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Contact Equivalence for Lagrangian Manifolds

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INTRODUCTION

Let Z be a manifold and X, Y_1, Y_2 equidimensional submanifolds, all intersecting at $0 \in X$. Y_1 and Y_2 are said to be *contact equivalent* (*with respect to* X) *at* 0 if there exists a germ of diffeomorphism $f: (Z, 0) \to (Z, 0)$ mapping X into X and Y_1 into Y_2. The notion of contact equivalence is due to John Mather and plays an important role in his theory of singularities of differentiable mappings. This paper has to do with a slightly modified notion of contact equivalence, namely Z is assumed to be a symplectic manifold; X, Y_1 and Y_2 are assumed to be Lagrangian submanifolds, and f is assumed to be a germ of a symplectic diffeomorphism. Our main theorem (Proposition 3.2) states that two Lagrangian submanifolds have the same contact with a third if certain algebraic data of contact (a local ring and a distinguished element) are isomorphic. This is reminiscent of a theorem of Mather [6, Section 2.2] for ordinary contact equivalence which, for motivational purposes we describe in Section 2. The proof of Proposition 3.2 requires some results from symplectic geometry which we describe in Section 1. In Section 4 we exploit the fact that to each function ϕ on a manifold X there is an associated Lagrangian submanifold, namely graph $d\phi$, in T^*X to reformulate in symplectic form a theorem about right equivalence due to Tougeron [8, p. 209].

In the last section we give some examples to show that the algebraic criteria for contact equivalence given in Section 3 can not be weakened. To conclude we note that this paper had its origins in an attempt (unsuccessful) on our part to find a simple formula for the order of the caustic associated to a (germ of a) Lagrangian manifold $(\Lambda, \lambda) \subset T^*X$ (see [1, Definition 1.6.1]). The results here show that this number is a

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contact invariant of Λ and the fiber of the cotangent bundle passing through λ.

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1. Symplectic Structures in the Neighborhood of Lagrangian Manifolds

Let Z be a symplectic manifold with 2-form $\Omega_Z = \Omega$ and X a Lagrangian submanifold of Z. For a proof of the following we refer the reader to [9].

Theorem 1.1 (Kostant–Weinstein). *There exists a* nghd M *of* X *in* Z, *a* nghd N *of the zero section,* X, *in* T^*X *and a symplectic diffeomorphism*: $M \underset{\rightarrow}{\cong} N$ *mapping* X *onto* X *as the identity.*

See [9, pg. 333, Theorem 4.1].

We will call a diffeomorphism of the type described in the theorem a *cotangent bundle structure* on M. On T^*X there is a canonical one-form $\sum \xi_i \, dx_i$. The pullback of this to M will be called the *one-form associated with the cotangent bundle structure*. Note that this form, call it α, has the following two properties:

$$\begin{aligned} &\text{(a)} \qquad \alpha_p = 0 \quad \text{when} \quad p \in X, \\ &\text{(b)} \qquad d\alpha = \Omega_Z . \end{aligned} \tag{1.2}$$

We will show that there is a one-to-one correspondence between one-forms with the two above properties and cotangent bundle structures. More precisely:

Theorem 1.3. *Let* α *be a one-form with domain of definition a* nghd *of* X *in* Z *and with properties* (a) *and* (b). *Then there exists a tubular* subnhgd M *of* X *in* Z, *a* nghd N *of the zero section,* X, *in* T^*X *and a unique vector bundle isomorphism* $f: (M, X) \underset{\rightarrow}{\cong} (N, X)$ *such that* f *is the identity on* X *and* $\alpha = f^*\alpha_0$, α_0 *being the canonical one form on* T^*X.

This theorem is due to Kostant and Sternberg (unpublished). A result similar to it is stated by T. Nagano in [7]. We will give a brief sketch of the proof.

LEMMA 1.4. *Let (V, Ω_0) be a $2n$-dimensional symplectic vector space, W a Lagrangian subspace, and $A\colon V \to V$ a linear map with the properties*

(i) $A \mid W = 0$,

(ii) $\Omega_0(Av, w) + \Omega_0(v, Aw) = \Omega_0(v, w), \quad v, w \in V.$

Then there is a Lagrangian complement W' of W invariant under A such that $A \mid W'$ is the identity.

Proof. For $w \in W$, $\Omega_0(Av, w) = \Omega_0(v, w)$; so, because of the non-degeneracy of Ω_0, A induces the identity map on V/W. Therefore, the generalized eigenspace of A associated with the eigenvalue 1 is exactly n dimensional. (It can't be more than n-dimensional since $A = 0$ on W.) Call this generalized eigenspace W'. Since $A =$ identity on V/W, $A =$ identity on W'; so for $v_1, v_2 \in W'$, $2\Omega_0(v_1, v_2) = \Omega_0(v_1, v_2)$ by (ii); so $\Omega_0(v_1, v_2) = 0$. Q.E.D.

Now let α be a one form on Z satisfying the hypotheses (1.2). Let Ξ_α be the vector field defined by the equation

$$\Xi_\alpha \rfloor \Omega = \alpha \tag{1.5}$$

By (a) of (1.2) $\Xi_\alpha(p) = 0$ when $p \in X$. From (1.5) we get

$$\mathscr{L}_{\Xi_\alpha}\Omega = d(\Xi_\alpha \rfloor \Omega) = \Omega$$

Now apply the lemma with $p \in X$, $V = T_pZ$, $W = T_pX$, and $A =$ the linear part of Ξ_α at p. By the lemma there exists a complementry Lagrangian space to T_pX in T_pZ on which the linear map A is "expanding" i.e. the real parts of its eigenvalues are > 0. (In fact they are all 1 by the lemma.) By a theorem of Hirsch, Pugh, and Shub on hyperbolic fixed point sets of flows (see [4]) there exists a tubular nghd, M, of X in Z and a (unique) fibration of M, $M \xrightarrow{\pi} X$, such that $X \hookrightarrow M \xrightarrow{\pi} X$ is the identity and Ξ_α is tangent to the fibers. This fibration is our candidate for the "cotangent bundle" associated with α. To show that it is indeed the cotangent bundle we make use of an observation of Weinstein. (See [9, Theorem 7.7].)

THEOREM 1.6. *Let Z be a symplectic manifold and $\pi\colon Z \to X$ a fibration whose fibers are Lagrangian submanifolds of Z. Then each fiber possesses a canonical trivialization of its tangent bundle.*

Proof. Let F_x be the fiber above x and let $z \in F_x$. Let V_z be the tangent space to F_x at z and let $V_z^{\perp}$ be the annihilator of V_z in T_z^*. Then $(d\pi_z)^*: T_x^* \to T_z^*$ maps T_x^* isomorphically onto $V_z^{\perp}$. Since F_x is Lagrangian the form Ω_z gives us an isomorphism between V_z and $V_z^{\perp}$ so we get a canonical map $V_z \cong T_x^*$, i.e., the tangent bundle of the fiber possesses a canonical trivialization. Q.E.D.

Now suppose there exists a section $\sigma: X \to Z$ such that the image of σ is a Lagrangian submanifold of Z. Since $V_{\sigma(x)}$ is a complementary Lagrangian space to $(d\sigma)(T_x)$ in $T_{\sigma(x)}$ we can canonically identify it with T_x^*X. On the other hand the connection on F_x defined by Theorem 1.6 supplies us with an "exp" map at $\sigma(x)$:

$$\exp: V_{\sigma(x)} \to F_x$$

Thus we get a diffeomorphism between a tubular nghd of $\sigma(X)$ in Z and a tubular nghd of the zero section in T^*X.

To apply this discussion to the fibration constructed above we have to show that the fibers, F_x, are Lagrangian manifolds. To do this let Φ_t be the flow associated with Ξ_α. Since $\mathscr{L}_{\Xi_\alpha}\Omega = \Omega$, $\Phi_t^*\Omega = e^t\Omega$ while, for a vector tangent to the "unstable manifold" F_x at $z \in F_x$, $|(d\Phi_t)_z v| = O(e^t)$. Therefore for two such vectors

$$\Phi_t^*\Omega(v, w) = e^t\Omega(v, w) = \Omega(d\Phi_t(v), d\Phi_t(w)) = O(e^{2t})$$

implying $\Omega(v, w) = 0$. This concludes our proof of Theorem 1.3.

One consequence of the theorem is the following.

COROLLARY 1.7. *Let α and β be one-forms satisfying* (1.2). *Then there exists a symplectic diffeomorphism $\rho: (Z, X) \to (Z, X)$ such that $\rho =$ identity on X and $\rho^*\beta = \alpha$.*

2. CONTACT EQUIVALENCE VIA THE GROUP OF DIFFEOMORPHISMS

In this section we review some results of John Mather on contact equivalence. The reader is refered to VII, Section 3 of [2] for a more leisurely exposition of this material.

Let Z be a manifold with X and Λ equidimensional submanifolds. Let p be in $\Lambda \cap X$. Consider germs of functions on Z near p which vanish to kth order on Λ. The restrictions of these functions to X forms an ideal $\mathscr{I}_k(X, \Lambda)$ in $C_p^\infty(X)$ (germs of smooth functions on X near p). Define

$$\mathscr{R}_k(X, \Lambda) = C_p^\infty(X)/\mathscr{I}_k(X, \Lambda) = \textit{local ring of contact of } \Lambda \textit{ with } X \text{ (to order } k). \tag{2.1}$$

The corresponding geometric notion is the following: let X, Λ_1, and Λ_2 be equidimensional submanifolds of Z with p in $X \cap \Lambda_1 \cap \Lambda_2$. Then Λ_1 and Λ_2 have the ***same contact with*** X if there exists a germ of a diffeomorphism $f\colon (Z, p) \to (Z, p)$ such that $f \mid X = \mathrm{id}_X$ (near p) and $f(\Lambda_1) = \Lambda_2$ (near p).

This notion and the following proposition are essentially due to Mather. See [6, Section 2.2].

PROPOSITION 2.1. *Λ_1 and Λ_2 have the same contact with X at p if $\mathscr{R}_1(X, \Lambda_1) = \mathscr{R}_1(X, \Lambda_2)$.*

The necessity of this condition is clear. We sketch a proof of the sufficiency. Choose coordinates $x_1, \ldots, x_k$ on X at p and a tubular neighborhood $U = \mathbb{R}^k \times \mathbb{R}^l$ of X near p so that X is identified with $\mathbb{R}^k \times \{0\}$ and p with 0. Do this so that Λ_1 and Λ_2 intersect $\{0\} \times \mathbb{R}^l$ transversely. Then locally we can write $\Lambda = \text{graph } b_i$ where $b_1, b_2\colon X \to \mathbb{R}^l$ are smooth functions. Let $y_1, \ldots, y_l$ be coordinates on $\mathbb{R}^l$ and let $b_i = (b_1^i, \ldots, b_l^i)$ in these coordinates. The functions $y_j - b_j^i(x)$ vanish on Λ_i for $j = 1, \ldots, l$ so that $\mathscr{I}_1(X, \Lambda_i) \supset (b_1^i, \ldots, b_l^i)$. An elementary argument shows equality; so $\mathscr{R}_1(X, \Lambda_i) = C_p^\infty(X)/(b_1^i, \ldots, b_l^i)$. The hypothesis that $\mathscr{R}_1(X, \Lambda_1) = \mathscr{R}_1(X, \Lambda_2)$ implies that the ideals $(b_1^1, \ldots, b_l^1) = (b_1^2, \ldots, b_l^2)$. Thus there exist smooth functions $g_{\alpha\beta}$ and $h_{\beta\gamma}$ where $1 \leqslant \alpha, \beta, \gamma \leqslant l$ so that

$$b_\alpha^1 = \sum_{\beta=1}^{l} g_{\alpha\beta} b_\beta^2 \qquad \text{and} \qquad b_\beta^2 = \sum_{\gamma=1}^{l} h_{\beta\gamma} b_\gamma^1.$$

Let G and H be the matrices $(g_{\alpha\beta})$ and $(h_{\beta\gamma})$. It is not hard to show that G and H can be chosen so that for all x near p, $G(x)$ and $H(x)$ are invertible. Now define $f\colon U \to U$ by $f(x, y) = H(x)\, y$. Then f is a diffeomorphism; since f is linear on fibers of U over X, $f \mid X = \mathrm{id}_X$; and f is constructed so that $f(\Lambda_1) = \Lambda_2$. So Λ_1 has the same contact with X at p as Λ_2.

Geometrically there is good reason to want to replace the above definition of contact equivalence by a slightly weaker definition. We will say that Λ_1 and Λ_2 have the *same contact with X at p in the generalized sense* if there is a germ of diffeomorphsim $f: (Z, p) \to (Z, p)$ such that $f(\Lambda_1) = \Lambda_2$ and $f(X) = X$ (rather than $f \mid X = \mathrm{id}_X$). Using this definition, if Λ_1 and Λ_2 have the same contact with X at p then $\mathscr{R}_1(X, \Lambda_1)$ is isomorphic to $\mathscr{R}_1(X, \Lambda_2)$, the isomorphism being induced by the pullback map f^* where $f^*\psi = \psi \circ f$ for ψ in $C_p^\infty(X)$. The converse is more difficult. It is not true that every isomorphism of $\mathscr{R}_1(X, \Lambda_1)$ into $\mathscr{R}_1(X, \Lambda_2)$ is realizable as the isomorphism induced by the pullback mapping f^* of some diffeomorphism f. On the other hand, if $\dim \mathscr{R}_1(X, \Lambda_i) < \infty$, $i = 1, 2$ then it is not hard to see that such a realization is possible. Given such an f, the proof of Proposition 2.2 sketched above goes through in this case also, and we have the following.

DEFINITION 2.3. *Λ has finite order of contact with X* at p if $\dim_{\mathbb{R}} \mathscr{R}_1(X, \Lambda) < \infty$.

PROPOSITION 2.4. *Let Λ_1 and Λ_2 have finite order of contact with X at p. Then Λ_1 and Λ_2 have the same contact with X at p (in the generalized sense) if and only if $\mathscr{R}_1(X, \Lambda_1) \simeq \mathscr{R}_1(X, \Lambda_2)$.*

Remark. To indicate what this finiteness hypothesis means, we mention the following result. If Λ has finite order of contact with X at p, then p is an isolated point of intersection of Λ with X.

It is natural to consider contact equivalence for restricted classes of submanifolds under pseudogroups other than the pseudogroup of all local diffeomorphisms. In the following section we consider the equivalence problem for Lagrangian submanifolds of symplectic manifolds with the pseudogroup being the pseudogroup of all local symplectic diffeomorphisms.

3. CONTACT EQUIVALENCE FOR THE SYMPLECTIC GROUP

Let Z be a symplectic manifold and X and Λ Lagrangian submanifolds tangent at 0 in Z. We will show that there is an element $\sigma \in \mathscr{R}_2 = \mathscr{R}_2(X, \Lambda)$ naturally associated with Λ. This element is not uniquely defined but it is uniquely defined up to an automorphism of $\mathscr{R}_2$. We denote the automorphism class by $\bar{\sigma}$ and call the pair $(\mathscr{R}_2, \bar{\sigma})$ the *symplectic contact data* at 0 associated with Λ. To define σ we choose a cotangent bundle structure on a neighborhood M of X in Z so that

$M = T^*X$, and $\Lambda = \text{graph } d\phi$ for some function ϕ on X. It is easy to see that

$$\mathscr{R}_i = C_0^\infty(X)\Big/\left(\frac{\partial\phi}{\partial x_1},\dots,\frac{\partial\phi}{\partial x_n}\right)^i$$

Now let σ be the image of ϕ in $\mathscr{R}_2$. We will prove that σ is well defined up to an automorphism of $\mathscr{R}_2$. Suppose we choose another cotangent bundle structure. Let α and β be the canonical one forms associated with the two cotangent bundle structures. Then $\alpha = \beta = 0$ on X and $\alpha - \beta$ is closed, so near X, $\alpha - \beta = dH$ for some H which vanishes to second order on X. Now let $k_{\alpha\beta}\colon X \to X$ be the diffeomorphism obtained by going from X to Λ via the α cotangent bundle structure then from Λ to X using the β structure. If $\Lambda = \text{graph } d\phi_\alpha$ in the α structure and $\Lambda = \text{graph } d\phi_\beta$ in the β structure then we claim:

LEMMA 3.1.

$$k_{\alpha\beta}^*\phi_\beta = \phi_\alpha + \sum h_{ij}(x, d\phi_\alpha)\frac{\partial\phi_\alpha}{\partial x_i}\frac{\partial\phi_\alpha}{\partial x_j}$$

where (in the α cotangent coord) $H = \sum h_{ij}(x, \xi)\,\xi_i\xi_j$.

Proof. Let $\tilde\phi_\alpha$ (resp. $\tilde\phi_\beta$) be the lift of ϕ_α (resp. ϕ_β) to Λ using the α (resp. β) structure. Note that the restriction of α to Λ is closed since Λ is a Lagrangian submanifold and that $\alpha = d\phi_\alpha$ on Λ. Similarly for β. Since $\beta - \alpha = dH$ we have that $\tilde\phi_\beta = \tilde\phi_\alpha + H$ on Λ. Now let p be in Λ and let p_α and p_β be the projections of p into X using the α and β cotangent bundle structures respectively. Then

$$\phi_\beta(p_\beta) = \phi_\alpha(p_\alpha) + \sum h_{ij}(p_\alpha, (d\phi_\alpha)(p_\alpha))\frac{\partial\phi_\alpha}{\partial x_i}(p_\alpha)\frac{\partial\phi_\alpha}{x_j}(p_\alpha)$$

which in view of the definition of $k_{\alpha,\beta}$ is the assertion above. Q.E.D.

By differentiating the equation above one sees that $k_{\alpha,\beta}^*$ maps the ideal of functions $(\partial\phi_\beta/\partial x_1,\dots,\partial\phi_\beta/\partial x_n)$ into the ideal $(\partial\phi_\alpha/\partial x_1,\dots,\partial\phi_\alpha/\partial x_n)$. But these ideals are identical, i.e., identical with the ideal $\mathscr{I}_1(X, \Lambda)$; so $k_{\alpha,\beta}^*$ induces an automorphism in $\mathscr{R}_2$ carrying σ_β, the representative of ϕ_β in $\mathscr{R}_2$ onto σ_α, the representative of ϕ_α. Q.E.D.

Let Λ and Λ' be Lagrangian manifolds that are tangent to X at 0, and let $(\mathscr{R}_2, \sigma)$ and $(\mathscr{R}_2', \sigma')$ be their contact data. We will show that if Λ and Λ' are contact equivalent via a symplectic diffeomorphism then their contact data are isomorphic; i.e., there exists a ring isomorphism $\gamma: \mathscr{R}_2 \cong \mathscr{R}_2'$ with $\gamma(\sigma) = \sigma'$. In fact, let $\rho: Z \to Z$ be a symplectic diffeomorphism leaving X fixed and carrying Λ into Λ'. If, for some cotangent bundle structure, $\rho = df^{\iota}$ where $f: X \to X$ is a diffeomorphism, then the symplectic data are isomorphic via the pullback isomorphism f^*. Therefore, we can assume ρ is the identity on X. Now let α be a one-form defining a cotangent bundle structure on a tubular nghd of X and let $\Lambda = \operatorname{graph} d\phi$ in this cotangent bundle structure. Then $\Lambda' = \operatorname{graph} d\phi$ in the cotangent bundle structure associated with $(\rho^{-1})^* \alpha$.

Our main result is that, in certain cases, the converse is true.

PROPOSITION 3.2. *Let Λ and Λ' be Lagrangian submanifolds of Z tangent to X at* 0. *Suppose Λ and Λ' have finite order of contact with X at* 0. *Suppose that their contact data are isomorphic, i.e., $(\mathscr{R}_2, \sigma) \cong (\mathscr{R}_2', \sigma')$. Then Λ and Λ' are contact equivalent.*

Proof. Since $\dim \mathscr{R}_2 < \infty$ any automorphism between $\mathscr{R}_2$ and $\mathscr{R}_2'$ can be realized by a diffeomorphism of X which in turn can be extended to a symplectic diffeomorphism of Z so we can assume, choosing a cotangent bundle structure on Z (i.e., $Z = T^*X$), that Λ is defined as graph $d\phi$ and Λ' as graph $d\phi'$ where

$$\phi'(x) = \phi(x) + \sum h_{ij}(x) \frac{\partial \phi}{\partial x_i}(x) \frac{\partial \phi}{\partial x_j}(x)$$

since $\mathscr{R}_2' = \mathscr{R}_2$ and $\phi' \bmod \mathscr{R}_2$.

Let α be the one form associated with the above cotangent bundle structure and define the function H on $Z = T^*X$ by $H(x, \zeta) = \sum h_{ij}(x) \zeta_i \zeta_j$. Now let $\beta = \alpha + dH$. Let $k_{\alpha\beta}$ be the diffeomorphism obtained by going from X to Λ via the α cotangent bundle structure and then from Λ to X via the β structure (note: there is no problem of β being a "graph" in the β structure because in both structures X is the zero section and Λ is tangent to X at 0.) By Lemma 3.1

$$k_{\alpha,\beta}^* \phi_\beta = \phi_\alpha + \sum h_{ij} \frac{\partial \phi_\alpha}{\partial x_i} \frac{\partial \phi_\alpha}{\partial x_j} = \phi + H = \phi'$$

where ϕ_β and $\phi = \phi_\alpha$ are the functions associated with Λ in the β and α cotangent bundle structures respectively.

Now let $\rho: Z \to Z$ be a symplectic mapping carrying β to α ($\beta = \rho^*\alpha$) and mapping X to X as the identity. Then ϕ_β is also the defining function on X for the Lagrangian manifold $\rho(\Lambda)$ using the α cotangent bundle structure. Finally if $\tau_{\alpha\beta}: Z \to Z$ is the map obtained by regarding Z as T^*X via the α cotangent bundle structure and inducing from $k_{\alpha\beta}: X \to X$, then $\phi' = k_{\alpha\beta}^*\phi_\beta$ is the function on X associated with $(\tau_{\alpha\beta} \circ \rho)(\Lambda)$ in the α cotangent bundle structure, i.e., $(\tau_{\alpha\beta} \circ \rho)(\Lambda) = \text{graph}\, d\phi'$ ("graph" means with respect to the α cotangent structure). But by assumption $\Lambda' = \text{graph}\, d\phi'$ so $\tau_{\alpha\beta} \circ \rho$ maps Λ to Λ'. Q.E.D.

In the next section we will investigate what contact equivalence with respect to the symplectic group means when the Lagrangian submanifolds are graph $d\phi$ and graph $d\psi$ in T^*X.

4. Right Equivalence

Let ϕ and ψ be germs of C^∞ functions at 0 in X with $\phi(0) = \psi(0) = d\phi(0) = d\psi(0) = 0$. Then ϕ and ψ are *right equivalent* if $\psi = \phi \cdot f$ where $f: (X, 0) \to (X, 0)$ is the germ of a diffeomorphism. In this section we will show that the problem of right equivalence for functions can be reduced to a problem of contact equivalence in symplectic geometry, thus recovering a result due to Tougeron [8, p. 209].

Given a germ ϕ in $C_0^\infty(X)$, let $\mathscr{I}_\phi$ be the ideal of first partials in $C_0^\infty(X)$; i.e., $\mathscr{I}_\phi = (\partial\phi/\partial x_1, \ldots, \partial\phi/\partial x_n)$. Let $\mathscr{R}_k(\phi) = C_0^\infty(X)/\mathscr{I}_\phi^k$ and let ϕ be the image of ϕ in this local ring. We say that ϕ satisfies the *Milnor Condition* if $\dim_{\mathscr{R}} \mathscr{R}_1(\phi) \leqslant \infty$.

Proposition 4.1. *Let ϕ and ψ be germs of functions at* 0 *in X satisfying the Milnor condition with $\phi(0) = \psi(0) = d\phi(0) = d\psi(0) = 0$. Then ϕ and ψ are right equivalent if*

(1) *The rank and signature of the Hessians $d^2\phi(0)$ and $d^2\psi(0)$ are equal, and*

(2) *There is an isomorphism $\gamma: \mathscr{R}_2(\phi) \to \mathscr{R}_2(\psi)$ such that $\gamma(\bar{\phi}) = \bar{\psi}$.*

The necessity is obvious. To prove the sufficiency we first make one reduction; namely we may assume that the Hessians $d^2\phi(0) = d^2\psi(0) = 0$. To see this we use the "relative Morse lemma" proved by Hormander [5, p. 138] and Gromoll and Meyers [3, p. 362]. Let k = signature of $d^2\phi(0)$ and l = rank of $d^2\phi(0)$. Then there exist coordinates $x_1, \ldots, x_n$ on X at 0 so that

$$\phi(x) = -(x_1^2 + \cdots + x_k^2) + x_{k+1}^2 + \cdots + x_l^2 + \phi'(x_{l+1}, \ldots, x_n)$$

where $d^2\phi'(0) = 0$. Similarly for ψ in some coordinate system $y_1, ..., y_n$. Thus $\psi \circ g(x) = -(x_1^2 + \cdots + x_k^2) + x_{k+1}^2 + \cdots + x_l^2 + \psi'(x_{l+1}, ..., x_n)$ where g is the change of coordinates from x to y. So ϕ and ψ are right equivalent if ϕ' and ψ' are right equivalent. Note that the isomorphism γ induces an isomorphism $\gamma'\colon \mathscr{R}_2(\phi') \to \mathscr{R}_2(\psi')$ such that $\gamma(\overline{\phi'}) = \overline{\psi'}$.

Note. $\mathscr{R}_2(\phi)$ may be used to recover the rank of $d^2\phi(0)$ (using the relative Morse lemma, for example) but cannot be used to recover the signature of $d^2\phi(0)$. Consider $\phi = x_1^2 + x_2^2$ and $\psi = x_1^2 - x_2^2$.

The following is sufficient to prove Proposition 4.1.

PROPOSITION 4.2. *Let ϕ and ψ be germs of smooth functions at 0 in X with $\phi(0) = \psi(0) = d\phi(0) = d\psi(0) = d^2\phi(0) = d^2\psi(0) = 0$. Then ϕ and ψ are right equivalent iff graph $d\phi$ and graph $d\psi$ have the same contact with the 0-section in T^*X with respect to the symplectic group.*

Proof of 4.1. Since ϕ and ψ satisfy the Milnor condition, graph $d\phi$ and graph $d\psi$ have finite order of contact with the 0-section at 0; and since $d^2\phi(0) = d^2\psi(0) = 0$, graph $d\phi$ and graph $d\psi$ are tangent at 0. Thus we may apply Proposition 3.2.

The proof which we shall give of Proposition 4.2 (and which we include here for completeness sake) is contained in Tougeron [8, p. 209] and Weinstein [10, Appendix].

Proof of 4.2. The necessity is obvious; so we just consider the sufficiency. Conjugating if necessary by a diffeomorphism of X we can assume (by Lemma 3.1) that

$$\psi = \phi + \sum h_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j}. \tag{4.3}$$

Let

$$\phi_t = \phi + t \sum h_{ij} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j}. \tag{4.4}$$

We will prove that there exists a germ of a diffeomorphism $f_t\colon (X, 0) \to (X, 0)$ depending smoothly on t such that

$$f_t^* \phi_t = \phi \qquad \text{for all} \quad 0 \leqslant t \leqslant 1. \tag{4.5}$$

Evaluating at $t = 1$ proves the proposition. Suppose that f_t exists. Then differentiating (4.5) with respect to t yields

$$\dot{\phi}_t(f_t) + \sum \frac{\partial \phi_t}{\partial x_i}(f_t) \dot{f}_t^i = 0 \tag{4.6}$$

where the dots indicate differentiation with respect to t and $f_t = (f_t^1, \ldots, f_t^n)$ in coordinates.

Evaluate (4.6) at f_t^{-1} to obtain

$$\phi_t + \sum \frac{\partial \phi_t}{\partial x_i} \dot{f}_t^i(f_t^{-1}) = 0. \tag{4.6'}$$

If we set

$$w(x, t) = \dot{f}_t(f_t^{-1}), \tag{4.7}$$

then the expression (4.6′) becomes

$$\dot{\phi}_t + \sum \frac{\partial \phi_t}{\partial x_i} w_i = 0. \tag{4.8}$$

Now note that (4.7) can be written as a system of ODE's

$$\dot{f}_t^i = w_i(f_t^1, \ldots, f_t^n, t) \qquad \text{for} \quad 1 \leqslant i \leqslant n$$

with initial data $f_0^i = x_i$. Thus if we can solve (4.8) for $w_i(x, t)$ with $w_i(0, t) = 0$, then this system can be solved for f_t on some nbhd of 0 in X and all t with $0 \leqslant t \leqslant 1$. With this f we get $(d/dt) f_t^* \phi_t = 0$, so $f_t^* \phi_t = f_0^* \phi_0 = \phi$.

We now return to (4.4) in order to find the w_i's needed for (4.8). Differentiating (4.4) with respect to t yields

$$\dot{\phi}_t = \sum h_{ij}(x) \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} \tag{4.9}$$

so $\dot{\phi}_t$ is in the ideal generated by $\partial \phi / \partial x_1, \ldots, \partial \phi / \partial x_n$.

We claim that this ideal is identical with the ideal generated by $\partial \phi_t / \partial x_1, \ldots, \partial \phi_t / \partial x_n$. In fact differentiate (4.4) with respect to x_l to get

$$\frac{\partial \phi_t}{\partial x_l} = \frac{\partial \phi}{\partial x_l} + t \sum \left[\frac{\partial h_{ij}}{\partial x_l} \frac{\partial \phi}{\partial x_i} \frac{\partial \phi}{\partial x_j} + 2 h_{ij} \frac{\partial^2 \phi}{\partial x_i \, \partial x_l} \frac{\partial \phi}{\partial x_j} \right] \tag{4.10}$$

and set

$$a_{lj} = \sum_{i=1}^{n} \left(\frac{\partial h_{ij}}{\partial x_l} \frac{\partial \phi}{\partial x_i} + 2 h_{ij} \frac{\partial^2 \phi}{\partial x_i \, \partial x_l} \right)$$

to obtain

$$\frac{\partial \phi_t}{\partial x_l} = \frac{\partial \phi}{\partial x_l} + t \sum_{j=1}^{n} a_{lj} \frac{\partial \phi}{\partial x_j} = \sum_{j=1}^{n} (\delta_{lj} + t a_{lj}) \frac{\partial \phi}{\partial x_j}. \tag{4.10'}$$

Note that $a_{ij}(0) = 0$ since the first and second derivatives of ϕ vanish at 0; so for x near 0 the matrix $(\delta_{ij} + ta_{ij})$ is invertible. Now we can write

$$\frac{\partial \phi}{\partial x_i} = \sum b_{ij}(x, t) \frac{\partial \phi_t}{\partial x_j} \qquad \text{for} \quad 0 \leqslant t \leqslant 1. \tag{4.11}$$

with $b_{ij}(0, t)$ being the identity matrix. This proves the claim.

Now substituting (4.11) into (4.9) yields

$$\dot{\phi}_t = \sum h_{jl} \frac{\partial \phi}{\partial x_j} b_{li} \frac{\partial \phi_t}{\partial x_i} \tag{4.12}$$

so defining

$$w_i(x, t) = -\sum h_{jl}(x) \frac{\partial \phi}{\partial x_j} b_{li}$$

yields (4.8). Note that $w_i(0, t) = 0$ since $(\partial\phi/\partial x_j)(0) = 0$. Q.E.D.

5. Examples

We now wish to give some examples to show that all of the conditions in Proposition 4.1 are necessary.

(I) Let $\phi(x, y) = x^5 + x^2y^2 + y^5$. Since ϕ has an isolated singularity at 0, $\dim_{\mathbb{R}} \mathscr{R}_1(\phi) < \infty$, so Proposition 4.1 applies. By a power series matching argument one can show that $\phi = a(\partial\phi/\partial x) + b(\partial\phi/\partial y)$ has no smooth solution; so $\bar{\phi} \neq 0$ in $\mathscr{R}_1(\phi)$. Let $\psi(x, y) = \phi(x, y(1 + x))$; by construction ϕ is right equivalent to ψ. A calculation shows that $\psi \neq \phi \bmod \mathscr{I}_\phi$. Thus in Proposition 4.1 one needs to assume $\mathscr{R}_2(\phi) \cong \mathscr{R}_2(\psi)$ not just $\mathscr{R}_2(\phi) = \mathscr{R}_2(\psi)$ since the constructed isomorphism is induced by $f(x, y) = (x, y(1 + x))$ and f^* does not induce the identity isomorphism on $\mathscr{R}_2(\psi)$. Also $\psi \neq \phi \bmod \mathscr{I}_\phi^2$ (since $\psi \neq \phi \bmod \mathscr{I}_\phi$); so one must consider orbits of $\bar{\phi}$ in $\mathscr{R}_2(\phi)$.

(II) Let $\phi(x, y) = x^4 + y^4$ and $\psi(x, y) = x^4 - y^4$. Clearly $\mathscr{R}_1(\phi) = \mathscr{R}_1(\psi)$ and $\dim_{\mathbb{R}} \mathscr{R}_1(\psi) < \infty$ so that Proposition 4.1 applies. By considering the zero sets of ϕ and ψ it is also clear that ϕ is not right equivalent to ψ. From this example we see that one must consider the second order information $\mathscr{R}_2(\psi)$ not just $\mathscr{R}_1(\psi)$. In particular $\bar{\phi}$ is not in the orbit of $\bar{\psi}$ in $\mathscr{R}_2(\psi)$.

(III) Finally for Lagrangian submanifolds, the problem of contact equivalence via diffeomorphisms is not the same as the problem of contact equivalence via symplectic diffeomorphisms. For example, let $\Lambda_\phi = \text{graph}(d\phi)$ and $\Lambda_\psi = \text{graph}(d\psi)$ where ϕ and ψ are as in (II). The contact rings $\mathscr{R}_1(\mathbb{R}^2, \Lambda_\phi)$ and $\mathscr{R}_1(\mathbb{R}^2, \Lambda_\psi)$ are the same so that Λ_ϕ and Λ_ψ are contact equivalent via diffeomorphisms using Proposition 2.4. However they are not contact equivalent via symplectic diffeomorphisms using Proposition 3.2.

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