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Bifurcations on Hemispheres

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Summary. It is now well known that the number of parameters and symmetries of an equation affects the bifurcation structure of that equation. The bifurcation behavior of reaction-diffusion equations on certain domains with certain boundary conditions is *nongeneric* in the sense that the bifurcation of steady states in these equations is not what would be expected if one considered only the number of parameters in the equations and the type of symmetries of the equations. This point was made previously in work by Fujii, Mimura, and Nishiura [6] and Armbruster and Dangelmayr [1], who considered reaction-diffusion equations on an interval with Neumann boundary conditions.

As was pointed out by Crawford et al. [5], the source of this nongenericity is that reaction-diffusion equations are invariant under translations and reflections of the domain and, depending on boundary conditions, may naturally and uniquely be extended to larger domains with *larger* symmetry groups. These extra symmetries are the source of the nongenericity. In this paper we consider in detail the steady-state bifurcations of reaction-diffusion equations defined on the hemisphere with Neumann boundary conditions along the equator. Such equations have a natural $\mathbf{O(2)}$-symmetry but may be extended to the full sphere where the natural symmetry group is $\mathbf{O(3)}$. We also determine a large class of partial differential equations and domains where this kind of extension is possible for both Neumann and Dirichlet boundary conditions.

Key words. bifurcation, symmetry, Neumann boundary conditions, Dirichlet boundary conditions, spherical symmetry
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1. Introduction

Considerable progress has been made in the bifurcation theory of nonlinear dynamical systems by starting from typical, or "generic", phenomena and investigating the effect of various degeneracies, such as multiple eigenvalues, which lead to more complicated phenomena. Symmetry, which is common in many applications, is one constraint that can create degeneracies, but it has become clear (see, for example, Golubitsky et al. [8]) that symmetry is best thought of as providing a new context in which to work. Thus, the "generic" phenomena in symmetric bifurcation theory are the typical phenomena *with the appropriate symmetry.*

It is not therefore sensible to appeal to genericity as an explanation of observed bifurcations unless one knows which context is appropriate. In order to apply the general methods of bifurcation theory, it is necessary to define precisely the context in which observed bifurcations are expected bifurcations. For example, the typical bifurcation of steady states is the saddle-node or limit-point bifurcation pictured in Figure 1(a). However, should the differential equation have a reflectional symmetry ρ, then steady-state bifurcations from a ρ-invariant steady state can be a pitchfork bifurcation, as pictured in Figure 1(b). Indeed, the pitchfork is expected when ρ acts nontrivially on the kernel of the linearized equation, while the limit-point bifurcation is expected otherwise.

General theory has shown that the types of bifurcations that are to be expected in systems of differential equations depend crucially on the number of parameters in the equations (that is, on the *codimension* of the bifurcation) and on the symmetries present in the system.

Here we are concerned with bifurcation of solutions of a partial differential equation (PDE), such as a reaction-diffusion equation, on a hemispherical domain. The natural symmetry of such a problem is circular: rotations and reflections of the domain. However, we will show that the expected bifurcations are governed not by circular symmetry but by spherical symmetry—subject to a final restriction back to the hemispherical domain. Thus, the problem is subject to more constraints than are at first apparent, and which can be understood as "additional" symmetries related to an extension of the domain.

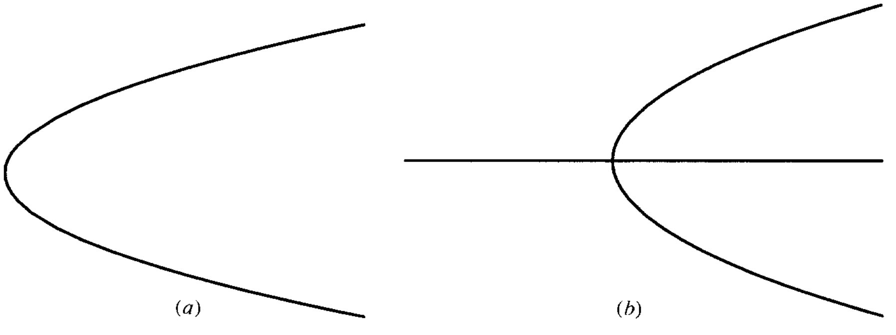


Fig. 1. Bifurcation diagrams for limit-point (a) and pitchfork (b) bifurcations

The realization that such a phenomenon can occur goes back to Fujii et al. [6] and Armbruster and Dangelmayr [1]. They pointed out that (higher-codimension) bifurcations of steady states of differential equations as common as reaction-diffusion equations on an interval behave in a manner that cannot be explained by their codimension and their (obvious) symmetries. To be precise, they considered the steady-state reaction-diffusion equation

$$\mathcal{P}(u) \equiv u'' + F(u, \lambda) = 0 \tag{1.1}$$

on an interval $N = [0, \pi]$ with Neumann boundary conditions

$$u'(0) = u'(\pi) = 0.$$

The only obvious symmetry is reflection about $\pi/2$, but the bifurcation behavior is *not* the generic behavior subject to this symmetry.

Crawford et al. [5] further amplified the surprises concerning the bifurcations in this equation by pointing out that even the simplest bifurcations are pitchforks, rather than the expected limit-point bifurcations. We expand briefly on this surprise.

The interval $[0, \pi]$ has one nontrivial symmetry, namely,

$$x \to \pi - x, \tag{1.2}$$

and both the equations and boundary conditions are invariant with respect to this symmetry. Eigenfunctions in the kernel of (1.1) come in two types: *even* and *odd*. The even eigenfunctions are invariant with respect to (1.2), while the odd ones are transformed to their negative by this symmetry. General theory tells us that when the eigenfunctions are odd, one expects a pitchfork, but that when the eigenfunctions are even, one expects a limit-point bifurcation. However, standard calculations show that in both cases the bifurcation is actually of pitchfork type.

The question is, why? The answer is based on a kind of "hidden symmetry" that is present in this equation. On account of the translational invariance of (1.1), solutions to the Neumann boundary value problem can be extended to the interval $M = [-\pi, \pi]$, and the extended solutions now satisfy periodic boundary conditions (this point follows from Theorem 3.25, p.266, in [15]). However, reaction-diffusion equations on M that satisfy periodic boundary conditions have as symmetries the larger group $\mathbf{O(2)}$ generated by translations modulo 2π as well as the reflection (1.2).

Summarizing, we see that the original Neumann boundary value problem on the smaller domain N with the single nontrivial symmetry (1.2) can be reformulated in terms of solutions on the larger domain M for an equation possessing $\mathbf{O(2)}$-invariance. The (distinctly nontrivial) effects of this additional symmetry on the bifurcation behavior of the Neumann boundary value problem was the point of the study by Armbruster and Dangelmayr [1].

Perhaps surprisingly, these seemingly abstract remarks have direct application to equations that are more complicated but have similar structure—such as those equations found in the numerical experiments of Cliffe and Mullin on the Couette-Taylor experiment and in the existence of modes in both the Faraday experiment and Rayleigh-Bénard convection (see [5]).

As noted in [5] (see also Gomes [9] and Gomes et al. [10]), the remarks concerning the bifurcation behavior of this simple PDE generalize in several ways:

- One can consider boundary conditions other than Neumann (in particular, Dirichlet boundary conditions).
- One can consider systems of equations (in particular, systems of equations with a mixture of Neumann and Dirichlet boundary conditions on different components).
- One can consider more complicated domains in higher dimensions.

In this article we focus on the third of these generalizations. First, we consider the bifurcation behavior of differential equations on the hemisphere that can be extended to the full sphere. On the hemisphere these equations have $\mathbf{O(2)}$-symmetry, while on the sphere they have $\mathbf{O(3)}$-symmetry. Second, we present a large class of pairs of manifolds $N \subset M$ where extra symmetries obtained from extensions to the larger manifold will change the expected bifurcation behavior on the smaller submanifold. The remainder of this introduction is devoted to the discussion of hemispheres. The abstract formulation of the extension problem is presented in Section 5.

Hemispheres

There is some interest in studying bifurcation problems on a hemispherical domain; in particular, there are applications to elastic buckling of hemispherical shells (see Bauer, Riess, and Keller [2]). The only obvious symmetry of a bifurcation problem defined on a hemisphere is the group $\mathbf{O(2)}$ of rotations and reflections that preserve the boundary. However, for a wide class of boundary value problems on the hemisphere, the expected behavior is not what one would predict from the theory of generic $\mathbf{O(2)}$-symmetric bifurcation. This is a subclass of those bifurcation problems that are defined on a hemisphere but whose equations extend naturally to the full sphere. The problems considered in [2] are examples. We explain here how the existence of an extension can change the expected bifurcation structure of solutions on the hemisphere. These results are, in fact, much more dramatic than those that we described for reaction-diffusion equations on the line. However, for pedagogical purposes, we shall think of the PDE as a steady-state reaction-diffusion equation, although the same considerations will apply to a larger class of equations (see Section 5 for details).

We denote the coordinates on $\mathbb{R}^3$ by (x_1, x_2, x_3) and let S denote the unit sphere in $\mathbb{R}^3$. Let $H = \{x \in S \mid x_3 \geq 0\}$ denote the upper hemisphere of S, and let Δ denote the Laplacian on S. Let $f : \mathbb{R}^2 \to \mathbb{R}$ be a smooth map. Consider the reaction-diffusion equation defined on H by

$$\Delta u + f(u, \lambda) = 0, \tag{1.3}$$

where $u : H \to \mathbb{R}$. Assume that (1.3) satisfies Neumann boundary conditions on $\partial H = \{(x_1, x_2, x_3) \in H \mid x_3 = 0\}$, that is,

$$\frac{\partial u}{\partial x_3}(x_1, x_2, 0) = 0 \quad \text{on } \partial H. \tag{1.4}$$

Solutions of (1.3) on H that satisfy the boundary conditions (1.4) can be extended to solutions of (1.3) on S by defining u on the lower hemisphere by reflection. More precisely, let $\tau : S \rightarrow S$ be the reflection across ∂H defined by

$$\tau(x_1, x_2, x_3) = (x_1, x_2, -x_3). \tag{1.5}$$

In symbols we can now define u on the lower hemisphere by

$$u(\tau(x)) \equiv u(x) \quad \forall x \in H. \tag{1.6}$$

We shall say that a function u on S is *τ-invariant* if

$$u(\tau(x)) = u(x) \quad \forall x \in S. \tag{1.7}$$

Note that the extension u that we have defined on S is τ-invariant. (In Theorem 5.18, we establish the regularity of the extended solution along ∂H.)

Conversely, suppose that u is a τ-invariant solution to the reaction-diffusion equation (1.3) on S. Then $u \mid H$ is a solution to the Neumann boundary value problem (1.4) on H. This point is most important. We can now find solutions to the Neumann problem on the hemisphere by first finding solutions to the extended problem on S that are τ-invariant. Indeed, this will be our approach.

Next, we observe that (1.3) defined on the hemisphere H and satisfying boundary conditions (1.4) has symmetry group $\mathbf{O(2)}$, whereas (1.3) defined on the entire sphere S has symmetry group $\mathbf{O(3)}$. The consequences of this extra symmetry for the bifurcation problem on the hemisphere are quite extensive, as we now explain.

Assume that

$$f(0, \lambda) \equiv 0; \tag{1.8}$$

that is, equation (1.3) has the trivial group-invariant steady state $u = 0$. Let V be the kernel of the linearization of (1.3). Theory states that the group of symmetries of the equation leaves the space V invariant, and that, moreover, generically the action of that symmetry group on V is irreducible. (See Proposition XIII, 3.2 in [8].)

We can now see one of the effects that the existence of this extra symmetry has on the expected bifurcations. Let V_S be the kernel of the linearization of (1.3) on the full sphere. Then V consists of those eigenfunctions in V_S that are τ-invariant. Now, the irreducible representations of $\mathbf{O(2)}$ are either one- or two-dimensional. Hence, a direct application of the general theory would imply that generically we should expect $\dim(V)$ to be one or two. However, the extension property of (1.3) implies that this equation has more structure than was taken into account by the general theory naively applied. Indeed, applying the genericity theory to the reaction-diffusion equation on the full sphere (where it seems to be applicable), one should expect the action of $\mathbf{O(3)}$ on V_S to be irreducible. Now there are irreducible representations of $\mathbf{O(3)}$ in each odd dimension, namely, the action of $\mathbf{O(3)}$ on the spherical harmonics of order ℓ. It can be checked that the vectors in V_S that are τ-invariant form a subspace of dimension approximately $\frac{1}{2}\dim(V_S)$. Thus, the kernel of the linearized equations may be of much higher dimension than would have been expected if one considered the bifurcation problem only on the hemisphere.

Indeed, even though this change of expected dimension of the kernel V is dramatic, it is fair to ask whether solutions to the hemisphere problem may now be expected that would not have been expected were the extension property not valid. We shall show in Sections 2–4 that the answer to this question is yes. We preview our results in the remainder of this subsection. As should be expected, the explicit results depend on the exact value of ℓ that appears in the solution of the linearized problem on the sphere.

In Section 2 we will recall some general properties of $\mathbf{O(3)}$-symmetric bifurcation problems. In particular, we will define the subgroups of $\mathbf{O(3)}$ that are known to support branches of steady-state solutions. This section reviews the work of [12] and [4]. In Section 3 we will show how $\mathbf{O(3)}$ symmetry affects the bifurcation to axisymmetric solutions on the hemisphere, and in Section 4 we will consider solutions with finite isotropy.

As noted previously, a solution to the $\mathbf{O(3)}$-symmetric equation (1.3) on S restricts to the hemisphere if and only if it is invariant under the reflection τ. Using our results from Section 2, we will be able to determine which group orbits of equilibria in generic $\mathbf{O(3)}$-equivariant bifurcations have representative solutions with isotropy containing τ. We will find that the isotropy groups associated to certain group orbits of equilibria never contain τ, and so elements of these group orbits can never restrict to give solutions of the Neumann boundary value problem on H. On the other hand, some group orbits of equilibria contain *multiple* equilibria with symmetry τ. Each of these equilibria then restrict to solutions of the Neumann boundary value problem on the hemisphere lying on *different* $\mathbf{O(2)}$-orbits. The realization of this point is one of the surprising features resulting from our analysis.

General $\mathbf{O(3)}$-equivariant bifurcation theory, as described in Section 2, shows that generically axisymmetric solutions of order ℓ are to be expected. In Figure 2(a), we picture the sphere deformed by an axisymmetric spherical harmonic of order $\ell = 6$. More specifically, since a spherical harmonic is a real-valued function on the sphere, we can picture it by deforming the sphere in the radial direction by an amount equal to the value of that spherical harmonic. The precise value of ℓ depends on the function f in (1.3). Assuming then that $\ell = 6$, the deformation pictured in Figure 2(a) is, up to first order, an accurate picture of the corresponding axisymmetric solution to (1.3).

In Theorem 3.1, we will prove that this axisymmetric solution can be sliced in two different ways to obtain solutions to the original equation posed on the hemisphere. Observe that the picture in Figure 2(b) has circular symmetry, while the picture in Figure 2(c) has only a reflectional symmetry.

Similarly, when $\ell = 5$, general theory predicts the existence of a solution having fivefold symmetry, as shown in Figure 3(a). In Theorem 4.3(a), we show that this solution may also be sliced in two distinct ways to obtain solutions to (1.3) on H. The first slice also has fivefold symmetry and is visualized by slicing off the obscured half of the deformed sphere in Figure 3(a). The other slice, which has only a reflectional symmetry, is shown in Figure 3(b).

The complete list of restricted solutions will be presented in Theorems 3.1 and 4.3.

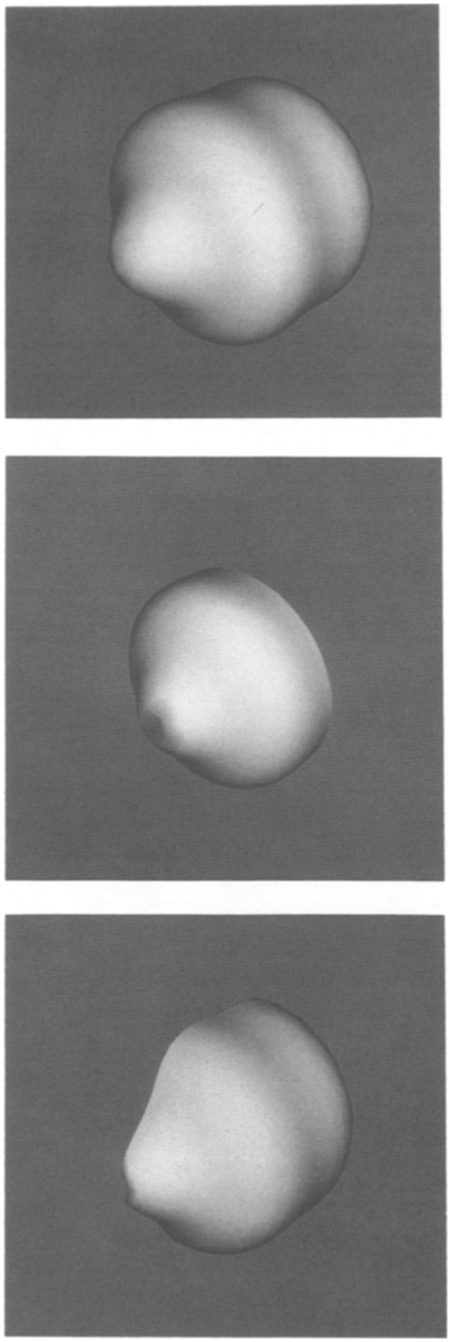
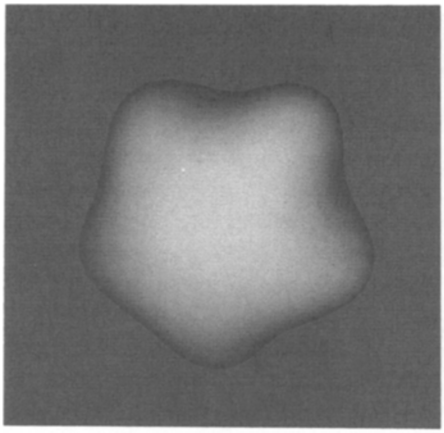


Fig. 2. (a) Axisymmetric deformation of sphere with spherical harmonics of order $\ell = 6$. (b) Restriction to hemisphere with circular symmetry and (c) reflectional symmetry.



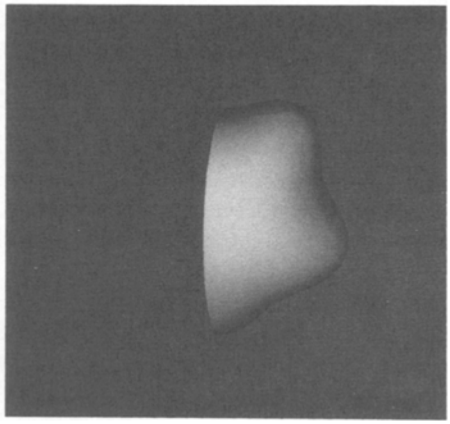


Fig. 3. Deformation of sphere with sector symmetry D_{10}^d. Restriction to hemisphere (a) with five-fold symmetry and (b) with no symmetry.

2. Bifurcation from O(3)-Invariant Solutions

We begin by recalling several facts concerning steady-state bifurcation in the presence of $\mathbf{O(3)}$ symmetry. Typically, we expect the kernel of the linearization about an $\mathbf{O(3)}$-invariant equilibrium to be an irreducible representation of $\mathbf{O(3)}$ (see Proposition XIII, 3.2, in [8]). Moreover, because the natural domains of PDEs are function spaces, we expect these irreducible representations to be isomorphic to V_ℓ, the spherical harmonics of order ℓ. There are two irreducible representations of $\mathbf{O(3)}$ in each odd dimension depending on whether $-I \in \mathbf{O(3)}$ acts trivially or acts as minus the identity on V_ℓ. There is, however, a natural action on V_ℓ, since V_ℓ consists of polynomial mappings $p : \mathbb{R}^3 \to \mathbb{R}$ that are homogeneous of degree ℓ. For such polynomials, the action of $-I$ is

$$p(-x) = (-1)^\ell p(x). \tag{2.1}$$

Thus, we assume that $-I$ acts trivially when ℓ is even and acts as minus the identity when ℓ is odd. (The other "nonstandard" representations could, of course, be dealt with in a similar manner.)

We use the notation and results of [12] and [8]. The closed subgroups of $\mathbf{SO}(3)$ are $\mathbf{Z}_m$ (cyclic group of order m), $\mathbf{D}_m$ (dihedral group of order $2m$), $\mathbf{T}$ (tetrahedral group), $\mathbf{O}$ (octahedral group), $\mathbf{I}$ (the icosahedral group), $\mathbf{SO}(2)$, and $\mathbf{O(2)}$. Subgroups of $\mathbf{O(3)} = \mathbf{SO}(3) \oplus \mathbf{Z}_2^c$ are of three types. Type I and type II subgroups are respectively of the form Σ and $\Sigma \oplus \mathbf{Z}_2^c$, where Σ is a subgroup of $\mathbf{SO}(3)$. Type III subgroups Σ of $\mathbf{O(3)}$ neither contain nor are contained in $\mathbf{SO}(3)$.

Type III subgroups of $\mathbf{O(3)}$ may be characterized by a pair $H \supset K$ of subgroups of $\mathbf{SO}(3)$ with K having index two in H. Indeed, we may take $K = \Sigma \cap \mathbf{SO}(3)$ and let H be the projection of Σ into $\mathbf{SO}(3)$. In this paper we shall be concerned with four type III subgroups, $\mathbf{O(2)}^-$, $\mathbf{O}^-$, $\mathbf{D}_{2m}^d$ and $\mathbf{D}_m^z$, which are defined by the pairs $\mathbf{O(2)} \supset \mathbf{SO}(2)$, $\mathbf{O} \supset \mathbf{T}$, $\mathbf{D}_{2m} \supset \mathbf{D}_m$ and $\mathbf{D}_m \supset \mathbf{Z}_m$. It is worth noting that this abstract definition of type III subgroups of $\mathbf{O(3)}$ can be given a more concrete realization. In particular, the subgroup Σ corresponding to $H \supset K$ is just

$$K \times \{I\} \cup (H \setminus K) \times \{-I\} \tag{2.2}$$

inside $\mathbf{O(3)} = \mathbf{SO}(3) \oplus \mathbf{Z}_2^c$.

Remark 2.1. The group of symmetries of the hemisphere is the subgroup of $\mathbf{O(3)}$ that leaves H invariant. This subgroup is just $\mathbf{O(2)}^-$.

Chossat, Lauterbach, and Melbourne [4] give a list of those conjugacy classes of isotropy subgroups that are known generically to support solutions for $\mathbf{O(3)}$-equivariant bifurcation problems. That is, for generic bifurcations, solutions with these isotropy groups must exist. Their result is reproduced in Table 1. With the exception of the $\ell = 5$ results for $\mathbf{D}_5^z$, $\mathbf{D}_4^z$, and $\mathbf{D}_3^z$, the isotropy groups displayed in Table 1 are maximal and have odd-dimensional fixed-point space (this follows from the computations of [12] and [8].) Hence, by degree theory, it follows that there always exist branches of solutions with the given isotropy type. This general theory implies that in $\mathbf{O(3)}$-equivariant bifurcation problems there are branches of solutions corresponding to each of the subgroups listed in Table 1 and indeed to every subgroup conjugate to one on this list. In this paper we consider which of these subgroups supports solutions of the hemisphere problem. We classify such solutions up to $\mathbf{O(2)}^-$, the symmetry group of the hemisphere.

Let τ denote the reflection defining the hemisphere (see equation (1.5)). Suppose that u is a solution to an $\mathbf{O(3)}$-equivariant bifurcation problem with isotropy Σ. It follows from Theorem 5.18 that u restricts to a solution to the Neumann problem on the hemisphere if and only if $\tau \in \Sigma$.

Since the groups $\mathbf{O}$ and $\mathbf{I}$ contain no orientation-reversing elements we have the following lemma.

Table 1. Isotropy subgroups generically supporting solutions

ℓ even:	
$\mathbf{O(2)} \oplus \mathbf{Z}_2^c$	all even ℓ
$\mathbf{O} \oplus \mathbf{Z}_2^c$	$\ell = 0, 4, 6, 8, 10, 14 \bmod 24$, $\ell \geq 4$
$\mathbf{I} \oplus \mathbf{Z}_2^c$	$\ell = 0, 6, 10, 12, 16, 18, 20, 22, 24, 26, 28, 32, 34, 38, 44 \bmod 60$, $\ell \geq 6$
ℓ odd:	
$\mathbf{O(2)}^-$	all odd ℓ
$\mathbf{O}^-$	$\ell = 3, 7, 9, 11, 13, 17 \bmod 24$
$\mathbf{O}$	$\ell = 9, 13, 15, 17, 19, 23 \bmod 24$
$\mathbf{I}$	$\ell = 15, 21, 25, 27, 31, 33, 35, 37, 39, 41, 43, 47, 49, 53, 59 \bmod 60$
$\mathbf{D}_{2m}^d$	$3 \leq m \leq \ell$
$\mathbf{D}_4^d$	$\ell = 1, 5, 15, 19, 21, 23 \bmod 24$, $\ell \geq 5$
$\mathbf{D}_5^z, \mathbf{D}_4^z, \mathbf{D}_2^z$	$\ell = 5$

Lemma 2.2. *The solutions to* $\mathbf{O}(3)$*-equivariant bifurcation problems with isotropy conjugate to* $\mathbf{O}$ *or* $\mathbf{I}$ *cannot restrict to solutions of the Neumann problem on the hemisphere.*

Following the terminology of Section 5, we say that an involution $\sigma \in \mathbf{O}(3)$ is a *reflection* if the fixed point set of σ is two-dimensional. The following elementary lemmas will be useful in determining which solutions to the $\mathbf{O}(3)$-equivariant bifurcation problem restrict to solutions of the hemisphere problem.

Lemma 2.3. *An involution* $\sigma \in \mathbf{O}(3)$ *is a reflection if and only if* σ *is conjugate to* τ. *That is, if and only if there exists* $\gamma \in \mathbf{O}(3)$ *such that*

$$\sigma = \gamma\tau\gamma^{-1}.$$

Proof. Observe that $\sigma \neq I$ is an involution if and only if σ has eigenvalues ± 1 and that σ is a reflection if and only if σ has precisely one eigenvalue equal to -1. Hence, if σ is a reflection, σ is conjugate to τ. □

Lemma 2.4. *Suppose that* $\sigma \in \mathbf{O}(3)$ *is an involution which is not a reflection and not equal to* $-I$. *Then* $-\sigma$ *is a reflection.*

Proof. The result is trivial. □

In the next section we will discuss solutions to the hemisphere problem having axisymmetric ($\mathbf{O}(2)$ or $\mathbf{O}(2)^-$) symmetry. In Section 4 we discuss sector ($\mathbf{D}_{2m}^d$ or $\mathbf{D}_m^z$), octahedral ($\mathbf{O}^-$ or $\mathbf{O} \oplus \mathbf{Z}_2^c$), and icosahedral ($\mathbf{I} \oplus \mathbf{Z}_2^c$) solutions.

3. Axisymmetric Solutions

Axisymmetric solutions are those that have an axis of rotation. That is, they have at least $\mathbf{SO}(2)$ symmetry. There are two types of isotropy subgroup that contain $\mathbf{SO}(2)$: $\mathbf{O}(2) \oplus \mathbf{Z}_2^c$ (when ℓ is even) and $\mathbf{O}(2)^-$ (when ℓ is odd).

Theorem 3.1. *Each orbit of axisymmetric solutions to the sphere problem restricts to solutions of the hemisphere problem as follows:*

(a) *When* ℓ *is even, an isolated axisymmetric solution with the* x_3*-axis as axis of symmetry.*
(b) *For all* ℓ, *a unique circle of solutions with isotropy* $\mathbf{Z}_2^-$ *(inside* $\mathbf{O}(2)^-$*).*

Proof. We must determine those subgroups of $\mathbf{O}(3)$ conjugate to either $\mathbf{O}(2) \oplus \mathbf{Z}_2^c$ (when ℓ is even) or $\mathbf{O}(2)^-$ (when ℓ is odd) that also contain τ.

Recall that $\mathbf{SO}(2)$ contains all rotations of the plane orthogonal to the axis of symmetry and that these rotations fix the axis of symmetry. Let σ be any reflection with fixed point set containing the axis of symmetry of $\mathbf{SO}(2)$. The subgroup $\mathbf{O}(2)$ is generated by $\mathbf{SO}(2)$ and $-\sigma$. The subgroup $\mathbf{O}(2)^-$ is generated by $\mathbf{SO}(2)$ and σ.

Let Σ be an isotropy subgroup containing $\mathbf{SO}(2)$, and assume that $\tau \in \Sigma$. Since τ reverses orientation, $\tau \notin \mathbf{SO}(2)$, but τ is in the normalizer of $\mathbf{SO}(2)$. Thus τ maps the axis of rotation of $\mathbf{SO}(2)$ into itself. It follows that this axis is either

(a) the x_3-axis or
(b) perpendicular to the x_3-axis.

In case (a), $\tau \notin \mathbf{O(2)}^-$, and so these solutions do not occur when ℓ is odd. If ℓ is even, $\tau \in \mathbf{O(2)} \oplus \mathbf{Z}_2^c$ when $-\tau \in \mathbf{O(2)}$. Observe that $-\tau$ is just the rotation through π in the x_1x_2-plane, so $-\tau \in \mathbf{SO}(2)$. Consequently, we obtain a solution for the hemisphere problem when ℓ is even.

In case (b), τ is always in Σ since τ is just a reflection in the plane of rotation associated with Σ that leaves the axis of rotation fixed. (That is, τ is in the appropriate conjugate of $\mathbf{O(2)}^-$.) Hence, in this case we always obtain solutions.

We remark that in case (a), $\Sigma \cap \mathbf{O(2)}^- = \mathbf{O(2)}^-$. In case (b), $\Sigma \cap \mathbf{O(2)}^-$ is the two-element group generated by $\tau \cdot \pi$, where π is the half-period rotation in the appropriate conjugate of $\mathbf{SO}(2)$. □

4. Solutions with Finite Isotropy

When the isotropy subgroup inside $\mathbf{O(3)}$ is finite, we adopt a different approach to finding those solutions which restrict to the hemisphere.

Before stating our first result, we recall that a group G is the *disjoint union* of subgroups G_i, $i \in I$, if $G = \cup_{i \in I} G_i$ and for all $i, j \in I$, $i \neq j$, $G_i \cap G_j$ is the identity element of G. We write $G = \dot{\cup}_i G_i$.

Lemma 4.1. *The finite subgroups of* $\mathbf{O(3)}$ *all have disjoint union decompositions. Specifically, we have*

$$\begin{aligned}
\mathbf{O} &= \dot{\cup}^3 \mathbf{Z}_4 \dot{\cup}^4 \mathbf{Z}_3 \dot{\cup}^6 \mathbf{Z}_2, \\
\mathbf{I} &= \dot{\cup}^6 \mathbf{Z}_5 \dot{\cup}^{10} \mathbf{Z}_3 \dot{\cup}^{15} \mathbf{Z}_2, \\
\mathbf{O}^- &= \dot{\cup}^3 \mathbf{Z}_4^- \dot{\cup}^4 \mathbf{Z}_3 \dot{\cup}^6 \mathbf{Z}_2^-, \\
\mathbf{D}_{2m}^d &= \mathbf{Z}_{2m}^- \dot{\cup}^m \mathbf{Z}_2^- \dot{\cup}^m \mathbf{Z}_2, \\
\mathbf{D}_m^z &= \mathbf{Z}_m \dot{\cup}^m \mathbf{Z}_2^- .
\end{aligned}$$

Proof. Proofs of these decompositions may be found in Ihrig and Golubitsky [12] or [8, pp.105, 123]. □

As an immediate consequence of Lemma 4.1 and Lemma 2.3, we have the following lemma.

Lemma 4.2. *With the notation of Lemma 4.1,*

- *The reflections in* $\mathbf{O}^-$ *are the order two elements of* $\mathbf{Z}_2^-$ *and* $\mathbf{Z}_4^-$.
- *The reflections in* $\mathbf{D}_{2m}^d$ *are the order two elements of* $\mathbf{Z}_2^-$ *and, in case m is odd, the order two element of* $\mathbf{Z}_{2m}^-$.
- *The reflections in* $\mathbf{D}_m^z$ *are the order two elements of* $\mathbf{Z}_2^-$.

Theorem 4.3. *Suppose that we have an* $\mathbf{O(3)}$*-orbit of solutions to the sphere problem with finite isotropy conjugate to* Σ. *On restriction, we obtain the following* $\mathbf{O(2)}^-$*-orbits of solutions to the hemisphere problem:*

$\Sigma = \mathbf{D}_{2m}^d$ **(Sector solutions)**

(a) *When m is odd, there is a unique circle of solutions with isotropy* $\mathbf{D}_m^z$ *(inside* $\mathbf{O(2)}^-$*).*

(b) *There are m circles of solutions with isotropy* $\mathbf{Z}_2^-$.

$\Sigma = \mathbf{D}_m^z$ **(Sector solutions)**

(c) *There are m circles of solutions with isotropy* $\mathbf{Z}_2^-$.

$\Sigma = \mathbf{O}^-$ **(Octahedral solutions)**

(d) *There are 6 circles of solutions with isotropy* $\mathbf{Z}_2^-$.

$\Sigma = \mathbf{O} \oplus \mathbf{Z}_2^c$ **(Octahedral solutions)**

(e) *There are 3 circles of solutions with isotropy* $\mathbf{D}_4^-$.

(f) *There are 6 circles of solutions with isotropy* $\mathbf{D}_2^-$.

$\Sigma = \mathbf{I} \oplus \mathbf{Z}_2^c$ **(Icosahedral solutions)**

(g) *There are 15 circles of solutions with isotropy* $\mathbf{Z}_2^-$.

Proof. We start by considering sector solutions with $\Sigma = \mathbf{D}_{2m}^d$. It follows from Lemma 4.2 that Σ has m distinct reflections in $\mathbf{Z}_2^-$ yielding the solutions (b). If m is odd, there is one reflection in $\mathbf{Z}_{2m}^-$ yielding the orbit of solutions described in (a). The existence of solutions described in (c) and (d) follows similarly.

The remaining cases have the form $\Sigma = \Delta \oplus \mathbf{Z}_2^c$. For these subgroups, reflections conjugate to τ are found by composing the nontrivial involutions in Δ with $-I$ (Lemma 2.4). For example, in $\mathbf{O}$ there are three nontrivial involutions in $\mathbf{Z}_4$ and six in $\mathbf{Z}_2$. Consequently, we derive the solutions described in (e) and (f). Similarly, one can read off the solutions stemming from icosahedral symmetry from the disjoint union decomposition in Lemma 4.1. □

5. The General Extension Problem

(a) The Abstract Formulation

The hemisphere is only one of a wide class of domains for which similar conclusions can be drawn. To describe this class we reformulate the observations concerning (1.1) in a more abstract setting and prove the appropriate extension theorems in some detail, since the existing literature tends to slide over this point.

Let M denote the circle defined by identifying the end points of $[-\pi, \pi]$. The symmetry group of M is $\mathbf{O(2)}$. The operator $\mathcal{P}$ in (1.1) induces a reaction-diffusion equation on M, which we continue to denote by $\mathcal{P}$. Clearly, $\mathcal{P}$ is $\mathbf{O(2)}$-invariant. Let τ denote the reflection of the circle M with fixed points $\{0, \pi\}$. The interval $[0, \pi]$ determines a submanifold N of M with $\partial N = \{0, \pi\}$. Suppose that u is a solution to the Neumann boundary value problem on N. We may extend u to M by defining $u \mid \tau(N) = u \cdot \tau$. The resulting function u on M is then a τ-invariant solution of $\mathcal{P}$ on M. Conversely, if u is a τ-invariant solution to $\mathcal{P}$ on M, then $u \mid N$ is a solution to the Neumann boundary value problem on N.

In this formulation M is a compact manifold without boundary. Similarly, the extension of H to S is from a manifold with boundary to a manifold without boundary. Finally, we note that in these examples the group of symmetries on M is just the group of isometries of M and the group of symmetries on N is just the subgroup of the symmetries on M that maps N into itself.

We now identify the pairs of manifolds $N \subset M$ and the kinds of differential operators $\mathcal{P}$ that admit the kind of extension that we have discussed previously for the specific examples. This generalization is based on using the group of isometries on M as the basic group of symmetries.

From now on we shall assume, unless indicated to the contrary, that maps and functions are smooth (that is, C^∞). We let M be a smooth, compact, connected, Riemannian n-dimensional manifold without boundary. We denote the group of isometries of M by $\mathrm{ISO}(M)$, and recall from Chapter VI, Theorems 3.3, 3.4 of [13] the result that $\mathrm{ISO}(M)$ is a compact Lie group of dimension $\leq n(n+1)/2$.

Definition of N

Denote the identity map of M by I_M. Suppose that $f : M \to M$, and let $\mathrm{Fix}(f) = \{x \in M \mid f(x) = x\}$ denote the fixed-point set of f.

Definition 5.1. A map $\sigma : M \to M$ is an *involution* if $\sigma \neq I_M$ and $\sigma^2 = I_M$.

Recall from Bredon [3, Chapter 6] that the fixed-point set of an involution σ of M is a smooth compact submanifold of M without boundary. In general, $\mathrm{Fix}(\sigma)$ may have connected components of differing dimensions. If all the components of $\mathrm{Fix}(\sigma)$ have the same dimension p, we shall say that $\mathrm{Fix}(\sigma)$ is a p-dimensional submanifold of M.

Definition 5.2. A map $\tau : M \to M$ is a *reflection* if

1. τ is an involution.
2. $\mathrm{Fix}(\tau)$ is an $(n-1)$-dimensional submanifold of M.
3. $M \setminus \mathrm{Fix}(\tau)$ has two connected components.

We note that even if M is an orientable manifold, conditions (1) and (2) of Definition 5.2 do *not* imply condition (3). However, if τ is any involution of M, then $M \setminus \mathrm{Fix}(\tau)$ has at most two connected components.

Definition 5.3. Let $1 \leq p \leq n+1$ and suppose that $\tau_1, \ldots, \tau_p$ are reflections of M. We say that $\{\tau_1, \ldots, \tau_p\}$ form an *admissible* set of reflections of M if

1. $\tau_i \tau_j = \tau_j \tau_i$, $1 \leq i, j \leq p$.
2. $\tau_i \neq \tau_j$, $i \neq j$.

Let $\mathcal{R} = \{\tau_1, \ldots, \tau_p\}$ be an admissible set of reflections of M. Clearly, $\mathcal{R}$ generates a finite group K of transformations of M isomorphic to $\mathbf{Z}_2^p$. Averaging any Riemannian metric on M over K, we may and shall assume that M is a Riemannian manifold and that K is a group of *isometries* of M.

We are interested in the geometric structure of the action of K on M. We describe the salient features of this structure in the following lemma. First, however, we need some notational conventions.

Suppose that G is any (Lie) group of transformations of M. Let $x \in M$. Denote the G-orbit through x by $G \cdot x$ and the isotropy subgroup of G at x by G_x. Thus, $G \cdot x = \{gx \mid g \in G\}$, and $G_x = \{g \in G \mid gx = x\}$.

Lemma 5.4. *Let $M_K = \{x \in M \mid K_x = \{I_M\}\}$. Then*

1. *M_K is an open and dense submanifold of M.*
2. *M_K has 2^p connected components.*
3. *K acts transitively on the set of connected components of M_K.*

The proof of this lemma will be given below. First, we examine some of its consequences.

Choose a connected component N of M_K. Let ∂N denote the boundary of N. If K is generated by a single reflection ($p = 1$), then ∂N will be a smooth submanifold of M of dimension $n-1$. If $p \geq 2$, ∂N may have corners. In the sequel, we sometimes refer to N as a *fundamental domain* for the action of K on M.

We shall show below that every isometry on N (for the Riemannian structure induced from M) extends uniquely to an isometry on M. Thus, we can think of the group of isometries on N as a subgroup of ISO(M). See Lemma 5.13.

Several examples of $N \subset S$ are as follows. Let S denote the unit sphere in $\mathbb{R}^3$. Let τ_1, τ_2 and τ_3 denote the reflections on S determined by reflection in the (x, y)-, (x, z)- and (y, z)-planes respectively. The group generated by τ_1 is isomorphic to $\mathbf{Z}_2$ and has fundamental domain the upper hemisphere. The group generated by τ_1 and τ_2 is isomorphic to $\mathbf{Z}_2^2$ and has fundamental domain $\{(x, y, z) \in S \mid z, y \geq 0\}$. The group generated by τ_1, τ_2 and τ_3 is isomorphic to $\mathbf{Z}_2^3$ and has an octant of the sphere as fundamental domain. We may similarly generate many examples from the compact orientable surfaces of genus greater than or equal to 1. Finally, we remark that our methods also work for the fundamental domains of irreducible finite reflection groups restricted to the unit sphere of the underlying representation space.

The operators $\mathcal{P}$

Abstractly, we may think of a differential operator $\mathcal{P}$ on M as a mapping on the smooth functions on M. That is, let $C^\infty(M)$ denote the space of smooth real-valued functions on M. Then $\mathcal{P} : C^\infty(M) \to C^\infty(M)$.

In our extension theory we consider only those operators that respect the symmetries on M and N. We define these operators as follows. There is a natural action of ISO(M) on $C^\infty(M)$ defined by $u \to g(u)$ where for $u \in C^\infty(M)$ and $g \in \mathrm{ISO}(M)$,

$$g(u)(x) = u(g^{-1}x). \tag{5.1}$$

The operator $\mathcal{P}$ is *ISO(M)-invariant* if for all $u \in C^\infty(M)$ and $g \in G$ we have

$$\mathcal{P}(g(u)) = g(\mathcal{P}(u)). \tag{5.2}$$

The best-known example of an ISO(M)-invariant is the Laplace operator associated to the Riemannian structure on M, denoted by Δ. It follows easily that the semi-linear elliptic operator $\mathcal{P}$ defined by

$$\mathcal{P}(u) = \Delta u + f(u), \tag{5.3}$$

where $f : \mathbb{R} \to \mathbb{R}$ is smooth, is also ISO(M)-invariant. Of course, this operator is the steady-state reaction-diffusion operator that we have discussed previously.

This example can be generalized as follows. Recall that $\Delta u = \mathrm{div\ grad}(u)$, where $\mathrm{grad}(u)$ is the gradient vector field of u. More generally, we say that a second order differential operator on M is in *divergence form* if we can write

$$\mathcal{P}(u) = \mathrm{div} A + B, \tag{5.4}$$

where A is a vector field on M depending smoothly on $x \in M$, u, and du (the differential of u) and B is a scalar function depending smoothly on x, u, and du.

Definition 5.5. A differential operator $\mathcal{P}$ is said to be a *second order quasilinear elliptic operator in divergence form* if $\mathcal{P}$ is an elliptic operator in the form (5.4).

Definition 5.6. Suppose that u is a solution of $\mathcal{P}$ on N.

1. We say that u satisfies Neumann boundary conditions (NBC) on N if for every $\tau \in \mathcal{R}$ and all $x \in \partial N \cap \mathrm{Fix}(\tau)$, we have

$$\frac{\partial u}{\partial n}(x) = 0,$$

   where n is the normal direction to $\mathrm{Fix}(\tau)$ at x.
2. We say that u satisfies Dirichlet boundary conditions (DBC) on N if $u \equiv 0$ on ∂N.

We prove (see Theorem 5.18) that if the operator $\mathcal{P}$ is a second order quasilinear elliptic operator in divergence form that is also K-invariant, then the following hold.

1. Every smooth K invariant solution u of $\mathcal{P}$ on M restricts to a smooth solution of the Neumann problem for $\mathcal{P}$ on N.
2. Suppose that u is a generalized solution to the Neumann problem for $\mathcal{P}$ on $\overline{N}$ (satisfying some minimal smoothness properties). Then u extends uniquely to a smooth K-invariant solution of $\mathcal{P}$ on M.

The situation for Dirichlet boundary conditions is slightly more complicated since this extension property depends on the operator having extra symmetries. We now describe the necessary modifications.

Definition 5.7. Given $\tau \in K$, we define sign(τ) to be -1 if τ is the product of an *odd* number of reflections and $+1$ if τ is either the identity or the product of an even number of reflections.

If we identify $\{-1, +1\}$ with $\mathbf{Z}_2$, it is clear that sign: $K \to \mathbf{Z}_2$ is a group homomorphism.

We define a new action of K on $C^\infty(M)$. by

$$\tau(u)(x) = \operatorname{sign}(\tau)u(\tau^{-1}x). \tag{5.5}$$

To avoid confusion with the original action of K on $C^\infty(M)$, we shall write $\hat{K}$ to denote K when using the new action of K on M. Thus, the operator $\mathcal{P}$ will be $\hat{K}$-invariant if, for all $u \in C^\infty(M)$ and $\tau \in K$, we have

$$\mathcal{P}(g(u)) = \operatorname{sign}(g)g(\mathcal{P}(u)). \tag{5.6}$$

We note that $\mathcal{P}$, the reaction-diffusion operator (5.3), is $\hat{K}$-invariant if and only if f is an odd function of u.

We also prove (see Theorem 5.19) that if the operator $\mathcal{P}$ is a second-order quasilinear elliptic operator in divergence form that is also $\hat{K}$-invariant, then the following hold.

1. Every smooth $\hat{K}$-invariant solution of $\mathcal{P}$ on M restricts to a smooth solution of the Dirichlet problem on N.
2. Suppose that u is a generalized solution to the Dirichlet problem on N (satisfying some minimal smoothness properties). Then u extends uniquely to a smooth $\hat{K}$-invariant solution of $\mathcal{P}$ on M.

It is worth noting that, in contrast to the extension from the hemisphere to the sphere defined for Neumann boundary conditions by (1.6), the extension for Dirichlet boundary conditions is defined by

$$u(\tau x) = -u(x) \quad \forall\, x \in H.$$

Even now, we have not formulated the ideas in their most general setting. For example, they apply to fourth order PDEs such as the von Kármán plate-buckling equations provided appropriate analogues of Neumann and Dirichlet boundary conditions are employed, and they also apply to equations in several variables with mixed Neumann and Dirichlet boundary conditions. However, we do not consider it worth elaborating a general theory beyond this point.

(b) The Proof of Lemma 5.4

Some of the results we describe are surely well known but, as we were unable to locate a suitable reference, it seemed worthwhile to provide details of the sometimes delicate proofs.

We continue with the notation and assumptions of Section 5(a). In particular, $\mathfrak{R} = \{\tau_1, \ldots, \tau_p\}$ will be an admissible set of reflections of M generating the group $K \cong \mathbf{Z}_2^p$ of isometries of M.

Lemma 5.8. *Let $\tau, \mu \in K$. Then*

$$\mu : \mathrm{Fix}(\tau) \to \mathrm{Fix}(\tau).$$

Proof. Suppose that $x \in \mathrm{Fix}(\tau)$. Since $\mu\tau = \tau\mu$, we have $\mu(x) = \mu(\tau(x)) = \tau(\mu(x))$. □

We recall that $M_K = \{x \in M \mid K_x = \{I_M\}\}$.

Lemma 5.9. *M_K is an open and dense subset of M and M_K/K is connected.*

Proof. We refer the reader to Chapter IV, Theorem 3.1, Bredon [3]. □

Lemma 5.10. *The number of connected components of M_K is less than or equal to 2^p.*

Proof. Let $M^\star = M/K$ denote the orbit space and $\pi : M \to M^\star$ the associated orbit map. We give $M^\star$ the quotient topology and note that since K is compact, π is a proper map ([3, Chapter I, Theorem 3.3].) Let N be any connected component of M_K. Since π is proper, it follows easily from Lemma 5.9 that $\pi(N) = M_K/K$. Let $x^\star \in M_K/K$. Since $\pi^{-1}(x^\star)$ is a K-orbit of points with trivial isotropy, we see that $\pi^{-1}(x^\star)$ contains 2^p distinct points. However, $x^\star$ lies in the π-image of every connected component of M_K and so M_K has at most 2^p connected components. □

Lemma 5.11.

(a) *K acts transitively on the set of connected components of M_K. In particular, M_K has precisely 2^p connected components.*

(b) *Given $1 \le i_1 < \cdots < i_s \le p$,*

$$\mathrm{Fix}(\tau_{i_1} \ldots \tau_{i_s}) = \bigcap_{j=1}^{s} \mathrm{Fix}(\tau_{i_j}).$$

(c) *If N is a connected component of M_K, then*

$$\partial N = \bigcup_{i=1}^{p} \overline{N} \cap \mathrm{Fix}(\tau_i).$$

Proof. Let $\mathcal{S}$ be a non-empty subset of $\mathfrak{R}$. Let $r = r(\mathcal{S})$ be the number of elements of $\mathcal{S}$. Let $K(\mathcal{S})$ denote the group generated by $\mathcal{S}$. Note that $K(\mathcal{S}) \cong \mathbf{Z}_2^r$. We shall prove by induction on $r = r(\mathcal{S})$ that statements (a,b,c) hold for $K(\mathcal{S})$, $1 \le r \le p$.

First, we remark that the case $r = 1$ is trivial. Suppose that we have proved the result for $r < n \le p$. Let $\mathcal{S} \subset \mathfrak{R}$, and suppose that $p(\mathcal{S}) = n$. Let

$$\mathcal{S} = \{\tau_{i_1}, \ldots, \tau_{i_n}\},$$

where $1 \leq i_1 < \cdots < i_n \leq p$. Let $\mathcal{S}'$ be the subset of $\mathcal{S}$ defined by removing one element of $\mathcal{S}$, say τ_{i_n}. Let K' denote the group generated by $\mathcal{S}'$ and K_n the group generated by $\mathcal{S}$. Let N' be a connected component of $M_{K'}$. We claim $N' \cap \mathrm{Fix}(\tau_{i_n}) \neq \emptyset$.

Suppose the contrary. It follows from Lemma 5.8 and (c) that for all $\tau \in K'$, $\tau N' \cap \mathrm{Fix}(\tau_{i_n}) = \emptyset$. Hence, $\mathrm{Fix}(\tau_{i_n}) \subset \bigcup_{j=1}^{n-1} \mathrm{Fix}(\tau_{i_j})$. Since $\mathrm{Fix}(\tau_{i_j})$ is an $(n-1)$-dimensional submanifold of M, $1 \leq j \leq n-1$, each connected component U of $\mathrm{Fix}(\tau_{i_n})$ will be contained in a component of some $\mathrm{Fix}(\tau)$, $\tau \in \mathcal{S}'$. Choosing a slice for the action of K_n at points of U, we find that $\mu = \tau\tau_{i_n}$ is equal to the identity map on a non-empty open subset of M. Since μ is an involution and M is connected, it follows that $\mu = I_M$. That is, $\tau = \tau_{i_n}$, contradicting our assumption that $\tau_{i_n} \notin \mathcal{S}'$.

Let $x \in \mathrm{Fix}(\tau_{i_n}) \cap N$. Choosing a slice for the action of K_n at x, we see there exist points $y \in N'$ such that $\tau_{i_n}(y) \in N'$ and $\tau_{i_n}(y) \neq y$. It follows that y and $\tau_{i_n}(y)$ lie in different connected components of $N' \setminus \mathrm{Fix}(\tau_{i_n})$. In particular, $N' \setminus \mathrm{Fix}(\tau_{i_n})$ has at least two connected components. Hence, $M_{K'} \setminus \mathrm{Fix}(\tau_{i_n})$ has at least 2^n connected components. However,

$$M_{K'} \setminus \mathrm{Fix}(\tau_{i_n}) \subset M_{K_n},$$

and so it follows from Lemma 5.10 that M_{K_n} has exactly 2^n connected components. Our inductive assumption (a) on K' implies that K' acts transitively on the connected components of $M_{K'}$. It follows that each component of $N' \setminus \mathrm{Fix}(\tau_{i_n})$ is a connected component of M_{K_n}. Hence, (b) is true for K_n. Finally, (c) follows trivially from (a) and (b) completing the inductive step. □

Lemma 5.12. *Let N be a connected component of M_K. Suppose that for some $x \in \partial N$ and $\tau \in K$ we have $\tau(x) \in \overline{N}$. Then $\tau(x) = x$.*

Proof. First, we note that if $x \in \partial N$ and $\tau(x) \in \overline{N}$, then, by the K-equivariance of τ, we must have $\tau(x) \in \partial N$.

We prove by induction on p. The result is trivial if $p = 1$. Now assume that the result holds for all groups K' generated by admissible sets of fewer than p reflections.

Let $x \in \partial N$ satisfy the hypotheses of the lemma. By Lemma 5.11(c), we may assume

$$x \in \bigcap_{j=1}^{s} \mathrm{Fix}(\tau_{i_j}),$$

where $1 \leq i_1 < \cdots < i_s \leq p$, $1 \leq s \leq p$. If $s = p$, the result is trivial. Suppose then that $s < p$. Either $\tau(x) = x$ or we may write $\tau(x) = \mu_1 \cdots \mu_k(x)$, where μ_i are distinct elements of $\mathcal{R} \setminus \{\tau_{i_1}, \ldots, \tau_{i_s}\}$ and $k < p$. Suppose the second condition holds. Applying the inductive hypothesis to the group K' generated by the μ_i we see, using Lemma 5.11, that if x lies in the closure of a connected component $N' \supset N$ of $M_{K'}$ then $\tau(x) \notin \overline{N}'$. A fortiori, $\tau(x) \notin \overline{N}$. Consequently, $\tau(x) = x$, completing the inductive step. □

We fix a connected component N of M_K and assume that M comes with a K-invariant Riemannian metric.

Lemma 5.13. *Every isometry of $\overline{N}$ extends uniquely to a K-equivariant isometry of M. Conversely, every K-equivariant isometry of M permutes the components of M_K.*

Proof. Let f be an isometry of $\overline{N}$. We extend f to M by setting

$$f \mid \tau(M) = \tau \cdot f \cdot \tau, (\tau \in K).$$

It follows from Lemma 5.12 that f is well defined and K-equivariant. In order to prove that f is a (smooth) isometry it suffices to show that f preserves the Riemannian distance $d(\,,\,)$ on M (see [11, Chapter I, Theorem 11.1]). Let $x, y \in M$, and let γ be a geodesic minimizing distance between x and y. Since the metric on M is assumed to be K-invariant, $\mathrm{Fix}(\tau_i)$ is a totally geodesic submanifold of M, $1 \leq i \leq p$. Hence, if $\tau \in \mathcal{R}$, Either γ meets $\mathrm{Fix}(\tau)$ in isolated points or γ is wholly contained within $\mathrm{Fix}(\tau)$. Let $z_1, \ldots, z_{k-1}$ denote the isolated points of intersection of γ with the fixed-point sets of the $\tau \in \mathcal{R}$, and order the z_i so that $d(x, z_{i+1}) > d(x, z_i)$, $i \geq 1$. Set $z_0 = x$, $z_k = y$. Then

$$d(x, y) = \sum_{i=0}^{k} d(z_i, z_{i+1}).$$

Since $f \mid \tau(\overline{N})$ is an isometry for all $\tau \in K$, we see that

$$d(f(x), f(y)) \leq \sum_{i=0}^{k} d(f(z_i), f(z_{i+1})) = d(x, y).$$

We claim we have equality. If not, we may reverse the argument and contradict the fact that γ was chosen to be a minimizing geodesic between x and y. Hence $d(f(x), f(y)) = d(x, y)$.

The converse of the lemma is trivial. □

(c) Smoothness of Extended Solutions

We begin our discussion of elliptic equations with some remarks.

Remarks 5.14.

1. We refer the reader to Gilbarg and Trudinger [7] and Ladyzhenskaya and Ural'tseva [14] for the theory of second order quasilinear elliptic operators. Our viewpoint is such that we do not need to be concerned with the (difficult) questions of existence or regularity for such operators. Consequently, in what follows we generally assume sufficient regularity of solutions. The reader whose main interest lies in reaction-diffusion equations may prefer to assume that $\mathcal{P}$ is a reaction-diffusion operator of the type described in equation (5.3).
2. The ellipticity assumption we make on $\mathcal{P}$ is weak. We do not need to assume uniform ellipticity (see [7, Chapter 9, p. 203], [14, Chapter 4, (2.1)]). Indeed, for our applications it suffices that if u is a solution satisfying appropriate regularity (see below), then $\mathcal{P}$ is elliptic with respect to u (see [7, Chapter 9, p. 203]).

For the remainder of the section we shall assume that $\mathcal{P}$ is a second order quasi-linear elliptic differential operator on M which is in divergence form. Since $\mathcal{P}$ is in divergence form, we may define generalized solutions of $\mathcal{P}(u) = 0$ having derivatives of the first order only (see [7], [14]). Let $C^{1,1}(M)$ denote the space of C^1 real-valued functions on M with Lipschitz continuous derivative. We similarly define $C^{1,1}(\overline{N})$.

Lemma 5.15. *Let $u \in C^{1,1}(M)$ be a generalized solution of $\mathcal{P}$. Then $u \in C^{\infty}(M)$.*

Proof. The result follows from the standard "bootstrapping" argument of nonlinear elliptic equations and uses only the ellipticity of $\mathcal{P}$ with respect to u. For details see [14, Chapter 4, Section 6] or [7, Chapter 8]. □

Remark 5.16. If $\mathcal{P}$ is of the form (5.3), it suffices that $u \in C^1(M)$ for the conclusion of Lemma 5.15 to hold.

Definition 5.17. Suppose that $u \in C^{1,1}(\overline{N})$ is a generalized solution of $\mathcal{P}$ on N.

1. We say that u satisfies Neumann boundary conditions (NBC) on N if for every $\tau \in \mathcal{R}$ and all $x \in \partial N \cap \mathrm{Fix}(\tau)$, we have
$$\frac{\partial u}{\partial n}(x) = 0,$$
where n is the normal direction to $\mathrm{Fix}(\tau)$ at x.
2. We say that u satisfies Dirichlet boundary conditions (DBC) on N if $u \equiv 0$ on ∂N.

In case (1) (respectively, (2)), we say u is a generalized solution to the Neumann problem (respectively, Dirichlet problem) defined by $\mathcal{P}$ on N. We omit the prefix "generalized" if u is of class at least C^2 and is a classical solution.

Theorem 5.18. *Suppose that the operator $\mathcal{P}$ is K-invariant. Then the following hold.*

1. *Every smooth K-invariant solution u of $\mathcal{P}$ on M restricts to a smooth solution of the Neumann problem for $\mathcal{P}$ on N.*
2. *Let $u \in C^{1,1}(\overline{N})$ be a generalized solution to the Neumann problem for $\mathcal{P}$ on $\overline{N}$. Then*
   (a) *u is smooth.*
   (b) *u extends uniquely to a smooth K-invariant solution of $\mathcal{P}$ on M.*

Proof of (1). Suppose that $u \in C^{\infty}(M)^K$. Let τ be one of the generators of K, and let $x \in \partial N \cap \mathrm{Fix}(\tau)$. It suffices to show that $\partial u/\partial n(x) = 0$, where n is the normal direction to $\mathrm{Fix}(\tau)$ at x. Choosing a slice for the action of τ at x (see Bredon [3, Chapter 6]), we may choose a local coordinate system $(x_1, \dots, x_n)$ at x such that
$$\tau(x_1, \dots, x_{n-1}, x_n) = (x_1, \dots, x_{n-1}, -x_n)$$

and $(0, \dots, 0, 1)$ defines the normal direction to $\mathrm{Fix}(\tau)$ at x. Since u is assumed K-invariant, u is τ-invariant, so

$$u(x_1, \dots, x_{n-1}, x_n) = u(x_1, \dots, x_{n-1}, -x_n).$$

It follows that $\partial u/\partial x_n(0) = 0$.

Proof of (2). Suppose that $u \in C^{1,1}(\overline{N})$ is a solution to $\mathcal{P}$ on $\overline{N}$ satisfying NBC. We may extend u to M by defining

$$u(\tau x) = u(x)$$

for all $\tau \in K$ and $x \in \overline{N}$. It follows from Lemma 5.12 that u is well defined on M. Obviously, u is C^1 and K-invariant. Moreover, it is easy to check that $u \in C^{1,1}(M)$. Since $\mathcal{P}$ is K-invariant, it follows that u is a generalized solution to $\mathcal{P}$ on M. By Lemma 5.15, $u \in C^\infty(M)$ and so the restriction of u to $\overline{N}$ is also smooth. □

Next, we wish to give a version of Theorem 5.18 appropriate for the study of the Dirichlet problem on N.

Theorem 5.19. *Suppose that $\mathcal{P}$ is $\hat{K}$-invariant. Then the following hold:*

1. *Every smooth $\hat{K}$-invariant solution of $\mathcal{P}$ on M restricts to a smooth solution of the Dirichlet problem on N.*
2. *Suppose that $u \in C^{1,1}(\overline{N})$ is a generalized solution to the Dirichlet problem on N. Then*
   (a) *u is smooth.*
   (b) *u extends to a smooth $\hat{K}$-invariant solution of $\mathcal{P}$ on M.*

Proof. The proof is similar to that of Theorem 5.18 except that for the proof of statement (2) we take the $\hat{K}$-invariant extension of u to M. □

(d) Multiplicity of Solutions

Theorems 5.18, 5.19 hold with minimal assumptions on the symmetry properties of the operator $\mathcal{P}$. We now wish to investigate what happens when $\mathcal{P}$ satisfies additional invariance properties. For simplicity, we concentrate on the Neumann problem and assume $\mathcal{P}$ is invariant by $\mathrm{ISO}(M)$. However, we may similarly consider the Dirichlet problem and/or suppose that $\mathcal{P}$ has invariance properties with respect to a compact Lie *subgroup* of isometries of M. We emphasize in what follows that we always assume $\mathrm{ISO}(M) \supset K$. In particular, following Lemma 5.13, we regard $\mathrm{ISO}(N)$ as a closed subgroup of $\mathrm{ISO}(M)$.

Theorem 5.20. *Let $\mathcal{P}$ be $\mathrm{ISO}(M)$-invariant. Suppose that u is a smooth solution of $\mathcal{P} = 0$ with isotropy group Σ. Let*

$$\Lambda_u = \{\gamma \in \mathrm{ISO}(M) \mid \gamma\Sigma\gamma^{-1} \supset K\}.$$

1. *If $\gamma \in \Lambda_u$, then $u_\gamma = \gamma(u) \mid \overline{N}$ is a smooth solution of the Neumann problem for $\mathcal{P}$ on N.*
2. *Suppose $\gamma, \phi \in \Lambda_u$. Then u_γ lies on the* ISO(N)*-orbit of ϕ if and only if there exists $h \in$* ISO(N) *and $\sigma \in \Sigma_u$ such that $\gamma = h\phi\sigma$. In particular, if $\Sigma \supset K$, then u_γ lies on the* ISO(N)*-orbit of u if and only if $\gamma \in$* ISO(N) $\cdot$ *u.*

Proof of (1). Since $\gamma\Sigma\gamma^{-1} \supset K$ and the isotropy group of the solution $\gamma(u)$ is $\gamma\Sigma\gamma^{-1}$, we see that $\gamma(u)$ is a K-invariant solution of $\mathcal{P}$. It follows by Theorem 5.18(1) that $\gamma(u)$ restricts to a smooth solution of the Neumann problem for $\mathcal{P}$ on N.

Proof of (2). Suppose that $\gamma = h\phi\sigma$, where $h \in$ ISO(N) and $\sigma \in \Sigma_u$. Then $\gamma(u) = h\phi(u)$ (since $\sigma(u) = u$) and so $u_\gamma = hu_\phi$. Conversely, suppose that $u_\gamma = hu_\phi$, for some $h \in$ ISO(N). Then $\phi^{-1}h^{-1}\gamma \in \Sigma_u$. Hence, there exists $\sigma \in \Sigma_u$ such that $\phi^{-1}h^{-1}\gamma = \sigma$. Thus, $\gamma = h\phi\sigma$. □

We should emphasize the implication of (2) of Theorem 5.20. The statement implies that a group orbit of solutions for $\mathcal{P}$ on M can yield two or more *different* group orbits of solutions for the Neumann problem on N.

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