. Using the coordinates of lemma 3.1 we see that

$$C = (A_0 + \varepsilon\tilde{B})|U_j = \begin{bmatrix} (a+\varepsilon)I & bI \\ cI & -(a+\varepsilon)I \end{bmatrix}. \tag{3.6}$$

The eigenvalues of C are just the eigenvalues of the 2×2 matrix

$$C_1 = \begin{bmatrix} (a+\varepsilon) & b \\ c & -(a+\varepsilon) \end{bmatrix}$$

repeated n times. Now

$$\det C_1 = -\varepsilon(2a+\varepsilon), \tag{3.7}$$

since the assumption that A_0 has zero eigenvalues implies that $a^2 + bc = 0$. Moreover $\varepsilon(\varepsilon + 2a) \neq 0$

for ε nonzero and sufficiently close to 0, regardless of the value of a.

Thus generically the generalized eigenspace E_0 consists of exactly one Γ-symplectic subspace (of the form W_k or $V_k \oplus V_k$). □

3.3. *The simplest zero eigenvalues*

For the remainder of this subsection we assume that the symplectic vector space Z under consideration is either W or $V \oplus V$ as above.

Proposition 3.4. Let A_λ be a generic 1-parameter family of matrices in $\mathfrak{sp}_\Gamma(Z)$ with $E_0(A_0) = Z$.

a) If $Z = W$, then $A_0 = 0$ and the eigenvalues of A_λ lie on the imaginary axis and cross through 0 with nonzero speed.

b) If $Z = V \oplus V$, then A_0 is nilpotent and the eigenvalues cross through 0 going from purely imaginary to real (or vice versa).

Proof.

a) As we observed in lemma 3.2, the matrices A_λ may be identified with the scalars $e(\lambda) \in \mathscr{D}$ (equal to $\mathbb{C}$ or $\mathbb{H}$). By assumption $e(0) = 0$, and $e(\lambda)$ for $\lambda \neq 0$ is purely imaginary and nonzero. Generically $\mathrm{d}e(\lambda)/\mathrm{d}\lambda \neq 0$.

b) Lemma 3.1b shows that

$$A_\lambda = \begin{bmatrix} a(\lambda)I & b(\lambda)I \\ c(\lambda)I & -a(\lambda)I \end{bmatrix},$$

where $\det A_0 = 0$. Generically, $A_0 \neq 0$, and after a Γ-equivariant linear change of coordinates we may assume

$$A_0 = \begin{bmatrix} 0 & I \\ 0 & 0 \end{bmatrix}.$$

Now

$$\det A_\lambda = (-1)^n \left(a(\lambda)^2 + b(\lambda)c(\lambda)\right)^n.$$

Generically, $\mathrm{d}[a(\lambda)^2 + b(\lambda)c(\lambda)]/\mathrm{d}\lambda = c'(0) \neq 0$, so the determinant of A_λ changes sign as λ crosses zero. Since $\mathrm{Tr}\, A_\lambda = 0$, the eigenvalues go from purely imaginary to real. □

4. Systems with one degree of freedom

In this section we consider several possibilities for Γ where $\dim Z = 2$. In this case, level surfaces of the Hamiltonian $H(p, q)$ determine the dynamics. Since right equivalences on H [10, 11] preserve the topology of the level sets, we can use them to put the Hamiltonian in normal form, even though such coordinate changes do not respect the symplectic structure.

The groups we consider are $\mathbb{1}$, $\mathbb{Z}_n$, and SO(2). The cases $\mathbb{Z}_2$ and $\mathbb{Z}_4$ are treated separately: the former because of differences in the representation theory, the latter because it has special features.

4.1. $\Gamma = \mathbb{1}$

The only irreducible representation of $\mathbb{1}$ is the trivial one $V = \mathbb{R}$, which is absolutely irreducible. Then $Z = V \oplus V$.

In this case if the Hamiltonian vector field has a linear part with a zero eigenvalue, then generically it has a nilpotent normal form

$$\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

and the Hamiltonian is

$$H(p, q) = \pm q^2 + \cdots .$$

Generically

$$H(p, q) = \pm p^3 \pm q^2 + \cdots \tag{4.1}$$

and the $\cdots$ in (4.1) may be eliminated by a right equivalence. The model 1-parameter family, in this case, has the normal form

$$H(p, q, \lambda) = p^3 + q^2 + \lambda p. \tag{4.2}$$

The level curves of (4.2) exhibit the well-known 'fish' structure pictured in fig. 1. See also Meyer [1].

4.2. $\Gamma = \mathbb{Z}_2$

The only nontrivial irreducible representation of $\mathbb{Z}_2$ is the 1-dimensional one $V = \mathbb{R}$ where $\mathbb{Z}_2$ acts

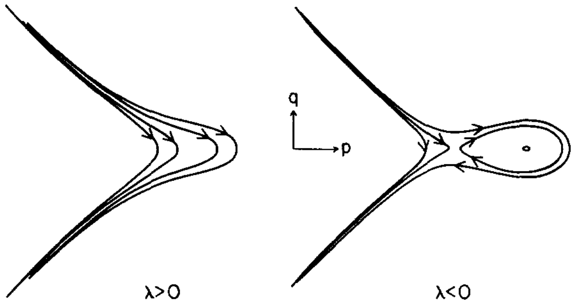

Fig. 1. The 'fish' in generic bifurcation of a Hamiltonian vector field with trivial symmetry.

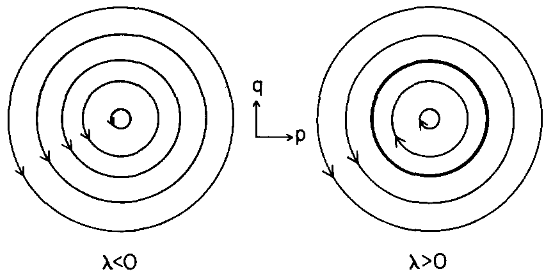

Fig. 3. A circle of fixed points in generic bifurcation of a Hamiltonian vector field with SO(2) symmetry.

as -1. Thus $Z = V \oplus V$ and the natural action is

$$-1 \cdot (p, q) = (-p, -q).$$

The model normal form in this case is

$$H(p, q, \lambda) = p^4 + q^2 + \lambda p^2. \tag{4.3}$$

The level curves of (4.3) are shown in fig. 2. See also Guckenheimer and Holmes [12] chap. 7. We emphasize that fig. 2 represents the *natural* transition in a Hamiltonian system when $\mathbf{Z}_2$ symmetry is present.

4.3. $\Gamma = SO(2)$

The only nontrivial irreducible actions of SO(2) are 2-dimensional and not absolutely irreducible, thus $Z = W$. Factoring out the kernel of the action we may (subject to suitable interpretation) assume the action is faithful. Then the only possible actions are

$$\theta \cdot z = e^{\pm i\theta} z, \tag{4.4}$$

where we identify Z with $\mathbf{C}$.

We note here that these two actions of SO(2) are not symplectically equivalent (that is, equivalent by an isomorphism preserving the symplectic structure), though this point is not important here.

The model Hamiltonian in this case is easily seen to be

$$H(p, q, \lambda) = (p^2 + q^2)^2 + \lambda(p^2 + q^2). \tag{4.5}$$

The level curves exhibit the transition in fig. 3, where a circle of fixed points appears for $\lambda > 0$.

4.4. $\Gamma = \mathbf{Z}_m$ ($m \geq 3, m \neq 4$)

Again, the only nontrivial irreducible actions of $\mathbf{Z}_m$ are 2-dimensional and not absolutely irreducible, thus $Z = W$. We assume the action is faithful (by factoring out the kernel if necessary): the results for non-faithful representations may easily be deduced from this case. The group action is generated by

$$\zeta \cdot z = e^{i\zeta} z, \tag{4.6}$$

where $\zeta = 2\pi/m$.

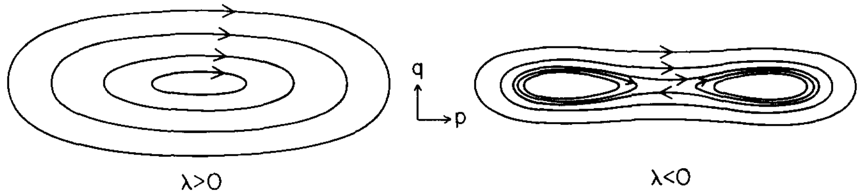

Fig. 2. The 'figure eight' in generic bifurcation of a Hamiltonian vector field with $\mathbf{Z}_2$ symmetry.

Wassermann [13] has classified singularities of functions – up to right equivalence – invariant under the group $\mathbb{Z}_m$. The universal unfoldings of the (topological) codimension-1 singularities are:

$$\begin{array}{lll} \text{(a)} & \varepsilon(z\bar{z})^2 + \alpha \mathrm{Re}\, z^m - \lambda z\bar{z} & (m \geq 5), \\ \text{(b)} & \beta(z\bar{z})^2 + \mathrm{Re}\, z^4 - \lambda z\bar{z} & (m \geq 4), \\ \text{(c)} & \mathrm{Re}\, z^3 - \lambda z\bar{z} & (m = 3), \end{array} \tag{4.7}$$

where $\varepsilon = \pm 1$, λ is the unfolding parameter, and $\alpha > 0, \beta \neq \pm 1$ are modal parameters. Recall that f and g are right equivalent if there exists a diffeomorphism φ such that

$$g(z) = f(\varphi(z)).$$

For $\mathbb{Z}_m$-right equivalence, we demand that φ be $\mathbb{Z}_m$-equivariant.

For our purposes we are interested only in drawing the level curves of f. Therefore, finding the level curves of $a \cdot f(z)$ is just as good as finding the level curves of f. In particular we may transform the normal forms $f(z, \lambda)$ in (4.7) by the scalings

$$af(bz, c\lambda)$$

to obtain the normal forms

$$\begin{array}{lll} \text{(a)} & (z\bar{z})^2 + \mathrm{Re}\, z^m - \lambda z\bar{z} & (m \geq 5), \\ \text{(b)} & (z\bar{z})^2 + \gamma \mathrm{Re}\, z^4 - \lambda z\bar{z} & (m = 4), \\ \text{(c)} & \mathrm{Re}\, z^3 - \lambda z\bar{z} & (m = 3), \end{array} \tag{4.8}$$

where the modal parameter $\gamma \neq 0, \pm 1$.

Remarks.

a) From our point of view, the importance of (4.8a) is that the modal parameter α and the sign ε have been eliminated from consideration. However, because $(z\bar{z})^2$ and $\mathrm{Re}\, z^4$ are of the same order, the modal parameter cannot be eliminated from (4.8b).

b) The term $(z\bar{z})^2$ is of higher order when $m = 3$ and could have been included in (4.8c) to conform to (4.8a).

Observe that when $\lambda < 0$, the level curves of (4.8a) are strongly elliptic near the origin, being

$$(-\lambda)z\bar{z} + (z\bar{z})^2 + \text{h.o.t.} = \text{const.}$$

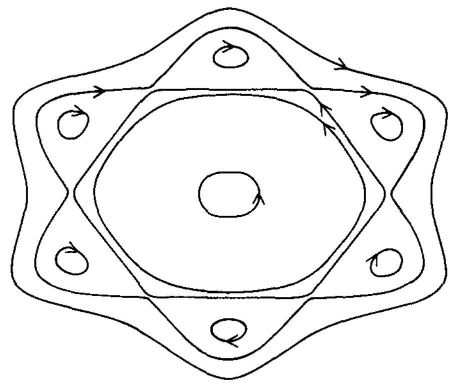

Fig. 4. Level curves of normal form (4.8a) in the typical case $m = 6$, when $\lambda > 0$.

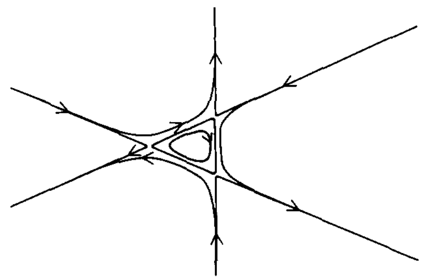

Fig. 5. Level curves of normal form (4.8c), $m = 3$, $\lambda > 0$.

These contours are identical with fig. 3 when $\lambda < 0$. However, when $\lambda > 0$ the symmetry-breaking term ($\mathrm{Re}\, z^m$) becomes important. The level surfaces (for $m = 6$ which is typical) are pictured in fig. 4.

When $m = 3$ the level curves for $\lambda < 0$ are obtained from those with $\lambda > 0$ by considering the reflection through the origin $z \mapsto -z$. The level curves for (4.8c) with $\lambda > 0$ are pictured in fig. 5.

4.5. $\Gamma = \mathbb{Z}_4$

We treat this case separately since it has special features: in particular, the modal parameter γ. The normal form is given by (4.8b), namely

$$(z\bar{z})^2 + \gamma \mathrm{Re}\, z^4 - \lambda z\bar{z}.$$

If γ is changed to $-\gamma$ the phase portrait is

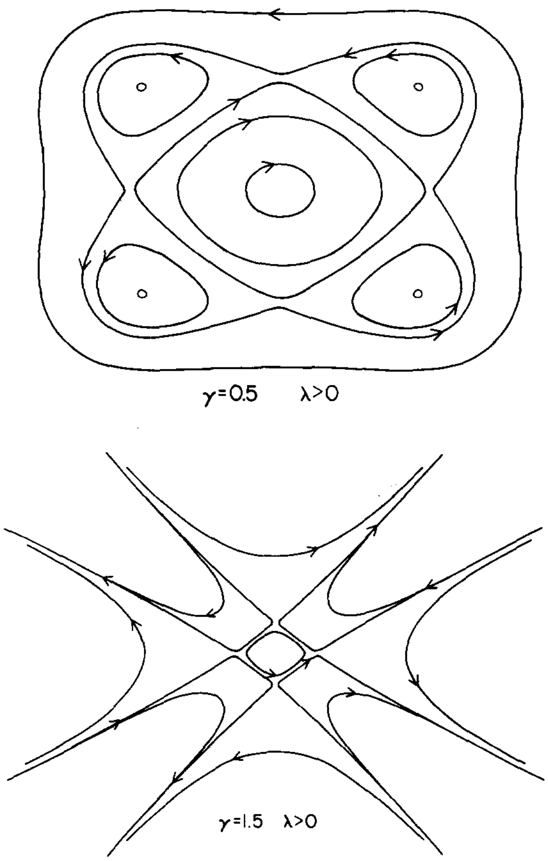

Fig. 6. Level curves for normal form (4.8b), corresponding to the group $\mathbf{Z}_4$. Each sequence is for a fixed value of γ and shows $\lambda < 0$, $\lambda = 0$, and $\lambda > 0$. The two rows show the cases $\gamma = 0.5$, $\gamma = 1.5$.

unchanged except for a 45° rotation. We therefore assume $\gamma \geq 0$. There are two exceptional values of γ, namely 0 and 1 (or -1) at which the topology of the phase portrait changes. The transition is shown in fig. 6 for typical values $\gamma = 0.5$ and 1.5. The value 0.5 is typical of the range $0 < \gamma < 1$; the value 1.5 is typical of the range $1 < \gamma < \infty$. (The value $\gamma = \infty$ can be interpreted as yielding the function $\operatorname{Re} z^4 - \lambda z\bar{z}$ and is then also exceptional.)

5. Several degrees of freedom

Let Γ act symplectically on the vector space Z and let $\Delta \subset \Gamma$ be a Lie subgroup. Define the *fixed-point subspace* of Δ to be

$$\operatorname{Fix}(\Delta) = \{ z \in Z | \delta z = z \text{ for all } \delta \in \Delta \}.$$

If the choice of Z is not clear from the context, write $\operatorname{Fix}_Z(\Delta)$ for this set. Observe that $\operatorname{Fix}(\Delta)$ is the sum of all irreducible subspaces on which Δ acts trivially. By theorem 2.1 applied to Δ, in particular (2.16), $\operatorname{Fix}(\Delta)$ is a symplectic subspace of Z. Our main result in this section is:

Proposition 5.1. Let H: $Z \to \mathbb{R}$ be a Γ-invariant Hamiltonian and let X_H be the associated Hamiltonian vector field. Then X_H leaves $\operatorname{Fix}(\Delta)$ invariant and $X_H | \operatorname{Fix}(\Delta)$ is a Hamiltonian vector field with Hamiltonian $H | \operatorname{Fix}(\Delta)$.

Proof. View X_H as a mapping $Z \to Z$. We claim that X_H commutes with Γ, from which it follows that X_H: $\operatorname{Fix}(\Delta) \to \operatorname{Fix}(\Delta)$. Since $\operatorname{Fix}(\Delta)$ is a symplectic subspace the proposition holds.

Of course X_H commutes with Γ provided that

$$\gamma_* X_H = X_H$$

for all $\gamma \in \Gamma$. Now X_H is defined uniquely by

$$dH = X_H \rfloor \Omega, \tag{5.1}$$

where Ω is the symplectic 2-form on Z. Apply γ_* to (5.1) to obtain

$$d(\gamma^* H) = (\gamma_* X_H) \rfloor \gamma^* \Omega.$$

But $\gamma^* \Omega = \Omega$ since Γ acts symplectically, and $\gamma^* H = H$ since H is Γ-invariant. Thus

$$dH = \gamma_* X_H \rfloor \Omega$$

and $\gamma_* X_H = X_H$ by the uniqueness of (5.1). □

Corollary 5.2. If $\dim \operatorname{Fix}(\Delta) = 2$, then the dynamics of $X_H | \operatorname{Fix}(\Delta)$ is determined by the methods of section 4. □

This corollary is the Hamiltonian analogue of the steady-state Equivariant Branching Lemma of

Vanderbauwhede [14] and Cicogna [15]. See Golubitsky, Stewart and Schaeffer [16].

Remarks.

i) $H|\mathrm{Fix}(\Delta)$ is not arbitrary. In particular, if $N(\Delta)$ is the normalizer of Δ in Γ then $N(\Delta)/\Delta$ acts on $\mathrm{Fix}(\Delta)$ and $H|\mathrm{Fix}(\Delta)$ is invariant under this action. However, there may be additional more subtle restrictions. A general discussion of the restrictions placed on $H|\mathrm{Fix}(\Delta)$ by the invariance of H under Γ is given in Golubitsky, Marsden, and Schaeffer [17].

ii) As discussed in section 3 the basic examples are $Z = V \oplus V$ where Γ acts absolutely irreducibly on V, and $Z = W$ where Γ acts non-absolutely irreducibly on W. In the first case

$$\dim \mathrm{Fix}_{V\oplus V}(\Delta) = 2 \cdot \dim \mathrm{Fix}_V(\Delta).$$

Therefore the application of corollary 5.2 in the case $Z = V \oplus V$ requires finding (isotropy) subgroups $\Delta \subset \Gamma$ for which

$$\dim \mathrm{Fix}_V(\Delta) = 1.$$

Such computations have been carried out for a number of examples – in particular for the case $\Gamma = \mathrm{O}(3)$ in any irreducible representation V, see Ihrig and Golubitsky [18].

The case $\Gamma = \mathrm{O}(2)$ on $\mathbb{R}^4$

We end this section with an example. Let $\Gamma = \mathrm{O}(2)$ act by its standard representation on $V = \mathbb{C}$ and let $Z = \mathbb{C} \oplus \mathbb{C}$. Let $\Delta = \mathbb{Z}_2 = \langle z \mapsto -z \rangle$. Then

$$\mathrm{Fix}(\mathbb{Z}_2) = \mathbb{R} \oplus \mathbb{R}$$

which is 2-dimensional. Note also that $N(\mathbb{Z}_2) = \mathbb{Z}_2^2$ and that $N(\mathbb{Z}_2)/\mathbb{Z}_2$ is generated by $z \mapsto e^{i\pi}z = -z$. By Remark (i) we know that $H|\mathrm{Fix}(\mathbb{Z}_2)$ is invariant under $\mathbb{Z}_2 \cong N(\mathbb{Z}_2)/\mathbb{Z}_2$, hence we expect the bifurcation picture of $H|\mathrm{Fix}(\mathbb{Z}_2)$ to be like that of figure 2.

To verify this last statement we must compute the general Hamiltonian on $\mathbb{C} \oplus \mathbb{C}$ commuting with O(2). This is easily done, yielding

$$H(z_1, z_2) = h\left(|z_1|^2, |z_2|^2, \mathrm{Re}(z_1\bar{z}_2)\right). \tag{5.2}$$

Restricting H to $\mathrm{Fix}(\mathbb{Z}_2) = \mathbb{R} \oplus \mathbb{R}$ yields

$$H|\mathrm{Fix}(\mathbb{Z}_2) = h\left(x_1^2, x_2^2, x_1x_2\right), \tag{5.3}$$

where $z_j = x_j + iy_j$. Of course (5.3) is the most general function invariant under the $\mathbb{Z}_2$-action on $\mathbb{R} \oplus \mathbb{R}$, taking (x_1, x_2) to $(-x_1, -x_2)$, discussed in section 4.2. Hence fig. 2 does give the generic bifurcation picture of $H|\mathrm{Fix}(\mathbb{Z}_2)$.

In particular, we observe

Proposition 5.3. Homoclinic and periodic orbits occur generically in the unfoldings of steady-state bifurcations of Hamiltonian systems with O(2) symmetry when O(2) acts on $Z = \mathbb{C} \oplus \mathbb{C}$.

Similar results may be obtained whenever there exist subgroups Δ of Γ acting absolutely irreducibly on V, for which

$$\dim \mathrm{Fix}_V(\Delta) = 1.$$

Thus our methods, while not giving complete information, yield some fairly delicate dynamics in a very direct and simple fashion, for a variety of complicated Hamiltonians.

We can also determine the stability of the equilibrium points occurring in this bifurcation, in the full 4-dimensional system. Assume (for compatibility with fig. 2) that the eigenvalues are purely imaginary for $\lambda > 0$ and pass through zero at $\lambda = 0$. By the O(2) symmetry the eigenvalues are double: for $\lambda > 0$ they are a purely imaginary complex conjugate pair $\pm i\omega$ of multiplicity 2; for $\lambda = 0$ there are four zero eigenvalues; and for $\lambda < 0$ there are two real eigenvalues $\pm\mu$ of multiplicity 2. Thus the origin makes a transition from being elliptic (4 purely imaginary eigenvalues) to hyperbolic (4 real).

The new equilibria that appear away from the origin when $\lambda < 0$ clearly have at least two imaginary eigenvalues, from fig. 2. We can determine the remaining two eigenvalues as follows. Since the isotropy group $\mathbb{Z}_2$ commutes with the linearization of the vector field, it follows from section 2 that the linearized matrix can be written in block form as

$$\begin{bmatrix} A & 0 \\ 0 & B \end{bmatrix},$$

where A and B are symplectic 2×2 matrices. The eigenvalues of A are purely imaginary (and nonzero). Since the orbit of the equilibrium under O(2) is 1-dimensional there is at least one zero eigenvalue, which must occur in B. But B, being symplectic, has trace zero and consequently has both eigenvalues 0. Thus the eigenvalues at the bifurcating equilibria are of the form $(\omega i, -\omega i, 0, 0)$. This represents "orbital ellipticity", that is, all eigenvalues not forced to zero by the group action are imaginary.

A more general theory of the constraints imposed on eigenvalues by symmetry, depending on the geometry of the momentum mapping, may be found in Montaldi, Roberts and Stewart [19]. The results are applied there to obtain equivariant versions of the Liapunov centre theorem and the Weinstein–Moser theorem.

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Appendix

Bifurcation, reduction, and normal forms
by J.E. Marsden

In this appendix, we wish to point out a use of the theory of reduction in Hamiltonian systems with symmetry (Marsden and Weinstein [20] and Abraham and Marsden [21]) to the bifurcation theory begun in this paper. Our goal is to give a simple exposition of a result of Cushman and Rod [22]; related references are Churchill, Kummer and Rod [23], Iwai [24], and van der Meer [25]. The result of Cushman and Rod concerns the bifurcation at the 1 : 1 semisimple resonance in which a pair of eigenvalues at a fixed point become double as they move vertically on the imaginary axis. The nonsemisimple 1 : − 1 resonance in which a pair of eigenvalues cross the imaginary axis in opposite directions is considered by van der Meer. (This is what Abraham and Marsden [21] p. 604 call the *Brown* or *Trojan bifurcation* since it occurs in the restricted three-body problem for a mass ratio near Routh's critical value. Van der Meer calls it the *Hamiltonian Hopf bifurcation*.) The semisimple case is considered by Iwai and can be done by the same procedure that we outline here, with SU(2) replaced by SU(1, 1) and the three-spheres replaced by hyperboloids. It is hoped that the ideas of reduction will be of use in the general theory of bifurcation of Hamiltonian systems with symmetry, and our aim is merely to bridge the gap between workers in these areas and update the Hamiltonian techniques a little.

Let P be a Poisson manifold, let Γ and G be two Lie groups acting canonically on P with equivariant momentum maps J_Γ and J_G:

$$\gamma^* \overset{J_\Gamma}{\leftarrow} P \overset{J_G}{\rightarrow} g^*,$$

where γ^* and g^* are the duals of the Lie algebras of Γ and G respectively. Suppose that Γ acts transitively on the level sets of J_G. (This assumption is related to the notion of a *dual pair*; see for example Marsden and Weinstein [26].) If $H: P \to \mathbb{R}$ is a Γ-invariant Hamiltonian, then by transitivity of the Γ-action on level sets of J_G, it is obvious that this Hamiltonian collectivizes, that is, is of the form $H = h \circ J_G$. (See Guillemin and Sternberg [27], Holmes and Marsden [28], and Marsden et al. [29] for some general properties of collective motion.) We refer to h as the *normal form* for H. In specific cases, the methods of singularity theory can be used to show that h is a smooth function. However, since J_G is a Poisson map (see the preceding references) the dynamics of H projects to the Lie–Poisson dynamics on $\boldsymbol{g}^*$, that is to Hamiltonian dynamics on $\boldsymbol{g}^*$ with the Lie–Poisson bracket

$$\{F, H\}(\mu) = \langle \mu, [\delta F/\delta\mu, \delta H/\delta\mu] \rangle,$$

where [,] denotes the Lie bracket on $\boldsymbol{g}$ and $\delta F/\delta\mu \in \boldsymbol{g}$ denotes the functional derivative of F, that is, the dual of the Fréchet derivative.

These remarks apply to give some of the results of Cushman and Rod [22] as follows. Let $P = \mathbb{R}^4 \equiv \mathbb{C} \times \mathbb{C}$, $\Gamma = S^1$, and $G = SU(2)$. Let $H_0 = J_\Gamma = \frac{1}{2}(x_1^2 + y_1^2 + x_2^2 + y_2^2)$ and let S^1 act by the flow of H_0. The group SU(2) acts on $\mathbb{R}^4$ by quaternionic multiplication; its momentum map is easily computed to be the Hopf map

$$J_G(x_1, y_1, x_2, y_2) = (W_1, W_2, W_3),$$

where g is identified with $\mathbb{R}^3$ and where

$$W_1 = 2(x_1x_2 + y_1y_2),$$
$$W_2 = 2(x_2y_1 - x_1y_2),$$
$$W_3 = (x_1^2 + y_1^2 - x_2^2 - y_2^2),$$

as in Cushman and Rod [22]. The level sets J_G are the circles in the Hopf fibration π: $S^3 \to S^2$, and so Γ acts transitively on them. Thus any S^1-invariant Hamiltonian (for example one that is obtained by averaging) collectivizes through the Hopf map. (We can also regard H as a function of J_G and H_0 since $H_0^2 = (w_1^2 + W_2^2 + W_3^2)/4$.) Thus the dynamics of such an H reduces to dynamics on $\mathbb{R}^3$, with respect to the rigid body Lie–Poisson structure

$$\{F, H\}(l) = \langle l, \nabla F \times \nabla H \rangle.$$

Since the symplectic leaves of this Poisson structure are the two-spheres, the bifurcation problem is reduced to studying the dynamics of a Hamiltonian on S^2, which is certainly more tractable than the original problem.

It seems that this procedure will work whenever the unperturbed system (here H_0) is completely integrable by the method of collectivization.

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