JOURNAL OF DIFFERENTIAL EQUATIONS **133**, 30–48 (1997)
ARTICLE NO. DE963201

An Example of Symmetry Breaking to Heteroclinic Cycles

Chuanze Hou and Martin Golubitsky

Department of Mathematics, University of Houston, Houston, Texas 77204-3476

Received August 9, 1995

1. INTRODUCTION

Lauterbach and Roberts [15] showed that when symmetry breaking terms are added to an equivariant differential equation with a group orbit of equilibria, heteroclinic cycles connecting equilibria on the perturbed group orbit may result.

More precisely, let $\Gamma \subset \mathbf{O}(\mathbf{n})$ be a Lie group acting on $\mathbf{R}^n$ and let

$$\dot{z} = f(z) \tag{1.1}$$

be a Γ-equivariant system of differential equations; that is,

$$f(\gamma z) = \gamma f(z)$$

for all $\gamma \in \Gamma$. Suppose that (1.1) has an equilibrium at z_0. Then equivariance implies that the manifold

$$X_0 = \Gamma z_0$$

is a group orbit of equilibria. We assume that this group orbit is normally hyperbolic; indeed, we assume that X_0 is orbitally asymptotically stable.

Suppose now that we consider a small system symmetry breaking perturbation of (1.1),

$$\dot{z} = f(z) + \varepsilon g(z), \tag{1.2}$$

where ε is small and g is only Δ-equivariant where $\Delta \subset \Gamma$ is a Lie subgroup. When ε is sufficiently small, normal hyperbolicity guarantees that there is a (perturbed) flow invariant manifold X_ε for the perturbed system (1.2) which is diffeomorphic to X_0 [9]. However, the dynamics of the perturbed flow on X_ε need not consist only of equilibria.

Indeed, when $\dim \Delta < \dim \Gamma$ the dynamics on the perturbed orbit X_ε will generally be more complicated than just consisting of equilibria. Lauterbach and Roberts [15] show that, depending on the pair Γ and Δ,

0022-0396/97 $25.00
Copyright © 1997 by Academic Press, Inc.
All rights of reproduction in any form reserved.

certain equilibria may be forced to occur on X_ε. In addition, one-dimensional flow invariant sets connecting these equilibria may be forced by the residual symmetry Δ, so that generically heteroclinic cycles connecting the equilibria on X_ε can be forced. Lauterbach and Roberts [15] give an example where heteroclinic cycles are forced by system symmetry breaking. In this example $\Gamma = \mathbf{O}(3)$ and $\Delta = \mathbb{T}$ (the group of symmetries of the tetrahedron). More recently, Lauterbach *et al.* [13, 14] have classified all pairs Γ and Δ which may result in heteroclinic cycles when Γ is either $\mathbf{O}(3)$ or $\mathbf{SO}(3)$ and Δ is any proper Lie subgroup. This investigation is completed when $\mathbf{R}^n$ is any of the natural irreducible representations of $\mathbf{O}(3)$.

We follow the work by Hou in [10] and continue this line of investigation by constructing an example where heteroclinic cycles are forced by system symmetry breaking which is simpler than those of Lauterbach and co-workers. In our example $\Gamma = \mathbf{D}_4 \dot{+} \mathbf{T}^2$, $\Delta = \mathbf{D}_2$, and $n = 4$. These groups occur when studying bifurcations of spatially periodic solutions to planar Euclidean equivariant systems of PDEs on a square lattice (see [4]).

Specifically, we write $\mathbf{R}^4 \cong \mathbf{C}^2$ and use complex coordinates $z = (z_1, z_2) \in \mathbf{C}^2$. The action of $\mathbf{D}_2$ on $\mathbf{C}^2$ is generated by the reflections

$$\kappa_1 \cdot (z_1, z_2) = (\bar{z}_1, z_2)$$
$$\kappa_2 \cdot (z_1, z_2) = (z_1, \bar{z}_2).$$

The action of $\mathbf{D}_4 \dot{+} \mathbf{T}^2$ on $\mathbf{C}^2$ is generated by κ_1 and

$$\kappa \cdot (z_1, z_2) = (z_2, z_1)$$
$$(\theta, \phi) \cdot (z_1, z_2) = (e^{i\theta} z_1, e^{i\phi} z_2)$$

where $(\theta, \phi) \in \mathbf{T}^2$.

The idea behind the heteroclinic cycle that we produce is quite simple. Suppose that there is an equilibrium to the unperturbed system (1.1) of the form $z_0 = (\mu, \mu)$ where $\mu > 0$ is real. We call such an equilibrium a *mixed mode* solution. Note that the isotropy subgroup of a mixed mode solution is the group $\mathbf{D}_4$ generated by κ and κ_1. Thus, the group orbit X_0 through z_0 is diffeomorphic to the 2-torus $\mathbf{T}^2$. The perturbed flow invariant manifold X_ε is invariant under the remaining symmetry $\mathbf{D}_2$. It is this residual symmetry that gives structure to the dynamics on X_ε. Since $X_\varepsilon \cong X_0$, we can use X_0 as a model for the restrictions on the perturbed dynamics given by $\Delta = \mathbf{D}_2$ symmetry.

In particular, the fixed-point subsets in X_0 of subgroups of Δ are flow invariant for the perturbed system. These fixed-point subsets are defined as follows. Let $\Upsilon \subset \Delta$ be a subgroup. Then

$$\mathrm{Fix}^0(\Upsilon) = \{z \in X_0 : vz = z \ \forall v \in \Upsilon\}.$$

It is easy to compute the action of $\mathbf{D}_2$ on $X_0 \cong \mathbf{T}^2$. Indeed, in the standard coordinates of $\mathbf{T}^2$, we have that

$$\kappa_1(\theta, \phi) = (-\theta, \phi)$$
$$\kappa_2(\theta, \phi) = (\theta, -\phi).$$

The fixed-point subsets of subgroups of $\mathbf{D}_2$ acting on X_0 are

$$\mathrm{Fix}^0(\kappa_1) = \{(0, \phi)\} \cup \{(\pi, \phi)\}$$
$$\mathrm{Fix}^0(\kappa_2) = \{(\theta, 0)\} \cup \{(\theta, \pi)\}$$
$$\mathrm{Fix}^0(\mathbf{D}_2) = \{(0, 0), (\pi, 0), (0, \pi), (\pi, \pi)\}.$$

So the fixed-point subset for each κ_j is the disjoint union of two circles; the fixed-point subset of $\mathbf{D}_2$ consists of four points, which we label A, B, C, D.

Thus the flow of the perturbed system on X_ε must have four equilibria (the points in $\mathrm{Fix}^0(\mathbf{D}_2)$). We call these four points the $\mathbf{D}_2$-*equilibria*. See Fig. 1. Similarly, symmetry forces the existence of the one-dimensional invariant circles $\mathrm{Fix}^0(\kappa_1)$ and $\mathrm{Fix}^0(\kappa_2)$ connecting these equilibria. If we can show that each of the equilibria are saddles with inflow and outflow directions as noted in Fig. 1 and if we can show that there are no other equilibria on these one-dimensional flow invariant manifolds, then

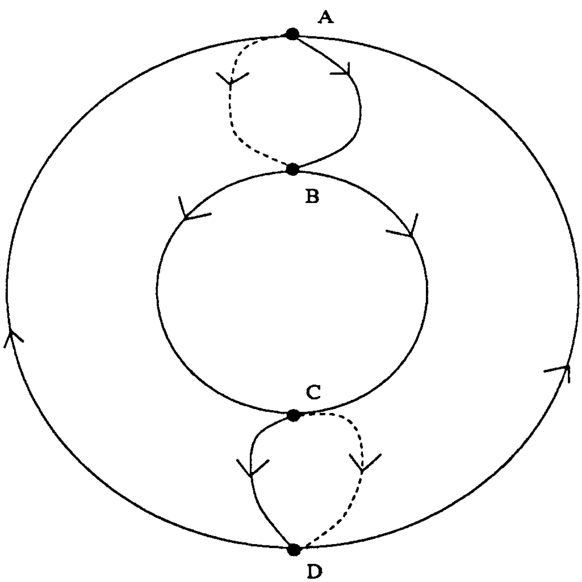

FIG. 1. Fixed-point subsets on the 2-tours X_ε.

we will have proved the existence of a heteroclinic cycle connecting the $\mathbf{D}_2$-equilibria.

To establish the existence of heteroclinic cycles of this form, we must prove the existence of orbitally stable mixed mode solutions in (1.1) and then show that there are perturbation terms $g(z)$, as in (1.2), such that the $\mathbf{D}_2$-equilibria are saddles and that there are no other equilibria.

We use equivariant bifurcation theory in the presence of $\mathbf{D}_4 \dot{+} \mathbf{T}^2$ symmetry to establish the existence of orbitally stable mixed mode equilibria. That is, we assume that the vector field f in (1.1) depends on a bifurcation parameter λ

$$\dot{z} = f(z, \lambda). \tag{1.3}$$

We show that under certain easily verifiable conditions on the lower order terms of f in (1.3), there exists a branch of orbitally asymptotically stable mixed mode equilibria. We then show that if the first derivatives of g at the origin satisfies certain inequalities, then for all fixed λ sufficiently near zero and all sufficiently small ε (depending on λ) the perturbed system has the dynamics shown in Fig. 1. That is, there exists a structurally stable, asymptotically stable, heteroclinic cycle.

Once we have established these conditions, we also show that the existence of this cycle depends only on the existence of the $\mathbf{D}_4 \dot{+} \mathbf{T}^2$-equivariant bifurcation to mixed mode solutions. That is, there exist an open set of perturbation terms g that force the existence of the desired cycle. We can then use these results to prove the existence of heteroclinic cycles in the dynamics of a reaction-diffusion system. See Hou [10].

In 1980 Field [6] proved that heteroclinic cycles appear generically in symmetric systems. More recently, such cycles have been shown to exist through spontaneous symmetry breaking where the symmetry of the equations never changes [8, 2, 16]. A discussion of heteroclinic cycles that are produced both by spontaneous symmetry breaking and by system symmetry breaking is given in Krupa [12]. Dynamics other than cycles is also observed through system symmetry breaking (cf. [1, 3]).

We end this section by outlining the structure of this paper. The existence and stability of mixed mode solutions obtained by bifurcation is discussed in the next section. In Section 3 we describe the perturbations g in (1.2) and present our main result (Theorem 3.2) on the existence of asymptotically stable heteroclinic cycles. The proof of this theorem is given in the next three sections. We discuss the stability of the saddles that make up the cycle in Section 4 and the asymptotic stability of the cycle itself in Section 6. The nonexistence of additional equilibria on the one-dimensional invariant manifolds that connect the saddles is established in Section 5. An example is discussed in Section 7.

TABLE I

Isotropy Subgroups of $\mathbf{D}_4 \dot{+} \mathbf{T}^2$ Acting on $\mathbf{C}^2$

Orbit representative	Isotropy subgroup	Fixed-point subspace	Dimension
$(0, 0)$	$\mathbf{D}_4 \dot{+} \mathbf{T}^2$	$(0, 0)$	0
$(x, 0), x \in \mathbf{R}$	$\mathbf{D}_2 \dot{+} \mathbf{S}^1$	$(x, 0)$	1
$(x, x), x \in \mathbf{R}$	$\mathbf{D}_4$	(x, x)	1
$(x, y), x, y \in \mathbf{R}$	$\mathbf{D}_2$	(x, y)	2

2. MIXED MODE SOLUTIONS

The group action of $\mathbf{D}_4 \dot{+} \mathbf{T}^2$ acting on $\mathbf{C}^2$ has the four (conjugacy classes of) isotropy subgroups listed in Table I. We call the points with isotropy subgroup $\mathbf{D}_2 \dot{+} \mathbf{S}^1$ *pure modes*; as noted previously, we call the points with isotropy subgroup $\mathbf{D}_4$ *mixed modes*.

The general form of the $\mathbf{D}_4 \dot{+} \mathbf{T}^2$-equivariant mapping is (see [7, 17])

$$f(z_1, z_2, \lambda) = (A(|z_1|^2, |z_2|^2, \lambda)\, z_1, A(|z_2|^2, |z_1|^2, \lambda)\, z_2), \tag{2.1}$$

where $A\colon \mathbf{R}^2 \times \mathbf{R} \to \mathbf{R}$. Assume $A_\lambda(0, 0, 0) \neq 0$. By rescaling λ we can assume

$$A(|z_1|^2, |z_2|^2, \lambda) = a\,|z_1|^2 + b\,|z_2|^2 - \lambda + \cdots,$$

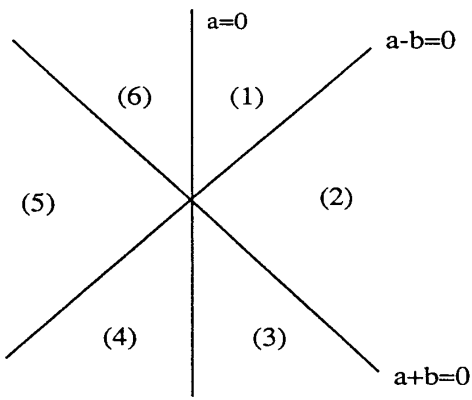

FIG. 2. Region division.

where $a, b \in \mathbf{R}$ and $\cdots$ stands for high order terms. Let

$$K(z, \lambda) = ((a\,|z_1|^2 + b\,|z_2|^2 - \lambda)\, z_1, (b\,|z_1|^2 + a\,|z_2|^2 - \lambda)\, z_2) \qquad (2.2)$$

be the third-order truncation of $f(z, \lambda)$. We call $K(z, \lambda)$ the *normal form* of $f(z, \lambda)$. If K satisfies certain nondegeneracy conditions, then the equilibria of the vector field $f(z, \lambda)$ and the linear stability of these equilibria are determined by K. More precisely:

DEFINITION 2.1. A $\mathbf{D}_4 \dot{+} \mathbf{T}^2$-equivariant vector field $f(z, \lambda)$ is nondegenerate if $a \neq 0$ and $a \neq \pm b$.

THEOREM 2.2. *A nondegenerate $\mathbf{D}_4 \dot{+} \mathbf{T}^2$-equivariant vector field $f(z, \lambda)$ is $\mathbf{D}_4 \dot{+} \mathbf{T}^2$-equivalent to $K(z, \lambda)$.*

Theorem 2.2 is proved in [17] and follows from [7] Proposition X2.3. The stability result follows from results in [11]. It follows that the two parameters a and b in (2.2) determine the branches of solutions of f near the bifurcation point and their orbital stability. Note that these nondegeneracy conditions divide the ab-plane into six regions, as shown in Fig. 2.

To simplify the computations, we assume that $f(z, \lambda)$ is in normal form. The method we use to discuss the branching and stability of solutions of the $\mathbf{D}_4 \dot{+} \mathbf{T}^2$-equivariant vector field K can also be used to discuss the same issues for the nondegenerate $\mathbf{D}_4 \dot{+} \mathbf{T}^2$-equivariant vector field f. We show later that the results for general f depend only on the third-order truncation K.

Now we solve $K(z, \lambda) = 0$ explicitly. There are three different solutions which are distinguished by their isotropy subgroups: the trivial, pure mode, and mixed mode solutions. See Table II for the branching equation.

To discuss the orbital stability of these solutions, we need only consider the restricted equations on $\mathrm{Fix}(\mathbf{D}_2)$. The $\mathbf{T}^2$ action guarantees that the two

TABLE II

Linear Stability

Solution	Subgroup	Subspace	Equation	Eigenvalues
Trivial	$\mathbf{D}_4 \dot{+} \mathbf{T}^2$	$(0, 0)$	$z_1 = 0 = z_2$	$-\lambda$ (twice)
Pure mode	$\mathbf{D}_2 \dot{+} \mathbf{S}^1$	$(x, 0)$	$\lambda = ax^2$	$2ax^2, (b-a)x^2$
Mixed mode	$\mathbf{D}_4$	(x, x)	$\lambda = (a+b)x^2$	$2x^2(a \pm b)$

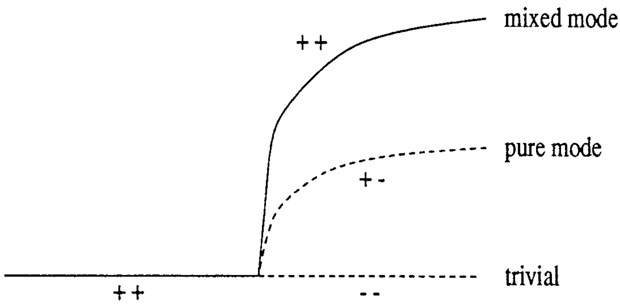

FIG. 3. Bifurcation diagram in region (2).

eigenvalues in directions transverse to $\mathrm{Fix}(\mathbf{D}_2)$ are zero. The restricted equations satisfy

$$(ax_1^2+bx_2^2-\lambda)\,x_1=0$$

$$(bx_1^2+ax_2^2-\lambda)\,x_2=0.$$

The linearization on $\mathrm{Fix}(\mathbf{D}_2)$ is

$$(dK)|_{\mathrm{Fix}(\mathbf{D}_2)}=\begin{pmatrix} 3ax_1^2+bx_2^2-\lambda & 2bx_1x_2 \\ 2bx_1x_2 & bx_1^2+3ax_2^2-\lambda \end{pmatrix}.$$

The computation of the eigenvalues of $(dK)|_{\mathrm{Fix}(\mathbf{D}_2)}$ is summarized in Table II.

Remark 2.3. We are interested only in bifurcations that produce orbitally stable mixed mode solutions. Stable mixed modes occur only in region (2) where $a>|b|$. In this region pure mode solutions are orbitally unstable, though both branches bifurcate supercritically. See Fig. 3.

3. THE PERTURBED BIFURCATION PROBLEM

In this section, we study the form of ODEs on C^2 that break symmetry from $\mathbf{D}_4 \dotplus \mathbf{T}^2$ to $\mathbf{D}_2$.

Assuming nondegeneracy conditions, the dynamics of the $\mathbf{D}_4 \dotplus \mathbf{T}^2$-equivariant vector field $f(z,\lambda)$ is well understood via singularity theory and local bifurcation theory [7]. The preceding section shows that when certain conditions on the coefficients up to the third-order terms of the Taylor expansion of $f(z,\lambda)$ are satisfied, the $\mathbf{D}_4 \dotplus \mathbf{T}^2$-equivariant vector field has an orbitally stable mixed mode solution which is a 2-torus, and this 2-torus is normally hyperbolic. We denote this 2-torus by X_0.

Now we perturb the vector field $f(z, \lambda)$ by the $\mathbf{D}_2$ equivariant mapping $\varepsilon g(z)$, where ε is small and $g(0)=0$. Let F: $C^2 \times \mathbf{R}^2 \to C^2$ be defined by

$$F(z, \lambda, \varepsilon) = f(z, \lambda) + \varepsilon g(z). \tag{3.1}$$

We consider the system of ODEs:

$$\frac{dz}{dt} + F(z, \lambda, \varepsilon) = 0. \tag{3.2}$$

Normal hyperbolicity implies that for fixed $\lambda > 0$ but near 0 and for small ε, (3.2) has a flow invariant 2-torus X_ε which is a perturbation of X_0.

In Section 2 we assumed that $f(z, \lambda)$ is in normal form. To simplify the computations, we make this assumption here. Thus we consider the following system of ODEs:

$$\frac{dz_1}{dt} + (a\,|z_1|^2 + b\,|z_2|^2 - \lambda)\, z_1 + \varepsilon g_1(z_1, z_2) = 0 \tag{3.3}$$

$$\frac{dz_2}{dt} + (b\,|z_1|^2 + a\,|z_2|^2 - \lambda)\, z_2 + \varepsilon g_2(z_1, z_2) = 0. \tag{3.4}$$

Here $g = (g_1, g_2)$ in coordinates. We assume that $a > |b|$ is valid, so that the mixed mode solution is orbitally stable. Note that since $g(0) = 0$, the trivial solution of $K(x, \lambda) = 0$ is still a solution in the perturbed system (3.2).

Remark 3.1. It can be checked that:

(a) a Hilbert basis of the $\mathbf{D}_2$-invariant functions is: $z_1 + \bar{z}_1$, $z_2 + \bar{z}_2$, $z_1\bar{z}_1$, $z_2\bar{z}_2$;

(b) a basis of the module of $\mathbf{D}_2$-equivariant mappings is: $(1, 0)$, $(0, 1)$, $(z_1, 0)$, $(0, z_2)$.

By Remark 3.1 we can rewrite the perturbation as:

$$\begin{aligned} g_1 &= a_1(z_1 + \bar{z}_1, z_2 + \bar{z}_2, z_1\bar{z}_1, z_2\bar{z}_2)\, z_1 + b_1(z_1 + \bar{z}_1, z_2 + \bar{z}_2, z_1\bar{z}_1, z_2\bar{z}_2) \\ g_2 &= a_2(z_1 + \bar{z}_1, z_2 + \bar{z}_2, z_1\bar{z}_1, z_2\bar{z}_2)\, z_2 + b_2(z_1 + \bar{z}_1, z_2 + \bar{z}_2, z_1\bar{z}_1, z_2\bar{z}_2) \end{aligned} \tag{3.5}$$

where a_1, a_2, b_1 and b_2 are maps from $\mathbf{R}^4$ to $\mathbf{R}$ and $b_1(0) = b_2(0) = 0$. Note that the form of g implies that

$$g_{1, z_2}(0, 0) = g_{1, \bar{z}_2}(0, 0) \qquad \text{and} \qquad g_{2, z_1}(0, 0) = g_{2, \bar{z}_1}(0, 0) \tag{3.6}$$

though these equalities could have been obtained directly using the κ_1 and κ_2 symmetries.

We now state our main theorem. Define

$$K_1 = g_{1,\bar{z}_1}(0,0) + g_{1,\bar{z}_2}(0,0)$$
$$K_2 = g_{1,\bar{z}_1}(0,0) - g_{1,\bar{z}_2}(0,0)$$
$$L_1 = g_{2,\bar{z}_2}(0,0) + g_{2,\bar{z}_1}(0,0)$$
$$L_2 = g_{2,\bar{z}_2}(0,0) - g_{2,\bar{z}_1}(0,0).$$

THEOREM 3.2. *Consider the system of ODEs* (3.3) *and* (3.4) *and assume that*:

$$a > |b| \tag{3.7}$$

$$\operatorname{sgn}(K_2) = -\operatorname{sgn}(K_1);\ \operatorname{sgn}(L_1) = -\operatorname{sgn}(K_1);\ \operatorname{sgn}(L_2) = \operatorname{sgn}(K_1). \tag{3.8}$$

Then for each fixed small $\lambda > 0$ and for every sufficiently small nonzero ε, there exist structurally stable heteroclinic cycles in (3.2) *connecting the* $\mathbf{D}_2$*-equilibria. When*

$$\operatorname{sgn}(\varepsilon) = \operatorname{sgn}(L_1 K_2 - K_1 L_2)\operatorname{sgn}(K_1) \tag{3.9}$$

the heteroclinic cycle is asymptotically stable.

Note that (3.8) implies:

$$|g_{1,\bar{z}_2}(0,0| < |g_{1,\bar{z}_1}(0,0)| \qquad \text{and} \qquad |g_{2,\bar{z}_1}(0,0| > |g_{2,\bar{z}_2}(0,0)|. \tag{3.10}$$

As we have seen (3.7) establishes the existence of orbitally stable 2-tori of mixed modes in the unperturbed bifurcation problem when $\lambda > 0$. In the

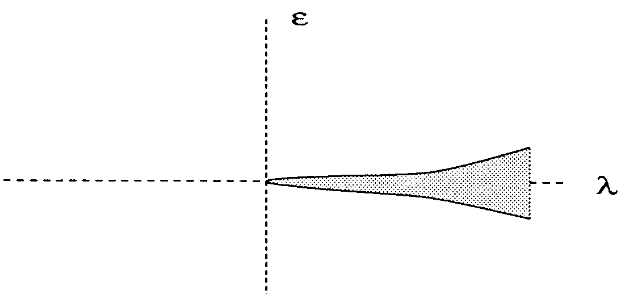

FIG. 4. Region of existence of heteroclinic cycles in the $\lambda\varepsilon$ parameter plane.

next section, we use (3.8) to show that the $\mathbf{D}_2$-equilibria are saddles with inflow and outflow manifolds consistent with a cycle. In Section 5, we use (3.10) to prove that there are no additional equilibria that would block the existence of the cycle. In Section 6 we use (3.9) to establish the asymptotic stability of the cycle. It is worth emphasizing that the region in the $\lambda\varepsilon$ parameter plane where the existence of the heteroclinic cycles is asserted is very small. See Fig. 4.

4. STABILITY OF PERTURBED EQUILIBRIA

For fixed $\lambda > 0$ and for ε small, (3.2) has four equilibria A, B, C, D—the $\mathbf{D}_2$-equilibria. In the unperturbed equation, the mixed mode equilibria that are also in $\mathrm{Fix}^0(\mathbf{D}_2)$ are $(\pm\mu, \pm\mu)$ where $\mu = \sqrt{\lambda/(a+b)}$. For the purpose of our discussion we set, when $\varepsilon = 0$,

$$A = (\mu, \mu) \quad B = (\mu, -\mu) \quad C = (-\mu, -\mu) \quad D = (-\mu, \mu).$$

The $\mathbf{D}_2$-equilibria are parameterized by the system symmetry breaking parameter ε. In our discussion, we assume that these equilibria are parameterized explicitly by $(\alpha(\varepsilon), \beta(\varepsilon))$.

Orbital stability of the mixed mode solution to the unperturbed problem guarantees that these four equilibria are stable in directions transverse to the 2-torus X_ε; that is, in directions in the subspace $\mathrm{Fix}_{\mathbf{C}^2}(\mathbf{D}_2)$. To prove the existence of a heteroclinic cycle on X_ε, we need to compute the eigenvalues corresponding to eigenvectors of dF in directions tangent to X_ε. Using the group structure, this computation can be done in a straightforward manner. The $\mathbf{D}_2$-isotypic decomposition of $\mathbf{C}^2$ is

$$\mathbf{C}^2 = \mathbf{V}_0 \oplus \mathbf{V}_1 \oplus \mathbf{V}_2,$$

where

$$\mathbf{V}_0 = \mathrm{Fix}(\mathbf{D}_2) = \{(x_1, x_2) : x_1, x_2 \in \mathbf{R}\}$$
$$\mathbf{V}_1 = \{(y_1 i, 0), y_1 \in \mathbf{R}\}$$
$$\mathbf{V}_2 = \{(0, y_2 i) : y_2 \in \mathbf{R}\}.$$

Observe that $\mathbf{V}_1$ and $\mathbf{V}_2$ are tangent to X_ε at each of the $\mathbf{D}_2$-equilibria. Since the Jacobian dF commutes with κ_1 and κ_2, it follows that $e_1 = (i, 0) \in \mathbf{V}_1$ and $e_2 = (0, i) \in \mathbf{V}_2$ are eigenvectors of dF. We denote the eigenvalues of dF in the e_1 and e_2 directions by

$$\sigma_1(\varepsilon) \qquad \text{and} \qquad \sigma_2(\varepsilon),$$

respectively. In this section we prove:

TABLE III

Eigenvalues in the Tangent Direction

Equilibria	$\operatorname{sgn}(\sigma_1(\varepsilon))$	$\operatorname{sgn}(\sigma_2(\varepsilon))$
A	$-\operatorname{sgn}(K_1\varepsilon)$	$-\operatorname{sgn}(L_1\varepsilon)$
B	$-\operatorname{sgn}(K_2\varepsilon)$	$-\operatorname{sgn}(L_2\varepsilon)$
C	$-\operatorname{sgn}(K_1\varepsilon)$	$-\operatorname{sgn}(L_1\varepsilon)$
D	$-\operatorname{sgn}(K_2\varepsilon)$	$-\operatorname{sgn}(L_2\varepsilon)$

THEOREM 4.1. *For fixed small λ and for small ε* (3.8) *are necessary and sufficient conditions for proving that the* $\mathbf{D}_2$*-equilibria on X_ε have inflow and outflow directions that are consistent with having a heteroclinic cycle.*

Proof. This theorem is proved by showing that the signs of the eigenvalues at the $\mathbf{D}_2$-equilibria are determined by the entries in Table III. The necessary and sufficient condition that each $\mathbf{D}_2$-equilibria is a saddle is

$$\sigma_1(\varepsilon)\,\sigma_2(\varepsilon) < 0.$$

From Table III, we can see that when (3.8) is valid, then for equilibria A and C

$$\operatorname{sgn}(\sigma_1(\varepsilon)\,\sigma_2(\varepsilon)) = \operatorname{sgn}(K_1 L_1) = -1,$$

and for equilibria B and D,

$$\operatorname{sgn}(\sigma_1(\varepsilon)\,\sigma_2(\varepsilon)) = \operatorname{sgn}(K_2 L_2) = -1.$$

In addition, the outflow direction from point A must be the inflow direction of point B and the inflow direction of point A must be the outflow direction of point D (or conversely). See Fig. 1. The remainder of this section is devoted to verifying the entries in this table. ∎

Remark 4.2. If one of the inequalities in (3.8) is invalid, then there must exist other equilibria on the invariant circles.

LEMMA 4.3. *At the* $\mathbf{D}_2$*-equilibria*

$$\sigma_1(\varepsilon) = F^1_{z_1} - F^1_{\bar{z}_1}$$

$$\sigma_2(\varepsilon) = F^2_{z_2} - F^2_{\bar{z}_2}$$

where the derivatives are evaluated at $(z_1, z_2) = (\alpha(\varepsilon), \beta(\varepsilon))$.

Proof. Using complex coordinates, we see that for every $\xi, \eta \in \mathbf{C}$

$$(dF)(\xi, \eta) = \begin{pmatrix} F^1_{z_1}\xi + F^1_{\bar{z}_1}\bar{\xi} + F^1_{z_2}\eta + F^1_{\bar{z}_2}\bar{\eta} \\ F^2_{z_1}\xi + F^2_{\bar{z}_1}\bar{\xi} + F^2_{z_2}\eta + F^2_{\bar{z}_2}\bar{\eta} \end{pmatrix}.$$

The commutativity condition

$$(dF)\cdot\kappa_1 = \kappa_1 \cdot (dF)$$

implies that

$$\begin{pmatrix} F^1_{z_1}\bar{\xi} + F^1_{\bar{z}_1}\xi + F^1_{z_2}\eta + F^1_{\bar{z}_2}\bar{\eta} \\ F^2_{z_1}\bar{\xi} + F^2_{\bar{z}_1}\xi + F^2_{z_2}\eta + F^2_{\bar{z}_2}\bar{\eta} \end{pmatrix} = \begin{pmatrix} \overline{F^1_{z_1}}\bar{\xi} + \overline{F^1_{\bar{z}_1}}\xi + \overline{F^2_{z_2}}\bar{\eta} + \overline{F^1_{\bar{z}_2}}\eta \\ F^2_{z_1}\xi + F^2_{\bar{z}_1}\bar{\xi} + F^2_{z_2}\eta + F^2_{\bar{z}_2}\bar{\eta} \end{pmatrix}.$$

Thus

$$F^2_{z_1} = F^2_{\bar{z}_1}, \qquad F^1_{\bar{z}_2} = \overline{F^1_{z_2}}, \qquad F^1_{z_1}, F^1_{\bar{z}_1} \in \mathbf{R}.$$

A similar argument, using the fact that (dF) commutes with κ_2, shows that

$$F^1_{z_2} = F^1_{\bar{z}_2}, \qquad F^2_{\bar{z}_1} = \overline{F^2_{z_1}}, \qquad F^2_{z_2}, F^2_{\bar{z}_2} \in \mathbf{R}.$$

Thus

$$F^2_{z_1} = \overline{F^2_{\bar{z}}} \qquad \text{and} \qquad F^1_{z_2} = \overline{F^1_{\bar{z}_2}}$$

$$F^1_{z_1}, F^1_{\bar{z}_1} \in \mathbf{R}, \qquad F^2_{z_1} = F^2_{\bar{z}_1} \in \mathbf{R} \qquad \text{and} \qquad F^1_{z_2} = F^1_{\bar{z}_2} \in \mathbf{R}.$$

Hence

$$(dF)(\xi, \eta) = \begin{pmatrix} F^1_{z_1}\xi + F^1_{\bar{z}_1}\bar{\xi} + 2F^1_{z_2}Re(\eta) \\ 2F^2_{z_1}Re(\xi) + F^2_{z_2}\eta + F^2_{\bar{z}_2}\bar{\eta} \end{pmatrix}.$$

We can now compute

$$(dF)\, e_1 = (F^1_{z_1} - F^1_{\bar{z}_1})\, e_1$$

$$(dF)\, e_2 = (F^2_{z_2} - F^2_{\bar{z}_2})\, e_2,$$

where the derivatives are evaluated at $(z_1, z_2) = (\alpha(\varepsilon), \beta(\varepsilon))$. ∎

LEMMA 4.4. *The signs of the two eigenvalues $\sigma_1(\varepsilon)$ and $\sigma_1(\varepsilon)$ at $\mathbf{D}_2$-equilibria in directions tangent to X_ε are those given on Table III.*

Proof.

$$F^1 = (a\,|z_1|^2 + b\,|z_2|^2 - \lambda)\,z_1 + \varepsilon g_1(z_1, z_2) \tag{4.1}$$

$$F^2 = (b\,|z_1|^2 + a\,|z_2|^2 - \lambda)\,z_2 + \varepsilon g_2(z_1, z_2). \tag{4.2}$$

Note that at a equilibrium point $F^1 = 0 = F^2$. Hence

$$a\,|z_1|^2 + b\,|z_2|^2 - \lambda = -\varepsilon\,\frac{g_1(z_1, z_2)}{z_1}$$

$$b\,|z_1|^2 + a\,|z_2|^2 - \lambda = -\varepsilon\,\frac{g_2(z_1, z_2)}{z_2}.$$

Using Lemma 4.3, we obtain

$$\begin{aligned}
\sigma_1(\varepsilon) &= a\,|z_1|^2 + b\,|z_2|^2 - \lambda + \varepsilon(g_{1,z_1} - g_{1,\bar{z}_1})\\
&= \varepsilon\left(-\frac{g_1}{z_1} + g_{1,z_1} - g_{1,\bar{z}_1}\right)\\
\sigma_2(\varepsilon) &= b\,|z_1|^2 + a\,|z_2|^2 - \lambda + \varepsilon(g_{2,z_2} - g_{2,\bar{z}_2})\\
&= \varepsilon\left(-\frac{g_1}{z_2} + g_{2,z_2} - g_{2,\bar{z}_2}\right)
\end{aligned}$$

where the right hand sides are evaluated at $(z_1, z_2) = (\alpha(\varepsilon), \beta(\varepsilon))$. Fixing $\lambda > 0$ (which is equivalent to fixing $\mu > 0$) we can compute the linear terms of σ_1 and σ_2 in ε, as follows. Write

$$\alpha(\varepsilon) = \delta_1 \mu + O(\varepsilon)$$

$$\beta(\varepsilon) = \delta_2 \mu + O(\varepsilon),$$

where $\delta_1 = \pm 1$ and $\delta_2 = \pm 1$. The choice of sign depends on which of the points $A - D$ are the base points for the calculation. Expanding in ε, we find

$$\sigma_1(\varepsilon) = \left(-\frac{g_1(\delta_1\mu, \delta_2\mu)}{\delta_1\mu} + g_{1,z_1}(\delta_1\mu, \delta_2\mu) - g_{1,\bar{z}_1}(\delta_1\mu, \delta_2\mu)\right)\varepsilon + O(\varepsilon^2) \tag{4.3}$$

$$\sigma_2(\varepsilon) = \left(-\frac{g_2(\delta_1\mu, \delta_2\mu)}{\delta_2\mu} + g_{2,z_2}(\delta_1\mu, \delta_2\mu) - g_{2,\bar{z}_2}(\delta_1\mu, \delta_2\mu)\right)\varepsilon + O(\varepsilon^2). \tag{4.4}$$

To determine the signs of $\sigma_1(\varepsilon)$ and $\sigma_2(\varepsilon)$ we need only determine the signs of the coefficients of the ε term in (4.3), (4.4). We can do this for small μ (that is, for λ near 0), as follows. Since $g_1(0,0)=0=g_2(0,0)$ we have

$$\begin{aligned} g_1(\delta_1\mu, \delta_2\mu) &= (\delta_1 g_{1,z_1}(0,0) + \delta_1 g_{1,\bar{z}_1}(0,0) + \delta_2 g_{1,z_2}(0,0) \\ &\quad + \delta_2 g_{1,\bar{z}_2}(0,0))\,\mu + O(\mu^2) \\ g_2(\delta_1\mu, \delta_2\mu) &= (\delta_1 g_{2,z_1}(0,0) + \delta_1 g_{2,\bar{z}_1}(0,0) + \delta_2 g_{2,z_2}(0,0) \\ &\quad + \delta_2 g_{2,\bar{z}_2}(0,0))\,\mu + O(\mu^2). \end{aligned}$$

Substituting into (4.3), (4.4), we see that

$$\begin{aligned} \sigma_1(\varepsilon) &= -[2g_{1,\bar{z}_1}(0,0) + \delta_1\delta_2 g_{1,z_2}(0,0) + \delta_1\delta_2 g_{1,\bar{z}_2}(0,0) + O(\mu)]\,\varepsilon + O(\varepsilon^2) \\ \sigma_2(\varepsilon) &= -[2g_{2,\bar{z}_2}(0,0) + \delta_1\delta_2 g_{2,z_1}(0,0) + \delta_1\delta_2 g_{2,\bar{z}_1}(0,0) + O(\mu)]\,\varepsilon + O(\varepsilon^2). \end{aligned}$$

Finally, we use (3.6) to verify that for each small $\mu>0$ and all sufficiently small ε (with the maximum size of ε depending on μ)

$$\begin{aligned} \operatorname{sgn}(\sigma_1(\varepsilon)) &= -\operatorname{sgn}(\varepsilon)\operatorname{sgn}(g_{1,\bar{z}_1}(0,0) + \delta_1\delta_2 g_{1,\bar{z}_2}(0,0)) \\ \operatorname{sgn}(\sigma_2(\varepsilon)) &= -\operatorname{sgn}(\varepsilon)\operatorname{sgn}(g_{2,\bar{z}_2}(0,0) + \delta_1\delta_2 g_{2,\bar{z}_1}(0,0)). \end{aligned}$$

It follows that the signs of the eigenvalues are as claimed in Table 3. ▮

5. NONEXISTENCE OF NONSYMMETRIC EQUILIBRIA

In this section we derive sufficient conditions for the existence of heteroclinic cycles. The remaining item left to prove is that there are no equilibria other than the $\mathbf{D}_2$-equilibria on the one-dimensional invariant manifolds on X_ε connecting the $\mathbf{D}_2$-equilibria.

As before we assume that (3.7) and (3.8) are valid. We claim that the validity of (3.10) is sufficient to establish that there are no other equilibria on the invariant circles, thus proving that heteroclinic cycles do exist. Specifically, we show that there are no additional equilibria in $\mathrm{Fix}^0(\kappa_1)$ and $\mathrm{Fix}^0(\kappa_2)$ on X_0, and then use continuity to establish the result for small ε.

Recall that the perturbed vector field (3.1) has the form

$$F(z,\lambda,\varepsilon) = f(z,\lambda) + \varepsilon g(z),$$

where recalling (2.1) and (3.5) we have:

$$\begin{aligned} f(z,\lambda) &= (A(|z_1|^2, |z_2|^2, \lambda)\, z_1, A(|z_2|^2, |z_1|^2, \lambda)\, z_2) \\ g(z) &= (a_1 z_1 + b_2, a_2 z_2 + b_2) \end{aligned}$$

Suppose that $(y, \lambda, \varepsilon)$ is an equilibrium of F that is in $\mathrm{Fix}(\kappa_2)$ but is not in $\mathrm{Fix}(\mathbf{D}_2)$. If there were an equilbium in X_ε blocking the heteroclinic cycle, it would have to be either in $\mathrm{Fix}(\kappa_1) \sim \mathrm{Fix}(D_2)$ or in $\mathrm{Fix}(\kappa_2) \sim \mathrm{Fix}(D_2)$. We show that there are no such equilibria which are near the group orbit of mixed mode solutions; that is, near X_0.

Let Y be the group orbit under $\mathbf{T}^2$ containing the point $y = (y_1, y_2)$ and let

$$\pi\colon \mathbf{C}^2 \to T_y Y$$

be orthogonal projection. In coordinates

$$\pi(w_1, w_2) = \left(w_1 - \frac{y_1}{|y_1|^2} \operatorname{Re}(w_1 \bar{y}), w_2 - \frac{y_2}{|y_2|^2} \operatorname{Re}(w_2 \bar{y}_2) \right).$$

Verify this formula by checking that $\pi(y_1, 0) = 0$, $\pi(0, y_2) = 0$, $\pi(iy_1, 0) = (iy_1, 0)$, and $\pi(0, iy_2) = (0, iy_2)$. That is, π vanishes on the directions normal to Y and is the identity on the tangent space directions.

Suppose that a and b are real numbers, then

$$\pi(ay_1, by_2) = 0.$$

Using the form of F, f, and g in it follows that

$$\begin{aligned}
\pi(F(y, \lambda, \varepsilon)) &= \left(b_1 - \frac{y_1}{|y_1|^2} \operatorname{Re}(b_1 \bar{y}_1), b_2 - \frac{y_2}{|y_2|^2} \operatorname{Re}(b_2 \bar{y}_2) \right) \\
&= \left(i \frac{y_1}{|y_1|^2} \operatorname{Im}(\bar{y}_1)\, b_1, i \frac{y_2}{|y_2|^2} \operatorname{Im}(\bar{y}_2)\, b_2 \right) \\
&= 0.
\end{aligned}$$

Note that the coefficient of b_1 vanishes precisely when $\operatorname{Im}(y_1) = 0$, which is just when $y \in \mathrm{Fix}(\kappa_1)$. Similarly, the coefficient of b_2 vanished precisely when $y \in \mathrm{Fix}(\kappa_2)$. Thus

$$\pi(g)|_{\mathrm{Fix}(\kappa_1)} = \left(0, i \frac{y_2}{|y_2|^2} \operatorname{Im}(\bar{y}_2)\, b_2 \right)$$

$$\pi(g)|_{\mathrm{Fix}(\kappa_2)} = \left(i \frac{y_1}{|y_1|^2} \operatorname{Im}(\bar{y}_1)\, b_1, 0 \right).$$

In addition, the coefficients of the b_j vanish precisely at the $\mathbf{D}_2$-equilibria. So if $y \notin \mathrm{Fix}(\kappa_2)$, then $b_1|_y = 0$. Similarly, if $y \notin \mathrm{Fix}(\kappa_1)$, then $b_2|_y = 0$.

So, if we can find conditions on b_1 and b_2 so that

$$b_1|_{\mathrm{Fix}^0(\kappa_2)} \qquad \text{and} \qquad b_2|_{\mathrm{Fix}^0(\kappa_1)}$$

are bounded away from zero on X_0, then the only zeros of F on X_ε for ε sufficiently small will be the $\mathbf{D}_2$-equilibria and the existence of the heteroclinic cycle will be proved.

Observe that

$$\mathrm{Fix}^0(\kappa_1) = \{(\mu, z_2) : |z_1| = \mu\}$$

$$\mathrm{Fix}^0(\kappa_2) = \{(z_1, \mu) : |z_1| = \mu\}.$$

Therefore,

$$b_2|_{\mathrm{Fix}^0(\kappa_1)} = b_2(2\mu, z_2 + \bar{z}_2, \mu^2, \mu^2)$$

$$b_1|_{\mathrm{Fix}^0(\kappa_2)} = b_1(z_1 + \bar{z}_1, 2\mu, \mu^2, \mu^2).$$

Next we write $z_1 = \mu e^{i\phi_1}$ and $z_2 = \mu e^{i\phi_2}$. Then

$$\begin{aligned} b_2|_{\mathrm{Fix}^0(\kappa_1)} &= b_2(2\mu, \mu\cos(\phi_2), \mu^2, \mu^2) \\ &= 2(b_{2,1}(0) + b_{2,2}(0)\cos(\phi_2))\,\mu + O(\mu^2) \\ b_1|_{\mathrm{Fix}^0(\kappa_2)} &= b_1(\mu\cos(\phi_1), 2\mu, \mu^2, \mu^2) \\ &= 2(b_{1,1}(0)\cos(\phi_1) + b_{1,2}(0))\,\mu + O(\mu^2). \end{aligned}$$

Hence for small μ these quantities are bounded away from zero uniformly in ϕ_1 and ϕ_2 if

$$|b_{2,1}(0)| > |b_{2,2}(0)|$$

$$|b_{1,2}(0)| > |b_{1,1}(0)|.$$

Using the form of g in (3.1), it follows that

$$b_{1,1}(0) = g_{1,\bar{z}_1}(0, 0) \qquad b_{1,2}(0) = g_{1,\bar{z}_2}(0, 0)$$

$$b_{2,1}(0) = g_{2,\bar{z}_1}(0, 0) \qquad b_{2,2}(0) = g_{2,\bar{z}_2}(0, 0).$$

Thus we see that the b_j are uniformly bounded away from zero when (3.10) is valid.

6. ASYMPTOTIC STABILITY OF CYCLE

We begin by summarizing our results. Suppose that (3.7) and (3.8) are valid. Then for $\lambda > 0$ and for all sufficiently small ε, the perturbed vector field has a flow invariant 2-torus X_ε. There are four saddles on X_ε—the $\mathbf{D}_2$-equilibria—and there are four flow invariant circles on X_ε—each circle intersects two other circles at $\mathbf{D}_2$-equilibria. Moreover, the dynamics on these circles forms a heteroclinic cycle. We now assume (3.9) and use a result of dos Reis [5] to prove that the heteroclinic cycle is asymptotically stable.

Proof. The eigenvalues of dF in the tangent directions of X_ε are σ_1 and σ_2. At each of the equilibria A–D, one of the σ_1 and σ_2 is attracting (the positive one), the other one is repelling (the negative one). Since X_ε is attracting in the normal directions, dos Reis' result implies that if the product of the four attracting eigenvalues is greater than the product of the four repelling eigenvalue, then the heteroclinic cycle is asymptotically stable. In the case $K_1 \varepsilon > 0$, the four attracting eigenvalues are σ_2^A, σ_1^B, σ_2^C, σ_1^D and the four repelling eigenvalues are σ_1^A, σ_2^B, σ_1^C, σ_2^D. Thus the heteroclinic cycle connecting A–D is asymptotically stable if

$$\sigma_2^A \sigma_1^B \sigma_2^C \sigma_1^D > |\sigma_1^A \sigma_2^B \sigma_1^C \sigma_2^D|.$$

For fixed μ small and for sufficiently small ε, this is equivalent to $L_1^2 K_2^2 - K_1^2 L_2^2 > 0$. Hypothesis (3.8) implies that $L_1 K_2 + K_1 L_2 > 0$. Thus, the condition for asymptotic stability is $L_1 K_2 - K_1 L_2 > 0$, which is just (3.9). If $K_1 \varepsilon < 0$, then the attracting and repelling eigenvalues are interchanged. Hence the inequality guaranteeing asymptotic stability is reversed and is $L_1 K_2 - K_1 L_2 < 0$. ∎

We end by noting that our results are independent of higher order terms in f. The reason is simple—the existence and stability of solutions on the mixed mode branch does not depend on terms of order higher than three. The existence and stability properties of the $\mathbf{D}_2$-equilibria depend only on terms in g and not on any terms in the unperturbed vector field f (beyond the orbital stability of the mixed modes).

7. AN EXAMPLE

An example of a $\mathbf{D}_2$-equivariant perturbation of $\mathbf{D}_4 \dot{+} \mathbf{T}^2$-equivariant vector field that satisfies the hypotheses of Theorem 3.2 is:

$$\frac{dz_1}{dt}+\left(|z_1|^2+\frac{1}{2}|z_2|^2-\lambda\right)z_1-\varepsilon\left[\frac{1}{2}(z_1+\bar{z}_1)+(z_2+\bar{z}_2)\right]=0 \tag{7.1}$$

$$\frac{dz_2}{dt}+\left(|z_2|^2+\frac{1}{2}|z_1|^2-\lambda\right)z_2+\varepsilon\left[(z_1+\bar{z}_1)-\frac{1}{2}(z_2+\bar{z}_2)\right]=0. \tag{7.2}$$

From (7.1) and (7.2) we see that $a=1$, $b=\frac{1}{2}$, and

$$K_1=-\tfrac{3}{2} \qquad K_2=\tfrac{1}{2} \qquad L_1=\tfrac{1}{2} \qquad L_2=-\tfrac{3}{2}.$$

Thus Theorem 3.2 guarantees that there is an asymptotically stable heteroclinic cycle on X_ε.

ACKNOWLEDGMENTS

We wish to thank Mike Field and Ian Melbourne for a number of helpful comments. This research was supported in part by NSF Grant DMS-9403624, ONR Grant N00014-94-1-0317, and the Texas Advanced Research Program (003652037).

REFERENCES

1. D. Armbruster and E. Ihrig, Topological constraints for explicit symmetry breaking, *Lect. Appl. Math.* **29** (1993), 37–47.
2. D. Armbruster, J. Guckenheimer, and P. Holmes, Heteroclinic cycles and modulated travelling waves in system with **O(2)** symmetry, *Physica D* **29** (1988), 257–282.
3. G. Dangelmayr and E. Knobloch, Hopf bifurcation with broken circular symmetry, *Nonlinearity* **4** (1991), 399–422.
4. B. Dionne and M. Golubitsky, Planforms in two and three dimensions, *ZAMP* **43** (1992), 43–62.
5. G. L. dos Reis, Structural stability of equivariant vector fields on two manifolds, *Trans. Amer. Math. Soc.* **283**, No. 2 (1984), 633–634.
6. M. Field, Equivariant dynamical systems, *Trans. Amer. Math. Soc.* **229**, No. 1 (1980), 189–205.
7. M. Golubitsky and D. G. Schaeffer, "Singularities and Groups in Bifurcation Theory, Vol. I," Appl. Math. Sci., Vol. 51, Springer-Verlag, New York, 1984.
8. J. Guckenheimer and P. Holmes, Structurally stable heteroclinic cycles, *Math. Proc. Cambridge Philos. Soc.* **103** (1988), 189–192.
9. M. W. Hirsch, C. C. Pugh, and M. Shub, "Invariant Manifolds," Lect. Notes Math., Vol. 583, Springer-Verlag, New York/Berlin, 1977.
10. C. Hou, "Symmetry Breaking and Heteroclinic Cycles," Thesis, Department of Mathematics, Univ. of Houston, 1995.
11. E. Ihrig and M. Golubitsky, Pattern selection with **O(3)** symmetry, *Physica D* **13** (1984), 1–33.
12. M. Krupa, Robust heteroclinic cycles, submitted for publication.
13. R. Lauterbach, Forced symmetry breaking from **O(3)**, *in* "Bifurcation and Symmetry" (E. Allgower, K. Böhmer, and M. Golubitsky, Eds.), ISNM, Vol. 104, pp. 253–262, Birkhäuser, Basel.

14. R. Lauterbach, S. Maier-Paape, and E. Reissner, A systematic study of heteroclinic cycles in dynamical systems with broken symmetries, *Proc. Roy. Soc. Edinburgh* **126A** (1996), 885–909.
15. R. Lauterbach and M. Roberts, Heteroclinic cycles in dynamical systems with broken spherical symmetry, *J. Differential Equations* **100** (1992), 22–48.
16. I. Melbourne, P. Chossat, and M. Golubitsky, Heteroclinic cycles involving periodic solutions in mode interactions with $\mathbf{O}(\mathbf{2})$ symmetry, *Proc. Roy. Soc. Edinburgh. Sect. A* **113** (1989), 313–345.
17. J. W. Swift, "Bifurcation and Symmetry in Convection," Thesis, Physics, U. C. Berkeley, 1984.