Abzweigung einer periodischen Lösung von einer stationaeren Lösung eines Differentialsystems, *Akad. Wiss*. **(Leipzig) 94 (1942), 3-22.**

Commentary

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Introduction

The Hopf Bifurcation Theorem provides the simplest criterion for a family of periodic solutions to bifurcate from a known family of equilibrium solutions of an evolution equation. A second theorem gives information about the stability or instability of the bifurcating branch of solutions. We do not know exactly what motivated Hopf to study these questions. When asked about it many years later, he could not recall how he come to them. However in his paper, he mentions that the qualitative phenomena they describe are well known in hydrodynamics, for example, periodic vortex shedding in flow past an obstacle when the velocity is large enough. Thus we might speculate that such problems were the origin of his interest in bifurcation.

The paper appeared in 1942 and in the introduction Hopf writes about the first theorem: "I scarcely think that there is anything new in the above theorem. The methods have been developed by Poincaré perhaps 50 years ago and belong today to the classical conceptual structure of the theory of periodic solutions in the small". In fact using such methods, Andronov obtained the bifurcation and stability results for two dimensional systems as can be seen already in his book, Andronov-Vitt-Khaikin (1937). Even earlier work of Andronov which was not available to us was cited by Arnold (1983). These points not withstanding, the Hopf Bifurcation Theorem has become a paradigm of a useful and elementary result that has been extremely influential. New proofs have been given and extensions have been made in many directions. There are now degenerate and equivariant and Hamiltonian and global and infinite dimensional versions of the theorem. Unexpected connections have been found to the much older Liapunov Center Theorem. Several numerical codes have been written to implement the theorem. And of course there are many physical applications. In what follows, we will discuss the two theorems, illustrating them in a simple setting and giving a sketch of a modern proof of the first. There will also be a brief discussion of the extensions and some applications.

1. The Hopf Theorem

Hopf considered the system of ordinary differential equations:

$$\dot{x} = F(x, \lambda), \quad x \in \mathbb{R}^n \tag{1.1}$$

where λ belongs to an interval $I \subset \mathbb{R}$ and (1.1) possesses a known family of equilibrium solutions $(x(\lambda), \lambda), \lambda \in I$. Making a change of variables, it can be assumed that $x(\lambda) \equiv 0$ and $0 \in I$. Let

$$J(\lambda) = d_x F|_{(0,\lambda)},$$

the Jacobian matrix or Fréchet derivative of F evaluated at $(0, \lambda)$. Suppose that

$$\begin{array}{l}(H_1) \quad J(\lambda) \text{ has a pair of simple complex conjugate eigenvalues} \\ \qquad a(\lambda), \overline{a}(\lambda) \text{ with } a(0) \text{ purely imaginary and } a'(0) \neq 0.\end{array}$$

The condition $a'(0) \neq 0$ is called the *eigenvalue crossing condition*. Assume also that

$$(H_2) \quad \pm\, a(0) \text{ are the only eigenvalues of } J(0) \text{ on the imaginary axis.}$$

Under assumptions (H1) and (H2), the first Hopf theorem asserts the existence of a one parameter family of solutions of (1.1) that are periodic in t (with period near $2\pi/|a'(0)|$) and bifurcate from the equilibrium solution $(0, 0)$. The theorem also contains some information about the form of the bifurcating solutions. In particular, roughly speaking, the solution is parametrized by its amplitude.

The simplest example of the result occurs in the planar linear system

$$\begin{aligned} \dot{x}_1 &= \lambda x_1 - x_2 \\ \dot{x}_2 &= x_1 + \lambda x_2. \end{aligned} \tag{1.2}$$

It is easy to check that the origin is a spiral sink when $\lambda < 0$ and a spiral source when $\lambda > 0$. When $\lambda = 0$, the origin is a center, and there is a continuous family of periodic solutions surrounding this center. See Figure 1.

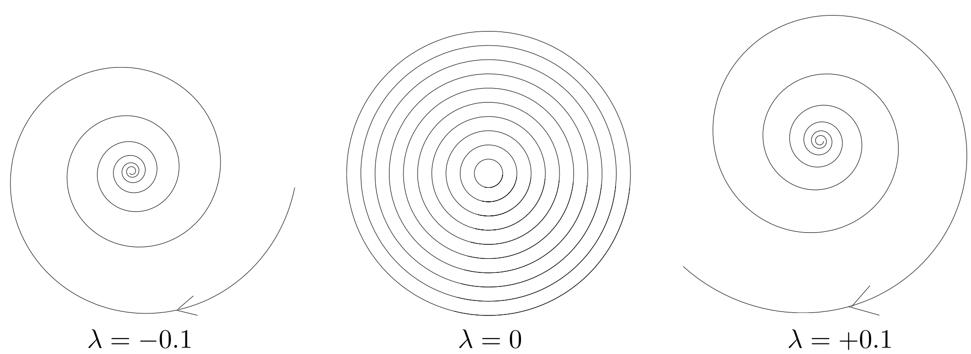


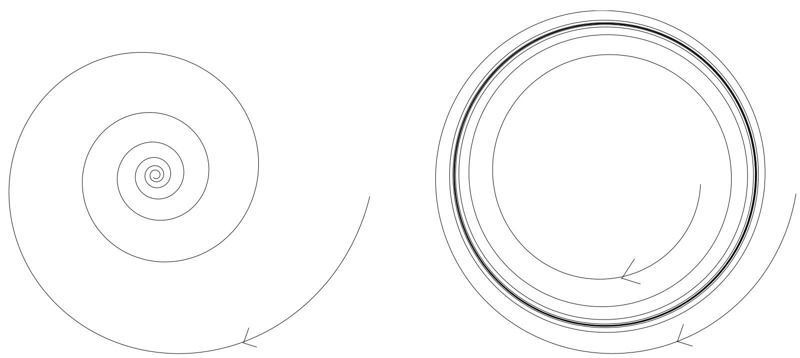
FIGURE 1. Phase planes for (1.2).

The second Hopf theorem which is sometimes referred to as the *exchange of stability theorem* provides information about the stability of the bifurcating branch. To simplify the discussion, suppose that the equilibria $(0, \lambda)$ with $\lambda < 0$, are asymptotically stable. Then the theorem states that if a certain number μ_2 (which can be computed explicitly from the linear, quadratic, and cubic terms of F) is nonzero, the bifurcating solutions are either supercritical (occur when $\lambda > 0$) and asymptotically stable or are subcritical (occur when $\lambda < 0$) and unstable.

A simple example of exchange of stability is given by adding a nonlinear term to the linear system (1.2).

$$\begin{aligned} \dot{x}_1 &= \lambda x_1 - x_2 - (x_1^2 + x_2^2) x_1 \\ \dot{x}_2 &= x_1 + \lambda x_2 - (x_1^2 + x_2^2) x_2. \end{aligned} \tag{1.3}$$

Note that when $\lambda < 0$ the origin is still a spiral sink and that the nonlinear terms $(x_1^2 + x_2^2)(x_1, x_2)^t$ point toward the origin, so that it is not surprising that globally solutions spiral toward the origin. However, when $\lambda > 0$, the origin is a spiral source but the nonlinear terms still point toward the origin. This interaction between the linear and nonlinear effects is resolved by the existence of a stable limit cycle.



$\lambda = -0.1$ $\lambda = +0.1$

FIGURE 2. Phase planes for (1.3).

Both Hopf theorems can be proved by two very different methods: (i) Liapunov-Schmidt reduction, or (ii) center manifold reduction coupled with Poincaré-Birkhoff normal form theory. It is noteworthy that the existence proof using Liapunov-Schmidt reduction is relatively straightforward, whereas the stability result is straightforward when the center manifold approach is invoked. In this review we sketch the existence proof by Liapunov-Schmidt reduction. This approach is due to Cesari and Hale (1969).

The Liapunov-Schmidt proof of the existence of periodic solutions proceeds in three steps. First, the ordinary differential equations in n dimensions are posed as an operator on infinite-dimensional loop space whose zeros are the desired periodic solutions. Second, the implicit function theorem is used to reduce the space from infinite dimensions back to two dimensions (the real eigenspace corresponding to the complex conjugate eigenvalues of $J(0)$). Finally, phase-shift $\mathbf{S}^1$ symmetry is used to simplify the search for zeros in two dimensions. Hopf's approach to the problem is related to the Liapunov-Schmidt reduction but, in line with the techniques of his day, uses a Poincaré map and the implicit function theorem in $\mathbf{R}^n$.

Step 1: Loop Space. To simplify the discussion, assume that the complex conjugate eigenvalues of $J(0)$ are $\pm i$. (Rescaling time in (1.1) will accomplish this task.) The linearized system

$$\dot{x} = J(0)x \tag{1.4}$$

then has periodic solutions of period 2π. The important observation that follows is that the bifurcating periodic solutions in the nonlinear system are parametrized by the periodic solutions for the linear system and that the period of the new periodic solutions of the nonlinear system are approximately 2π. Rescaling time allows us to search only for periodic solutions that have period exactly 2π. This clever trick is what makes the approach work.

Introduce the loop space $\mathcal{C}^0_{2\pi}$ consisting of $\mathcal{C}^0$ maps $\mathbf{S}^1 \to \mathbb{R}^n$ and let $\mathcal{C}^1_{2\pi}$ be the corresponding subspace of $\mathcal{C}^1$ maps. Observe that zeros of the operator equation

$$\mathcal{F} : \mathcal{C}^1_{2\pi} \times \mathbb{R} \times \mathbb{R} \to \mathcal{C}^0_{2\pi}$$

where

$$\mathcal{F}(u, \lambda, \tau) = (1+\tau)\frac{du}{ds} - f(u, \lambda),$$

correspond to $2\pi/(1+\tau)$ periodic solutions to the original system (1.1); so we think of τ as the perturbed period parameter.

Step 2: Reduction to Two Dimensions. The linearization of $\mathcal{F}$ at the origin is just the linear system of differential equations

$$\mathcal{L}(u) = \frac{du}{ds} - J(0)u.$$

The eigenvalue assumptions on $J(0)$ imply that $\mathcal{K} = \ker \mathcal{L}$ is the two-dimensional space of 2π periodic solutions to the linear differential equation (1.4). Let $\mathcal{R}$ denote the range of $\mathcal{L}$ and let $P : \mathcal{C}^0_{2\pi} \to \mathcal{R}$ be a projection. The Fredholm alternative can be used to show that $\ker P$ is also two-dimensional.

It follows that solving the nonlinear operator equation $\mathcal{F} = 0$ can be divided into two parts

$$\begin{aligned} P\mathcal{F} &= 0 \\ (I-P)\mathcal{F} &= 0 \end{aligned}$$

The first equation can be solved near the origin using the implicit function theorem. Let $\mathcal{W}$ be a complement to the kernel $\mathcal{K}$ in the domain space $\mathcal{C}^1_{2\pi}$ and observe that $\mathcal{L}|_{\mathcal{W}} : \mathcal{W} \to \mathcal{R}$ is invertible. Thus, there exists an implicit function $W : \mathcal{K}\times\mathbb{R}\times\mathbb{R} \to \mathcal{W}$ such that

$$P\mathcal{F}(k, W(k,\lambda,\tau), \lambda, \tau) \equiv 0.$$

It now follows that periodic solutions to (1.1) near the origin and with period near 2π are parametrized by zeros of the equation

$$G(k, \lambda, \tau) = (I-P)\mathcal{F}(k, W(k,\lambda,\tau), \lambda, \tau)$$

where $G : \mathcal{K} \times \mathbb{R} \times \mathbb{R} \to \ker P$. We can identify the two-dimensional subspaces $\mathcal{K}$ and $\ker P$ with $\mathbf{C}$ and the proof then reduces to finding zeros of a map

$$g : \mathbf{C} \times \mathbb{R} \times \mathbb{R} \to \mathbf{C}$$

near the origin. Since g is defined only implicitly and since simultaneously solving two nonlinear equations in two variables that depend on two parameters is not straightforward, what remains is still a difficult problem. However, phase-shift symmetry comes to the rescue.

Step 3: The Use of $\mathbf{S}^1$ *Phase Shift Symmetry.* The circle group $\mathbf{S}^1$ acts naturally on a periodic function $u(t)$ by

$$\theta u(t) = u(t-\theta).$$

It is easy to check that the operator $\mathcal{F}$ commutes with this action of $\mathbf{S}^1$. It is also possible to set up the implicit function theorem (correct choices for $\mathcal{W}$ and $\ker P$) so that the $\mathbf{S}^1$ action survives reduction. That is, we may assume that

$$g(e^{i\theta}z, \lambda, \tau) = e^{i\theta}g(z, \lambda, \tau).$$

It follows from invariant theory that g has the form

$$g(z,\lambda,\tau) = p(|z|^2,\lambda,\tau)z + q(|z|^2,\lambda,\tau)iz$$

where p and q are real-valued functions.

It is now possible to solve for the zeros of g. Since we are looking only for trajectories of periodic solutions, we can apply $\mathbf{S}^1$ symmetry to assume that $z \in \mathbb{R}$. Observe that solving $g = 0$ is equivalent to solving $p = q = 0$. Finally some calculations (based on implicit differentiation) are needed. These calculations do require substantial work to complete. In particular $p_\lambda(0,0,0) = a'(0)$, $p_\tau(0,0,0) = 0$, and $q_\tau(0,0,0) = -1$ where $a'(0)$ is the speed with which the critical eigenvalues of $J(\lambda)$ cross the imaginary axis.

We can now apply the implicit function theorem to obtain a function $\tau(|z|^2,\lambda)$ such that

$$q(|z|^2,\lambda,\tau(|z|^2,\lambda)) \equiv 0.$$

So our desired periodic solutions are obtained by solving

$$A(|z|^2,\lambda) \equiv p(|z|^2,\lambda,\tau(|z|^2,\lambda)) = 0.$$

Second $A_\lambda(0) = a'(0)$, so we can now apply the implicit function theorem (for the third time) to obtain a branch of solutions to $A = 0$ parametrized by $|z|^2$. These calculations complete the proof of the first Hopf theorem.

Finally, turning to the second theorem, let $\mu_2 = A_{|z|^2}(0)$. Recall that μ_2 was the number alluded to above that determined whether the branch of new periodic solutions were supercritical or subcritical. To lowest order

$$A(|z|^2,\lambda) = \mu_2|z|^2 + a'(0)\lambda + \cdots .$$

When $\mu_2 \neq 0$ we see that the branch has the form

$$\lambda = -\frac{\mu_2}{a'(0)}|z|^2 + \cdots .$$

which decides super- or subcriticality. This is roughly the computation Hopf made.

The discussion of stability of solutions requires Floquet theory and careful control of Floquet exponents in the reduction process. To be a bit more precise, returning to $\mathcal{F}(u,\lambda,\tau) = 0$ or equivalently

$$\frac{du}{ds} = \frac{1}{1+\tau} f(u,\lambda), \tag{1.5}$$

the stability of the solution $u(s,z)$ depends on the Floquet exponents of the linearization of (1.5) about $u(s,z)$. In a physical problem where $(x(\lambda),\lambda)$ is actually observed e.g. for $\lambda < 0$, it is implicit that this equilibrium solution is stable. Hence at $(x(0),0) = (0,0)$, we expect that $n-2$ Floquet exponents of

$$\frac{dv}{ds} = f_u(0,0)v \tag{1.6}$$

have negative real parts and 0 is a Floquet exponent of multiplicity 2. The Floquet exponents of (1.6) are the values $-\kappa$ for which

$$\frac{dw}{ds} - f_u(0,0)w = \kappa w, \quad w(0) = w(2\pi) \tag{1.7}$$

has a nontrivial solution. Differentiating (1.5) with respect to s shows that $w = \frac{du}{ds}$ satisfies (1.6) with $\kappa = 0$, i.e. 0 continues as a Floquet exponent along the bifurcating branch, $\mathcal{B}$, of solutions. Since the $n-2$ Floquet exponents with negative real parts will also continue to be negative on $\mathcal{B}$ near the bifurcation point, the stability of the solutions on $\mathcal{B}$ is governed by how the second zero exponent of (1.7) continues along $\mathcal{B}$. It turns out that $\kappa(z)$ and $\lambda'(z)$ have the same zeroes and whenever $\lambda'(z) \neq 0$, $\kappa(z)$ and — $a'(0)z\lambda'(z)$ have the same sign. In particular, if $a'(0) < 0$ and $z\lambda'(z) > 0$ if $z \neq 0$ (i.e. bifurcation is supercritical), then the solutions on $\mathcal{B}$ near the bifurcation point are stable. This generalizes the remarks made about μ_2 above. Such results can be found in Joseph-Nield (1975), Weinberger (1977), and Crandall-Rabinowitz (1977).

2. Some Extensions

The Hopf Bifurcation Theorem is a local theorem; it describes the structure of the branch of periodic solutions of (1.1) near the bifurcation point $(x, \lambda) = (0, 0)$. Using topological methods, Alexander and Yorke (1978) have given a global version of the theorem. While they pose their result for an n-manifold, in the setting of the Hopf theorem, they allow more general spectral conditions than (H_1) – (H_2). In particular (H_2) is dropped and (H_1) is replaced by a milder condition that will not be made explicit here. To describe the conclusions, let $G(\lambda, t, x_0)$ denote the solution of (1.1) with $G(\lambda, 0, x_0) = x_0$. Thus equilibrium solutions have $G(\lambda, t, x_0) = x_0$ for all $t \in \mathbb{R}$. If $x(t) = G(\lambda, t, x_0)$ is not an equilibrium solution and there is a $T > 0$ such that $x(0) = x(T)$, then $x(t)$ is a nontrivial T-periodic solution of (1.1). Set

$$\begin{aligned}\mathcal{N} \;&= \{(\lambda, T, x_0) \in \mathbb{R} \times (0, \infty) \times \mathbb{R}^n | G(\lambda, T, x_0) = x_0 \\ &\quad \text{and } \ x(t) \ \text{ is a nontrivial T-periodic solution of (1.1) } \},\end{aligned}$$

i.e. $\mathcal{N}$ corresponds to the set of nonequilibrium periodic solutions of (1.1). The main result of Alexander and Yorke (1978) is that $\mathcal{N} \cup \{(0, \frac{2\pi i}{a(0)}, 0)\}$ contains a connected subset $\mathcal{N}_0$ which is either unbounded in $\mathbb{R} \times [0, \infty) \times \mathbb{R}^n$ or meets $(\overline{\lambda}, \overline{T}, \overline{x}) \in \overline{\mathcal{N}}_0 \backslash \mathcal{N}_0$ with $G(\overline{\lambda}, \overline{T}, \overline{x})$ an equilibrium solution of (1.1). Stated more informally, there is a global branch of periodic solutions of (1.1) which is either unbounded in the triple (λ, T, x_0) or meets an equilibrium solution other than $(0, \frac{2\pi i}{a(0)}, 0)$. Thus bifurcation here, like bifurcation from equilibrium to equilibrium solutions as described by the so-called Global Bifurcation Theorem (Rabinowitz (1971)) is not a local but a global phenomenon.

The Liapunov Center Theorem (Liapunov (1907)) is an early bifurcation theorem that predates the Hopf Theorem. It considers the n dimensional autonomous system:

$$\dot{x} = Ax + g(x) \tag{2.1}$$

where A is an n matrix having eigenvalues $\pm i\beta, \lambda_3, \ldots, \lambda_n$, with $\beta \neq 0$ and $g(x) = o(|x|)$ as $x \to 0$. Assume (2.1) has an integral I, (that is, $I(z(t)) \equiv$ constant for any solution $z(t)$ of (2.1)) with

$$I(x) = \frac{1}{2} x \cdot Sx + o(|x|^2)$$

as $x \to 0$, S being symmetric and nonsingular. Suppose further that $\lambda_j / i\beta \notin \mathbf{Z}$ for $j = 3, \ldots, n$. Then the Liapunov Theorem states that (2.1) possesses a 1-parameter family of solutions $x(t, s)$ of period $T(s)$ for s near 0 with $x(t, 0) = 0$ and $T(0) = 2\pi/\beta$.

Being autonomous, (2.1) looks rather different from (1.1), but as was observed by Schmidt (1976) – see also Alexander-Yorke (1978) – by a nice trick one can prove the Liapunov Theorem by a simple application of the Hopf theorem. To see how, consider

$$\dot{x} = F(\lambda, x) \equiv Ax + g(x) + \lambda \ \text{grad} \ I(x) \tag{2.2}$$

For this choice of F, it is not difficult to verify that $J(\lambda) = A + \lambda S$ and satisfies $(H_1) - (H_2)$ so by (a small generalization of) the Hopf theorem, there is a branch of solutions of (2.2) $(x(t,s), \lambda(s))$, with $x(\cdot, s)$ periodic in t, bifurcating from $(0,0)$ in $\mathbb{R} \times \mathbb{R}^n$. Therefore

$$\frac{dI(x)}{dt} = \ \text{grad} \ I(x) \cdot (Ax + g(x) + \lambda \ \text{grad} \ I(x)) \tag{2.3}$$

with $x = x(t,s)$. Since I is an integral for (2.1), for all $z \in \mathbb{R}^n$,

$$\text{grad} \ I(z) \cdot (Az + g(z)) = 0. \tag{2.4}$$

Hence

$$\frac{dI(x)}{dt} = \lambda |\text{grad}\, I(x)|^2 \tag{2.5}$$

Since $I(x(t,s))$ is periodic in t for $s \neq 0$, the right hand side of (2.5) must equal 0 for all t. The form of I shows grad $I(x(t,s)) \not\equiv 0$ so $\lambda(s) \equiv 0$, i.e. $x(t,s)$ is a solution of (2.1).

There have been many infinite dimensional versions of the Hopf theorem. The first that we know of appeared in the 1970's motivated by attempts to establish the bifurcation of periodic solutions of the Navier-Stokes equations. See Iudovich (1971), Sattinger (1971), Iooss (1972), Joseph and Sattinger (1972), Crandall-Rabinowitz (1977), ... Subsequently there have been applications in many other directions such as reaction-diffusion problems (Henry (1981)), vortex shedding (Provansal, Mathis, and Boyer (1987)), convection in binary fluids (Knobloch (1986)) and in double-diffusion systems (Knobloch and Proctor (1981)), panel flutter (Holmes-Marsden (1978)), predator prey problems, ..., and much more. Considerably different technical settings and tools are required to treat these problems. Often, however, after a nontrivial Liapunov-Schmidt or center manifold reduction, they play back to the two basic approaches to the Hopf setting.

The finite-dimensional Hopf bifurcation theorems can be generalized to include degenerate cases ($a'(0) = 0$ or $\mu_2 = 0$ or both). These degeneracies appear in systems with several parameters and can lead to the existence of multiple periodic solutions for a given λ. See Kielhofer (1979) and Golubitsky-Langford (1981). Symmetry, which is often present in fluid mechanics problems, can force the critical eigenvalues to be multiple and can lead to multiple branches of periodic solutions with intriguing spatio-temporal symmetry. See Chossat-Iooss (1985) and Golubitsky-Stewart (1985).

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