Bifurcations on Fully Inhomogeneous Networks

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Abstract

Center manifold reduction is a standard technique in bifurcation theory, reducing the essential features of local bifurcations to equations in a small number of variables corresponding to critical eigenvalues. This method can be applied to admissible differential equations for a network, but it bears no obvious relation to the network structure. A fully inhomogeneous network is one in which all nodes and couplings can be different. For this class of networks there are general circumstances in which the center manifold reduced equations inherit a network structure of their own. This structure arises by decomposing the network into path components, which connect to each other in a feedforward manner. Critical eigenvalues can then be associated with specific components, and the network structure on the center manifold depends on how these critical components connect within the network. This observation is used to analyze codimension one and two local bifurcations. For codimension-1 only one critical component is involved, and generic local bifurcations are saddle-node and standard Hopf. For codimension two, we focus on the case when one component is downstream from the other in the feedforward structure. This gives rise to four cases: steady or Hopf upstream combined with steady or Hopf downstream. Here the generic bifurcations, within the realm of network-admissible equations, differ significantly from generic codimension-2 bifurcations in a general dynamical system. In each case we derive singularity-theoretic normal forms and unfoldings, present bifurcation diagrams, and tabulate the bifurcating states and their stabilities.

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1 Introduction

The structure and behavior of networks is a rapidly developing area with applications to many branches of science: Stewart (2004); Newman *et al.* (2006); Lu *et al.* (2016). The research literature now extends to many thousands of papers. The mathematical methods employed include graph theory, algebra, probability, combinatorics, topology, and extensive computer simulations.

A topic of considerable interest is the study of dynamics and bifurcations for networks of coupled dynamical systems, Stewart *et al.* (2003); Golubitsky *et al.* (2005); Golubitsky and Stewart (2006). In this context, a network is a directed graph whose edges are classified into distinct types, one for each type of coupling. In many examples, the network structure influences the dynamics that can be expected to occur generically, leading to behavior that does not arise generically in a general dynamical system. Therefore the methodology of modern nonlinear dynamics, Guckenheimer and Holmes (1983), although widely applicable, often has to be adapted to the network context before it can be used.

Each network architecture determines a class of *admissible maps* and associated *admissible differential equations (ODEs)*, which respect the structure of the network. The dynamics of each node is determined by the node itself, and all nodes from which it receives inputs. The type of coupling involved in the inputs is also taken into account. In particular, issues such as symmetry and synchrony can be studied systematically using this formalism.

One of the powerful methods of bifurcation theory is center manifold reduction, Carr (1981). This makes it possible to analyze steady-state and Hopf bifurcations analytically, using coordinate changes to determine local polynomial approximations that capture the bifurcation behavior. In general these coordinate changes are not well adapted to the network structure of coupled systems. In fact, at first sight there seems to be little connection between the network structure of a dynamical system and the structure of a center manifold reduction. We show that in certain specific circumstances such a connection exists, and it sometimes leads to unexpected bifurcations and dynamics. We examine this phenomenon in detail, with rigorous proofs based on singularity-theoretic normal forms; see for example Martinet (1982); Golubitsky and Schaeffer (1985).

A similar observation has been made for a very different class of networks. Rink and Sanders (2015, 2014a,b); Nijholt *et al.* (2016, 2017); Nijholt (2018) have developed an elegant approach to synchrony in networks, and technical issues concerning center manifold reduction, based on graph fibrations (see Boldi and Vigna (2002); Deville and Lerman (2015)). In particular, their results show that in some cases a center manifold reduction of a network system has a network structure of its own, inherited from the original network. Their viewpoint is algebraic, and it is most effective for a special class of homogeneous networks: networks where every node receives exactly one input of each of a specific list of types. Moreover, their strongest results apply to feedforward networks, in which there are no closed directed cycles.

Here we consider a class of networks that is almost the exact opposite of this special class: *fully inhomogeneous* networks, in which all nodes and arrows have distinct types, Golubitsky and Stewart (2017). The graph structure of such networks carries with it dynamical notions, notably admissible maps and systems of ODEs. For a fully inhomogeneous n-node network, the admissible ODEs are determined by the connections, and take the form

$$\dot{x}_j = f_j(x_j, x_{\sigma_j(1)}, \ldots, x_{\sigma_j(s_j)}) \qquad j = 1, \ldots, n \tag{1.1}$$

where $\sigma_j(1), \ldots, \sigma_j(s_j)$ enumerate the s_j nodes that connect to node j. Because all edges of the network have different types, the function f_j is arbitrary, subject to having the appropriate domain and range, and the x_j are elements of finite-dimensional real vector spaces.

Biochemical networks, gene regulatory networks, and food webs are examples of such systems. Therefore the dynamics and bifurcations for this class of networks deserve attention. Here we consider the two standard types of local bifurcations, steady-state and Hopf, on any dynamical system that is admissible for a fully inhomogeneous network. We also consider mode interactions, where two local bifurcations occur simultaneously at the same parameter values. We show that in mode interactions the center manifold determined by the critical eigenvalues inherits its own network structure.

These observations makes it possible to apply an appropriate version of singularity theory, adapted to the network of the center manifold and the type of local bifurcation, determining a normal form for the bifurcation and computing and its universal unfolding — a parametrized family of perturbations that captures the structure of all such families in a sense explained in Section 7. We do not include a distinguished bifurcation parameter as in Golubitsky and Schaeffer (1985), which would complicate the calculations considerably. Instead, the bifurcation parameters are included as universal unfolding parameters. It is also convenient to work with the special case in which all nodes have a one-dimensional phase space, which we take to be the real line $\mathbb{R}$. However, Appendix 11.4 shows that a network with multidimensional nodes can be reduced to one with one-dimensional nodes and the same admissible maps. Using this reduction, our results can be transferred directly to fully inhomogeneous networks for which node phase spaces have any finite dimension.

Organization of Paper

The main principle underlying this paper is introduced in Section 2. Local bifurcation in the dynamics of (1.1) occurs when the Jacobian has critical eigenvalues, which can be associated with specific path components. See Lemma 2.3. Center manifolds are in general not unique, but any choice captures the bifurcation structure. We show that with a suitable choice, the dynamics on the center manifold can be interpreted as a dynamical system for a simplified network. The remainder of the paper analyzes the most common local bifurcations, those of codimension one or two. The codimension is the minimal number of parameters for such a bifurcation to occur generically in a parametrized family, (Guckenheimer and Holmes, 1983, p. 122).

The generic codimension-1 steady-state and Hopf bifurcations are described in Section 3, without proofs at this stage. Theorem 3.1 describes steady-state bifurcation and Theorem 3.7 describes Hopf bifurcation. Abstractly these bifurcations are the same as those expected for a network with general vector fields, that is, with *all-to-all* coupling of the variables. This result is plausible, but the proofs involve some subtleties because a path component need not be all-to-all connected. We therefore postpone proofs to Section 6. The impact of the bifurcations on the full network is also considered; unsurprisingly, only the nodes within or downstream from the critical path component feel this influence. Thus, there are two kinds of codimension-1 bifurcation for each path component in the network - one for steady-state bifurcation and one for Hopf bifurcation.

The structure of center manifolds for the codimension-2 mode interaction bifurcations that we consider is described in Section 4. The main result is Theorem 4.3, which states that the dynamics on the center manifold is that of a two-node feedforward network.

The results of the bifurcation analyzes on the center manifold are stated in Section 5. The proofs are again postponed to later sections. The four mode interactions (steady-state/steady-state, Hopf/steady-state, steady-state/Hopf and Hopf/Hopf) are discussed in successive subsections. In each subsection we state the singularity-theoretic normal form for the mode interaction, compute its codimension, and obtain a universal unfolding. The equilibria and/or periodic states involved are tabulated, along with their stabilities, and bifurcation diagrams are presented.

Section 6 gives a proof of the codimension-1 bifurcation results. We first prove that, within the class of admissible maps, the eigenvalues of the Jacobian are generically simple (multiplicity 1). The proofs are presented for steady-state bifurcation in Section 6.2, and for Hopf bifurcation in Section 6.3. Both cases involve the construction of suitable admissible perturbations of the vector field for the linear analysis, and use Liapunov-Schmidt reduction to control nonlinear terms up to the relevant order.

In the codimension-1 case the singularity theory required to deduce the normal form is straightforward. The codimension-2 case requires more sophisticated ideas from singularity theory, because of the feedforward structure of the center manifold dynamics. We therefore outline the singularity theory needed for the codimension-2 theorems in Section 7. The proofs for the four possible codimension-2 mode interactions are given in Sections 8 – 11. They consist mainly of singularity-theoretic calculations of restricted tangent spaces (for the normal form) and tangent spaces (for the unfoldings).

Finally, Appendix A describes, in the context of fully inhomogeneous networks, a general construction that converts any network with higher-dimensional node phase spaces into an 'expanded' network with one-dimensional node phase spaces, without changing the space of admissible maps. This construction justifies our running assumption that node phase spaces are one dimensional, and implies that the same results are valid for general node phase spaces.

2 Path Components and Feedforward Structure

Our strategy is to give a systematic and general description of the constraints on center manifold reduced equations that are associated with mode interactions in fully inhomogeneous networks. To that end we enumerate a set of 'critical components' and associated 'central networks' that capture the possible bifurcations that can occur generically for any given inhomogeneous network. Throughout this paper, 'path' refers to a directed path.

Definition 2.1. Node q is *downstream* from node p if there exists a path from p to q. Node p is *upstream* from node q if q is downstream from p. Nodes p and q are *path equivalent*, denoted $p \sim q$, if node p is both upstream and downstream from node q.

Definition 2.2. A *path component* is an equivalence class of nodes under path equivalence. Path component Q is *downstream* from path component P if there exist a node p in component P and a node q in component Q such that q is downstream from p. Path component P is *upstream* from component Q if Q is downstream from P.

The notions of *downstream* and *upstream* are relational concepts that play a key role in determining the central network.

The path components are connected in a feedforward manner (the graph-theoretic term is 'acyclic': no closed path). This is well known in the theory of directed graphs, Schröder (2002). The directed graph induced on the components is called the *component graph* or *condensation* of the original network, Eppstein (2016). The proof is simple. Write $i \preceq j$ if there is a path from node i to node j (including the trivial path from i to itself). Then $\preceq$ is a preorder. The relation $i \sim j$ defined by $i \preceq j$ and $j \preceq i$ is an equivalence relation on nodes, and the path components are the equivalence classes. Now $\preceq$ induces a partial order on the set of equivalence classes. It is easy to prove inductively that there exists a total order on the nodes that is compatible with $\preceq$. That is, $i \preceq j$ implies $i \leq j$.

Consider a fully inhomogeneous n-node network with components $C_1, \ldots, C_m$ ordered in this manner. The partial order determines a feedforward structure on the path components. We can then order the nodes so that the Jacobian matrix of (1.1) is block lower triangular. To see this let $X_j \in \mathbb{R}^{\alpha_j}$ be the coordinates in the j-th path component, where α_j is the number of nodes in C_j. The coordinates of an admissible vector field for a network with m path components has the form

$$\dot{X}_j = F_j(X_j, X_1, \ldots, X_{j-1}) \qquad j = 1, \ldots, m \tag{2.1}$$

In general the F_j are not arbitrary, since they arise from (1.1) by collecting variables, nor are the F_j defined uniquely from the f_j. Lemma 2.3 then follows.

Lemma 2.3. The Jacobian matrix J of (2.1) at any point is block lower triangular with the form

$$J = \begin{bmatrix} J_1 & & & \\ * & J_2 & \mathbf{0} & \\ * & * & \ddots & \\ * & * & \cdots * & J_m \end{bmatrix} \tag{2.2}$$

where J_j is the $\alpha_j \times \alpha_j$ Jacobian matrix of the j^{th} path component.

The blocks in this decomposition are unique up to reordering nodes within each path component. The ordering of the blocks need not be unique, but it must be compatible with the feedforward partial order.

2.1 Critical Components

By (1.1), translation preserves admissibility, so we may translate coordinates so that any given equilibrium of (2.1) is at the origin. The form of J in (2.2) implies that the eigenvalues of J are the union of the eigenvalues of $J_1, \ldots, J_m$ (including multiplicities). Since the J_i are unique up to reordering of the component nodes, they are unique up to similarity. Therefore the critical eigenvalues are invariants.

Definition 2.4. At an equilibrium, the path component C_j is *critical* if an eigenvalue of J_j is on the imaginary axis.

For example, Lemma 2.3 implies that a codimension-2 Hopf/steady-state bifurcation can be associated with either one or two critical components. Moreover, when there are two critical components, we know which is Hopf and which is steady-state.

This paper classifies the behavior of all codimension-1 and certain codimension-2 local bifurcations on a given fully inhomogeneous network. Codimension-2 bifurcations can occur in two types: nonlinear degeneracies of a codimension-1 bifurcation or mode interactions occurring from the nonlinear interaction of two codimension-1 bifurcations. Moreover, in networks, mode interactions can occur in several ways, related to how the critical components lie within the network. Specifically, mode interactions can occur with two critical eigenvalues

- in the same critical component,
- in two critical components where one is downstream of the other,
- in two critical components where neither component is downstream of the other.

In this paper we consider in detail only the second possibility. We show that these codimension-2 bifurcations are qualitatively different from mode interactions in general systems of differential equations. The third possibility is easy to analyze: the two codimension-1 bifurcations are independent. We believe, but have not proved, that the first possibility behaves just like codimension-2 bifurcations in general systems.

2.2 The Central Network

For a given bifurcation from an equilibrium in a fully inhomogeneous network, we construct a central network and show that the dynamics on the center manifold (Carr (1981)) of the full network are isomorphic to the dynamics on the center manifold of the central network. See Theorem 2.10.

Definition 2.5. Suppose that the network $\mathcal{G}$ has at least one critical path component. The *central network* $\mathcal{C}$ of $\mathcal{G}$ is defined by the following:

(a) The path components of $\mathcal{C}$ are the path components of $\mathcal{G}$ that are both upstream from some critical component and downstream from some critical component.

(b) The arrows in $\mathcal{C}$ are the arrows of $\mathcal{G}$ that connect nodes in $\mathcal{C}$.

Remark 2.6. If $\mathcal{G}$ has only one critical path component, the central network $\mathcal{C}$ consists of the nodes in the critical component, and the arrows that connect nodes in that critical component. This follows since nodes are both upstream and downstream from the same path component if and only if they lie in that path component.

Lemma 2.7. *The central network can be constructed as follows:*

(a) Let $\mathcal{X}$ be the union of all path components that are not downstream from any critical path component.

(b) Let $\mathcal{Z}$ be the union of all path components in $\mathcal{G} \setminus \mathcal{X}$ that are not upstream from any critical component.

Then the nodes in the central network $\mathcal{C}$ consist of nodes in $\mathcal{G}$ that are not in $\mathcal{X} \cup \mathcal{Z}$. The arrows are those whose head and tail are in $\mathcal{C}$. The nodes in $\mathcal{G}$ decompose as a disjoint union

$$\mathcal{G} = \mathcal{X} \dot{\cup} \mathcal{C} \dot{\cup} \mathcal{Z} \tag{2.3}$$

Proof. By (a), $\mathcal{X} \cap \mathcal{C} = \emptyset$. By (b), $\mathcal{Z} \cap \mathcal{C} = \emptyset$. Hence the nodes in the central network are contained in the complement of nodes in $\mathcal{X} \cup \mathcal{Z}$. Conversely, nodes in the complement of $\mathcal{X}$ and $\mathcal{Z}$ are both upstream and downstream from some critical components, hence in $\mathcal{C}$. Finally, (b) implies that $\mathcal{X} \cap \mathcal{Z} = \emptyset$, so (2.3) holds. □

Using the notation in Lemma 2.7 we have:

Lemma 2.8. *(a) Tails of arrows in $\mathcal{G}$ whose heads are in $\mathcal{X}$ must also be in $\mathcal{X}$.*

(b) Tails of arrows in $\mathcal{G}$ whose heads are in $\mathcal{C}$ must be in either $\mathcal{C}$ or $\mathcal{X}$.

(c) Tails of arrows in $\mathcal{G}$ whose heads are in $\mathcal{Z}$ can be in any node in $\mathcal{G}$.

Label the nodes of $\mathcal{G}$ so that the first n_x are in $\mathcal{X}$, the last n_z are in $\mathcal{Z}$, and the remaining nodes are all in the central network $\mathcal{C}$. Then Lemma 2.8 implies that an admissible ODE for $\mathcal{G}$ has the form

$$\dot{X} = F(X) \tag{2.4a}$$
$$\dot{Y} = G(X, Y) \tag{2.4b}$$
$$\dot{Z} = H(X, Y, Z), \tag{2.4c}$$

where $X \in \mathbb{R}^{n_x}$, $Y \in \mathbb{R}^{n_y}$ and $Z \in \mathbb{R}^{n_z}$. Since all critical components are in the central network, the eigenvalues of the Jacobians $D_X F(0)$ and $D_Z H(0)$ all have nonzero real part.

Relabeling coordinates, we can assume that the bifurcation point of (2.4) is at $(0, 0, 0)$. Specifically, we assume $F(0) = 0$. Now (2.4) implies:

Theorem 2.9. *A center manifold of* (2.4) *is contained in the subspace* $X = 0$*. That is, the coordinates of nodes that are not downstream from any critical node are equal to* 0*.*

We will prove that the center manifold dynamics of the central network system

$$\dot{Y} = G(0, Y)$$

is conjugate to the center manifold dynamics of the vector field

$$\begin{aligned} \dot{X} &= 0 \\ \dot{Y} &= G(0, Y) \\ \dot{Z} &= H(0, Y, Z) \end{aligned} \tag{2.5}$$

on $\mathcal{G}$. Without loss of generality we can drop the dependence of G, H on the zero coordinates, leading to:

$$\dot{Y} = G(Y) \tag{2.6a}$$

$$\dot{Z} = H(Y, Z). \tag{2.6b}$$

Our goal is to prove that the dynamics on the center manifold of (2.6a) is conjugate to the dynamics of the center manifold of (2.6).

Denote the m-dimensional center subspace of (2.6) by $E^c_{y,z}$, and denote the m-dimensional center subspace of the central network with the vector field (2.6a) by E^c_y. Let $\pi_y : \mathbb{R}^{n_y} \times \mathbb{R}^{n_z} \to \mathbb{R}^{n_y}$ be projection, $\pi_y(Y, Z) = Y$. Denote an m-dimensional center manifold for (2.6) by $\mathcal{W}^c_{y,z}$ and let π be the restriction of π_y to $\mathcal{W}^c_{y,z}$.

Theorem 2.10. *(a) The projection of the center subspace for the original network is the center subspace of the central network. That is,* $\pi_y(E^c_{y,z}) = E^c_y$*.*

(b) The projection of a center manifold for the original network is a center manifold for the central network. That is, $\mathcal{W}^c_y \equiv \pi(\mathcal{W}^c_{y,z})$ *is a center manifold for the central network equations* (2.6a)*.*

(c) The dynamics on the central network center manifold $\mathcal{W}^c_y$ *are conjugate to the dynamics on the center manifold of the original network* $\mathcal{W}^c_{y,z}$*.*

Proof. Since all critical components are in the central network, $E^c_{y,z} \cap (\{0\} \times \mathbb{R}^{n_z}) = \{0\} \times \{0\}$. Hence, $(d\pi)_0 = \pi_y|_{E^c_{y,z}}$ is injective. Since $E^c_{y,z}$ and E^c_y have the same dimension, $\pi_y : E^c_{y,z} \to E^c_y$ is an isomorphism.

Injectivity of $(d\pi)_0$ implies that π is locally injective, so $\mathcal{W}^c_y \equiv \pi(\mathcal{W}^c_{y,z})$ is locally a submanifold. We claim that $\mathcal{W}^c_y$ is a center manifold for the central network. This is proved in two steps. First, we show that the tangent space of $\mathcal{W}^c_y$ at the origin is E^c_y. This follows from

$$T_0\mathcal{W}^c_y = T_0\pi(\mathcal{W}^c_{y,z}) = (d\pi)_0(T_0\mathcal{W}^c_{y,z}) = (d\pi)_0(E^c_{y,z}) = E^c_y,$$

where T_0 denotes the tangent space of a manifold at the origin.

Next, we show that $\mathcal{W}_y^c$ is flow-invariant under the vector field on the central network (2.6a). Let $\eta = (G, H)|_{\mathcal{W}_{y,z}^c}$ be the vector field (2.6) restricted to the chosen center manifold for the original network. By construction, the pushforward $\pi_*(\eta)$ leaves the submanifold $\mathcal{W}_y^c$ flow-invariant. By direct calculation $\pi_*(\eta) = G|_{\mathcal{W}_y^c}$.

Let Ψ_t be the flow of the vector field η on the center manifold $\mathcal{W}_{y,z}^c$ for the original network. Let Φ_t be the flow of $\pi_*\eta$ on the center manifold $\mathcal{W}_y^c$ for the central network. By the definition of pushforward, $\Phi_t = \pi\Psi_t\pi^{-1}$, so Φ and Ψ are conjugate. $\square$

2.3 Interpretation of Results

In applications of networks of coupled dynamical systems, what matters most is not the abstract nature of the dynamics (steady, periodic, quasiperiodic, chaotic, and so on) of the entire system, but the dynamics of individual nodes. Indeed, a major feature that distinguishes network dynamics from general dynamical systems theory is the presence of distinguished node variables.

We sketch the implications of our results in terms of the dynamics of individual nodes of the network. The discussion can be placed in the context of pattern formation: what is the pattern of dynamic behavior, described from the viewpoint of the nodes?

We give a fairly complete answer (with proofs) for the codimension-1 bifurcations, steady-state and Hopf. See Theorems 3.3 and 3.7. The result for Hopf bifurcation (all downstream nodes from the critical component oscillate) is consistent with the results in Golubitsky et al. (2010, 2012); Joly (2012). For the codimension-2 mode interactions considered in this paper, with two critical components, one being downstream from the other, we base our description on some plausible but sometimes unproved conjectures about the extent to which observing a single node can accurately reflect the dynamics of the system.

Filling in the details rigorously (and correcting them if necessary) offers much scope for future work. One feature of this paper deserves emphasis: the results apply to arbitrarily large (fully inhomogeneous) networks. Even when the number of nodes, hence state variables, is large, the most likely bifurcations have low codimension; roughly speaking, the smaller the codimension, the more common the bifurcation is likely to be.

Our results show that typical mode interactions when one critical component is downstream from the other differ from the corresponding mode interactions in general dynamical systems. In a general system, a steady-state/steady-state (Takens-Bogdanov) mode interaction can create periodic solutions. A Hopf/steady-state mode interaction can lead to 2-tori. A Hopf/Hopf mode interaction can lead to 3-tori. Since a center manifold reduction captures all of the dynamics of the full system of ODEs near the bifurcation point, our analysis shows that these extra frequency motions do not occur when the critical components are related in the feedforward manner assumed here.

This assertion follows from two principles. The main principle is heuristically reasonable: if a particular node A has a certain qualitative kind of dynamic behavior, then generically every node B downstream from A receives (either direct or indirectly) a signal from A. For example, if A oscillates periodically, then B receives an input with the same period. If the other inputs to B have dynamics similar to A, then B receives no other *conflicting* signals, so we expect B to oscillate periodically as well. (In effect, we can think of B as being 'forced' by its inputs, and consider the case where all inputs produce a consistent type of forcing. Steady inputs act like parameters and do not conflict with each other, or with periodic ones.) Although effects such as resonance might cause the period of B to differ from that of A, these would normally require higher codimension behavior, so we ignore such possibilities here. In the case of a codimension-1 Hopf bifurcation, which creates a periodic state of the whole system, the period of B should be the same as that of A. For similar reasons, the growth rates of particular states along bifurcating branches in a codimension-1 steady-state bifurcation should also be the same for A and B.

Potential complications arise when B is downstream from both critical components, because the incoming signals can interact. However, sufficiently close to the bifurcation point, solutions should be well described by appropriate linearized eigenfunctions, and this remark applies to the entire network. For example, if a node is forced by a periodic signal from the upstream critical component, and a quasiperiodic signal from the other component, these signals share a common frequency. This suggests that no node in the full network should exhibit more complicated dynamics than occurs in the normal form. In particular, steady states continue to act like parameters, so the difficult case is a Hopf-Hopf mode interaction. Generically, the two periods are incommensurable, so we expect B to behave quasiperiodically (invariant 2-torus) near the bifurcation point.

Consider first a codimension-1 bifurcation with critical component C. We partition nodes into two kinds: those downstream from C (including those in C) and the rest. Denote these sets of nodes by $\mathcal{D}$ and $\mathcal{R}$ respectively.

Nodes in $\mathcal{R}$ receive no signals from the critical components, so they do not 'feel' the bifurcation. Our standing assumption is that bifurcation occurs from an equilibrium, which is hyperbolic away from the center subspace. Therefore nodes in $\mathcal{R}$ remain steady, and the Implicit Function Theorem implies that when projected onto each such node, the equilibrium state moves smoothly with the bifurcation parameter λ. Thus the state of each node in $\mathcal{R}$ moves along a smooth path parametrized by λ, with typical growth rate $|\lambda|$.

Nodes in $\mathcal{D}$, on the other hand, do 'feel' the bifurcation. Consider first a steady-state bifurcation. Since we prove that generically this is a saddle-node, we expect a bifurcation diagram resembling a saddle-node in each node of $\mathcal{D}$, so the growth rate is $\sqrt{|\lambda|}$ and the branch folds over on itself like a parabola. For a Hopf bifurcation, all nodes in $\mathcal{D}$ should begin to oscillate, with a common period (determined by the relevant conjugate pair of imaginary eigenvalues).

In the codimension-1 case we can read off this behavior from the entries in a critical eigenvector, as in Theorem 3.3 for steady-state bifurcation and Theorem 3.7 for Hopf bifurcation. Thus we can make the above description rigorous in these cases.

We now come to codimension-2 mode interactions, with our standing assumption that one critical component C_2 is downstream from the other C_1. Now we partition nodes into four disjoint subsets:

(a) Subset $\mathcal{R}$: Nodes not downstream from either C_1 or C_2.

(b) Subset $\mathcal{D}_1$: Nodes downstream from C_1 but not from C_2.

(c) Subset $\mathcal{D}_2$: Nodes downstream from C_2 (which must therefore be downstream from C_1), but not downstream from C_1 by any directed path not passing through C_2.

(d) Subset $\mathcal{B}$: Nodes downstream from both C_1 and C_2, where some directed path from C_1 does not pass through C_2.

The definition implies that each of these sets is a union of transitive components of the network. See Figure 1, where for simplicity the only transitive components with more than one node are C_1, C_2, and a set of three white nodes at the top of the figure.

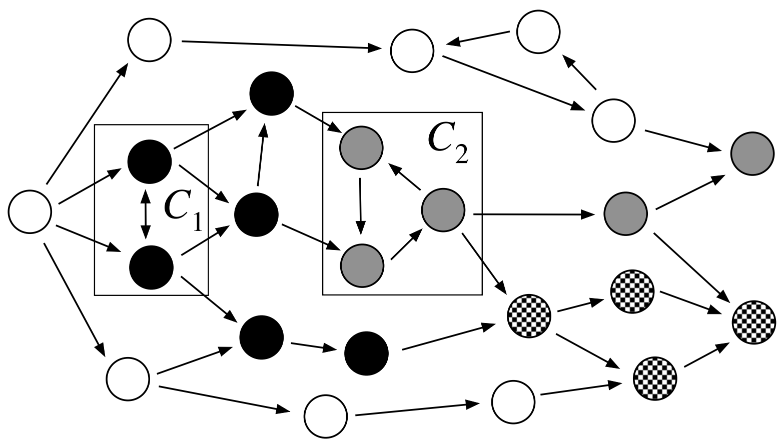

Figure 1: Partition of a 23-node network into 20 transitive components and four disjoint subsets indicated by color. White: $\mathcal{R}$. Black: $\mathcal{D}_1$. Grey: $\mathcal{D}_2$. Checkered: $\mathcal{B}$.

Now the (heuristic and unproved) principles that we assume govern the behavior are:

($\mathcal{R}$) Nodes in $\mathcal{R}$ are unaffected by the bifurcation. They thus remain in a steady state, with typical growth rate $|\lambda|$, because of the Implicit Function Theorem.

($\mathcal{D}_1$) Nodes in $\mathcal{D}_1$ all have the same qualitative behavior. If C_1 is steady-state, the behavior is like the codimension-1 steady-state case. If C_1 is Hopf, the behavior is like the codimension-1 Hopf case.

($\mathcal{D}_2$) Nodes in $\mathcal{D}_2$ all have the same qualitative behavior. This is described by the C_2-coordinate of the appropriate normal form. The signal from C_1 is built into the C_1-component of the normal form, and affects the C_2-component via the function occurring in the normal form.

($\mathcal{B}$) Nodes in $\mathcal{B}$ receive 'independent' signals from C_1 and C_2. Steady-state signals act like parameters. Periodic signals from the same subset of nodes induce periodic states of the same period, because the relevant Hopf bifurcation coordinates these signals; except that in the Hopf-Hopf case, nodes in $\mathcal{B}$ receive two different (coordinated sets of) periodic signals. In this case the most natural outcome (sufficiently close to the bifurcation point) is a two-period quasiperiodic state (invariant 2-torus).

Typically, in general systems of differential equations, universal unfoldings of codimension-2 mode interactions lead to solutions with additional frequencies. For example, Takens-Bogdanov singularities can perturb to periodic solutions and steady-state / Hopf mode interactions can lead to two frequency solutions. However, we show that solutions with these additional frequencies are not to be expected in network mode interaction unfoldings. This is a (perhaps surprising) expectation based on the theorems stated in Section 5.

For any given node in $\mathcal{D}_1$ or $\mathcal{D}_2$ the growth rate is expected to be the same as that given by the normal form, for any specific branch. For $\mathcal{B}$ in the Hopf-Hopf case, the growth rate is proportional to $\sqrt{|\lambda|}$ for each component oscillation, so the torus should have that growth rate in each direction. We do not expect anomalous growth rates of the kind discussed in Stewart and Golubitsky (2011); Stewart (2014) for steady-state bifurcation and Elmhirst and Golubitsky (2006); Golubitsky and Postlethwaite (2012) for Hopf bifurcation, because the networks concerned are homogeneous. (Also, in the steady-state case, they are highly artificial, with a small number of nodes connected by arrows with large multiplicities.)

In summary: our general results make it possible (conjecturally but plausibly) to predict the general type of dynamic behavior on each node of the network: whether it is steady, periodic, or quasiperiodic; how the periods concerned are related; and the growth rate of any particular branch. The ingredients for the prediction are the type of mode interaction and the associated singularity-theoretic normal form.

In any specific model, the components of the critical eigenvectors add further quantitative information. For example, when Hopf bifurcation is involved, these components determine the initial relative amplitudes and phases of the bifurcating branches. As observed earlier, the interpretation of our results for the behavior of individual nodes can be viewed as a description of the types of dynamic pattern formation that are associated with codimension-1 and codimension-2 bifurcations of the feedforward type considered in this paper.

3 Codimension One Bifurcations

We prove that for codimension-1 bifurcations, the eigenvalues at bifurcation are simple and the central network is always a single critical path component. The proof that the eigenvalues are simple requires careful analysis, discussed in Theorem 6.1 and Corollary 6.2. The nonlinear analysis is described in two distinct cases: steady-state (Section 3.1) and Hopf (Section 3.2).

3.1 Codimension One Steady-State Bifurcations

In Section 6.2 we prove two generic results about codimension-1 steady-state bifurcations of admissible differential equations, stated here as Theorems 3.1 and 3.3. First, these bifurcations are saddle-node; second, the growth rate of the equilibrium solution when viewed within any particular node is determined by the network architecture.

Theorem 3.1. *Generically, codimension-1 steady-state bifurcations on a fully inhomogeneous network are saddle-node bifurcations.*

Remarks 3.2. (a) The curve of equilibria emanating from a saddle-node bifurcation is tangent to the eigenvector v of the Jacobian at the bifurcation point. In particular, if a coordinate in v is nonzero, and assuming without loss of generality that the bifurcation occurs at $\lambda = 0$, then the growth rate of the zeros in that coordinate is of order $\sqrt{|\lambda|}$, where λ is the bifurcation parameter.

(b) This result is not always valid for networks that are not fully inhomogeneous; that is, where some arrows or nodes have the same type. The case of regular networks (all nodes and arrows are identical and each node has the same number of input arrows) is discussed in Leite and Golubitsky (2006); Golubitsky and Stewart (2011); Stewart (2014).

Theorem 3.3. *Assume the bifurcation occurs at $\lambda = 0$. For all of nodes within the critical path component, or nodes downstream from those, the growth rate of the equilibrium is $\sqrt{|\lambda|}$. The growth rate is at most $|\lambda|$ in all other components.*

Remark 3.4. It follows from Theorem 3.1 and Theorem 3.3 that generically the two solutions bifurcating from the saddle-node bifurcation have different coordinates on all nodes downstream from the critical components and the same values on all other nodes. In this sense there is a pattern hidden in the bifurcation based on which component is the critical component.

3.2 Codimension One Hopf Bifurcations

In Section 6.3 we prove two generic results about codimension-1 Hopf bifurcations of admissible differential equations for a fully inhomogeneous network, stated here as Theorems 3.6 and 3.7.

Definition 3.5. The system $\dot{X} = F(X, \lambda)$ has a *nondegenerate* Hopf bifurcation at the equilibrium X_0 if:

(a) The Jacobian $J = (d_X F)_{X_0,\lambda_0}$ has a complex conjugate pair of simple purely imaginary eigenvalues with all other eigenvalues off of the imaginary axis.

(b) The growth rate of the small amplitude periodic solutions is $\sqrt{\lambda}$.

The two theorems show that codimension-1 Hopf bifurcations are nondegenerate, and only the nodes downstream from a critical path component experience periodic motion.

Theorem 3.6. *Generically, codimension-1 Hopf bifurcation on a fully inhomogeneous network is nondegenerate.*

Theorem 3.7. *Hopf bifurcation yields periodic motion in all nodes in the critical path component H and any node downstream from H. The amplitude of this periodic motion has growth rate $\sqrt{|\lambda|}$. All other nodes remain constant and experience at most $|\lambda|$ growth rate.*

Remark 3.8. By Theorem 3.7, there is a unique type of Hopf bifurcation associated to each path component that can be critical, in the sense that the Jacobian on that path component can have imaginary eigenvalues. (This would not be possible, for example, if the component has a single node with one-dimensional phase space.) Theorem 3.7 implies that the type of Hopf bifurcation defines the set of nodes that generically oscillate. More specifically, bifurcating periodic solutions generically oscillate on on all nodes downstream from the critical component and are constant on all other nodes.

4 Center Manifolds for Codimension Two Bifurcations

There are three possible central networks for codimension-2 bifurcations: (1) a single critical path component, (2) two disconnected critical path components, and (3) two critical path components with one strictly downstream of the other and possibly nodes in between. As stated in the Introduction, we focus on case (3).

For case (3), suppose the network has two critical path components C_1 and C_2, with C_2 downstream from C_1. By Theorem 2.9, $\mathcal{W}^c$ for the full network is independent of coordinates of nodes that are not downstream from any critical node. Hence these non-downstream nodes can be eliminated by fixing the corresponding coordinates at equilibrium 0. Ignoring the non-upstream nodes, the admissible vector field on the central network takes the general form

$$\dot{x}_1 = f_1(x_1) \tag{4.1a}$$
$$\dot{x}_2 = f_2(x_1, x_2) \tag{4.1b}$$
$$\dot{x}_3 = f_3(x_1, x_2, x_3) \tag{4.1c}$$

where for convenience we have dropped the dependence of F_i on the zero coordinates for the non-downstream nodes. Here $x_1 \in \mathbb{R}^{m_1}$ and $x_3 \in \mathbb{R}^{m_3}$ are coordinates for the nodes in C_1 and C_2, respectively, while coordinates $x_2 \in \mathbb{R}^{m_2}$ correspond to nodes downstream from C_1 and upstream from C_2. By assumption, $f_1(0) = f_2(0,0) = f_3(0,0,0) = 0$. Throughout, we write $\mathrm{D}f$ for the derivative of a map f and $\mathrm{D}_j f$ for the partial derivative with respect to the jth variable. Sometimes we also denote the relevant variable by a subscript, as in $\mathrm{D}_x f$, which is $\partial f/\partial x$. The linearization about 0 is

$$J = \begin{bmatrix} J_1 & 0 & 0 \\ \mathrm{D}_1 f_2 & J_2 & 0 \\ \mathrm{D}_1 f_3 & \mathrm{D}_2 f_3 & J_3 \end{bmatrix}$$

where both $J_1 = \mathrm{D}f_1(0)$ and $J_3 = \mathrm{D}_3 f_3(0,0,0)$ are singular with critical eigenvalues while $J_2 = \mathrm{D}_2 f_2(0,0)$ is nonsingular. In the following, we first focus on the central network, and then explore how dynamics on the central network affects nodes downstream. We show that the flow restricted to the center manifold of (4.1) has a feedforward structure and depends only on the two critical components. This is proved in Theorem 4.3, but first we need two lemmas.

Lemma 4.1. *The flow of* (4.1) *can be written as*

$$\Phi_t(x_1, x_2, x_3) = (\phi_{1t}(x_1), \phi_{2t}(x_1, x_2), \phi_{3t}(x_1, x_2, x_3))$$

Proof. The feedforward structure of (4.1) implies that the flow of x_1 is independent of x_2 and x_3, while the flow of x_2 is independent of x_3. □

Assume that the center subspaces of J_1 and J_3 are the n_1-dimensional subspace E_1^c and the n_2-dimensional subspace E_2^c, respectively. Let $\pi_1(x_1, x_2, x_3) = (x_1, 0, 0)$ be projection onto the first coordinate of $\mathbb{R}^{m_1} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3}$.

Let $\nu_1^c \times \{0\} \times \{0\}$ be an n_1-dimensional center manifold of (4.1a) in $\mathbb{R}^{m_1} \times \{0\} \times \{0\}$. It follows from Lemma 4.1 and the fact that ν_1^c is flow-invariant for (4.1a) that $\pi_1^{-1}(\nu_1^c \times \{0\} \times \{0\}) = \nu_1^c \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3}$ is flow-invariant for (4.1). Therefore we can choose an (n_1+n_2)-dimensional center manifold $\mathcal{W}^c$ for (4.1) in $\pi_1^{-1}(\nu_1^c \times \{0\} \times \{0\})$ such that $\pi_1(\mathcal{W}^c) = \nu_1^c \times \{0\} \times \{0\}$.

Let ν_2^c be an n_2-dimensional center manifold of (4.1c) on $x_1 = x_2 = 0$. Since (4.1a)-(4.1b) have 0 as fixed point, $\{0\} \times \{0\} \times \nu_2^c$ is flow-invariant. We can choose $\mathcal{W}^c$ so that

$$\{0\} \times \{0\} \times \nu_2^c \subseteq \mathcal{W}^c$$

is a submanifold.

Lemma 4.2. *The manifold $\nu_1^c \times \{0\} \times \{0\}$ is a submanifold of $\mathcal{W}^c$, and $\mathcal{W}^c$ is a fiber bundle over the base $\nu_1^c \times \{0\} \times \{0\}$ with fibers isomorphic to $\{0\} \times \{0\} \times \nu_2^c$.*

Proof. First, we show

$$\nu_1^c \times \{0\} \times \{0\} \subseteq \mathcal{W}^c$$

by verifying that

$$\nu_1^c \times \{0\} \times \{0\} = \mathcal{W}^c \cap (\mathbb{R}^{m_1} \times \{0\} \times \{0\}).$$

To this end, we define

$$\hat{\nu}_1^c = \mathcal{W}^c \cap (\mathbb{R}^{m_1} \times \{0\} \times \{0\})$$

and note that $\hat{\nu}_1^c$ is n_1-dimensional manifold. Indeed, the center manifold theorem lets us coordinatize the center manifold $\mathcal{W}^c$ of the network by its center subspace E^c. Now $\hat{\nu}_1^c$ is the slice of $\mathcal{W}^c$ along the direction that contains the n_1-dimensional subspace E_1^c, so $\dim(\hat{\nu}_1^c) = n_1$.

Moreover, π_1 is the identity on $\mathbb{R}^{m_1} \times \{0\} \times \{0\}$, so $\pi_1(\hat{\nu}_1^c) = \hat{\nu}_1^c$. Therefore

$$\begin{aligned} \hat{\nu}_1^c &= \pi_1(\mathcal{W}^c \cap (\mathbb{R}^{m_1} \times \{0\} \times \{0\})) \\ &\subseteq \pi_1(\mathcal{W}^c) \cap \pi_1((\mathbb{R}^{m_1} \times \{0\} \times \{0\})) \\ &= (\nu_1^c \times \{0\} \times \{0\}) \cap ((\mathbb{R}^{m_1} \times \{0\} \times \{0\})) \\ &= \nu_1^c \times \{0\} \times \{0\} \end{aligned} \tag{4.2}$$

Since $\hat{\nu}_1^c$ and $\nu_1^c \times \{0\} \times \{0\}$ are manifolds with the same dimension, they must be the same, so $\nu_1^c \times \{0\} \times \{0\} \subseteq \mathcal{W}^c$.

Second, choose $(z_1, 0, 0)$ and $(0, 0, z_2)$ on $\nu_1^c \times \{0\} \times \{0\}$ and $\{0\} \times \{0\} \times \nu_2^c$, respectively. Write $\mathcal{W}^c$ as a fiber bundle with base $\nu_1^c \times \{0\} \times \{0\}$. For each z_1, define the fiber over z_1 as

$$\mathcal{U}_{z_1} = \mathcal{W}^c \cap (\{z_1\} \times \mathbb{R}^{m_2} \times \mathbb{R}^{m_3}) = \{z \in \mathcal{W}^c : \pi_1(z) = (z_1, 0, 0)\}.$$

□

Since $\mathcal{W}^c$ is a fiber bundle, for each z_1 there exists a map

$$\rho : (z_1, \{0\} \times \{0\} \times \nu_2^c) \to \mathcal{U}_{z_1}$$

so $\rho(z_1, z_2) \in \mathcal{U}_{z_1}$. We have $\rho(0, z_2) = \{0\} \times z_2$ and $\rho(z_1, 0) = \{0\} \times \{0\}$. While $\rho(z_1, z_2) \in \mathcal{U}_{z_1}$ is isomorphic to $\{0\} \times \{0\} \times \nu_2^c$, it may have component in coordinate x_2, which we denote by $\rho_2(z_1, z_2)$. The following result is analogous to one proved in Golubitsky and Postlethwaite (2012):

Theorem 4.3. *The dynamics on the center manifold $\mathcal{W}^c$ of (4.1) can be written on $\nu_1^c \times \nu_2^c$ as*

$$\dot{z}_1 = g_1(z_1) \tag{4.3a}$$
$$\dot{z}_2 = g_2(z_1, z_2) \tag{4.3b}$$

for some functions g_1 and g_2 and coordinates $z_1 \in \nu_1^c$ and $z_2 \in \nu_2^c$.

Proof. Coordinatize the flow on $\mathcal{W}^c$ with the map $P : \nu_1^c \times \nu_2^c \to \mathcal{W}^c$ defined by

$$P(z_1, z_2) = (z_1, \rho(z_1, z_2))$$

where $\rho(0, z_2) = \{0\} \times \{0\} \times z_2$. Clearly P is invertible, with inverse

$$P^{-1}(z_1, \tilde{z}_2) = (z_1, \sigma(z_1, \tilde{z}_2))$$

where σ satisfies

$$\sigma(z_1, \rho(z_1, z_2)) = z_2.$$

In particular,

$$\sigma(0, \rho(0, z_2)) = \sigma(0, 0, z_2) = z_2.$$

Denote the flow on $\nu_1^c \times \nu_2^c$ by $\Psi_t(z_1, z_2)$. Then

$$\begin{array}{rcl} \Psi_t(z_1, z_2) & = & P^{-1}\Phi_t P(z_1, z_2) \\ & = & P^{-1}\Phi_t(z_1, \rho(z_1, z_2)) \\ & = & P^{-1}(\phi_{1t}(z_1), \phi_{2t}(z_1, \rho_2(z_1, z_2)), \phi_{3t}(z_1, \rho(z_1, z_2))) \\ & = & (\phi_{1t}(z_1), \sigma(\phi_{1t}(z_1), (\phi_{2t}(z_1, \rho_2(z_1, z_2)), \phi_{3t}(z_1, \rho(z_1, z_2))))). \end{array} \tag{4.4}$$

The flow of the first coordinate is independent of z_2, as required. □

5 Codimension Two Mode Interactions

In this section we summarize the main results for codimension-2 bifurcations when the central network contains two critical path components with one downstream of the other (and possibly nodes in between). Throughout this section we analyze the dynamics of each system using the center manifold network associated to the given central network. Section 4 shows that the center manifold network inherits the feedforward structure of the critical components in the central network. This leads to four possible mode interactions, defined by whether the eigenvalues of each critical component corresponds to steady-state or Hopf bifurcation.

The feedforward structure of the center manifold network leads to generic behavior of the mode interactions that is different from generic behavior in the context of general vector fields (which arise in networks with all-to-all coupling).

Remark 5.1. One manifestation of this difference is the existence of a new type of solution in the center manifold network for the four mode interactions. In the steady-state / steady-state and steady-state / Hopf mode interactions these solutions are invariant sets where the coordinate of the upstream node is constant and the coordinates of the downstream node are not. The flow-invariant set on which these solutions exist can act as a boundary that other trajectories cannot cross, thereby partitioning phase space. $\diamond$

For each of the four mode interactions described in this section, we begin with the center manifold vector field, identify a singularity-theoretic normal form and its universal unfolding, and classify the small amplitude steady-state and periodic solutions as a function of unfolding parameters. We discuss singularity theory in Section 7, which summarizes all the required concepts and results. However, we give brief indications of the key steps as we proceed.

5.1 Steady-State/Steady-State Mode Interaction

In the steady-state/steady-state mode interaction, the Jacobian associated with each critical component of the original network has a single zero eigenvalue, and each critical component corresponds to a one-dimensional phase space on the center manifold. In this case the vector field on the center manifold has the form:

$$\begin{aligned} \dot{x} &= f(x) \\ \dot{y} &= g(x, y) \end{aligned} \tag{5.1}$$

where $x, y \in \mathbb{R}$. We assume that the origin is an equilibrium, so $f(0) = g(0, 0) = 0$. The Jacobian of (5.1) has two zero eigenvalues at the origin, so $f_x(0) = g_y(0, 0) = 0$.

The goal is to identify a normal form and show that any vector field of the form (5.1), with the associated defining conditions, is equivalent to that normal form, assuming suitable nondegeneracy conditions. We define equivalence in terms of transformations that preserve the center manifold structure; that is, the variable dependence of the functions for each center manifold node. To apply singularity theory we must define when two bifurcation problems in (5.1) are (strongly) equivalent.

Definition 5.2. Maps $F(x,y) = (f(x), g(x,y))$ and $\hat{F}(x,y) = (\hat{f}(x), \hat{g}(x,y))$ are *strongly equivalent* if there exist $a(x), \phi(x), b(x,y), c(x,y), \psi(x,y)$ such that

$$\begin{bmatrix} \hat{f}(x) \\ \hat{g}(x,y) \end{bmatrix} = \begin{bmatrix} a(x) & 0 \\ b(x,y) & c(x,y) \end{bmatrix} \begin{bmatrix} f(\phi(x)) \\ g(\phi(x), \psi(x,y)) \end{bmatrix} \tag{5.2}$$

where $\phi(0) = \psi(0,0) = 0$ and $a(0), c(0,0), \phi_x(0), \psi_y(0,0) > 0$.

In Section 8 we prove the following:

Theorem 5.3. *Assume that* (5.1) *satisfies the defining conditions*

$$f(0) = f_x(0) = g(0,0) = g_y(0,0) = 0 \tag{5.3}$$

and the nondegeneracy conditions

$$g_x(0,0) \neq 0,\ f_{xx}(0) \neq 0,\ g_{yy}(0,0) \neq 0. \tag{5.4}$$

Then $F(x,y) = (f(x), g(x,y))$ is strongly equivalent to the normal form $\hat{F}(x,y)$ given by

$$\begin{array}{rcl} \hat{f}(x) & = & \varepsilon_p x^2 \\ \hat{g}(x,y) & = & \varepsilon_s x + \varepsilon_t y^2, \end{array} \tag{5.5}$$

where

$$\varepsilon_s = \operatorname{sign}(g_x(0,0)),\ \varepsilon_p = \operatorname{sign}(f_{xx}(0)),\ \varepsilon_t = \operatorname{sign}(g_{yy}(0,0)).$$

Theorem 5.3 is useful only if the set of admissible vector fields with a steady-state / steady-state mode interaction that satisfies the mode interaction degeneracy conditions (5.3) includes vector fields that satisfy the nondegeneracy conditions (5.4). We prove more:

Proposition 5.4. *Consider two path components in a network such that one is downstream of the other and a pair of functions $f : \mathbb{R} \to \mathbb{R}$ and $g : \mathbb{R}^2 \to \mathbb{R}$ such that $f(0) = f'(0) = 0$ and $g(0,0) = g_y(0,0) = 0$. Then there exists an admissible vector field F whose center manifold restriction is given by* (5.1)*, where f is the component of the vector field associated to the upstream path component and g is the component of the vector field associated to the downstream path component.*

Proof. Given two path components with one downstream of the other, there exists a path consisting of $n+1$ distinct nodes that connects a node p_0 in the upstream path component to a node p_n in the downstream path component. We denote the nodes along this path as p_i for $i = 1, \ldots, n-1$ and denote the remaining nodes in the network by p_s where $s > n$.

We construct an admissible vector field in the following way. Associate the coordinate $z_j \in \mathbb{R}$ with the node p_j and define the admissible vector field F by

$$\begin{array}{rcll} \dot{z}_0 & = & f(z_0) & \\ \dot{z}_i & = & -z_i + z_{i-1} + f(z_i) & 1 \leq i < n \\ \dot{z}_n & = & g(z_{n-1}, z_n) & \\ \dot{z}_s & = & -z_s & s > n \end{array} \tag{5.6}$$

for these nodes.

We have assumed that the origin is an equilibrium; that is, $f(0) = 0$ and $g(0,0) = 0$. We have also assumed that the Jacobian of F evaluated at the origin is lower triangular with two zero eigenvalues (that is, $f'(0) = g_{z_n}(0,0) = 0$) and the remaining eigenvalues equal to -1. The center subspace E^c is spanned by a vector w with single nonzero component $w_n = 1$ and a vector v with nonzero components $v_i = 1$ for $i = 0, \ldots, n-1$. The center subspace can be parametrized by coordinates $u^c = xv + yw \in E^c$ where $x, y \in \mathbb{R}$.

By construction the admissible vector field F leaves the center subspace invariant; hence the center subspace is in fact a center manifold for F. Indeed, the restriction of F to E^c is precisely the vector field (5.1). □

Remark 5.5. Proposition 5.4 implies that generically admissible vector fields that satisfy the degeneracy conditions (5.3) also satisfy the nondegeneracy conditions (5.4).

Universal unfoldings classify perturbations up to equivalence. Recall that when computing universal unfoldings, we relax the restrictions in Definition 5.2 that $\phi(0)$ and $\psi(0,0)$ vanish.

Theorem 5.6. *The normal form* (5.5) *has codimension two and a universal unfolding is*

$$\begin{aligned} f(x,\lambda) &= \lambda + \varepsilon_p x^2 \\ g(x,y,\mu) &= \mu + \varepsilon_s x + \varepsilon_t y^2. \end{aligned} \tag{5.7}$$

Remark 5.7. 1) Since the Jacobian is always lower triangular, Hopf bifurcation cannot occur in the universal unfolding (5.7). This contrasts with the codimension-2 Takens-Bogdanov singularity (a steady-state steady-state mode interaction) where we expect both periodic solutions and homoclinic orbits to occur, Guckenheimer and Holmes (1983). Neither of these solution types appears in (5.7).

2) On the other hand, solutions of the type described in Remark 5.1 exist in the universal unfolding (5.7). These solutions are invariant lines where $x(t)$ is constant and $y(t)$ is not. They appear in pairs where one is stable in the x-direction and the other is unstable in that direction. These pairs of solutions are 'heteroclinic-like' in that solutions with initial conditions between the two invariant lines converge to one line in forward time and to the other line in backward time.

To state the next result we must briefly discuss the concept of a transition variety. Given a k-parameter family of ODEs parametrized by $\lambda = (\lambda_1, \ldots, \lambda_k) \in \mathbb{R}^k$, generically the set of equilibria or periodic states has the same topology throughout a neighborhood of a point λ_0. Bifurcations occur when this statement is false. It is often possible to classify the relevant points λ_0 according to the type of bifurcation that occurs. For each type of bifurcation, the relevant points λ_0 form the corresponding *transition variety* or *bifurcation set*. See Golubitsky and Schaeffer (1985) Section III.5 in the similar context of qualitative changes to bifurcation diagrams.

We now compute the steady-state solutions as a function of the parameters μ and λ. Changing $\varepsilon_p, \varepsilon_s, \varepsilon_t$ to $-\varepsilon_p, -\varepsilon_s, -\varepsilon_t$ rotates the transition variety in the $\mu\lambda$-plane by 180° and changes the stability of each steady-state in both the x and y directions. Hence we can assume $\varepsilon_p = -1$ and consider the four cases $\varepsilon_s = \pm 1$, $\varepsilon_t = \pm 1$.

Case 1: $\varepsilon_p = \varepsilon_s = \varepsilon_t = -1$. The saddle-node part of the transition variety in parameter space occurs when the Jacobian matrix

$$J = \begin{bmatrix} -2x & 0 \\ -1 & -2y \end{bmatrix}$$

at an equilibrium has a zero eigenvalue. The first case is $x = 0$, which leads to the half line $\lambda = 0, \mu \geq 0$. The second case $y = 0$ leads to the parabola $\lambda = \mu^2$, Figure 2 (left). Each component of the transition variety corresponds to a saddle-node bifurcation.

There are no steady states for $\lambda < 0$; two steady states inside the parabola; and four steady states for parameters in the region between the parabola and the half line. These steady states and their stabilities are listed in Table 1.

	Region	Equilibria (x, y)	Eigenvalues
I_-	$0 < \lambda < \mu^2$	$(\sqrt{\lambda}, y)$	$-*$
	$\mu < 0$; $y \in \mathbb{R}$	$(-\sqrt{\lambda}, y)$	$+*$
II	$0 < \lambda < \mu^2$	$\left(\sqrt{\lambda}, +\sqrt{\mu - \sqrt{\lambda}}\right)$	$--$
	$0 < \mu$	$\left(\sqrt{\lambda}, -\sqrt{\mu - \sqrt{\lambda}}\right)$	$-+$
		$\left(-\sqrt{\lambda}, +\sqrt{\mu + \sqrt{\lambda}}\right)$	$+-$
		$\left(-\sqrt{\lambda}, -\sqrt{\mu + \sqrt{\lambda}}\right)$	$++$
III	$\mu^2 < \lambda$	$\left(-\sqrt{\lambda}, +\sqrt{\mu + \sqrt{\lambda}}\right)$	$+-$
		$\left(-\sqrt{\lambda}, -\sqrt{\mu + \sqrt{\lambda}}\right)$	$++$

Table 1: Steady-state/Steady-state mode interaction: Case 1. List of equilibria (column 3) and their stabilities (column 4) in each of the regions (column 1) in Figure 2 (left). The entries in I_- column 3 list the invariant lines.

The transition varieties here are the boundaries between the four regions. The transitions across these boundaries are conveniently illustrated by the *circulant (or gyrant) bifurcation diagram* of Figure 2 (right). To obtain this diagram we start in region $\mathbf{I}_+$ ($\lambda < 0$) and plot the equilibria, following a counterclockwise circle around the origin in the $\mu\lambda$-plane. It is also helpful to track x-invariant sets defined by $f(x) = 0$, in addition to the steady states.

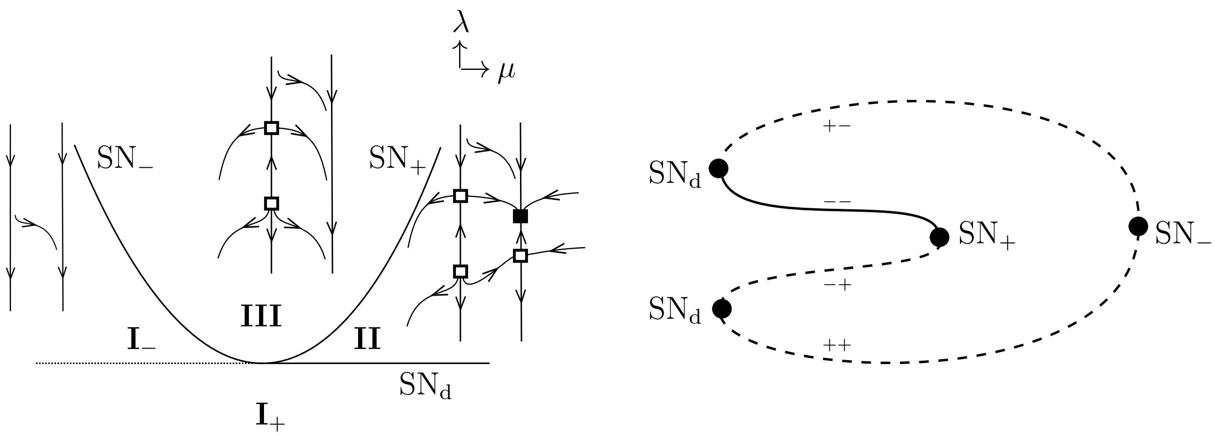

Figure 2: Steady-state/steady-state mode interactions: Case 1: $\varepsilon_p = \varepsilon_s = \varepsilon_t = -1$. *Left*: Transition variety in the $\mu\lambda$ parameter plane, with phase portraits in the xy phase plane in each connected region of parameter space. Solid curves $\mathrm{SN_d}$, SN_+ and SN_- represent a double saddle-node and two single saddle-node transition curves for (5.7). The straight vertical lines in the phase portraits indicate the existence of invariant lines defined by $f(x) = 0$. The dotted half-line indicates a saddle-node of invariant lines. *Right*: Circulant bifurcation diagram as a small circle is traversed counterclockwise around the origin in the $\mu\lambda$-plane. The plus and minus signs show the stability of each branch in the x and y directions.

In region I_+ there are no steady states or x-invariant sets in the phase space. Moving from region I_+ to region II across the transition line $\mathrm{SN_d}$, two pairs of steady states appear, each pair appearing simultaneously through a saddle-node bifurcation. This transition results in four steady-state solutions, Table 1 column 3. The set of four equilibria is invariant under reflection $y \to -y$.

Moving from region II to region III across the transition line SN_+ results in the loss of the two equilibria with $x > 0$ through a saddle-node bifurcation. At the transition curve SN_+ there are three steady-state solutions. In region III, only the two steady states remain; however, an x-invariant line defined by $x = \sqrt{-\lambda}$ persists as a remnant of the pair of steady states from region II that have been lost.

The remaining pair of steady-state solutions in region III disappears through a saddle-node bifurcation when crossing the transition curve SN_-. Along SN_- both a single equilibrium at $(-\sqrt{\lambda}, 0)$ and an x-invariant line given by $x = \sqrt{\lambda}$ exist. In I_- no steady states exist. However, in this region two x-invariant lines given by $x = \pm\sqrt{\lambda}$ partition the phase space into three regions. Finally, as λ decreases through 0, the two x-invariant lines disappear through a saddle-node of $f(x)$ (not a saddle-node of the entire system).

Case 2: $\varepsilon_p = -1$, $\varepsilon_s = \varepsilon_t = -1$. Reversing the signs of ε_s and ε_t maps $\mu \to -\mu$ in the bifurcation diagram; that is, reflects the 2-parameter bifurcation diagram in Figure 2 about the λ-axis. The set of equilibria is preserved under the transformation $\varepsilon_s \to -\varepsilon_s$, $\varepsilon_t \to -\varepsilon_t$ and $\mu \to -\mu$, but the stability of these fixed points changes in the y-direction, while remaining the same in the x-direction.

Case 3: $\varepsilon_p = -1$, $\varepsilon_s = 1$, $\varepsilon_t = -1$. $\varepsilon_s \to -\varepsilon_s$ does not affect the bifurcation structure in the $\mu\lambda$-plane, nor the stability of any equilibrium.

Case 4: $\varepsilon_p = -1$, $\varepsilon_s = -1$, $\varepsilon_t = 1$. $\varepsilon_t \to -\varepsilon_t$ corresponds to transforming $\mu \to -\mu$ in the bifurcation diagram.

5.2 Hopf/Steady-State Mode Interaction

In a Hopf/steady-state mode interaction, the Jacobian associated with the critical component where the Hopf bifurcation occurs has a pair of purely imaginary eigenvalues, whereas the Jacobian associated with the other component has a single zero eigenvalue. On the center manifold, the component associated with the Hopf bifurcation has a two-dimensional phase space. In this case the center manifold vector field has the form

$$\begin{aligned} \dot{X} &= f(X) \\ \dot{y} &= g(X, y) \end{aligned} \tag{5.8}$$

where $X \in \mathbb{R}^2$ and $y \in \mathbb{R}$. Assume that the origin is an equilibrium so that $f(0) = g(0, 0) = 0$. At the origin, the Jacobian of (5.8) has a pair of purely imaginary eigenvalues and a zero eigenvalue, so that $\mathrm{D}f(0)$ has eigenvalues $\pm i$ and $g_y(0,0)=0$. We show in Section 9 that under these assumptions, Liapunov-Schmidt reduction leads to a two-dimensional map F with $\mathbf{Z}_2$-symmetry $\sigma(x,y)=(-x,y)$. We obtain:

Theorem 5.8. *Assume* (5.8) *satisfies the defining conditions* $f(0)=g(0,0)=0$*,* $Df(0)$ *has eigenvalues* $\pm i$ *and* $g_y(0,0)=0$*. Then there exists a smooth map*

$$F(x,y)=\left[\begin{array}{c} r(u)x \\ g(u,y) \end{array}\right], \quad u=x^2, \tag{5.9}$$

where $r(0)=0$ *and* $g(0,0)=0$*, such that locally, solutions to* $F(x,y)=0$ *with* $x\geq 0$ *are in one-to-one correspondence with small amplitude periodic solutions to* (5.8) *with period near* 2π*.*

Theorem 5.8 reduces finding periodic solutions of (5.8) to finding the zeros of a two-dimensional system (5.9) with $\mathbf{Z}_2$-symmetry. The goal is then to identify a normal form and to show that all vector fields of the form (5.9) satisfying the associated defining and nondegeneracy conditions are equivalent. For this we need a $\mathbf{Z}_2$-symmetric version of Definition 5.2:

Definition 5.9. A map $F(x,y)=(r(u)x,g(u,y))$ and $G(x,y)$ are *strongly* $\mathbf{Z}_2$*-equivalent* if

$$G(x,y)=\left[\begin{array}{cc} a(u) & 0 \\ b(u,y)x & c(u,y) \end{array}\right]\left[\begin{array}{c} r(\phi(u)^2u)\phi(u)x \\ g(\phi(u)^2u,\psi(u,y)) \end{array}\right] \tag{5.10}$$

where $\psi(0,0)=0$ and $a(0)$, $c(0,0)$, $\phi(0)$, $\psi_y(0,0)>0$.

Remark 5.10. The difference between equivalence in Definition 5.2, and $\mathbf{Z}_2$-equivalence in Definition 5.9, is that the change of coordinates is $\mathbf{Z}_2$-equivariant. That is, $\gamma G(x,y)= G(\gamma(x,y)$ where $\gamma(x,y)=(-x,y)$. This implies

$$\begin{array}{rcl} \gamma S(x,y) & = & S(\gamma(x,y))\gamma \\ \gamma\Phi(x,y) & = & \Phi(\gamma(x,y)) \end{array}$$

leading to the form (5.10) for strong $\mathbf{Z}_2$-equivalence. Requirements on ϕ, ψ, a and c follow from the restrictions on Φ, S as in Definition 5.2.

Theorem 5.11. *Assume that* (5.9) *satisfies the defining conditions*

$$r(0)=g(0,0)=g_y(0,0)=0 \tag{5.11}$$

and the nondegeneracy conditions

$$r_u(0)\neq 0,\ g_u(0,0)\neq 0,\ g_{yy}(0,0)\neq 0. \tag{5.12}$$

Then $F(x,y)=(r(u)x,g(u,y))$ *is strongly* $\mathbf{Z}_2$*-equivalent to the normal form* $\hat{F}$ *given by*

$$\begin{array}{rcl} \hat{r}(u)x & = & \varepsilon_p ux \\ \hat{g}(u,y) & = & \varepsilon_q u+\varepsilon_t y^2 \end{array} \tag{5.13}$$

where

$$\varepsilon_p=\mathrm{sign}(r_u(0)),\ \varepsilon_q=\mathrm{sign}(g_u(0,0)),\ \varepsilon_t=\mathrm{sign}(g_{yy}(0,0)).$$

Theorem 5.11 is useful only if Hopf/steady-state mode interaction admissible vector fields satisfying the mode interaction defining conditions (5.11) are rich enough to satisfy the nondegeneracy conditions (5.12). We would like to prove a result analogous to the steady-state/steady-state Proposition 5.4 for the Hopf/steady-state mode interaction. At the very least, we would like to exhibit one example of an admissible vector field for a given network that satisfies both the defining conditions (5.11) and nondegeneracy conditions (5.12). If we can do so, the desired result can be obtained by arguments based on algebraic geometry. What we do know so far is that, given any network, we can construct an admissible vector field that satisfies the defining conditions by considering a subnetwork consisting of a directed ring with a feed forward chain coming off of one of the nodes.

Because the critical path component associated with the steady-state bifurcation (zero eigenvalue) is downstream from the one associated with the Hopf bifurcation (complex conjugate pair of pure imaginary eigenvalues), there exits a path from a node s in the downstream critical component to a node h in the upstream critical component. Moreover, because the upstream component is associated with Hopf bifurcation, it contains at least two nodes. We can therefore construct a path connecting node h to another node in the upstream critical component going in both directions. A directed ring exists within this bidirectional path and can be found by clipping sections of the path beyond nodes that are contained along both the path to and from h. Any admissible vector field on this ring-and-chain subnetwork along with dynamics on other nodes o defined by $\dot{x}_o = -x_o$ will be admissible on the full network.

To classify perturbations up to equivalence, we use universal unfoldings. Again, we relax the restrictions in Definition 5.9 that $\psi(0,0)$ vanishes to compute universal unfoldings.

Theorem 5.12. *The normal form* (5.13) *has codimension two. A universal unfolding is*

$$\begin{aligned} r(u,\lambda)x &= (\lambda + \varepsilon_p u)x \\ g(x,y,\mu) &= \mu + \varepsilon_q u + \varepsilon_t y^2. \end{aligned} \tag{5.14}$$

Remark 5.13. In Hopf/steady-state mode interaction, the x-invariant solutions described in Remark 5.1 appear in the universal unfolding (5.14). The invariant line with $x > 0$ on the Liapunov-Schmidt reduced space corresponds to a flow-invariant solid cylinder aligned along the y-axis in the center manifold phase space. When present, this invariant set partitions phase space, and 'heteroclinic-like' orbits appear that connect the flow-invariant line at the center of the cylinder $x = 0$ with the boundary of the solid cylinder: see region I_+ in Figure 3 (right). In one direction these trajectories approach helical oscillation on the boundary of the cylinder, and in the other they approach the central line of the cylinder.

It is now straightforward to compute the steady-state solutions of (5.14) as a function of the parameters μ and λ. Locally these equilibria are in one-to-one correspondence with periodic solutions of (5.8) by Theorem 5.8. We are interested only in nonnegative solutions for the variable x, which relate to the amplitude of periodic orbits of (5.8). As in Section 5.1, it is sufficient to assume $\varepsilon_p = -1$ and consider the four cases $\varepsilon_t = \pm 1$, $\varepsilon_q = \pm 1$.

Case 1: $\varepsilon_p = \varepsilon_q = \varepsilon_t = -1$**.** The bifurcation in the $\mu\lambda$ parameter space occurs when the Jacobian

$$J = \begin{bmatrix} \lambda - 3x^2 & 0 \\ -2x & -2y \end{bmatrix}$$

at an equilibrium has a zero eigenvalue. The first case is $\lambda - 3x^2 = 0$, which leads to the half line $\lambda = 0$, $\mu \geq 0$ corresponding to a pitchfork bifurcation (Hopf bifurcation in the full three-dimensional system (5.8)). The second case $y = 0$ leads to two saddle-node bifurcations given by $\mu = 0$ and the half line $\mu - \lambda = 0$, $\lambda \geq 0$, where the former corresponds to a saddle-node of steady states in (5.8) and the latter to a saddle-node bifurcation of periodic orbits (SNPO) in (5.8). See Figure 3 (left).

There are no steady states for $\mu < 0$; two steady states for $\lambda < 0$, $\mu > 0$ and $\lambda > \mu > 0$; and four steady states for $\mu > \lambda > 0$. Unlike the steady-state/steady-state mode interaction, x-invariant lines do not always appear in pairs; in fact, one x-invariant line given by $x = 0$ persists throughout the full parameter space. These steady states, x-invariant solutions and their stability are listed in Table 2.

	Region	Equilibria (x, y)	Eigenvalues
II	$\lambda < 0$	$\left(0, +\sqrt{\mu}\right)$	$--$
	$0 < \mu$	$\left(0, -\sqrt{\mu}\right)$	$-+$
III	$0 < \lambda$	$\left(0, +\sqrt{\mu}\right)$	$+-$
	$\lambda < \mu$	$\left(0, -\sqrt{\mu}\right)$	$++$
		$\left(\sqrt{\lambda}, +\sqrt{\mu - \lambda}\right)$	$--$
		$\left(\sqrt{\lambda}, -\sqrt{\mu - \lambda}\right)$	$-+$
IV	$\mu < \lambda$	$\left(0, +\sqrt{\mu}\right)$	$-+$
	$0 < \mu$	$\left(0, -\sqrt{\mu}\right)$	$++$
I_+	$0 < \lambda$	$(0, *)$	$+*$
	$\mu < 0$	$\left(\sqrt{\lambda}, *\right)$	$-*$
I_-	$\lambda < 0$	$(0, *)$	$-*$
	$\mu < 0$		

Table 2: List of equilibria (column 3) and their stabilities (column 4) in each of the regions (column 1) in Figure 3 (left). The entries in I_+ and I_- column 3 list the invariant lines.

The circulant diagram in Figure 3 (right) shows transitions across the boundaries between the five regions. Begin in region I_-; there are no steady states but there is one x-invariant line $x = 0$, which partitions the phase space into two regions. Moving from region I_- to region II across the transition line SN_- (with $\lambda < 0$), the first pair of steady states, Table 2 column 3, appears through a saddle-node bifurcation. At the transition line SN_- there is one steady-state solution $(x, y) = (0, 0)$.

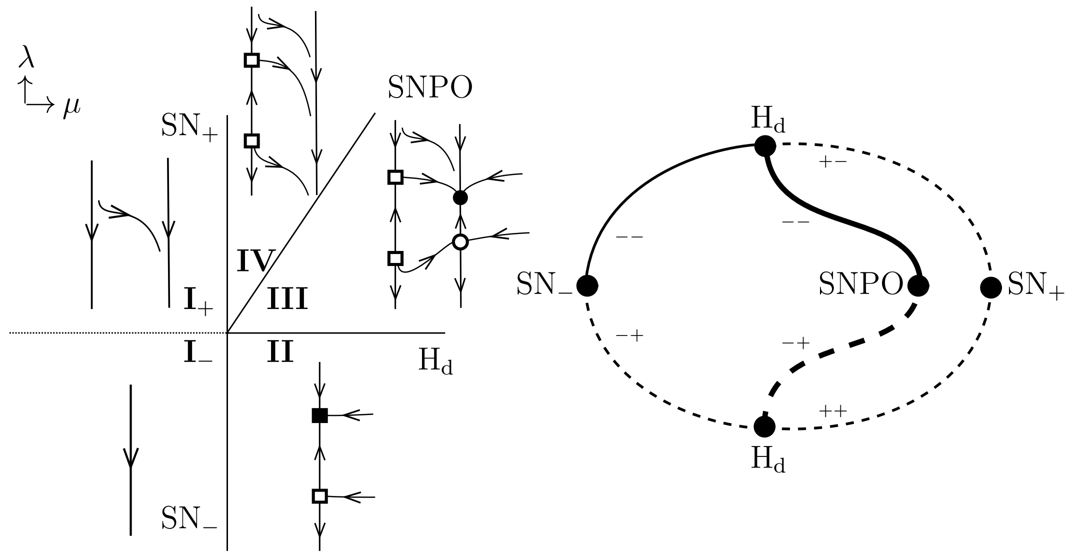

Figure 3: Hopf/steady-state mode interaction: case 1: $\varepsilon_p = \varepsilon_q = \varepsilon_t = -1$. *Left*: Transition variety in the $\mu\lambda$ parameter plane, with phase portraits in the xy phase plane in each connected region of parameter space. Solid curves SN_-, SN_+, SNPO and $\mathrm{H_d}$ represent three saddle-node and a double pitchfork transition curves for (5.14), and correspond to two saddle-nodes, a saddle-node of periodic orbits, and a double Hopf bifurcation for (5.8). In the phase portraits, the square and circle symbols correspond to the steady states and the periodic solutions of the full three-dimensional system (5.8), while the straight vertical lines indicate x-invariant lines defined by $f(x) = 0$. *Right*: Circulant bifurcation diagram as a small circle is traversed counterclockwise around the origin in the $\mu\lambda$-plane. The plus and minus signs show the stability of each branch in the x and y directions. Thicker lines indicate the periodic solutions of (5.8). The dotted half-line in the left-hand figure indicates a 'pitchfork' of the invariant line $x = 0$ to the invariant lines $x = 0$ and $x = \sqrt{\lambda}$ as λ increases through 0.

Moving from region II to region III across the transition line $\mathrm{H_d}$ (with $\mu > 0$), each of the two steady states that appeared at $\mu = 0$ splits into three steady states through a pair of pitchfork bifurcations. This creates six steady states. Four have nonnegative x values; see Table 2 column 3 and Figure 3 (left). The two steady states with positive x correspond to the two periodic solutions of (5.8) born at Hopf bifurcations.

Moving from region III to region IV across the transition line SNPO results in the loss of the equilibria with $x \neq 0$ through a saddle-node bifurcation. At the transition curve SNPO there are three steady-state solutions $(0, 0)$, $(0, \pm\sqrt{\mu})$. Only the two equilibria at $(0, \pm\sqrt{\mu})$ remain in region IV; these disappear through a saddle-node bifurcation when crossing the transition curve SN_+. Along SN_+ there is one steady-state solution at $(0, 0)$. In region I_+ no steady states exist. However, in this region there are two x-invariant lines $x = 0$ (unstable in the x-direction) and $x = \sqrt{\lambda}$ (stable in the x-direction). These two solutions are 'heteroclinic-like' as discussed in Section 5.1, and they partition the two dimensional phase space into three regions. Finally, as λ decreases through 0, the x-invariant line given by $x = \sqrt{\lambda}$ disappears through a pitchfork bifurcation of $f(x)$ (not a pitchfork in the entire system) and only $x = 0$ persists.

Case 2: $\varepsilon_p = -1$**,** $\varepsilon_q = +1$**,** $\varepsilon_t = -1$**.** Reversing the sign of ε_q changes the SNPO transition line to $\lambda + \mu = 0$, $\lambda \geq 0$, which is to the left of the line SN_+. In contrast to case 1, the two periodic solutions of (5.8) that appeared at the transition line $\mathrm{H_d}$ persist after the two steady states disappear at SN_+, going counterclockwise in a circle around the origin in the $\mu\lambda$-plane. Changing $\varepsilon_q \to -\varepsilon_q$ does not affect the stability of any equilibria. See Figure 4.

Case 3: $\varepsilon_p = -1$**,** $\varepsilon_q = +1$**,** $\varepsilon_t = +1$**.** Reversing the sign of ε_q and ε_t (from Case 1) is equivalent to a transformation $\lambda \to -\lambda$ and a change of stability in the y direction.

Case 4: $\varepsilon_p = -1$**,** $\varepsilon_q = -1$**,** $\varepsilon_t = +1$**.** Reversing the sign of ε_q and ε_t (from Case 2) is equivalent to a transformation of $\lambda \to -\lambda$ and a change of stability in the y direction.

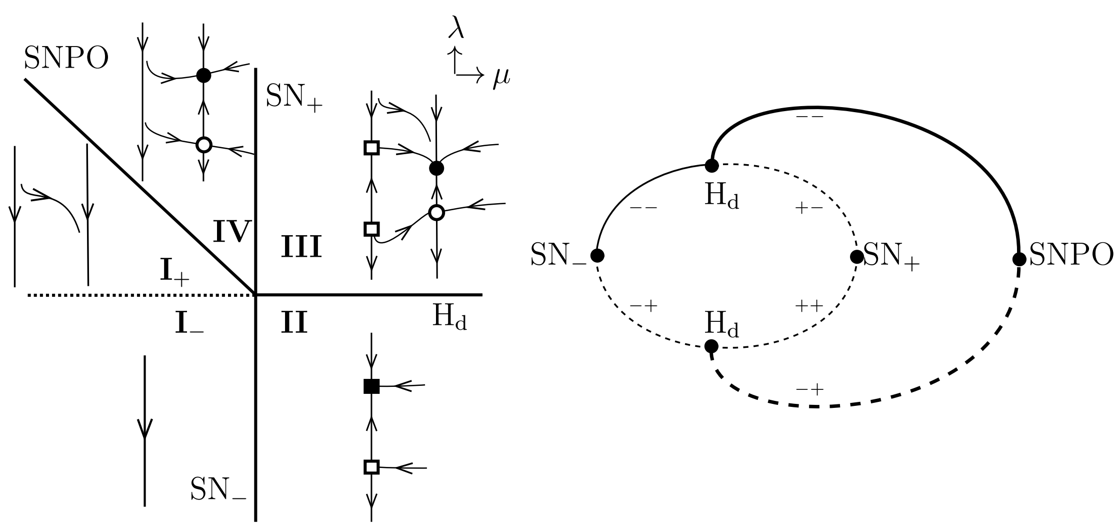

Figure 4: Hopf/steady-state mode interaction: Case 2. Labels and symbol codings have the same meanings as in Figure 3. *Left:* Transition varieties in the $\mu\lambda$ parameter plane and phase portraits in the xy phase plane for the regions they determine. *Right*: circulant bifurcation diagram.

5.3 Steady-State/Hopf Mode Interaction

Assume, as in the previous two subsections, that there are two critical components with one downstream of the other. The steady-state/Hopf mode interaction differs from the Hopf/steady-state mode interaction: the Jacobian associated with the upstream critical component of the original network has a single zero eigenvalue, whereas the Jacobian associated with the downstream critical component has a complex conjugate pair of pure imaginary eigenvalues. The vector field on the center manifold has the form:

$$\begin{array}{rcl} \dot{x} & = & f(x) \\ \dot{Y} & = & g(x, Y) \end{array} \tag{5.15}$$

where $x \in \mathbb{R}$, $Y \in \mathbb{R}^2$, $f(0) = 0$, and $g(0, 0) = 0$.

At the origin, the Jacobian of (5.15) has a zero eigenvalue and a pair of purely imaginary eigenvalues, namely:

$$f_x(0) = 0 \qquad \operatorname{tr} \mathrm{D}_Y g(0, 0) = 0 \qquad \det \mathrm{D}_Y g(0, 0) > 0.$$

Rescaling time, we can assume $\det \mathrm{D}_Y g(0, 0) = 1$ and the eigenvalues are $\pm i$.

Section 9 employs Liapunov-Schmidt reduction to reduce finding steady-states and periodic orbits of (5.15) to finding zeros of a map associated with two one-dimensional nodes. The goal is to identify a normal form for the reduced map (5.16) below, using equivalences that preserve the network structure of the center manifold.

Theorem 5.14. *Assume that* (5.15) *has an equilibrium at the origin that undergoes a steady state/Hopf mode interaction. The defining conditions are*

$$f(0) = 0 \qquad g(0, 0) = 0 \qquad f_x(0) = 0 \qquad \operatorname{tr} \mathrm{D}_Y g(0, 0) = 0 \qquad \det \mathrm{D}_Y g(0, 0) = 1.$$

Then there exists a smooth map on $\mathbb{R} \times \mathbb{R}$ of the form

$$F(x, y) = (f(x), r(x, v)y) \tag{5.16}$$

where $v = y^2$ and $r(0, 0) = 0$, such that locally solutions to $F(x, y) = 0$ with $y \geq 0$ are in one-to-one correspondence with small amplitude periodic solutions to (5.15) *with period near 2π.*

On the reduced system (5.16), the transformations that define equivalence must respect $\mathbf{Z}_2$-symmetry in addition to the network structure of the center manifold.

Definition 5.15. Maps $F(x, y) = (f(x), r(x, v)y)$ and $\hat{F}(x, y) = (\hat{f}(x), \hat{r}(x, v)y)$ are *strongly $\mathbf{Z}_2$-equivalent* if there exist $a(x)$, $\phi(x)$, $b(x, v)$, $c(x, v)$, $\phi(x, v)$ such that

$$\hat{F}(x, v) = \begin{bmatrix} a(x) & 0 \\ b(x, v)y & c(x, v) \end{bmatrix} \begin{bmatrix} f(\phi(x)) \\ r(\phi(x), \psi(x, v)y) \end{bmatrix} \tag{5.17}$$

where $v = y^2$, $\phi(0) = 0$, and $a(0)$, $c(0, 0)$, $\phi_x(0)$, $\psi(0, 0) > 0$.

Remark 5.16. This differs from equivalence as in Definition 5.9 because it requires equivariance under a different representation of $\mathbf{Z}_2$, namely $\gamma G(x,y) = G(\gamma(x,y))$ where $\gamma(x,y) = (x,-y)$.

We now state the normal form in terms of the Liapunov-Schmidt reduced system (5.16), where the variable y represents the amplitude of the periodic orbit in the Y coordinates of (5.15) on the center manifold.

Theorem 5.17. *Assume* (5.16) *satisfies the defining conditions*

$$f(0) = f_x(0) = r(0,0) = 0,$$

and the nondegeneracy conditions

$$f_{xx}(0) \neq 0,\ r_v(0,0) \neq 0,\ r_x(0,0) \neq 0.$$

Then $F(x,y) = (f(x), r(x,v)y)$ *where* $v = y^2$ *is strongly* $\mathbf{Z}_2$*-equivalent, as defined by* (5.15)*, to the normal form* $\hat{F}(x,y) = (\hat{f}(x), \hat{r}(x,v)y)$ *given by*

$$\begin{aligned} \hat{f}(x) &= \varepsilon_p x^2 \\ \hat{r}(x,v)y &= (\varepsilon_t v + \varepsilon_s x)\, y \end{aligned} \tag{5.18}$$

where

$$\varepsilon_p = \mathrm{sign}(f_{xx}(0)),\ \varepsilon_t = \mathrm{sign}(r_v(0,0)),\ \varepsilon_s = \mathrm{sign}(r_x(0,0)).$$

As usual we compute a universal unfolding by relaxing the restriction in Definition 5.15 that $\phi(0)$ vanishes.

Theorem 5.18. *The normal form* (5.18) *has codimension two. A universal unfolding is*

$$\begin{aligned} f(x) &= \lambda + \varepsilon_p x^2 \\ r(x,v)y &= (\mu + \varepsilon_s x + \varepsilon_t v)\, y. \end{aligned} \tag{5.19}$$

Remark 5.19. Both the steady-state/Hopf mode interaction (5.19) and the the Hopf/steady-state mode interaction (5.14) are very different from what is observed for the analogous mode interaction in general vector fields. In particular, 2-tori, which are observed in general vector fields, are not possible because of the feedforward structure of the network.

Remark 5.20. In steady-state/Hopf mode interaction, the x-invariant solutions described in Remark 5.1 exist in the universal unfolding (5.19). These invariant lines on the Liapunov-Schmidt reduced space correspond to the flow-invariant planes perpendicular to the x-axis in the center manifold phase space. Similar to the steady-state/steady-state mode interaction, they appear in pairs where one is stable in the x-direction and the other is unstable in that direction. When present, the invariant sets partition phase space, and 'heteroclinic-like' orbits appear that connect the two flow-invariant planes. In one direction these trajectories approach one plane, and in the other they approach the other plane.

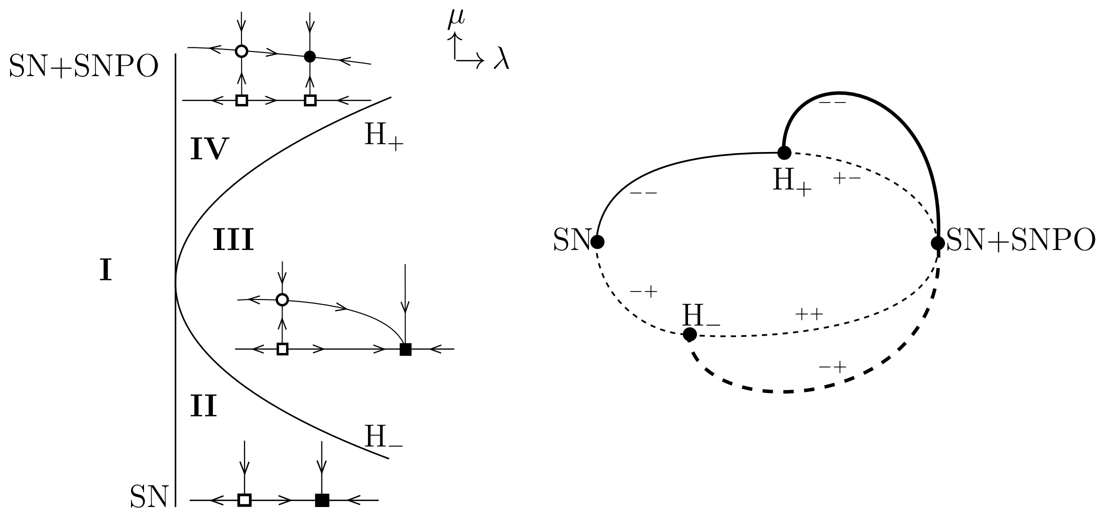

Figure 5: Steady-state/Hopf mode interaction: Case 1: $\varepsilon_p = -1$, $\varepsilon_s = -1$, $\varepsilon_q = -1$. *Left*: Transition variety in the $\lambda\mu$-plane, with phase portraits in the xy phase plane in each connected region of parameter space. Solid curves SN, SN + SNPO, H$_+$ and H$_-$ represent saddle-node, double saddle-node, and pitchfork transition curves for (5.19). They correspond to saddle-node, saddle-node and saddle-node of periodic orbits, and Hopf bifurcation curves for (5.15). *Right*: Circulant bifurcation diagram. Plus and minus signs show the stability of each branch in the x and y directions. Thicker lines indicate the periodic solutions of (5.8).

We discuss all possible cases described by (5.19) in terms of the signs of the coefficients, as follows. We fix $\varepsilon_p = -1$ and consider the four cases with $\varepsilon_s = \pm 1$ and $\varepsilon_t = \pm 1$. The remaining four possibilities are obtained from these by noting that flipping the sign of all three coefficients is equivalent to changing the sign of the parameters μ and λ, and changing the stability of each equilibrium in both the x and y direction. We provide a detailed discussion when $\varepsilon_p = \varepsilon_s = \varepsilon_t = -1$ (Table 3, Figure 5), and present the other three cases in terms of the signs of coefficients ε_s and ε_t.

Case 1: $\varepsilon_p = -1$, $\varepsilon_s = -1$, $\varepsilon_t = -1$. The Jacobian of (5.19) linearized about an equilibrium (x, y) is

$$J = \begin{bmatrix} -2x & 0 \\ -y & \mu - 3y^2 - x \end{bmatrix}.$$

The eigenvalues can be read off rectly as $-2x$ and $\mu - 3y^2 - x$. For all parameter values the line $y = 0$ is flow-invariant in the two-dimensional Liapunov-Schmidt reduced space. In the three-dimensional center manifold this line corresponds to the line $(x, 0, 0)$ (see (5.15)). When $\lambda > 0$, a pair of equilibria $(x, y) = (\pm\sqrt{\lambda}, 0)$ exist; they are created at a saddle-node

	Region	Equilibria (x,y)	Type	Eigenvalues
I-IV		$(*,0)$	FIS	
II-IV	$0<\lambda$	$\left(+\sqrt{\lambda},*\right)$	FIS	$-*$
		$\left(-\sqrt{\lambda},*\right)$	FIS	$+*$
II	$0<\lambda$	$\left(+\sqrt{\lambda},0\right)$	SS	$--$
	$\mu<-\sqrt{\lambda}$	$\left(-\sqrt{\lambda},0\right)$	SS	$+-$
III	$\mu^2<\lambda$	$\left(+\sqrt{\lambda},0\right)$	SS	$--$
		$\left(-\sqrt{\lambda},0\right)$	SS	$++$
		$\left(-\sqrt{\lambda},\sqrt{\mu+\sqrt{\lambda}}\right)$	PO	$+-$
IV	$0<\lambda$	$\left(+\sqrt{\lambda},0\right)$	SS	$-+$
		$\left(+\sqrt{\lambda},\sqrt{\mu-\sqrt{\lambda}}\right)$	PO	$--$
	$\sqrt{\lambda}<\mu$	$\left(-\sqrt{\lambda},0\right)$	SS	$++$
		$\left(-\sqrt{\lambda},\sqrt{\mu+\sqrt{\lambda}}\right)$	PO	$+-$

Table 3: Steady-state/Hopf mode interaction: Case 1. List of steady-state equilibria (SS), periodic orbits (PO) and flow-invariant sets (FIS) with $y \geq 0$. The region in parameter space where each exists is given in column 1 in terms of the four regions identified in Figure 5 (left) and in terms of parameter regimes in column 2. The coordinates and type are given in columns 3 and 4, respectively, and the stabilities of the equilbria are in column 5.

bifurcation along this y-invariant line at $\lambda = 0$. Each of these two equilibria exists as part of an x-invariant line in the two-dimensional space that corresponds to an x-invariant plane in the three-dimensional center manifold. If additionally $\mu > 0$, there is a saddle-node of periodic orbits bifurcating simultaneously at $\lambda = 0$. These periodic orbits correspond to the pair of steady-state equilibria

$$(x,y) = \left(\pm\sqrt{\lambda}, \sqrt{\mu \mp \sqrt{\lambda}}\right)$$

in the two-dimensional space. Each of the two periodic orbits created through this bifurcation exists within the x-invariant plane containing the equilibrium point with the same x-coordinate. The curve $\lambda = \mu^2$ defines two families of Hopf bifurcations, each connecting the steady state $y = 0$ to the periodic orbit with the same x-coordinate. Figure 5 (left) shows these transition varieties in the (λ, μ)-plane, along with representative phase portraits within each connected region of the complement.

The circulant diagram of Figure 5 (right) shows transitions across the boundaries between the four regions in the $\lambda\mu$-plane. The dynamics along the x-direction is y-invariant, and the line $y = 0$ (corresponding to the line $Y = 0$ in the three-dimensional center manifold) is flow-invariant and persists for all parameter values. In region I there are no equilibria, since $\mu < 0$; trajectories approach $-\infty$ along the x-direction. A saddle-node bifurcation occurs as we cross from region I into region II where a stable and an unstable steady-state equilibrium exist on the invariant line $y = 0$. A heteroclinic trajectory along $y = 0$ connects the unstable steady state to the stable steady state. Because the flow along the x-direction is y-invariant, a pair of x-invariant lines (corresponding to flow-invariant planes in the center manifold) is also created along with the equilibria. Each x-invariant subspace contains one of the steady states, and the unstable steady state is in fact stable when restricted to its x-invariant subspace. Moreover, these x-invariant subspaces partition the phase space so that trajectories between them remain trapped for all time. In contrast to the previous two mode interactions, the x-invariant subspaces never exist independently of equilibria. This can be traced back to the persistence of the y-invariant subspace defined by $y = 0$ for all parameter values.

Going from region II to region III, a pitchfork bifurcation of the unstable steady state occurs. This corresponds to a Hopf bifurcation in the center manifold, and creates an unstable periodic orbit contained in the x-invariant plane of the unstable steady state. The orbit is unstable to perturbations along the x-direction, but stable within the flow-invariant plane that contains it. We also expect a heteroclinic connection from the periodic orbit to the stable steady state. Crossing from region III to region IV also results in a pitchfork bifurcation, corresponding to a Hopf bifurcation in the center manifold space, but this time of the stable steady state. The resulting periodic orbit is stable and is contained in the x-invariant subspace of the steady state from which it emerges. In general, the two periodic orbits do not have the same amplitude y. A heteroclinic orbit connects them, which corresponds to a trajectory that approaches the frequency of the unstable periodic orbit as $t \to -\infty$, and the frequency of the stable periodic orbit as $t \to \infty$.

Going from region IV back into region I, the two x-invariant subspaces undergo a kind of saddle-node bifurcation in which both the pair of steady states and the pair of periodic orbits annihilate. At this transition, the two periodic orbits therefore have the same amplitude. Finally we return to the original configuration with no equilibria and a flow-invariant line $y = 0$.

Remark 5.21. The steady-state/Hopf mode interaction (5.19) differs from the Hopf/steady-state mode interaction (5.14). Instead of the simultaneous Hopf bifurcations observed in the Hopf/steady-state case, simultaneous saddle-node and saddle-node of periodic orbits occur. [1]

[1] The following was commented out: Need to say something about this unfolding: (1) compare to general vector field SS-H mode interaction, and (2) compare to H-SS in previous section.

Case 2: $\varepsilon_p = -1$**,** $\varepsilon_t = -1$**,** $\varepsilon_s = 1$**.** Reversing the sign of ε_s is equivalent to reflecting the phase space across the x-axis and changing stability along the x-direction. The two Hopf transition lines H_+ and H_- exchange location, but the number of equilibria within each region remains unchanged.

Case 3: $\varepsilon_p = -1$**,** $\varepsilon_t = 1$**,** $\varepsilon_s = -1$**.** Reversing the sign of ε_t maps $\mu \to -\mu$ in the bifurcation diagram; that is, reflects the 2-parameter bifurcation diagram of Figure 5 in the λ-axis. In addition, the x-axis is reflected and stability along the y direction for the equilibria is changed.

Case 4: $\varepsilon_p = -1$**,** $\varepsilon_t = 1$**,** $\varepsilon_s = 1$**.** Reversing the signs of ε_t and ε_s maps $\mu \to -\mu$ as in the previous case. Stability along the y direction is also changed, but the orientation of the x axis remains unchanged.

5.4 Hopf/Hopf Mode Interaction

In the Hopf/Hopf mode interaction, the Jacobian associated with each of the two critical components of the original network has a complex conjugate pair of pure imaginary eigenvalues. We assume that one critical component is downstream from the other, so that the center manifold vector field has the form:

$$\begin{aligned} \dot{X} &= f(X) \\ \dot{Y} &= g(X,Y) \end{aligned} \tag{5.20}$$

where $X, Y \in \mathbb{R}^2$. Assume that the origin of (5.20) is a steady state, so that $f(0) = 0$ and $g(0,0) = 0$, and that the Jacobian of (5.15) has two distinct pairs of purely imaginary eigenvalues; namely,

$$\operatorname{tr} \mathrm{D}_X f(0) = 0 \qquad \det \mathrm{D}_X f(0) > 0 \qquad \operatorname{tr} \mathrm{D}_Y g(0,0) = 0 \qquad \det \mathrm{D}_Y g(0,0) > 0,$$

with $\det \mathrm{D}_Y g(0,0) \neq \det \mathrm{D}_X f(0)$. Let the eigenvalues associated with the upstream and downstream critical component be $\pm i\omega$ and $\pm i\nu$ respectively. We restrict the following discussion to the nonresonant case in which ω and ν are not rationally related, though the results turn out to also be valid for sufficiently weak resonance. The reason for this restriction is that in the nonresonant case the Birkhoff normal form of the Hopf/Hopf mode interaction commutes with the 2-torus $\mathbb{T}^2$ acting on $\mathbb{R}^4$ by

$$\vec{R}(\theta_x, \theta_y)(X, Y) = (R(\theta_x)X, R(\theta_y)Y) \tag{5.21}$$

where $R(\theta)$ acts on $\mathbb{R}^2$ by counterclockwise rotation through θ.

The goal of this section is more limited in scope than in the previous sections. We begin by assuming that(5.20) is in Birkhoff normal form, and use phase-amplitude coordinates to reduce finding steady-states and periodic orbits in (5.20) to finding zeros of a map with two one-dimensional nodes that describes the amplitude dynamics of the two-dimensional nodes. We then identify a normal form for the reduced map (5.22) below using equivalences that preserve the feedforward structure of the center manifold network.

Theorem 5.22. *Assume that* (5.20) *has an equilibrium at the origin that undergoes a non-resonant Hopf/Hopf mode interaction. The defining conditions are*

$$\operatorname{tr} \mathrm{D}_X f(0) = 0 \qquad \det \mathrm{D}_X f(0) = \omega^2 \qquad \operatorname{tr} \mathrm{D}_Y g(0,0) = 0 \qquad \det \mathrm{D}_Y g(0,0) = \nu^2,$$

with ω and ν nonzero and irrationally related. Further assume that (5.20) *is in Birkhoff normal form, so $F = (f, g)$ commutes with $\mathbb{T}^2$ under the action* (5.21)*. Then there exists a smooth map on $\mathbb{R} \times \mathbb{R}$ of the form*

$$F(x, y) = (r(u)x, s(u, v)y) \tag{5.22}$$

where $u = x^2$, $v = y^2$, $r(0) = 0$ and $s(0, 0) = 0$, such that locally solutions to $F(x, y) = 0$ with $y \geq 0$ and $x \geq 0$ are in one-to-one correspondence with solutions to (5.20) *of the four types in Table 4.*

Upstream Node	Downstream Node	Equilibrium Type
$x = 0$	$y = 0$	steady state
$x \neq 0$, $r = 0$	$y = 0$	periodic orbit with period near $2\pi/\omega$
$x = 0$	$y \neq 0$, $s = 0$	periodic orbit with period near $2\pi/\nu$
$x \neq 0$, $r = 0$	$y \neq 0$, $s = 0$	invariant two-torus

Table 4: Four possible types of equilibria in feedforward Hopf/Hopf mode interaction classified by amplitude of the upstream and downstream nodes.

On the reduced system (5.22), the transformations defining equivalence must respect $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-symmetry, in addition to the network structure of the center manifold:

Definition 5.23. Maps $F(x, y) = (r(u)x, s(u, v)y)$ and $\hat{F}(x, y) = (\hat{r}(u)x, \hat{s}(u, v)y)$ are *strongly* $(\mathbf{Z}_2 \oplus \mathbf{Z}_2)$*-equivalent* if there exist $a(u)$, $\phi(u)$, $b(u, v)$, $c(u, v)$, $\psi(u, v)$ such that

$$\hat{F}(x, y) = \begin{bmatrix} a(u) & 0 \\ b(u, v)xy & c(u, v) \end{bmatrix} \begin{bmatrix} r(\phi^2(u)u)\phi(u)x \\ s(\phi^2(u)u, \psi^2(u, v)v)\psi(u, v)y \end{bmatrix} \tag{5.23}$$

where $u = x^2$, $v = y^2$ and $a(0)$, $c(0, 0)$, $\phi(0)$, $\psi(0, 0) > 0$.

The normal form is stated in terms of the reduced system (5.22), where the variables x and y represent the amplitudes of periodic motions in the X and Y coordinates of (5.20) on the center manifold, assuming Birkhoff normal form.

Theorem 5.24. *Assume* (5.22) *satisfies the defining conditions*

$$r(0) = s(0,0) = 0,$$

and the nondegeneracy conditions

$$r_u(0) \neq 0,\ s_u(0,0) \neq 0,\ s_v(0,0) \neq 0.$$

Then $F(x, y) = (r(u)x, s(u, v)y)$ *where* $u = x^2$ *and* $v = y^2$ *is strongly* $\mathbf{Z}_2 \oplus \mathbf{Z}_2$*-equivalent to the normal form* $\hat{F}(x, y) = (\hat{r}(u)x, \hat{s}(u, v)y)$ *given by*

$$\begin{array}{rcl} \hat{r}(u)x & = & \varepsilon_p ux \\ \hat{s}(u,v)y & = & (\varepsilon_q u + \varepsilon_t v)\, y \end{array} \tag{5.24}$$

where

$$\varepsilon_p = \mathrm{sign}(r_u(0)),\ \varepsilon_q = \mathrm{sign}(s_u(0,0)),\ \varepsilon_t = \mathrm{sign}(s_v(0,0)).$$

To classify all possible perturbations of the normal form (5.24) up to equivalence, we compute a universal unfolding.

Theorem 5.25. *The normal form* (5.24) *has codimension two. A universal unfolding is*

$$\begin{array}{rcl} r(u)x & = & (\lambda + \varepsilon_p u)\, x \\ s(u,v)y & = & (\mu + \varepsilon_q u + \varepsilon_t v)\, y \end{array} \tag{5.25}$$

It is now straightforward to compute the steady-state solutions of (5.25) as a function of the parameters μ and λ. Solutions with one of x or y equal to zero correspond to periodic solutions of (5.20), and those where both x and y are non-zero correspond to invariant tori. Changing $\varepsilon_p, \varepsilon_q, \varepsilon_t$ to $-\varepsilon_p, -\varepsilon_q, -\varepsilon_t$ rotates the transition variety in the $\mu\lambda$-plane by 180° and changes the stability of all solutions in both the x and y directions. Hence we assume $\varepsilon_p = -1$, and consider the resulting four cases.

Case 1: $\varepsilon_p = \varepsilon_q = \varepsilon_t = -1$ The Jacobian at an equilibrium (x, y) is

$$J = \begin{bmatrix} \lambda - 3x^2 & 0 \\ -2xy & \mu - 3x^2 - 3y^2 \end{bmatrix}$$

There is an equilibrium $x = y = 0$ for all values of μ and λ, with eigenvalues λ and μ. The lines $\mu = 0$ and $\lambda = 0$ correspond to Hopf bifurcations in the full system, creating periodic solutions with $y = \sqrt{\mu}$ and $x = \sqrt{\lambda}$ respectively. When $\lambda > 0$ there is an invariant cylinder $x = \sqrt{\lambda}$. There are also two torus bifurcations. One occurs on the half-line $\lambda = 0$, $\mu \geq 0$, and the other on the half-line $\lambda = \mu$, $\mu \geq 0$. The steady states and their stabilities are listed in Table 5 and shown in Figure 6.

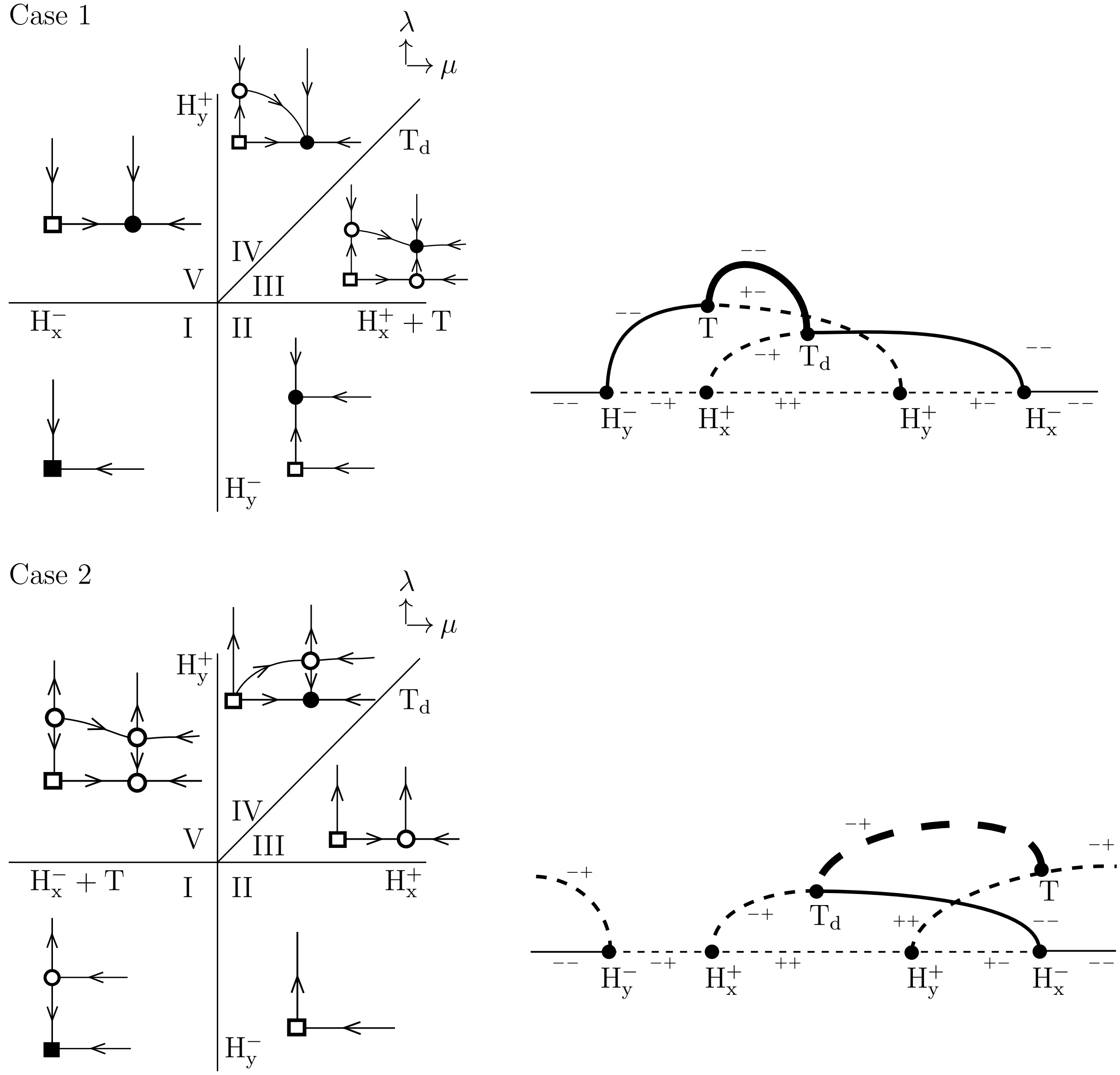

Figure 6: Hopf/Hopf mode interaction: Cases 1 and 2. *Left*: bifurcation sets (bold lines) in the $\mu\lambda$-plane, with phase portraits in the xy plane for each region of parameter space. *Right*: bifurcation diagram as a small circle is traversed around the origin in the $\mu\lambda$-plane, starting in the third quadrant. Dots indicate bifurcations and small plus and minus signs show the stability of each branch in the x and y directions. Lines of medium thickness (solid or dashed) indicate periodic solutions in the full four-dimensional system. Very thick lines linking these branches represent the invariant torus. In case 1, solid lines H_y^-, H_y^+, H_x^-, and T_d represent pitchfork, bifurcations for (5.25) and $H_x^+ + T$ represents a double pitchfork bifurcation for (5.25). They correspond to Hopf, Hopf, Hopf, torus and Hopf plus torus bifurcations respectively for (5.20). In case 2, labels are the same except the torus bifurcation T occurs with H_x^- rather than H_x^+.

	Region	Equilibria (x,y)	Eigenvalues
I	$\lambda < 0$	$(0,0)$	$--$
	$\mu < 0$		
II	$\lambda < 0$	$(0,0)$	$-+$
	$0 < \mu$	$(0, \sqrt{\mu})$	$--$
III	$0 < \lambda$	$(0,0)$	$++$
	$\lambda < \mu$	$(0, \sqrt{\mu})$	$+-$
		$\left(\sqrt{\lambda}, 0\right)$	$-+$
		$\left(\sqrt{\lambda}, \sqrt{\mu - \lambda}\right)$	$--$
IV	$0 < \mu$	$(0,0)$	$++$
	$\mu < \lambda$	$(0, \sqrt{\mu})$	$+-$
		$\left(\sqrt{\lambda}, 0\right)$	$--$
V	$\mu < 0$	$(0,0)$	$+-$
	$0 < \lambda$	$\left(\sqrt{\lambda}, 0\right)$	$--$

Table 5: Hopf/Hopf mode interaction: Case 1. List of equilibria (column 3) and their stabilities (column 4) in each of the regions (column 1) in Figure 6 (left).

Case 2: $\varepsilon_p = \varepsilon_q = -1$, $\varepsilon_t = 1$ Changing the sign of ε_t (from case 1) changes the criticality of the bifurcations which create the equilibria at $(x,y) = \left(0, \sqrt{\frac{-\mu}{\varepsilon_t}}\right)$ and $\left(\sqrt{\lambda}, \sqrt{\frac{-(\varepsilon_q \lambda + \mu)}{\varepsilon_t}}\right)$. Bifurcation curves however remain unchanged. See Figure 6.

Case 3: $\varepsilon_p = -1$, $\varepsilon_t = \varepsilon_q = 1$ Changing the sign of ε_q and ε_t (from case 1) is equivalent to changing the sign of μ together with a change of stability in the y-direction.

Case 4: $\varepsilon_p = \varepsilon_t = -1$, $\varepsilon_q = 1$ Changing the sign of ε_q and ε_t (from case 2) is equivalent to changing the sign of μ together with a change of stability in the y-direction.

6 Proofs of Codimension One Theorems

Section 6.2 below provides proofs of the main theorems on codimension-1 steady-state bifurcation, and Section 6.3 provides proofs of the main theorems on codimension-1 Hopf bifurcation. Neither proof is trivial. First, we show in Section 6.1 that generically, within the class of admissible maps, the Jacobian has distinct eigenvalues.

It is useful to introduce the following notion. A *shape space* $\mathcal{S}$ is a vector space of all $n \times n$ matrices A having a certain set of nondiagonal matrix entries equal to zero, Golubitsky and Stewart (2017). By (1.1) the linear admissible maps for a fully inhomogeneous network form a shape space; a zero entry in the (i, j)th slot indicates that node j is not directly connected to node i. The same equation obviously implies that at *any* point, the Jacobian of an admissible map for a fully inhomogeneous network $\mathcal{G}$ lies in the shape space corresponding to $\mathcal{G}$.

6.1 Simple Eigenvalues are Generic

We begin with a technical lemma. Let $\|\cdot\|$ denote any norm on $\mathbb{R}^n$ (all norms on $\mathbb{R}^n$ are equivalent).

Theorem 6.1. *Let A be an $n \times n$ matrix in a shape space $\mathcal{S}$ that includes all diagonal matrices, with characteristic polynomial $p(t) = \det(A - tI)$. Let*

$$p(t) = t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0.$$

Then there exists $\varepsilon > 0$ such that for any $(b_{n-1}, \ldots, b_0) \in \mathbb{R}^n$ satisfying

$$\|(b_{n-1} - a_{n-1}, \ldots, b_0 - a_0)\| < \varepsilon$$

there exist ε_j such that the perturbed matrix

$$B = A + \operatorname{diag}(\varepsilon_1, \ldots, \varepsilon_n) \in \mathcal{S}$$

has characteristic polynomial

$$q(t) = t^n + b_{n-1}t^{n-1} + \cdots + b_1 t + b_0.$$

Proof. We start from the standard version of Jacobi's formula, Wikipedia (2018):

$$\det(K + \varepsilon X) = \det K + \operatorname{tr}(\operatorname{adj}(K)X)\varepsilon + O(\varepsilon^2)$$

for $n \times n$ matrices K, X. Here $\operatorname{adj}(K)$ is the adjugate (or adjoint) matrix — the transpose of the matrix of cofactors $C_{ij} = (-1)^{i+j} \det M_{ij}$ where M_{ij} is the minor obtained by deleting row i and column j from K. This formula follows directly from the standard formula for the determinant as a sum over permutations σ of products of the form $\operatorname{sign}(\sigma)a_{i,\sigma(i)}$.

The first step is to perturb the diagonal so that all diagonal entries a_{ii} are distinct. Having pre-prepared A in this manner, we proceed as follows:

Let e_{ij} be the elementary matrix with 1 in the (i, j) position and 0 everywhere else. Put $K = A - tI$ and $X = e_{11}$. The only contribution to the trace of $\operatorname{adj}(K)X$ comes from the $(1, 1)$ position, so we want the cofactor C_{11} for $A - tI$. Since $(-1)^{1+1} = 1$, this is

$$\det[A^{[1]} - tI]$$

where $A^{[1]}$ is A with row and column 1 deleted. This is the characteristic polynomial of $A^{[1]}$.

Let $p^{[i]}$ be the characteristic polynomial of $A^{[i]}$, which is A with row and column i deleted. The same calculation (think of the Taylor expansion or just take $X = \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$) yields

$$q(t) = \det(A - tI + \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)) = p(t) + \sum_i \varepsilon_i p^{[i]}(t) + O(2)$$

where $O(2)$ is of order 2 in the ε_i.

By the Implicit Function Theorem it is enough to prove the theorem neglecting the $O(2)$ terms. So we have to prove that generically (that is, after a small enough perturbation) the polynomials $p^{[1]}, \ldots, p^{[n]}$ are linearly independent. (The ε_i are independent, and there are n of them, the same as the number of coefficients in each $p^{[i]}$, including the leading term $1 \cdot t^{m-1}$.

Expanding and collecting coefficients of powers of t, each coefficient of each $p^{[i]}$ is a polynomial in the entries a_{ij} of A. The condition for linear independence is that the determinant Δ of these coefficients (including the leading term with coefficient 1) should not vanish.

We claim that Δ defines a codimension-1 subvariety. This follows provided Δ does not vanish identically on the shape space $\mathcal{S}$ of $\mathcal{G}$. Suppose for a contradiction that it does vanish. Then it vanishes on the diagonal matrices, since these are contained in $\mathcal{S}$, so some nontrivial linear combination vanishes:

$$\sum_i \mu_i p^{[i]}(t) \equiv 0. \tag{6.1}$$

However, when A is diagonal,

$$p^{[i]}(t) = \prod_{j \neq i} (t - a_{jj}).$$

We initially perturbed the diagonal of A (say by $\varepsilon/2$) so that all diagonal elements a_{ii} are distinct. Now for each i we can substitute $t = a_{ii}$ in (6.1). All terms vanish except possibly

$$\mu_i p^{[i]}(a_{ii}) = \prod_{j \neq i} (a_{ii} - a_{jj}),$$

so $\mu_i = 0$ for all i, contradiction.

Therefore the $p^{[i]}$ are linearly independent off the codimension-1 variety $\Delta = 0$, so small enough ε_j give characteristic polynomials filling an entire neighbourhood of $p(t)$. □

Corollary 6.2. *All sets of eigenvalues sufficiently close to those of A can be obtained by a diagonal perturbation $A + \mathrm{diag}(\varepsilon_1, \ldots, \varepsilon_n)$.*

Proof. Take $q(t) = (t - \lambda_1) \cdots (t - \lambda_n)$ to be the perturbed polynomial in Theorem 6.1, where the λ_j are the required perturbed eigenvalues. □

6.2 Proofs for Codimension One Steady-State Bifurcation

Proof of Theorem 3.1 Fix a fully inhomogeneous network and consider an admissible system

$$\dot{y} = F(y, \lambda)$$

for the network, where $y \in \mathbb{R}^n, \lambda \in \mathbb{R}$. Assume $F(0,0) = 0$ so that $y = 0$ is an equilibrium when $\lambda = 0$. Because the network is fully inhomogeneous, Corollary 6.2 implies that at a codimension-1 steady-state bifurcation the Jacobian $J = (\mathrm{D}_y F)_{(0,0)}$ generically has a simple zero eigenvalue. We call the corresponding eigenvector $v \neq 0$, so $Jv = 0$.

Let $v^* \neq 0$ be a null vector for the adjoint J^*. The range of J is the orthogonal complement of v^* since $\langle v^*, Jw \rangle = \langle J^* v^*, w \rangle = 0$ for any $w \in \mathbb{R}^n$, and the range of J is $(n-1)$-dimensional. We claim that

$$\langle v^*, v \rangle \neq 0. \tag{6.2}$$

To prove the claim assume, for a contradiction, that $\langle v^*, v \rangle = 0$. This implies $v \in \mathrm{range}(J)$, so there exists $u \neq 0$ such that $Ju = v$ and $J^2 u = Jv = 0$. Since u and v are linearly independent, zero is not a simple eigenvalue of J, a contradiction.

Since $\mathrm{rank}(J) = n-1$, Liapunov-Schmidt reduction shows that the zeros of $F(y, \lambda)$ near the bifurcation are in one-to-one correspondence with zeros of a single equation $g(x, \lambda)$, where $x \in \mathbb{R}$. The Liapunov-Schmidt procedure implies that $g_x(0,0) = 0$. Moreover, $g_{xx}(0,0) g_\lambda(0,0) \neq 0$ if and only if the resulting bifurcation is a saddle-node bifurcation.

The formulas for computing $g_{xx}(0,0)$ and $g_\lambda(0,0)$ are standard (Golubitsky and Schaeffer, 1985, p. 33). In particular,

$$\begin{aligned} g_{xx}(0,0) &= \langle v^*, \mathrm{D}^2 F(v,v) \rangle \\ g_\lambda(0,0) &= \langle v^*, F_\lambda \rangle \end{aligned}$$

where when $v = (v_1, \ldots, v_n)$ and $w = (w_1, \ldots, w_n)$ the k^{th} component of $\mathrm{D}^2 F$ is

$$[\mathrm{D}^2 F(v,w)]_k = \sum_{i,j=1}^{n} \frac{\partial^2 f_k}{\partial x_i \partial x_j}(0,0) v_i w_j.$$

In order to show that saddle-node bifurcations are generic, we must consider the case when $g_{xx}(0,0) = 0$ and show that a generic homogeneous quadratic perturbation of F leads to a new vector field $G = F + \varepsilon \Phi$ where

$$g^{\varepsilon}_{xx}(0,0) = \langle v^*, \mathrm{D}^2 G(v,v) \rangle = \varepsilon \langle v^*, \mathrm{D}^2 \Phi(v,v) \rangle$$

is nonzero. The Jacobian J and therefore v and v^* remain unchanged by this perturbation because we assume Φ to be homogeneous quadratic. Indeed, because the map $\Phi \mapsto \langle v^*, d^2 \Phi(v,v) \rangle$ is linear, it is enough to show that

$$\langle v^*, \mathrm{D}^2 \Phi(v,v) \rangle \neq 0 \tag{6.3}$$

for some admissible Φ, in order to satisfy (6.3) for almost all admissible Φ.

The quadratic $\Phi = (\phi_1, \ldots, \phi_n)$ is admissible if

$$\frac{\partial \phi_j}{\partial x_i} = 0$$

whenever node j is not connected to node i and $j \neq i$. In particular, a quadratic Φ of the form $\partial\phi_j/\partial x_i = 0$ for $i \neq j$ is admissible for any network.

By (6.2) there is a component k such that both $v_k^* \neq 0$ and $v_k \neq 0$. We can therefore choose $\phi_k = x_k^2/2$ and $\phi_j = 0$ for $j \neq k$ so that for this Φ we have

$$g_{xx}^{\varepsilon}(0,0) = \varepsilon v_k^* v_k^2 \neq 0$$

The bifurcation of the perturbed vector field is therefore a saddle-node. $\square$

Lemma 6.3. *Fix a fully inhomogeneous network with shape space $\mathcal{S}$. Let $\hat{\mathcal{S}}$ be the set of matrices $J \in \mathcal{S}$ that have a simple zero eigenvalue. Define $\hat{\mathcal{S}}_i \subseteq \hat{\mathcal{S}}$ to be the set of matrices $J \in \hat{\mathcal{S}}$ with null vector $v = (v_1, \ldots, v_n)$ such that $v_i \neq 0$. Let*

$$\hat{\mathcal{T}}_{ij} = \{J \in \hat{\mathcal{S}}_i : \; v_j \neq 0\}.$$

Then for any node j downstream of node i, $\hat{\mathcal{T}}_{ij}$ is open and dense in $\hat{\mathcal{S}}_i$.

Proof. The set of $J \in \hat{\mathcal{S}}_i$ that lead to a nonzero v_j is open by the continuous movement of the null vector v. So it is enough to show that this subset is dense in $\hat{\mathcal{S}}_i$ for all j downstream of i.

Fix $J \in \hat{\mathcal{S}}_i$. If $v_j \neq 0$, the proof is complete, so we can suppose $v_j = 0$. Because node j is downstream of node i, there exists a path of length m such that $k_0 \to k_1 \to \cdots \to k_m$ where $k_0 = i$, $k_m = j$, and arrows indicate connections between the corresponding nodes.

Given that component $v_{k_0} = v_i$ is nonzero, the proof proceeds by constructing a series of m perturbations that sequentially makes each k_ℓ component of the null vector v nonzero, to achieve the desired result $v_j \neq 0$.

Suppose we have found perturbations 1 through ℓ so that $v_{k_0}, \ldots., v_{k_\ell}$ are all nonzero. We show how to make an arbitrarily small perturbation of J, denoted by $\tilde{J}$, that makes $v_{k_{\ell+1}} \neq 0$. We choose the perturbation small enough so that the perturbed Jacobian still lies in $\hat{\mathcal{S}}_i$, and the nonzero components of v remain nonzero. For convenience, relabel nodes so that $k_{\ell+1}$ is 1 and k_ℓ is 2. Now $v_1 = 0$ and node 1 receives input from node 2 with $v_2 \neq 0$. Each perturbation is constructed in two stages, as we now describe.

First perturbation: Given the above labeling, let

$$J = \begin{bmatrix} A & B \\ C & D \end{bmatrix}$$

where A is a scalar and D is an $(n-1) \times (n-1)$ matrix. By assumption $v = (0 \; z)^T$ for $z \in \mathbb{R}^{n-1}$. The condition that v is a null vector becomes

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix} \begin{bmatrix} 0 \\ z \end{bmatrix} = J \begin{bmatrix} 0 \\ z \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}. \tag{6.4}$$

Now (6.4) implies $Bz = 0$ and $Dz = 0$, so D is singular with null vector z.

We claim there exists a perturbation of J so that the zero eigenvalue of D is simple. By Theorem 6.1 we can choose an admissible $n \times n$ perturbation matrix

$$\Psi_\Delta = \begin{bmatrix} 0 & 0 \\ 0 & \Delta \end{bmatrix},$$

where Δ is diagonal, so that the perturbed matrix $J + \Psi_\Delta$ has a simple eigenvalue λ close to zero. Then the perturbed matrix

$$\tilde{J} = J + \Psi_\Delta - \lambda I = \begin{bmatrix} \tilde{A} & B \\ C & \tilde{D} \end{bmatrix},$$

where $\tilde{A} = A - \lambda$ and $\tilde{D} = D + \Delta - \lambda I$, has a simple zero eigenvalue. Moreover, we can choose the perturbation Ψ_Δ small enough so that the nonzero components of v remain nonzero and $\tilde{J} \in \hat{\mathcal{S}}_i$.

If the new null vector of $\tilde{J}$ has $v_1 \neq 0$, we are done. So we may assume that the null vector still has the form $\tilde{v} = (0\ \tilde{z})^T$, which implies that the simple zero eigenvalue is associated with $\tilde{D}$. The claim is verified by dropping the tildes on $\tilde{A}$, $\tilde{D}$, $\tilde{J}$, $\tilde{v}$ and $\tilde{z}$.

Second perturbation: Let

$$\Phi_E = \begin{bmatrix} 0 & E \\ 0 & 0 \end{bmatrix},$$

where $E = (\varepsilon, 0, \ldots, 0)$. Since node 2 connects to node 1, Φ_E is a small admissible matrix. Moreover, $Ez \neq 0$. Consider the small perturbation $\tilde{J} = J + \Phi_E - \rho I$ of J, where ρ is the simple eigenvalue of the matrix $J + \Phi_E$ near zero. Thus $\tilde{J}$ has a simple zero eigenvalue with null vector $\tilde{v} = [\tilde{y}\ \tilde{z}]^T$ and

$$\tilde{J} \begin{bmatrix} \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} A - \rho I & B + E \\ C & D - \rho I \end{bmatrix} \begin{bmatrix} \tilde{y} \\ \tilde{z} \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{6.5}$$

We claim that $\tilde{y} \neq 0$. We argue by contradiction; suppose $\tilde{y} = 0$. Then (6.5) reduces to

$$\begin{aligned} B\tilde{z} + E\tilde{z} &= 0 \\ D\tilde{z} &= \rho\tilde{z} \end{aligned} \tag{6.6}$$

Since ρ is near 0 and D has a simple eigenvalue at 0 with all other eigenvalues bounded away from 0, it follows that $\rho = 0$ and we can take $\tilde{z} = z$. From (6.4) we know that $Bz = 0$; hence (6.6) implies $Ez = 0$. This is a contradiction, so $\tilde{y} \neq 0$. □

The proof of Lemma 6.3 does not require the eigenvalues or the eigenvectors of J to be real. Hence Lemma 6.3 also holds for matrices in $\mathcal{S}$ that have a pair of simple purely imaginary eigenvalues. This adaptation, stated without proof in the next lemma, is needed in Section 3.2.

Lemma 6.4. *Fix a fully inhomogeneous network with shape space $\mathcal{S}$. Let $\hat{\mathcal{S}}$ be the set of matrices $J \in \mathcal{S}$ that have a pair of simple purely imaginary eigenvalues. Define $\hat{\mathcal{S}}_i \subseteq \hat{\mathcal{S}}$ to be the set of matrices $J \in \hat{\mathcal{S}}$ with critical eigenvector $v = (v_1, \ldots, v_n)$ such that $v_i \neq 0$. Let*

$$\hat{\mathcal{T}}_{ij} = \{J \in \hat{\mathcal{S}}_i : \; v_j \neq 0\}.$$

Then for any node j downstream of node i, $\hat{\mathcal{T}}_{ij}$ is open and dense in $\hat{\mathcal{S}}_i$. □

Lemma 6.5. *Let C be the critical path component associated with the saddle-node bifurcation. Let v be the associated critical eigenvector. Then the coordinates of v on nodes that are not downstream from C are zero. Generically, the coordinates of v on all nodes that are downstream from C (including C) are nonzero.*

Proof. Because the zero eigenvalue at the saddle-node bifurcation is simple, it is associated with a unique path component C. By Theorem 2.9, the corresponding zero eigenvector v has zero components on nodes that are not downstream from C. Since v has at least one nonzero component on C, Lemma 6.3 implies that v has nonzero components on C and on all nodes downstream. □

Proof of Theorem 3.3 The growth rates follow from Lemma 6.5 and Remark 3.2. □

6.3 Proofs for Codimension One Hopf Bifurcation

We first consider the special case of a directed ring, and then parlay this case into a proof of the general result.

Lemma 6.6. *Nondegenerate Hopf bifurcation can occur for suitable admissible vector fields in a directed ring with more than one node.*

Proof. Consider a directed ring of nodes $1, \ldots, m$ with $1 \to 2, \ldots, m \to 1$. Admissible vector fields for this ring have the form

$$\begin{array}{rcl} \dot{x}_1 & = & f_1(x_1, x_m) \\ \dot{x}_2 & = & f_2(x_2, x_1) \\ & \vdots & \\ \dot{x}_m & = & f_m(x_m, x_{m-1}) \end{array} \tag{6.7}$$

Assume that (6.7) has an equilibrium at the origin; that is, $f_j(0) = 0$ for all j. We claim that the $m \times m$ Jacobian of (6.7) at the origin can be chosen to have a pair of simple complex conjugate purely imaginary eigenvalues, and no other imaginary eigenvalues.

In block form let L be the $m \times m$ matrix

$$L = \left[\begin{array}{cc} 0 & 1 \\ I_{m-1} & 0 \end{array} \right].$$

The characteristic polynomial of L is $p(\lambda) = \det(\lambda I_m - L) = \lambda^m - 1$ and the eigenvalues of L are the m^{th} roots of unity. For each m the matrix L has simple complex conjugate eigenvalues, so there exists μ such that $J = L - \mu I_m$ has simple purely imaginary eigenvalues and no other imaginary eigenvalues. The standard Hopf bifurcation theorem implies that adding λI_m to the vector field leads to a nondegenerate Hopf bifurcation; that is, to the desired $\sqrt{\lambda}$ growth rate of small amplitude periodic solutions. □

Lemma 6.7. *Hopf bifurcation in a path component H, at a pair of simple complex conjugate purely imaginary eigenvalues, is possible for some admissible map if and only if the number of nodes in that component satisfies $n_H > 1$.*

Proof. For $n_H = 1$, Hopf bifurcation is not possible. We therefore show that for $n_H > 1$, there exists an admissible Jacobian J with one pair of simple purely imaginary eigenvalues and all other eigenvalues off of the imaginary axis.

Fix a path component H with $n_H > 1$ nodes. We construct a directed ring within that path component as follows. Given any two distinct nodes ℓ and k in H, there exists a directed loop $\ell \to \cdots \to k$ and $k \to \cdots \to \ell$. Consider a loop of minimal length m. If any node occurs twice (except where the ends join) the segment in between is a smaller loop. So a minimal loop consists of distinct nodes, forming a closed ring. In particular, there are no connections between distinct nodes in the ring, except for the unidirectional nearest neighbor ones.

Order the nodes in the ring by $1, \ldots, m$. Consider admissible vector fields such that the coordinate function $f_j \equiv 0$ when $j > m$ and f_j has the form in (6.7) for $1 \leq j \leq m$. By Lemma 6.6 these admissible vector fields can have an equilibrium at which the Jacobian J has simple purely imaginary eigenvalues. However, 0 occurs $n - m$ times as an eigenvalue of J.

Theorem 6.1 implies that we can perturb the diagonal entries of J to make the 0 eigenvalues nonzero while fixing the purely imaginary pair of eigenvalues. The Jacobian J constructed in this way is admissible, and it has exactly one pair of simple purely imaginary eigenvalues and no other imaginary eigenvalues. □

Proof of Theorem 3.6 Fix a fully inhomogeneous network and consider the network admissible system

$$\dot{y} = F(y, \lambda)$$

for $y \in \mathbb{R}^n, \lambda \in \mathbb{R}$. Assume $F(0, \lambda) = 0$ so that $y = 0$ is a steady-state solution for all λ. By Lemma 6.7, purely imaginary eigenvalues associated with each path component H are possible as long as $n_H > 1$. Moreover, we can assume these eigenvalues are simple and all other eigenvalues are off the imaginary axis. Without loss of generality we can assume at codimension-1 Hopf bifurcation that $J = (\mathrm{D}_y F)_{(0,0)}$ has simple eigenvalues $\pm i$ and no other imaginary eigenvalues.

Define the eigenvectors c and d by $Jc = -ic$ and $J^T d = id$, where the superscript T denotes transpose. Using the inner product

$$\langle w, v \rangle = \bar{w}^T v, \tag{6.8}$$

where the overbar denotes complex conjugate, we can choose d such that $\langle d, c\rangle = 2$ (Golubitsky and Schaeffer, 1985, p. 346). In particular,

$$\langle d, c\rangle \neq 0.$$

Since dim ker$(J) = 2$, Liapunov-Schmidt reduction shows that near bifurcation the small amplitude periodic orbits of $\dot{y} = F(y, \lambda)$ are in one-to-one correspondence with zeros of a single equation $g(x, \lambda) = r(x^2, \lambda)x = 0$, where $x \in \mathbb{R}$. By the Liapunov-Schmidt procedure, $r_z(0, 0)r_\lambda(0, 0) \neq 0$ (where $z = x^2$) if and only if the resulting bifurcation is a nondegenerate Hopf bifurcation. The formulas for computing $r_z(0, 0)$ and $r_\lambda(0, 0)$ are standard (Golubitsky and Schaeffer, 1985, p. 352), and we assume that $r_\lambda(0, 0) \neq 0$.

To show that Hopf bifurcations are nondegenerate, we consider the case $r_z(0, 0) = 0$, and prove that a generic homogeneous cubic perturbation of F leads to a new vector field $G = F + \varepsilon\Psi$ such that the new cubic coefficient in the reduction $r^\varepsilon_z(0, 0) \neq 0$. In this case the coefficient can be computed as

$$r^\varepsilon_z(0, 0) = \frac{1}{16}\mathrm{Re}\,\langle d, (\mathrm{D}^3G)(c, c, \bar{c})\rangle = \frac{\varepsilon}{16}\mathrm{Re}\,\langle d, (\mathrm{D}^3\Psi)(c, c, \bar{c})\rangle$$

The Jacobian J and therefore c and d remain unchanged by the perturbation because we assume Ψ to be homogeneous cubic.

Because $\mathrm{Re}\langle d, c\rangle = 2$, there must be some node k such that $\mathrm{Re}(\bar{d}_k c_k) \neq 0$ and $|c_k| \neq 0$. Choose $\Psi_k = \frac{1}{6}x_k^3$ and $\Psi_j = 0$ for $j \neq k$. Then

$$r^\varepsilon_z(0, 0) = \frac{\varepsilon}{16}\mathrm{Re}(\bar{d}_k c_k c_k \bar{c}_k) = \frac{\varepsilon}{16}|c_k|^2\mathrm{Re}(\bar{d}_k c_k) \neq 0$$

as desired. This perturbation is admissible, since for every node j the variable x_j appears in f_j in (1.1). □

Lemma 6.8. *Let H be the critical path component associated with a nondegenerate Hopf bifurcation, and let v be the associated critical eigenvector. Then the coordinates of v on nodes that are not downstream from H are zero. Generically, the coordinates of v on all nodes that are downstream from H are nonzero.*

Proof. Since the purely imaginary eigenvalues are simple at the Hopf bifurcation, we can associate it with a unique path component H. The fact that the coordinates of v that are not downstream from H are zero follows from Theorem 2.9. By Lemma 6.4, generically all components of v that are downstream from H are nonzero. □

Proof of Theorem 3.7 This follows immediately from Lemma 6.8. □

7 Overview of Singularity Theory

The analysis of codimension-2 mode interactions in the next four sections relies on techniques from singularity theory. We summarize the main concepts and results here. We follow the approach to bifurcation problems in Golubitsky and Schaeffer (1985); Golubitsky, Stewart and Schaeffer (1988), but we do so without a distinguished parameter. These sources should be consulted for further details and proofs.

The analysis of the four codimension-2 mode interactions (steady-state/steady-state, steady-state/Hopf, Hopf/steady-state, and Hopf/Hopf) reduce to functions $F : \mathbb{R}^2 \to \mathbb{R}^2$ that satisfy a feedforward structure $F(x, y) = (f(x), g(x, y))$. In addition, F commutes with the action of a symmetry group on $\mathbb{R}^2$ in the three interactions involving Hopf modes.

Singularity theory is about the local topological structure of classes of $\mathcal{C}^\infty$ smooth maps

$$F : \mathbb{R}^m \to \mathbb{R}^n$$

near some point. By translation, we take this point to be $0 \in \mathbb{R}^m$ and assume $F(0) = 0$. The local structure is captured by introducing the following notion. Two such maps F, G are *germ-equivalent* if their restrictions to some open neighborhood $U \subseteq \mathbb{R}^m$ of 0 are equal; that is, $F(X) = G(X)$ for all $X \in U$. A *germ* is a germ-equivalence class. We define a germ by specifying a representative map, and identify the germ with this map, bearing in mind that only local information near 0 is meaningful. In particular, derivatives $\mathrm{D}^k F|_{X=0}$ of F at 0 are meaningful concepts for the germ of F, and so is the Taylor series of F near 0.

With this understood, we can henceforth omit 'germ' and refer to maps and functions. All of these are assumed smooth, and we mainly require the case $m = n = 2$.

Singularity theory uses changes of coordinates to simplify the form of F, where possible. These changes of coordinates preserve the number of solutions (zeros of F), and the type of solutions (if symmetry is present). To do this, define two problems $F, G : \mathbb{R}^2 \to \mathbb{R}^2$ to be *contact equivalent* if

$$G(X) = S(X)F(\Phi(X))$$

where the smooth map $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is a diffeomorphism and the smooth map

$$S : \mathbb{R}^2 \to \mathbb{GL}(2)$$

where $\mathbb{GL}(2)$ is the group of invertible real 2×2 matrices. The equivalence is *strong* if $\Phi(0) = 0$.

Contact equivalence preserves the topology of the zero set of F. It is the most general form of equivalence with this property, and it has technical advantages over any stronger form of equivalence.

The methods of singularity theory usually have to be adapted to any specific context, imposing extra conditions to ensure that the equivalences preserve any relevant structure. As we see in the next four sections, contact equivalence must be suitably modified in each of the four mode interactions to preserve the feedforward structure and the relevant symmetry.

Normal form theory: The first main objective is to use an suitable equivalence to transform a given map F into a simple polynomial map, a *normal form*. This is not always possible, but it can be done for 'almost all' maps, namely, those of finite codimension, see (7.1). To achieve this we consider 'infinitesimal' perturbations. Consider a one-parameter family of strong equivalences

$$G(X, \varepsilon) = S(X, \varepsilon)F(\Phi(X, \varepsilon))$$

where $\varepsilon \in \mathbb{R}$ is small. Differentiate with respect to ε (shown by a dot) and evaluate at $\varepsilon = 0$. We get

$$\dot{G}(x, 0) = \dot{S}(X, 0)F(X) + (\mathrm{D}F)_X\dot{\Phi}(X, 0)$$

We therefore define the *restricted tangent space* of F to consist of all possible $\dot{G}$; that is,

$$RT(F) = \{SF + (\mathrm{D}F)\Phi\}$$

where $S(X)$ is an arbitrary 2×2 matrix for each X and $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$ is an arbitrary map that satisfies $\Phi(0) = 0$. We now have:

Theorem 7.1 (Tangent Space Constant Theorem)**.** *Let F be a vector field. Suppose there exists $p : \mathbb{R}^2 \to \mathbb{R}^2$ such that*

$$RT(F + \varepsilon p) = RT(F)$$

for all $\varepsilon \in [0, 1]$. Then $F + \varepsilon p$ is strongly equivalent to F for all $\varepsilon \in [0, 1]$.

See Golubitsky and Schaeffer (1985) Chapter II Theorem 2.2 when $n = 1$, and Golubitsky, Stewart and Schaeffer (1988) Chapter XIV Theorem 3.1 for the general case. We can apply Theorem 7.1 to construct normal forms, and to solve the *recognition problem*: using conditions on Taylor coefficients to characterize when F has the normal form concerned.

The proof of Theorem 7.1 can be adapted to prove analogous theorems for each of the mode interactions. Alternatively, the appropriate tangent space constant theorem for each mode interaction follows from general results of Damon (1988). The principal difficulty in applying Theorem 7.1 is the computation of $RT(F)$. This computation is simplified by using its algebraic structure (a module over a system of rings) and Nakayama's Lemma (Golubitsky and Guillemin (1973); Gibson (1979); Martinet (1982)), which we briefly recall:

Lemma 7.2 (Nakayama's Lemma)**.** *Let R be a commutative ring with unit, with an ideal I such that whenever $r \in I$ the element $1 + r$ is invertible in R. Let M be a finitely generated R-module, with a submodule N. Then the condition*

$$N + IM = M$$

implies that $N = M$.

The next four sections carry out these calculations under the assumption that the corresponding tangent space constant theorem is valid.

Unfolding theory: The other main topic we need is unfolding theory, which determines all possible perturbations of F (of finite codimension) in a sense we now explain. A *k-parameter unfolding* of F is a smooth map

$$\widetilde{F} : \mathbb{R}^2 \times \mathbb{R}^k \to \mathbb{R}^2$$

such that

$$\widetilde{F}(X, 0) = F(0)$$

Let $\widetilde{H}(X, \beta)$ be an l-parameter unfolding of F. Then $\widetilde{H}$ *factors through* $\widetilde{F}$ if

$$\widetilde{H}(X, \beta) = S(X, \beta)\widetilde{F}(\Phi(X, \beta), A(\beta))$$

where $A : \mathbb{R}^l \to \mathbb{R}^k$, $A(0) = 0$, and $S(X, 0) = I, \Phi(X, 0) = X$. An unfolding is *versal* if every unfolding factors through it. It is *universal* if it is versal and the number of parameters is minimal among all versal unfoldings. This minimal number is the *codimension*

$$\operatorname{codim}(F). \tag{7.1}$$

The *tangent space* of F is

$$T(F) = \{SF + (\mathrm{D}F)\Phi\},$$

where the diffeomorphism Φ is an equivalence, but not necessarily a strong equivalence. That is, $\Phi(0)$ need not equal 0.

The codimension of F is equal to the codimension of the *tangent space* $T(F)$, which contains $RT(F)$ but may be larger. We refer to Golubitsky, Stewart and Schaeffer (1988) Chapter XV Section 2 for a discussion and definition.

Finally, we state a criterion for a universal unfolding to exist:

Theorem 7.3. *A family $\widetilde{F}$ is a universal unfolding of F if and only if*

$$\vec{\mathcal{E}}_X = T(F) \oplus \mathbb{R}\{\widetilde{F}_{\alpha_1}(X, 0), \ldots, \widetilde{F}_{\alpha_k}(X, 0)\}.$$

Corollary 7.4. *The codimension of F is equal to the codimension of $T(F)$ in $\vec{\mathcal{E}}_X$.*

The proofs of the theorems for the four mode interaction cases that are analogous to Theorem 7.3 follow from Damon (1988) and these corresponding theorems are used in the next four sections. Generally speaking, a singularity theory analysis proceeds by computing $RT(F)$, determining a normal form $\hat{F}$ of F, computing $T(F)$ based on the computation of $RT(\hat{F})$, and finally determining a universal unfolding of $\hat{F}$.

8 Steady-State/Steady-State Mode Interaction: Proofs

This section outlines proofs based on singularity theory, as reviewed in Section 7, of the main results in Section 5.1 on steady-state/steady-state mode interaction. We first compute the restricted tangent space for the center manifold dynamics (5.1) associated with this mode interaction, and use this result to prove Theorem 5.3: that (5.1) can be transformed to the normal form (5.5). We then use a complement of the unrestricted tangent space of (5.5) to identify the universal unfolding (5.7), proving Theorem 5.6.

8.1 Restricted Tangent Space for SS/SS Mode Interaction

The restricted tangent space of a map F, denoted $RT(F)$, is obtained from $\frac{d}{d\tau}\Gamma_\tau(F)|_{\tau=0}$, where Γ_τ is a one-parameter family of strong equivalences (as in Definition 5.2) with $\Gamma_0(x,y) = (x,y)$.

For technical reasons we use a version of singularity theory adapted to maps of the form $(f(x), g(x,y))$. These maps are analyzed using a special case of the general concept of a *system of rings* and an associated *system of modules*, as defined in Damon (1984) p. 242–243. In this case the key step is to work with a pair of rings $(\mathcal{E}_x, \mathcal{E}_{x,y})$ instead of a single ring. In place of a module over a ring, we use a direct sum $M_1 \oplus M_2$ where M_1 is a module over $\mathcal{E}_x$ and M_2 is a module over $\mathcal{E}_{x,y}$. Tangent spaces and restricted tangent spaces are defined by analogy with the case of a single ring and module.

In Lemma 8.1 we show that $RT(F)$ is a system of modules over the system of rings $(\mathcal{E}_x, \mathcal{E}_{x,y})$. The tangent space constant theorem that is analogous to Theorem 7.1 states that if

$$RT(F + \tau p) = RT(F)$$

for all $\tau > 0$, then $F + \tau p$ is strongly equivalent to F. In the context of systems of rings, this theorem follows from Damon (1984, 1988). See also Dangelmayr and Stewart (1985).

Lemma 8.1. *Let $F = (f(x), g(x,y))$ be a map in $(\mathcal{E}_x, \mathcal{E}_{x,y})$. A map $G \in (\mathcal{E}_x, \mathcal{E}_{x,y})$ is in $RT(F)$ if and only if there exist maps $P_i(x) \in \mathcal{E}_x$ and $Q_j(x,y) \in \mathcal{E}_{x,y}$ such that*

$$G(x,y) = P_1 \begin{bmatrix} f \\ 0 \end{bmatrix} + P_2 \begin{bmatrix} xf_x \\ xg_x \end{bmatrix} + Q_1 \begin{bmatrix} 0 \\ f \end{bmatrix} + Q_2 \begin{bmatrix} 0 \\ g \end{bmatrix} + Q_3 \begin{bmatrix} 0 \\ xg_y \end{bmatrix} + Q_4 \begin{bmatrix} 0 \\ yg_y \end{bmatrix}$$

Proof. The general form of strong equivalence is given in (5.2). Define a one-parameter family of strong equivalences by

$$\Gamma_\tau(F)(x,y) = \begin{bmatrix} a(x,\tau) & 0 \\ b(x,y,\tau) & c(x,y,\tau) \end{bmatrix} \begin{bmatrix} f\left(\phi(x,\tau)\right) \\ g\left(\phi(x,\tau), \psi(x,y,\tau)\right) \end{bmatrix}, \tag{8.1}$$

where Γ_0 is the identity and $\Gamma_\tau(0,0) = (0,0)$. Then

$$\begin{gathered} a(x,0) = 1 \quad b(x,y,0) = 0 \quad c(x,y,0) = 1 \quad \phi(x,0) = x \quad \psi(x,y,0) = y \\ \phi(0,\tau) = 0 \quad \psi(0,0,\tau) = 0 \end{gathered} \tag{8.2}$$

We compute the restricted tangent space by differentiating (8.1) with respect to τ (indicated by a dot) and evaluating at $\tau = 0$, obtaining

$$\begin{aligned} \dot{\Gamma}_0(x,y) = \quad & \dot{a}(x,0) \begin{bmatrix} f\,(x) \\ 0 \end{bmatrix} + \dot{b}(x,y,0) \begin{bmatrix} 0 \\ f\,(x) \end{bmatrix} + \dot{c}(x,y,0) \begin{bmatrix} 0 \\ g\,(x,y) \end{bmatrix} \\ & + \dot{\phi}(x,0) \begin{bmatrix} f_x\,(x) \\ g_x\,(x,y) \end{bmatrix} + \dot{\psi}(x,y,0) \begin{bmatrix} 0 \\ g_y\,(x,y) \end{bmatrix} \end{aligned} \tag{8.3}$$

We use (8.2) to conclude that $\dot{a}(x,0)$, $\dot{b}(x,y,0)$ and $\dot{c}(x,y,0)$ are arbitrary, whereas $\dot{\phi}(x,0) = x\eta(x)$ and $\dot{\psi}(x,y,0) = x\sigma(x,y) + y\nu(x,y)$ for arbitrary functions η, σ, ν. The restricted tangent space is therefore

$$\left\langle \begin{bmatrix} f \\ 0 \end{bmatrix}, \begin{bmatrix} xf_x \\ xg_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ f \end{bmatrix}, \begin{bmatrix} 0 \\ g \end{bmatrix}, \begin{bmatrix} 0 \\ xg_y \end{bmatrix}, \begin{bmatrix} 0 \\ yg_y \end{bmatrix} \right\rangle_{\{x,y\}}$$

Here the notations $\langle \cdots \rangle_{\{x\}}$ and $\langle \cdots \rangle_{\{x,y\}}$ indicate generators of a module over the rings $\mathcal{E}_x$ and $\mathcal{E}_{x,y}$, respectively. □

8.2 Normal Form for SS/SS Mode Interaction

We prove Theorem 5.3 by showing that F can be transformed to the normal form (5.5). We do this in two steps. First, in Lemma 8.2 we explicitly transform the linear and quadratic terms of F into the normal form (5.5); then we use the tangent space constant theorem to transform away terms of order three and higher.

The defining conditions for a steady-state steady-state mode interaction imply that to quadratic order F takes the form $F_2 = (f_2, g_2)$, where

$$\begin{aligned} f_2(x) &= px^2 \\ g_2(x,y) &= qx + rx^2 + sxy + ty^2. \end{aligned} \tag{8.4}$$

Lemma 8.2. *Any map of the form* (8.4) *with* $p, q, t \neq 0$ *is strongly equivalent to the normal form* $\hat{F} = (\hat{f}, \hat{g})$ *where*

$$\begin{aligned} \hat{f}(x) &= \varepsilon_p x^2 \\ \hat{g}(x,y) &= \varepsilon_q x + \varepsilon_t y^2, \end{aligned} \tag{8.5}$$

and $\varepsilon_* = sign(*)$.

Proof. As we are interested only in terms up to second order in x and y, we take the truncated forms of the transformation functions used to define equivalence in (5.2) to be

$$\phi(x) = \alpha x \quad \psi(x,y) = \beta x + \gamma y \quad a(x) = \delta \quad b(x,y) = \sigma \quad c(x,y) = \rho.$$

Now

$$\begin{bmatrix} \hat{f}(x) \\ \hat{g}(x,y) \end{bmatrix} = \begin{bmatrix} \delta p \alpha^2 x^2 \\ \sigma p \alpha^2 x^2 + \rho\left(q\alpha x + r\alpha^2 x^2 + s\alpha x(\beta x + \gamma y) + t(\beta x + \gamma y)^2\right) \end{bmatrix} \tag{8.6}$$

Combining terms in (8.6), the transformed coefficients are

$$\begin{aligned} \hat{p} &= \delta\alpha^2 p \\ \hat{q} &= \rho\alpha q \\ \hat{r} &= \sigma\alpha^2 p + \rho\alpha^2 r + \rho\alpha\beta s + \rho\beta^2 t \\ \hat{s} &= \rho\alpha\gamma s + 2\rho\beta\gamma t \\ \hat{t} &= \rho\gamma^2 t \end{aligned}$$

We assumed that $p \neq 0$, $t \neq 0$, $q \neq 0$ in f and g. Thus we can simplify the system so that $\hat{p} = \varepsilon_p$, $\hat{q} = \varepsilon_q$, $\hat{t} = \varepsilon_t$ where $\varepsilon_{(*)} = \text{sign}(*)$. Additionally, we can impose the conditions $\hat{r} = 0$ and $\hat{s} = 0$, leading to the transformation

$$\begin{aligned}
\rho &= \frac{\varepsilon_t}{\gamma^2 t} \\
\alpha &= \frac{\varepsilon_q}{\rho q} = \frac{\varepsilon_q \gamma^2 t}{\varepsilon_t q} \\
\delta &= \frac{\varepsilon_p}{\alpha^2 p} = \frac{\varepsilon_t^2 \varepsilon_p q^2}{\varepsilon_p^2 \gamma^4 t^2 p} \\
\beta &= -\frac{\alpha s}{2t} = -\frac{\varepsilon_q \gamma^2 s}{2\varepsilon_t q} \\
\sigma &= -\frac{\rho}{\alpha^2 p}\left(\alpha^2 r + \alpha\beta s + \beta^2 t\right) = -\frac{\varepsilon_t}{\gamma^2 t p}\left(r - \frac{s^2}{4t}\right).
\end{aligned}$$

Here $\gamma > 0$ is a free parameter. We require $\delta, \rho, \alpha, \gamma > 0$ so that the transformation preserves the stabilities of steady states. Were we free to choose the signs of δ, α, ρ, we could have transformed $\hat{p}, \hat{q}, \hat{t}$ to $+1$. Applying the transformation specified above to (8.4) produces (8.5). $\square$

Proof of Theorem 5.3: By Lemma 8.2 a general map $F(x, y) = (f(x), g(x, y))$ satisfying the defining and nondegeneracy conditions is strongly equivalent to (5.5) modulo terms of order three or higher. That is, F is equivalent to $\tilde{F} = \hat{F} + \cdots$, where $\cdots$ indicates terms of order three or higher. Using the tangent space constant theorem we may also remove terms of order three and higher by a suitable transformation. Specifically, we show that $RT(\tilde{F}) = RT(\hat{F})$.

First, we claim that the restricted tangent space of the normal form $(\hat{f}, \hat{g})$ is

$$RT(\hat{F}) = \begin{bmatrix} \mathcal{M}_x^2 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathcal{M}_{xy}^2 + \mathbb{R}\{x\} \end{bmatrix}, \tag{8.7}$$

which is a system of modules over the system of rings $(\mathcal{E}_x, \mathcal{E}_{xy})$. By Lemma 8.1, the restricted tangent space for the normal form $\hat{F}$ in (5.5) is

$$RT(\hat{F}) = \left\langle \begin{bmatrix} \varepsilon_p x^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2\varepsilon_p x^2 \\ \varepsilon_q x \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ \varepsilon_p x^2 \end{bmatrix}, \begin{bmatrix} 0 \\ \varepsilon_q x + \varepsilon_t y^2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2\varepsilon_t xy \end{bmatrix}, \begin{bmatrix} 0 \\ 2\varepsilon_t y^2 \end{bmatrix} \right\rangle_{\{x,y\}}.$$

By linear combinations of the vectors, we can reduce this to

$$RT(\hat{F}) = \left\langle \begin{bmatrix} x^2 \\ 0 \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ x^2 \end{bmatrix}, \begin{bmatrix} 0 \\ y^2 \end{bmatrix}, \begin{bmatrix} 0 \\ xy \end{bmatrix} \right\rangle_{\{x,y\}}.$$

The restricted tangent space is therefore as in (8.7).

Next, we consider higher order maps $n \in \mathcal{M}_x^3$ and $m \in \mathcal{M}_{xy}^3$; that is, $\tilde{F} = \hat{F} + (n, m)$. We use Nakayama's Lemma (Lemma 7.2) to prove that $RT(\tilde{F}) = RT(\hat{F})$. It follows that $\tilde{F} = (\hat{f} + n, \hat{g} + m)$ is strongly equivalent to the normal form $\hat{F}$ and hence that F is strongly equivalent to $\hat{F}$, as desired. Specifically, we set

$$\begin{aligned} \tilde{f}(x) &= \varepsilon_p x^2 + n(x) \\ \tilde{g}(x,y) &= \varepsilon_q x + \varepsilon_t y^2 + m(x,y) \end{aligned} \tag{8.8}$$

and compute

$$\begin{aligned} &RT(\tilde{F}) = \left\langle \begin{bmatrix} \varepsilon_p x^2 + n \\ 0 \end{bmatrix}, \begin{bmatrix} 2\varepsilon_p x^2 + x n_x \\ \varepsilon_q x + x m_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \\ &\left\langle \begin{bmatrix} 0 \\ \varepsilon_p x^2 + n \end{bmatrix}, \begin{bmatrix} 0 \\ \varepsilon_q x + \varepsilon_t y^2 + m \end{bmatrix}, \begin{bmatrix} 0 \\ 2\varepsilon_t xy + x m_y \end{bmatrix}, \begin{bmatrix} 0 \\ 2\varepsilon_t y^2 + y m_y \end{bmatrix} \right\rangle_{\{x,y\}}. \end{aligned} \tag{8.9}$$

By (8.7) each generator of $RT(\tilde{F})$ in (8.9) is in $RT(\hat{F})$. Hence $RT(\tilde{F}) \subseteq RT(\hat{F})$. Next we apply Nakayama's Lemma to prove $RT(\hat{F}) \subseteq RT(\tilde{F})$, for which we must show that$RT(\hat{F}) \subseteq RT(\tilde{F}) + (\mathcal{M}_x, \mathcal{M}_{xy})RT(\hat{F})$. The generators of $RT(\hat{F})$ over the system of rings are

$$\begin{bmatrix} x^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ y^2 \end{bmatrix} = \begin{bmatrix} \mathcal{M}_x^2 \\ \mathcal{M}_{xy}^2 + \mathbb{R}\{x\} \end{bmatrix}$$

and

$$(\mathcal{M}_x, \mathcal{M}_{xy})RT(\hat{F}) = \begin{bmatrix} \mathcal{M}_x^3 \\ \mathcal{M}_{xy}^3 + \mathcal{M}_{xy}\{x\} \end{bmatrix}.$$

Thus we must show that

$$\left\langle \begin{bmatrix} x^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x \end{bmatrix}, \begin{bmatrix} 0 \\ y^2 \end{bmatrix} \right\rangle \subseteq RT(\tilde{F}) + \left\langle \begin{bmatrix} x^3 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ x^2 \end{bmatrix}, \begin{bmatrix} 0 \\ xy \end{bmatrix}, \begin{bmatrix} 0 \\ y^3 \end{bmatrix} \right\rangle,$$

which follows from

$$\begin{aligned} \begin{bmatrix} x^2 \\ 0 \end{bmatrix} &= \begin{bmatrix} x^2 + n \\ 0 \end{bmatrix} - \begin{bmatrix} n \\ 0 \end{bmatrix} \in RT(\tilde{F}) + \begin{bmatrix} \mathcal{M}_x^3 \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ y^2 \end{bmatrix} &= \begin{bmatrix} 0 \\ y^2 + y m_y \end{bmatrix} - \begin{bmatrix} 0 \\ y m_y \end{bmatrix} \in RT(\tilde{F}) + \begin{bmatrix} 0 \\ \mathcal{M}_{xy}^3 \oplus \mathcal{M}_{xy} \langle x \rangle \end{bmatrix} \\ \begin{bmatrix} 0 \\ x \end{bmatrix} &= \begin{bmatrix} 0 \\ x + y^2 + m \end{bmatrix} - \begin{bmatrix} 0 \\ y^2 \end{bmatrix} - \begin{bmatrix} 0 \\ m \end{bmatrix} \in RT(\tilde{F}) + \begin{bmatrix} 0 \\ \mathcal{M}_{xy}^3 \oplus \mathcal{M}_{xy} \langle x \rangle \end{bmatrix}. \end{aligned}$$

Therefore the restricted tangent space of the perturbed system is identical to the restricted tangent space of the original system, so the two are equivalent. □

The next step is to compute the codimension of the restricted tangent space to ensure that it is finite. A complement of the restricted tangent space of (5.5) in $(\mathcal{E}_x, \mathcal{E}_{x,y})$ is

$$\mathbb{R}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\}$$

so the codimension of $RT(\hat{F})$ is 4.

8.3 Tangent Spaces for SS/SS Mode Interaction

In order to find a universal unfolding we must compute a complement of the unrestricted tangent space, which is generated by relaxing the constraint that the origin remains fixed under coordinate transformations.

Lemma 8.3. *The tangent space of a map $F = (f, g)$ where $f \in \mathcal{E}_x$ and $g \in \mathcal{E}_{x,y}$ is*

$$T(F) = RT(F) \oplus \mathbb{R}\left\{ \begin{bmatrix} f_x(x) \\ g_x(x,y) \end{bmatrix}, \begin{bmatrix} 0 \\ g_y(x,y) \end{bmatrix} \right\}. \tag{8.10}$$

Proof. Computing the tangent space is similar to computing the restricted tangent space as in Lemma 8.1, except that now we do not require the origin to be fixed by the coordinate transformation. That is, $\dot{\phi}(x,0)$ and $\dot{\psi}(x,y,0)$ in (8.3) can be arbitrary functions. Hence the tangent space is

$$\left\langle \begin{bmatrix} f \\ 0 \end{bmatrix}, \begin{bmatrix} f_x \\ g_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ f \end{bmatrix}, \begin{bmatrix} 0 \\ g \end{bmatrix}, \begin{bmatrix} 0 \\ g_y \end{bmatrix} \right\rangle_{\{x,y\}}.$$

Equation (8.10) follows from

$$\begin{aligned} \left\langle \begin{bmatrix} f_x \\ g_x \end{bmatrix} \right\rangle_{\{x\}} &= \left\langle \begin{bmatrix} xf_x \\ xg_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \mathbb{R}\left\{ \begin{bmatrix} f_x \\ g_x \end{bmatrix} \right\} \\ \left\langle \begin{bmatrix} 0 \\ g_y \end{bmatrix} \right\rangle_{\{x,y\}} &= \left\langle \begin{bmatrix} 0 \\ xg_y \end{bmatrix}, \begin{bmatrix} 0 \\ yg_y \end{bmatrix} \right\rangle_{\{x,y\}} \oplus \mathbb{R}\left\{ \begin{bmatrix} 0 \\ g_y \end{bmatrix} \right\}. \end{aligned}$$

□

We are now in a position to compute a universal unfolding of the normal form (5.5) using the analog of Theorem 7.3.

Proof of the Universal Unfolding Theorem 5.6: Compute

$$\left\{ \begin{bmatrix} f_x(x) \\ g_x(x,y) \end{bmatrix}, \begin{bmatrix} 0 \\ g_y(x,y) \end{bmatrix} \right\} = \left\{ \begin{bmatrix} 2\varepsilon_p x \\ \varepsilon_q \end{bmatrix}, \begin{bmatrix} 0 \\ 2\varepsilon_t y \end{bmatrix} \right\}.$$

The tangent space is therefore

$$T(F) = \begin{bmatrix} \mathcal{M}_x^2 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathcal{M}_{xy} \end{bmatrix} \oplus \mathbb{R}\left\{ \begin{bmatrix} 2\varepsilon_p x \\ \varepsilon_q \end{bmatrix} \right\},$$

and a complement to $T(F)$ is a two dimensional complement to $(2\varepsilon_p x, \varepsilon_q)$ in the span

$$\mathbb{R}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right\}.$$

The complement

$$\mathbb{R}\left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\}$$

leads to the universal unfolding (5.7).

9 Hopf/Steady-State Mode Interaction: Proofs

This section outlines proofs of the main results in Section 5.2 on Hopf/steady-state mode interaction. To apply singularity theory, we first use Liapunov-Schmidt reduction on the three-dimensional center manifold to construct a two-dimensional network whose zeros are in one-to-one correspondence with the equilibria of the center manifold network. We compute the restricted tangent space for the Liapunov-Schmidt reduced network of the vector field (5.8), and use this result to prove Theorem 5.11, which states that (5.13) is a normal form. We then use the complement of the tangent space of (5.13) to identify the universal unfolding (5.14), proving Theorem 5.12.

9.1 Liapunov-Schmidt Reduction for H/SS Mode Interaction

In this subsection, we prove Theorem 5.8 using the standard 'loop space' approach to Hopf bifurcation via Liapunov-Schmidt reduction, Golubitsky and Schaeffer (1985) Chapter VIII. First we construct, from the center manifold vector field (5.8), an operator Φ with the property that solutions to $\Phi = 0$ correspond to periodic solutions of (5.8) with period approximately 2π. Then we apply Liapunov-Schmidt reduction to Φ to prove Theorem 5.8.

Proof of Theorem 5.8: We seek periodic solutions of (5.8) with period approximately 2π, for which we introduce τ corresponding to a rescaled time $s = t/(1+\tau)$. In terms of s, (5.8) can be rewritten as

$$\begin{bmatrix} \frac{dX}{ds} - (1+\tau)f(X) \\ \frac{dy}{ds} - (1+\tau)g(X,y) \end{bmatrix} = 0. \tag{9.1}$$

A 2π-periodic solution of (9.1) corresponds to a periodic solution of (5.8) with period $2\pi(1+\tau)$, which is close to 2π for $\tau \approx 0$. We define

$$\Phi : \mathcal{C}^1_{2\pi}(\mathbb{R}^2) \times \mathcal{C}^1_{2\pi}(\mathbb{R}) \times \mathbb{R} \to \mathcal{C}_{2\pi}(\mathbb{R}^2) \times \mathcal{C}_{2\pi}(\mathbb{R}) \times \mathbb{R}$$

by the left hand side of (9.1). Then the zeros of $\Phi(X, y, \tau)$ characterizes periodic solutions of (5.8) with period approximately 2π.

We now apply the Liapunov-Schmidt reduction to Φ to find a reduced map ϕ in the coordinates (9.3) on $\ker(d\Phi)$. Then we use its properties to derive the theorem. The linearization of Φ about $(X, y, \tau) = (0, 0, 0)$ is

$$d\Phi = \begin{bmatrix} \frac{d}{ds} - \mathrm{Df} & 0 \\ -\nabla_X g & \frac{d}{ds} \end{bmatrix}, \tag{9.2}$$

whose kernel is 3-dimensional:

$$\ker(d\Phi) = \left\langle \begin{bmatrix} c \\ -i(\nabla_X g)\cdot c \end{bmatrix} e^{is}, \begin{bmatrix} \bar{c} \\ i(\nabla_X g)\cdot \bar{c} \end{bmatrix} e^{-is}, \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right\rangle. \tag{9.3}$$

Here c is a 2-dimensional complex eigenvector that satisfies Df $c = ic$. Identify $\ker(d\Phi)$ with $\mathbb{R}^3$ via the map

$$(x_1, x_2, y) \to x_1 \mathrm{Re}[we^{is}] + x_2 \mathrm{Im}[we^{is}] + ye_3, \tag{9.4}$$

where $w = (c, -i(\nabla_u g) \cdot c)$ and $e_3 = (0, 0, 1)$. The circle group $\mathbb{S}^1$ acts on $\ker(d\Phi)$ by

$$\gamma(\theta) = \begin{bmatrix} R(\theta) & 0 \\ 0 & 1 \end{bmatrix}$$

where $R(\theta)$ acts on $\mathbb{R}^2$ by rotation counterclockwise through the angle θ.

In the coordinates (9.4) on $\ker(d\Phi)$, the reduced map ϕ has the form

$$\phi(x_1, x_2, y, \tau) = \ p(x_1^2 + x_2^2, \tau) \begin{bmatrix} x_1 \\ x_2 \\ 0 \end{bmatrix} + q(x_1^2 + x_2^2, \tau) \begin{bmatrix} -x_2 \\ x_1 \\ 0 \end{bmatrix} + \sigma(x_1^2 + x_2^2, y, \tau) \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

because ϕ commutes with the action of $\mathbb{S}^1$. Formulas for the derivatives of the reduced function (Golubitsky and Schaeffer, 1985, p. 295) imply that $p(0, 0) = q(0, 0) = \sigma(0, 0, 0) = p_\tau(0, \tau) = \sigma_\tau(0, 0, \tau) = 0$ and $q_\tau(0, \tau) = -1$. Solutions to $\phi = 0$ locally are in one-to-one correspondence with periodic solutions of (5.8).

The rotational symmetry lets us assume that $x_2 = 0$, $x_1 \geq 0$, and the implicit function theorem lets us solve $q(x_1^2, \tau) = 0$ for $\tau = \tau(x_1^2)$ (Golubitsky and Schaeffer, 1985, p. 345). Now all solutions to $\phi = 0$ may be obtained from zeros of

$$F(x_1, y) = \begin{bmatrix} r(x_1^2)x_1 \\ g(x_1^2, y) \end{bmatrix}$$

where $r(z) = p(z, \tau(z))$ and $g(z, y) = \sigma(z, y, \tau(z))$ and $x_1 \geq 0$. □

9.2 Restricted Tangent Space for H/SS Mode Interaction

The restricted tangent space of a map F in the context of symmetry, denoted by $RT(F)$, is obtained from $\frac{d}{d\tau}\Gamma_\tau(F)|_{\tau=0}$, where Γ_τ is a one-parameter family of strong $\mathbf{Z}_2$-equivalences (Definition 5.9) with $\Gamma_0(x, y) = (x, y)$.

Lemma 9.1. *Let $F = (r(u)x, g(u, y))$ be a map in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,y})$. A map $G \in (\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,y})$ lies in $RT(F)$ if and only if there exist maps $P_i(u) \in \mathcal{E}_u$ and $Q_j(u, y) \in \mathcal{E}_{u,y}$ such that*

$$G(x, y) = P_1 \begin{bmatrix} rx \\ 0 \end{bmatrix} + P_2 \begin{bmatrix} 2r_u ux + rx \\ 2g_u u \end{bmatrix} + Q_1 \begin{bmatrix} 0 \\ ru \end{bmatrix} + Q_2 \begin{bmatrix} 0 \\ g \end{bmatrix} + Q_3 \begin{bmatrix} 0 \\ ug_y \end{bmatrix} + Q_4 \begin{bmatrix} 0 \\ yg_y \end{bmatrix}.$$

Proof. The general form of strong $\mathbf{Z}_2$-equivalence is (5.10). Define a one-parameter family of strong $\mathbf{Z}_2$-equivalences by

$$\Gamma_\tau(F)(x, y) = \begin{bmatrix} a(u, \tau) & 0 \\ b(u, y, \tau)x & c(u, y, \tau) \end{bmatrix} \begin{bmatrix} r(\phi^2(u, \tau)u)\phi(u, \tau)x \\ g(\phi^2(u, \tau)u, \psi(u, y, \tau)) \end{bmatrix}, \tag{9.5}$$

where Γ_0 is the identity and $\Gamma_\tau(F)(0,0) = (0,0)$. Then

$$a(u,0) = 1,\; b(u,y,0) = 0,\; c(u,y,0) = 1,\; \phi(u,0) = 1,\; \psi(u,y,0) = y,\; \psi(0,0,\tau) = 0. \quad (9.6)$$

To compute the restricted tangent space, differentiate (9.5) with respect to τ and evaluate at $\tau = 0$, to obtain

$$\begin{aligned} \dot{\Gamma}_0(x,y) &= \dot{a}(u,0)\begin{bmatrix} r(u)x \\ 0 \end{bmatrix} + \dot{b}(u,y,0)\begin{bmatrix} 0 \\ r(u)u \end{bmatrix} + \dot{c}(u,y,0)\begin{bmatrix} 0 \\ g(u,y) \end{bmatrix} \\ &\quad + \dot{\phi}(u,0)\begin{bmatrix} 2r_u(u)ux + r(u)x \\ 2g_u(u,y)u \end{bmatrix} + \dot{\psi}(u,y,0)\begin{bmatrix} 0 \\ g_y(u,y) \end{bmatrix}. \end{aligned} \quad (9.7)$$

Conditions (9.6) imply that $\dot{a}(u,0)$, $\dot{b}(u,y,0)$, $\dot{c}(u,y,0)$ and $\dot{\phi}(u,0)$ are arbitrary, whereas $\dot{\psi}(u,y,0) = u\sigma(u,y) + y\nu(u,y)$ for arbitrary functions σ and ν. The restricted tangent space is therefore spanned by

$$\left\langle \begin{bmatrix} rx \\ 0 \end{bmatrix}, \begin{bmatrix} 2r_u ux + rx \\ 2g_u u \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ ru \end{bmatrix}, \begin{bmatrix} 0 \\ g \end{bmatrix}, \begin{bmatrix} 0 \\ ug_y \end{bmatrix}, \begin{bmatrix} 0 \\ yg_y \end{bmatrix} \right\rangle_{\{u,y\}}.$$

□

9.3 Normal Form for H/SS Mode Interaction

The proof of Theorem 5.11 is similar to the proof of Theorem 5.3, and is carried out in two steps. First, in Lemma 9.2 we explicitly transform the lower order terms in $r(u)$ and $g(u,y)$ in $F = (r(u)x, g(u,y))$ into the normal form (5.13). Second, we use the tangent space constant theorem for the Hopf/steady-state mode interaction that is analogous to Theorem 7.1 to transform away higher order terms.

The defining conditions for a Hopf/steady-state mode interaction imply that to first order in u and quadratic order in y, the functions $r(u)$ and $g(u,y)$ take the forms pu and $qu + ty^2$. Hence F takes the form $\bar{F} = (\bar{r}x, \bar{g})$ where

$$\begin{aligned} \bar{r}(u)x &= pux \\ \bar{g}(u,y) &= qu + ty^2 \end{aligned} \quad (9.8)$$

Lemma 9.2. *Any map of the form (9.8) with $p, q, t \neq 0$ is strongly $\mathbf{Z}_2$-equivalent to the normal form $\hat{F} = (\hat{r}x, \hat{g})$ where*

$$\begin{aligned} \hat{r}(u)x &= \varepsilon_p ux \\ \hat{g}(u,y) &= \varepsilon_q u + \varepsilon_t y^2 \end{aligned} \quad (9.9)$$

and $\varepsilon_* = sign(*)$.

Proof. We are interested only in terms of $r(u)$ and $g(u,y)$ up to first order in u and second order in y, so we compute the truncated forms of the transformation functions used to define $\mathbf{Z}_2$-equivalence in (5.10), obtaining

$$\phi(u) = \alpha \quad \psi(u,y) = \gamma y \quad a(u) = \delta \quad b(u,y) = 0 \quad c(u,y) = \rho.$$

Now

$$\begin{bmatrix} \hat{r}(u)x \\ \hat{g}(u,y) \end{bmatrix} = \begin{bmatrix} \delta p \alpha^3 ux \\ \rho q \alpha^2 u + \rho t \gamma^2 y^2 \end{bmatrix}. \tag{9.10}$$

Combining like terms, the transformed coefficients are

$$\begin{array}{rcl} \hat{p} &=& \delta\alpha^3 p \\ \hat{q} &=& \rho\alpha^2 q \\ \hat{t} &=& \rho\gamma^2 t. \end{array} \tag{9.11}$$

By assumption, $p \neq 0$, $q \neq 0$, $t \neq 0$ in $\bar{r}$ and $\bar{g}$. Thus we can impose the conditions $\hat{p} = \varepsilon_p$, $\hat{q} = \varepsilon_q$ and $\hat{t} = \varepsilon_t$, where $\varepsilon_* = \text{sign}(*)$, by making the transformation

$$\begin{array}{rcl} \delta &=& \frac{\varepsilon_p}{p\alpha^3} \\ \rho &=& \frac{\varepsilon_q}{\alpha^2 q} \\ \gamma^2 &=& \frac{\alpha^2 q \varepsilon_t}{\varepsilon_q t}. \end{array}$$

Here $\alpha > 0$ is a free parameter. We require δ, ρ, α, $\gamma > 0$ to preserve the stabilities of steady states. Were we free to choose the signs of δ, ρ, we could have transformed $\hat{p}$, $\hat{q}$, $\hat{t}$ to be $+1$. Applying the above transformation to (9.8) produces (9.9). $\square$

Proof of Theorem 5.11: By Lemma 9.2 a general map $F(x,y) = (r(u)x, g(u,y))$ satisfying the defining and nondegeneracy conditions is strongly $\mathbf{Z}_2$-equivalent to $\tilde{F} = \hat{F} + \cdots$, where $\hat{F}$ is the normal form (5.13) and $\cdots$ indicates terms of higher order. Using the tangent space constant theorem we remove higher order terms associated to $\hat{F}$ by a suitable transformation. Specifically, we show that $RT(\tilde{F}) = RT(\hat{F})$.

First, we claim that the restricted tangent space of the normal form $(\hat{r}x, \hat{g})$ is

$$RT(\hat{F}) = \begin{bmatrix} \mathcal{M}_u \langle x \rangle \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathcal{M}_{uy}^2 + \mathbb{R}\{u\} \end{bmatrix}, \tag{9.12}$$

which is a system of modules over the system of rings $(\mathcal{E}_u, \mathcal{E}_{u,y})$. By Lemma 9.1, the restricted tangent space for the normal form (5.13) is

$$RT(\hat{F}) = \left\langle \begin{bmatrix} \varepsilon_p ux \\ 0 \end{bmatrix}, \begin{bmatrix} 3\varepsilon_p ux \\ 2\varepsilon_q u \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ \varepsilon_p u^2 \end{bmatrix}, \begin{bmatrix} 0 \\ \varepsilon_q u + \varepsilon_t y^2 \end{bmatrix}, \begin{bmatrix} 0 \\ 2\varepsilon_t uy \end{bmatrix}, \begin{bmatrix} 0 \\ 2\varepsilon_t y^2 \end{bmatrix} \right\rangle_{\{u,y\}}.$$

Taking linear combinations, this reduces to

$$RT(\hat{F}) = \left\langle \begin{bmatrix} ux \\ 0 \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ u \end{bmatrix}, \begin{bmatrix} 0 \\ y^2 \end{bmatrix}, \begin{bmatrix} 0 \\ uy \end{bmatrix} \right\rangle_{\{u,y\}}.$$

The restricted tangent space is therefore given by (9.12).

Next, consider higher order maps $n \in \mathcal{M}_u^2$, and $m \in \mathcal{M}_{uy}^3 \oplus \mathcal{M}_{uy}\langle u\rangle$. That is, let $\tilde{F} = (\tilde{r}x, \tilde{g}) = \hat{F} + (nx, m)$, where

$$\begin{aligned} \tilde{r}(u)x &= (\varepsilon_p u + n(u))x \\ \tilde{g}(u,y) &= \varepsilon_q u + \varepsilon_t y^2 + m(u,y). \end{aligned}$$

We use Nakayama's Lemma to prove that $RT(\tilde{F}) = RT(\hat{F})$. Then $\tilde{F} = ((\hat{r}+n)x, \hat{g}+m)$ is strongly $\mathbf{Z}_2$-equivalent to the normal form $\hat{F}$, so F is strongly $\mathbf{Z}_2$-equivalent to $\hat{F}$, as desired.

Observe that

$$RT(\tilde{F}) = \left\langle \begin{bmatrix} \varepsilon_p ux + nx \\ 0 \end{bmatrix}, \begin{bmatrix} 3\varepsilon_p ux + 2n_u ux + nx \\ 2\varepsilon_q u + 2m_u u \end{bmatrix} \right\rangle_{\{u\}} \oplus$$
$$\left\langle \begin{bmatrix} 0 \\ \varepsilon_p u^2 + nu \end{bmatrix}, \begin{bmatrix} 0 \\ \varepsilon_q u + \varepsilon_t y^2 + m \end{bmatrix}, \begin{bmatrix} 0 \\ 2\varepsilon_t uy + um_y \end{bmatrix}, \begin{bmatrix} 0 \\ 2\varepsilon_t y^2 + ym_y \end{bmatrix} \right\rangle_{\{u,y\}}. \tag{9.13}$$

By (9.12), each generator of $RT(\tilde{F})$ in (9.13) lies in $RT(\hat{F})$. Hence $RT(\tilde{F}) \subseteq RT(\hat{F})$. Next, we apply Nakayama's Lemma to prove that $RT(\hat{F}) \subseteq RT(\tilde{F})$, for which we need to show that $RT(\hat{F}) \subseteq RT(\tilde{F}) + (\mathcal{M}_u, \mathcal{M}_{uy})RT(\hat{F})$. The generators of $RT(\hat{F})$ over the system of rings $(\mathcal{E}_u, \mathcal{E}_{u,y})$ are

$$\left\{ \begin{bmatrix} ux \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u \end{bmatrix}, \begin{bmatrix} 0 \\ y^2 \end{bmatrix} \right\}.$$

Therefore

$$(\mathcal{M}_u, \mathcal{M}_{uy})RT(\hat{F}) = \left\langle \begin{bmatrix} u^2x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u^2 \end{bmatrix}, \begin{bmatrix} 0 \\ uy \end{bmatrix}, \begin{bmatrix} 0 \\ y^3 \end{bmatrix} \right\rangle = \begin{bmatrix} \mathcal{M}_u^2\langle x\rangle \\ \mathcal{M}_{uy}^3 + \mathcal{M}_{uy}\langle u\rangle \end{bmatrix}.$$

So we need to show that

$$\left\langle \begin{bmatrix} ux \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u \end{bmatrix}, \begin{bmatrix} 0 \\ y^2 \end{bmatrix} \right\rangle \subseteq RT(\tilde{F}) + \left\langle \begin{bmatrix} u^2x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ u^2 \end{bmatrix}, \begin{bmatrix} 0 \\ uy \end{bmatrix}, \begin{bmatrix} 0 \\ y^3 \end{bmatrix} \right\rangle,$$

which follows from

$$\begin{aligned} \begin{bmatrix} ux \\ 0 \end{bmatrix} &= \begin{bmatrix} ux + nx \\ 0 \end{bmatrix} - \begin{bmatrix} nx \\ 0 \end{bmatrix} \in RT(\tilde{F}) + \begin{bmatrix} \mathcal{M}_u^2\langle x\rangle \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ y^2 \end{bmatrix} &= \begin{bmatrix} 0 \\ y^2 + ym_y \end{bmatrix} - \begin{bmatrix} 0 \\ ym_y \end{bmatrix} \in RT(\tilde{F}) + \begin{bmatrix} 0 \\ \mathcal{M}_{uy}^3 \oplus \mathcal{M}_{uy}\{u\} \end{bmatrix} \\ \begin{bmatrix} 0 \\ u \end{bmatrix} &= \begin{bmatrix} 0 \\ u + y^2 + m \end{bmatrix} - \begin{bmatrix} 0 \\ y^2 \end{bmatrix} - \begin{bmatrix} 0 \\ m \end{bmatrix} \in RT(\tilde{F}) + \begin{bmatrix} 0 \\ \mathcal{M}_{uy}^3 \oplus \mathcal{M}_{uy}\langle u\rangle \end{bmatrix}. \end{aligned}$$

Therefore the restricted tangent space of $\tilde{F}$ is equal to the restricted tangent space of the original system F, so the two are equivalent.

Remark 9.3. A complement in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,y})$ of the restricted tangent space of the normal form (5.13) is

$$\mathbb{R}\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\}$$

so the codimension of $RT(\hat{F})$ is 3.

9.4 Tangent Space for H/SS Mode Interaction

We follow the standard procedure used in previous cases, starting with

Lemma 9.4. *The tangent space $T(F)$ of a map $F = (r(u)x, g(u,y))$ in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,y})$ is*

$$T(F) = RT(F) \oplus \mathbb{R}\left\{\begin{bmatrix} 0 \\ g_y(u,y) \end{bmatrix}\right\}. \tag{9.14}$$

Proof. Computing the tangent space is similar to computing the restricted tangent space in Lemma 9.1, except that now we do not require the origin to be fixed by the coordinate transformation. That is, $\dot{\psi}(u,y,0)$ in (9.7) can be an arbitrary function. The tangent space is therefore

$$\left\langle \begin{bmatrix} r(u)x \\ 0 \end{bmatrix}, \begin{bmatrix} 2r_u(u)ux + r(u)x \\ 2g_u(u,y)u \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ r(u)u \end{bmatrix}, \begin{bmatrix} 0 \\ g(u,y) \end{bmatrix}, \begin{bmatrix} 0 \\ g_y(u,y) \end{bmatrix} \right\rangle_{\{u,y\}}. \tag{9.15}$$

Equation (9.14) follows from

$$\left\langle \begin{bmatrix} 0 \\ g_y \end{bmatrix} \right\rangle_{\{u,y\}} = \left\langle \begin{bmatrix} 0 \\ ug_y \end{bmatrix}, \begin{bmatrix} 0 \\ yg_y \end{bmatrix} \right\rangle_{\{u,y\}} \oplus \mathbb{R}\left\{\begin{bmatrix} 0 \\ g_y \end{bmatrix}\right\}.$$

□

We now find a universal unfolding of the normal form (5.13) according to the analog of Theorem 7.3.

Proof of the Universal Unfolding Theorem 5.12: Compute

$$\begin{bmatrix} 0 \\ g_y(x,y) \end{bmatrix} = \begin{bmatrix} 0 \\ 2\varepsilon_t y \end{bmatrix}. \tag{9.16}$$

The tangent space is therefore

$$T(F) = \begin{bmatrix} \mathcal{M}_u \langle x \rangle \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathcal{M}_{uy} \end{bmatrix},$$

and the complement to $T(F)$ is two dimensional:

$$\mathbb{R}\left\{\begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ 1 \end{bmatrix}\right\}.$$

This leads to the universal unfolding (5.14).

10 Steady-State/Hopf Mode Interaction: Proofs

This section outlines proofs for the main results presented in Section 5.3 for steady-state/Hopf mode interaction. The basic strategy is the same as in Section 9, though some differences appear in the details of the proofs. In order to apply the methods of singularity theory, we use Liapunov-Schmidt reduction on the three-dimensional center manifold to construct a two-dimensional network whose zeros are in one-to-one correspondence with the equilibria of the center manifold network. We compute the restricted tangent space for this reduced network for the vector field (5.15), and use this to prove Theorem 5.17 that (5.18) is a normal form. We then use the complement of the tangent space of (5.18) to identify the universal unfolding (5.19), proving Theorem 5.18.

10.1 Liapunov-Schmidt Reduction for SS/H Mode Interaction

We outline a proof of Theorem 5.14. Begin with the system (5.15) that describes the center manifold dynamics for the steady-state Hopf mode interaction. Assume that the origin is an equilibrium, so that $f(0) = g(0, 0) = 0$; that f is associated with steady-state bifurcation, so that $f_x(0) = 0$; and that a Hopf bifurcation is associated with g, so that $D_Y g$ has eigenvalues $\pm i$ at the origin. We seek periodic solutions and rescale time by $t = (1+\tau)s$ to set the period to 2π. We can now define a map

$$\Phi : \mathcal{C}^1_{2\pi}(\mathbb{R}) \times \mathcal{C}^1_{2\pi}(\mathbb{R}^2) \times \mathbb{R} \to \mathcal{C}_{2\pi}(\mathbb{R}) \times \mathcal{C}_{2\pi}(\mathbb{R}^2) \times \mathbb{R}$$

given by

$$\begin{aligned} \Phi_1(u, v, \tau) &= \tfrac{du}{ds} - (1 + \tau)f(u) \\ \Phi_2(u, v, \tau) &= \tfrac{dv}{ds} - (1 + \tau)g(u, v) \end{aligned}$$

where $u \in \mathcal{C}^1_{2\pi}(\mathbb{R})$ and $v \in \mathcal{C}^1_{2\pi}(\mathbb{R}^2)$ are once-differentiable 2π-periodic functions on $\mathbb{R}$ and $\mathbb{R}^2$ respectively, and $\tau \in \mathbb{R}$.

The zeros of Φ correspond to periodic solutions for the center manifold vector field. The linearization of Φ about $(u, v, \tau) = (0, 0, 0)$ is

$$d\Phi = \begin{bmatrix} \frac{d}{ds} & 0 \\ -g_u & \frac{d}{ds} - D_v g \end{bmatrix}$$

and an element of the kernel, $\eta(s) \in \ker(d\Phi)$, is

$$\eta(s) = x \begin{bmatrix} 1 \\ (D_v g)^{-1} g_u \end{bmatrix} + \mathrm{Re}\left[z \begin{bmatrix} 0 \\ c \end{bmatrix} e^{is} \right],$$

for coordinates $x \in \mathbb{R}$ and $z \in \mathbb{C}$, where $c \in \mathbb{C}^2$ is the complex eigenvector defined by $(D_v g)c = ic$. Moreover, the $\mathbb{S}^1$-action is

$$\gamma(\theta) = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\theta} \end{bmatrix}$$

The reduction now proceeds as in standard Hopf bifurcation, Golubitsky and Schaeffer (1985). Periodic solutions of the vector field are locally in one-to-one correspondence with zeros of the function $F(x,y) = (f(x), r(x,y^2)y)$ on $\mathbb{R}^2$.

10.2 Restricted Tangent Space for SS/H Mode Interaction

The proof of Theorem 5.17 requires computing the restricted tangent space $RT(F)$ of the map (5.18). Let Γ_τ be a one-parameter family of strong $\mathbf{Z}_2$-equivalences as in (5.15), with $\Gamma_0(x,y) = (x,y)$. A typical element of $RT(F)$ is

$$\frac{d}{d\tau}\Gamma_\tau(F)\bigg|_{\tau=0}$$

The analog of the tangent space constant theorem, Theorem 7.1, lets us prove equivalence of maps. Here we use restricted tangent spaces in the $\mathbf{Z}_2$-symmetric context.

Lemma 10.1. *Let $F = (f(x), r(x,v)y)$ be a map in $(\mathcal{E}_x, \mathcal{E}_{x,v}\cdot\{y\})$. A map $G \in (\mathcal{E}_x, \mathcal{E}_{x,v}\cdot\{y\})$ is in $RT(F)$ if and only if there exist maps $P_i(x) \in \mathcal{E}_x$ and $Q_j(x,v) \in \mathcal{E}_{x,v}$ such that*

$$G(x,y) = \quad P_1\begin{bmatrix} f \\ 0 \end{bmatrix} + P_2\begin{bmatrix} xf_x \\ xyr_x \end{bmatrix} + Q_1\begin{bmatrix} 0 \\ yf \end{bmatrix} + Q_2\begin{bmatrix} 0 \\ yr \end{bmatrix} + Q_3\begin{bmatrix} 0 \\ yvr_v \end{bmatrix}.$$

Proof. Define a parametrized family of near-identity transformations (5.17), generating an orbit of strongly equivalent systems near the original vector field $F(x,y)$ for all small τ, by

$$\Gamma_\tau(F)(x,y) = \begin{bmatrix} a(x,\tau) & 0 \\ b(x,v,\tau)y & c(x,v,\tau) \end{bmatrix}\begin{bmatrix} f(\phi(x,\tau)) \\ r\left(\phi(x,\tau), \psi(x,v,\tau)^2 v\right)\psi(x,v,\tau)y \end{bmatrix}, \tag{10.1}$$

where Γ_0 is the identity and $\Gamma_\tau(0,0) = (0,0)$. Then

$$a(x,0) = 1 \quad b(x,v,0) = 0 \quad c(x,v,0) = 1 \quad \phi(x,0) = x \quad \psi(x,v,0) = 1. \tag{10.2}$$

Compute the restricted tangent space by differentiating (10.1):

$$\begin{aligned}\dot\Gamma_0(x,y) = \quad & \dot a(x,0)\begin{bmatrix} f(x) \\ 0 \end{bmatrix} + \dot b(x,v,0)\begin{bmatrix} 0 \\ f(x)y \end{bmatrix} + \dot c(x,v,0)\begin{bmatrix} 0 \\ r(x,v)y \end{bmatrix} \\ & + \dot\chi(x,0)\begin{bmatrix} f_x(x)x \\ r_x(x,v)xy \end{bmatrix} + \dot\psi(x,v,0)\begin{bmatrix} 0 \\ \left(2r_v(x,v)v + r(x,v)\right)y \end{bmatrix},\end{aligned}$$

where $\phi(x,0) = \chi(x,0)x$ and $\chi(x,0) = 1$. Equations (10.2) imply that

$$\dot a(x,0) \quad \dot b(x,v,0) \quad \dot c(x,v,0) \quad \dot\chi(x,0) \quad \dot\psi(x,v,0)$$

are arbitrary functions. The restricted tangent space is therefore spanned by

$$\left\langle \begin{bmatrix} f \\ 0 \end{bmatrix}, \begin{bmatrix} xf_x \\ yxr_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ yf \end{bmatrix}, \begin{bmatrix} 0 \\ yr \end{bmatrix}, \begin{bmatrix} 0 \\ yvr_v \end{bmatrix} \right\rangle_{\{x,v\}}.$$

□

10.3 Normal Form for SS/H Mode Interaction

We prove that (5.18) is a normal form by showing that a given admissible $\mathbf{Z}_2$-equivariant F of the form (5.16) satisfying the defining and nondegeneracy conditions of Theorem 5.17 can be transformed to (5.18) via a transformation of the form (5.17). Consider a map $\bar{F}(x,y) = (\bar{f}(x), \bar{r}(x,v)y)$, where

$$\begin{array}{rcl} \bar{f}(x) & = & px^2 \\ \bar{r}(x,v)y & = & (qv+sx)y, \end{array} \tag{10.3}$$

and $v = y^2$. Aapply the tangent space constant theorem to show that all other nonlinear terms can be removed by a suitable transformation. Specifically, we compute $RT(\bar{F})$ in Lemma 10.2 and then show that $RT(F) = RT(\bar{F})$ for the given F. Finally, by an appropriate (orientation preserving) rescaling of $\bar{F}$, we obtain the normal form (5.18).

Lemma 10.2. *The restricted tangent space of* $(\bar{f}, \bar{r}y)$ *given by* (10.3) *is*

$$RT(\bar{F}) = \begin{bmatrix} \mathcal{M}_x^2 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathcal{M}_{x,v}\langle y\rangle \end{bmatrix},$$

which is a system of modules over the system of rings $(\mathcal{E}_x, \mathcal{E}_{x,v})$.

Proof. By Lemma 10.1 the restricted tangent space for (10.3) is

$$RT(\bar{F}) = \left\langle \begin{bmatrix} px^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 2px^2 \\ sxy \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ px^2y \end{bmatrix}, \begin{bmatrix} 0 \\ (qv+sx)y \end{bmatrix}, \begin{bmatrix} 0 \\ (3qv+sx)\,y \end{bmatrix} \right\rangle_{\{x,v\}}.$$

Taking linear combinations, this reduces to

$$RT(\bar{F}) = \left\langle \begin{bmatrix} x^2 \\ 0 \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ vy \end{bmatrix}, \begin{bmatrix} 0 \\ xy \end{bmatrix} \right\rangle_{\{x,v\}}.$$

The restricted tangent space is therefore (10.2). □

Proof of Theorem 5.17: The restricted tangent space of (10.3) is given by Lemma 10.2. We show that a general $\mathbf{Z}_2$-equivariant map $\tilde{F} = \bar{F} + \cdots$, where $\bar{F}$ is given by (10.3), and $\cdots$ indicates higher-order admissible perturbations. We use Nakayama's Lemma to show that $RT(\tilde{F}) = RT(\bar{F})$, so the tangent space constant theorem guarantees that the two maps $\tilde{F}$ and $\bar{F}$ are strongly $\mathbf{Z}_2$-equivalent as in Definition 5.15.

Let $\tilde{F}(x,y) = (\tilde{f}(x), \tilde{r}(x,v)y)$, where

$$\begin{array}{rcl} \tilde{f}(x) & = & px^2 + n(x) \\ \tilde{r}(x,v)y & = & \big(qv+sx+m(x,v)\big)y, \end{array}$$

and $n \in \mathcal{M}_x^3$, $m \in \mathcal{M}_{xv}^2$. Lemma 10.1 shows that

$$RT(\tilde{F}) = \left\langle \begin{bmatrix} px^2 + n \\ 0 \end{bmatrix}, \begin{bmatrix} 2px^2 + xn_x \\ (s + m_x)xy \end{bmatrix} \right\rangle_{\{x\}} \oplus \\ \left\langle \begin{bmatrix} 0 \\ (px^2 + n)y \end{bmatrix}, \begin{bmatrix} 0 \\ (qv + sx + m)y \end{bmatrix}, \begin{bmatrix} 0 \\ (3qv + sx + 2vm_v + m)\,y \end{bmatrix} \right\rangle_{\{x,v\}}. \tag{10.4}$$

By Lemma 10.2, each generator of $RT(\tilde{F})$ in (10.4) lies in $RT(\bar{F})$, so $RT(\tilde{F}) \subseteq RT(\bar{F})$. Next we apply Nakayama's Lemma to prove $RT(\bar{F}) \subseteq RT(\tilde{F})$, for which we need to show $RT(\bar{F}) \subseteq RT(\tilde{F}) + (\mathcal{M}_x, \mathcal{M}_{xv})RT(\bar{F})$. A set of generators of $RT(\bar{F})$ over the system of rings $(\mathcal{E}_x, \mathcal{E}_{xv})$ is

$$\left\{ \begin{bmatrix} x^2 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ vy \end{bmatrix}, \begin{bmatrix} 0 \\ xy \end{bmatrix} \right\}.$$

Therefore

$$(\mathcal{M}_x, \mathcal{M}_{xv})RT(\bar{F}) = \left\langle \begin{bmatrix} x^3 \\ 0 \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ xvy \end{bmatrix}, \begin{bmatrix} 0 \\ v^2y \end{bmatrix}, \begin{bmatrix} 0 \\ x^2y \end{bmatrix} \right\rangle_{\{x,v\}} = \begin{bmatrix} \mathcal{M}_x^3 \\ \mathcal{M}_{xv}^2 \langle y \rangle \end{bmatrix},$$

and we need to show that

$$\left\langle \begin{bmatrix} x^2 \\ 0 \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ vy \end{bmatrix}, \begin{bmatrix} 0 \\ xy \end{bmatrix} \right\rangle_{\{x,v\}} \subseteq RT(\tilde{F}) + \begin{bmatrix} \mathcal{M}_x^3 \\ \mathcal{M}_{xv}^2 \langle y \rangle \end{bmatrix}.$$

This follows from

$$\begin{aligned}
\begin{bmatrix} x^2 \\ 0 \end{bmatrix} &= \tfrac{1}{p} \begin{bmatrix} px^2 + n \\ 0 \end{bmatrix} - \tfrac{1}{p} \begin{bmatrix} n \\ 0 \end{bmatrix} \in RT(\tilde{F}) + \begin{bmatrix} \mathcal{M}_x^3 \\ 0 \end{bmatrix} \\
\begin{bmatrix} 0 \\ vy \end{bmatrix} &= \tfrac{1}{2q} \left(\begin{bmatrix} 0 \\ (3qv + sx + 2vm_v + m)y \end{bmatrix} - \begin{bmatrix} 0 \\ (qv + sx + m)y \end{bmatrix} \right) \\
&\quad - \tfrac{1}{q} \begin{bmatrix} 0 \\ vym_v \end{bmatrix} \in RT(\tilde{F}) + \begin{bmatrix} 0 \\ \mathcal{M}_{xv}^2 \langle y \rangle \end{bmatrix} \\
\begin{bmatrix} 0 \\ xy \end{bmatrix} &= \tfrac{1}{2s} \left(3 \begin{bmatrix} 0 \\ (qv + sx + m)y \end{bmatrix} - \begin{bmatrix} 0 \\ (3qv + sx + 2vm_v + m)y \end{bmatrix} \right) \\
&\quad - \tfrac{1}{s} \begin{bmatrix} 0 \\ ym - yvm_v \end{bmatrix} \in RT(\tilde{F}) + \begin{bmatrix} 0 \\ \mathcal{M}_{xv}^2 \langle y \rangle \end{bmatrix}.
\end{aligned}$$

Therefore the restricted tangent space of $\tilde{F}$ is equal to the restricted tangent space of $\bar{F}$, so $\tilde{F}$ and $\bar{F}$ are equivalent. Moreover the transformation $f \to f/|p|$, $r \to r/|q|$ and $x \to |q|x/|s|$ takes $\bar{F}$ to the normal form (5.18).

Remark 10.3. The restricted tangent space has finite codimension. The complement of the restricted tangent space of the normal form (5.18) in $(\mathcal{E}_x, \mathcal{E}_{x,v} \cdot \{y\})$ is

$$\mathbb{R}\left\{\begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix}\right\},$$

so the codimension of $RT(\hat{F})$ is 3.

10.4 Tangent Space for SS/H Mode Interaction

As usual we first specify the relevant tangent space:

Lemma 10.4. *The tangent space $T(F)$ of a map $F = (f(x), r(x,v)y)$ in $(\mathcal{E}_x, \mathcal{E}_{x,v} \cdot \{y\})$ is*

$$T(F) = RT(F) \oplus \mathbb{R}\left\{\begin{bmatrix} f_x \\ yr_x \end{bmatrix}\right\}. \tag{10.5}$$

Proof. Computing the tangent space is similar to computing the restricted tangent space in Lemma 9.1, except that now we do not require the origin to be fixed by the coordinate transformation. This means that we no longer enforce $\dot{\phi}(x,0) = \dot{\chi}(x,0)x$, and instead take $\dot{\phi}(x,0)$ to be an arbitrary function. Hence the tangent space is

$$\left\langle \begin{bmatrix} f \\ 0 \end{bmatrix}, \begin{bmatrix} f_x \\ yr_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \left\langle \begin{bmatrix} 0 \\ yf \end{bmatrix}, \begin{bmatrix} 0 \\ yr \end{bmatrix}, \begin{bmatrix} 0 \\ y\left(2vr_v + r\right) \end{bmatrix} \right\rangle_{\{x,v\}}.$$

Relaxing the restriction of fixing the origin modifies the second element of the first span compared to the calculation for the restricted tangent space. In fact we can write the span of this modified vector in terms of a span of vectors of $RT(F)$ as

$$\left\langle \begin{bmatrix} f_x \\ yr_x \end{bmatrix} \right\rangle_{\{x\}} = \left\langle \begin{bmatrix} xf_x \\ xyr_x \end{bmatrix} \right\rangle_{\{x\}} \oplus \mathbb{R}\left\{\begin{bmatrix} f_x \\ yr_x \end{bmatrix}\right\}.$$

The tangent space is therefore given by (10.5). □

By computing the complement of $T(\hat{F})$, we derive a universal unfolding of the normal form (5.18) using the analog of Theorem 7.3.

Proof of the Universal Unfolding Theorem 5.18: The restricted tangent space $RT(\hat{F}) = RT(\bar{F})$ is given by Lemma 10.2. We must therefore compute

$$\begin{bmatrix} \hat{f}_x(x) \\ \hat{r}_x(x,v)y \end{bmatrix} = \begin{bmatrix} 2\varepsilon_p x \\ \varepsilon_s y \end{bmatrix}.$$

The tangent space is

$$T(F) = \begin{bmatrix} \mathcal{M}_x^2 \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathcal{M}_{xv}\left\langle y \right\rangle \end{bmatrix} \oplus \mathbb{R}\left\{\begin{bmatrix} 2\varepsilon_p x \\ \varepsilon_s y \end{bmatrix}\right\}.$$

A two-dimensional complement of $T(F)$ can be spanned either by

$$\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} x \\ 0 \end{bmatrix} \right\rangle$$

or by

$$\left\langle \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\rangle .$$

A universal unfolding corresponding to the latter choice is (5.19).

11 Hopf/Hopf Mode Interaction: Proofs

This section outlines proofs for the main results presented in Section 5.4 for Hopf/Hopf mode interaction. In order to apply the methods of singularity theory, we assume the four-dimensional center manifold dynamics is in Birkhoff normal form. We can then reduce it to the dynamics of a two-dimensional network, whose vector field (5.22) commutes with the standard action of $\mathbf{Z}_2 \oplus \mathbf{Z}_2$ in the plane. We compute the restricted tangent space for this reduced network, and use the result to prove Theorem 5.24, which states that (5.24) is a normal form.

11.1 Amplitude Reduction for H/H Mode Interaction

Here we outline a proof of Theorem 5.22. Begin with the system (5.20) that describes the center manifold dynamics for the Hopf/Hopf mode interaction. Assume that the origin is an equilibrium so that $f(0) = g(0,0) = 0$, and that the linear part of (f,g) is nonresonant, so that $\mathrm{D}_X f$ and $\mathrm{D}_Y g$ have two distinct pairs of complex conjugate purely imaginary eigenvalues $\pm i\omega$ and $\pm i\nu$ at the origin, with ω and ν irrationally related. Assume also that (5.20) is in Birkhoff normal form, so that (f,g) commutes with the two-torus $\mathbb{T}^2$ whose action on $\mathbb{R}^4$ is (5.21). Equivalently, $\mathbb{T}^2$ acts on $\mathbb{C}^2$ by

$$(\psi_1, \psi_2)(z_1, z_2) = (e^{i\psi_1} z_1, e^{i\psi_2} z_2) \tag{11.1}$$

where $(\psi_1, \psi_2) \in \mathbb{T}^2$ and $(z_1, z_2) \in \mathbb{C}^2$. Now

$$(\psi_1, \psi_2)(f(z_1), g(z_1, z_2)) = (f(e^{i\psi_1} z_1), g(e^{i\psi_1} z_1, e^{i\psi_2} z_2)),$$

which implies

$$(f(z_1), g(z_1, z_2)) = (P_1(|z_1|^2) z_1, P_2(|z_1|^2, |z_2|^2) z_2) \ , \tag{11.2}$$

where $P_1(0) = \omega i$ and $P_2(0,0) = \nu i$.

Set $z_1 = x e^{i\theta_1}$ and $z_2 = y e^{i\theta_2}$. Using (11.2) we can reduce the Birkhoff normal form (f,g) to the amplitude equations (5.22), which we write as

$$F(x,y) = (p_1(x^2) x, p_2(x^2, y^2) y) \ .$$

Here p_j is the real part of P_j for $j = 1, 2$, so that $p_1(0) = p_2(0,0) = 0$.

11.2 Restricted Tangent Space for H/H Mode Interaction

The proof of Theorem 5.24 requires computing the restricted tangent space $RT(F)$ of the map F in (5.24). Let Γ_τ be a one-parameter family of strong $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivalences (as in Definition 5.23) with $\Gamma_0(x, y) = (x, y)$. A typical element of $RT(F)$ is

$$\frac{d}{d\tau}\Gamma_\tau(F)\bigg|_{\tau=0}.$$

The analog of the tangent space constant theorem, Theorem 7.1, lets us prove equivalence of the relevant maps. The restricted tangent spaces involved are computed in the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-symmetric context using Lemma 11.1.

Lemma 11.1. *Let $F = (r(u)x, s(u,v)y)$ be a map in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,v} \cdot \{y\})$. A map $G \in (\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,v} \cdot \{y\})$ is in $RT(F)$ if and only if there exist maps $P_i(u) \in \mathcal{E}_u$ and $Q_j(u, v) \in \mathcal{E}_{u,v}$ such that*

$$G(x, y) = \quad P_1 \begin{bmatrix} xr \\ 0 \end{bmatrix} + P_2 \begin{bmatrix} xr_u u \\ ys_u u \end{bmatrix} + Q_1 \begin{bmatrix} 0 \\ ys \end{bmatrix} + Q_2 \begin{bmatrix} 0 \\ yru \end{bmatrix} + Q_3 \begin{bmatrix} 0 \\ ys_v v \end{bmatrix}.$$

Proof. Define a parametrized family of near-identity transformations (5.23), generating an orbit of strongly equivalent systems near the original vector field $F(x, y)$ for all small τ, by

$$\Gamma_\tau(F)(x,y) = \begin{bmatrix} a(u,\tau) & 0 \\ b(u,v,\tau)xy & c(u,v,\tau) \end{bmatrix} \begin{bmatrix} r(\phi^2(u,\tau)u)\phi(u,\tau)x \\ s\left(\phi^2(u,\tau)u, \psi^2(u,v,\tau)v\right)\psi(u,v,\tau)y \end{bmatrix}, \tag{11.3}$$

where Γ_0 is the identity and $\Gamma_\tau(0, 0) = (0, 0)$. Then

$$a(u,0) = 1 \quad b(u,v,0) = 0 \quad c(u,v,0) = 1 \quad \phi(u,0) = 1 \quad \psi(u,v,0) = 1 \tag{11.4}$$

Compute the restricted tangent space by differentiating (11.3):

$$\begin{aligned} \dot{\Gamma}_0(x,y) = \quad & \dot{a}(u,0) \begin{bmatrix} r(u)x \\ 0 \end{bmatrix} + \dot{b}(u,v,0) \begin{bmatrix} 0 \\ r(u)uy \end{bmatrix} + \dot{c}(u,v,0) \begin{bmatrix} 0 \\ s(u,v)y \end{bmatrix} \\ & + \dot{\phi}(u,0) \begin{bmatrix} (2r_u(u)u + r(u))\,x \\ 2s_u(u,v)uy \end{bmatrix} + \dot{\psi}(u,v,0) \begin{bmatrix} 0 \\ (2s_v(x,v)v + s(u,v))\,y \end{bmatrix}. \end{aligned}$$

Equations (11.4) imply that

$$\dot{a}(u,0) \quad \dot{b}(u,v,0) \quad \dot{c}(u,v,0) \quad \dot{\phi}(u,0) \quad \dot{\psi}(u,v,0)$$

are arbitrary functions. The restricted tangent space is therefore spanned by

$$\left\langle \begin{bmatrix} xr \\ 0 \end{bmatrix}, \begin{bmatrix} xr_u u \\ ys_u u \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ ys \end{bmatrix}, \begin{bmatrix} 0 \\ yru \end{bmatrix}, \begin{bmatrix} 0 \\ ys_v v \end{bmatrix} \right\rangle_{\{u,v\}}.$$

□

11.3 Normal form for H/H Mode Interaction

We prove that (5.24) is a normal form by showing that a given admissible $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivariant F of the form (5.22), satisfying the defining and nondegeneracy conditions of Theorem 5.24, can be transformed to (5.24) via a transformation of the form (5.23).

The defining conditions for Hopf/Hopf mode interaction imply that to first order in u and v, the functions $r(u)$ and $s(u,v)$ take the forms pu and $qu+tv$. Therefore F takes the form $\bar{F}(x,y) = (\bar{r}(u)x, \bar{s}(u,v)y)$ with

$$\begin{array}{rcl} \bar{r}(u)x & = & pux \\ \bar{s}(u,v)y & = & (qu+tv)y, \end{array} \tag{11.5}$$

where $u = x^2$, $v = y^2$. We prove Theorem 5.24 in two steps. First, we apply the tangent space constant theorem to transform away all other higher order terms by showing that $RT(F) = RT(\bar{F})$. Second, we rescale $\bar{F}$ to obtain the normal form (5.24).

Lemma 11.2. *The restricted tangent space of* $(\bar{r}x, \bar{s}y)$ *given by* (11.5) *is*

$$RT(\bar{F}) = \begin{bmatrix} \mathcal{M}_u\langle x\rangle \\ 0 \end{bmatrix} \oplus \begin{bmatrix} 0 \\ \mathcal{M}_{u,v}\langle y\rangle \end{bmatrix},$$

which is a system of modules over the system of rings $(\mathcal{E}_u, \mathcal{E}_{u,v})$.

Proof. By Lemma 11.1 the restricted tangent space for (11.5) is

$$RT(\bar{F}) = \left\langle \begin{bmatrix} xpu \\ 0 \end{bmatrix}, \begin{bmatrix} xpu \\ yqu \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ y(qu+tv) \end{bmatrix}, \begin{bmatrix} 0 \\ ypu^2 \end{bmatrix}, \begin{bmatrix} 0 \\ ytv \end{bmatrix} \right\rangle_{\{u,v\}}.$$

Taking linear combinations, this reduces to

$$RT(\bar{F}) = \left\langle \begin{bmatrix} xu \\ 0 \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ yv \end{bmatrix}, \begin{bmatrix} 0 \\ yu \end{bmatrix} \right\rangle_{\{u,v\}}.$$

The restricted tangent space is therefore given by (11.2). □

Proof of Theorem 5.24: Now $F = \bar{F} + \cdots$ where $\bar{F}$ is given by (11.5) and $\cdots$ indicates admissible higher-order perturbations. We use Nakayama's Lemma to show that $RT(F) = RT(\bar{F})$, so the tangent space constant theorem guarantees that the maps F and $\bar{F}$ are strongly $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-equivalent, as in Definition 5.23. Then we rescale $\bar{F}$ to obtain the normal form (5.24).

We can write $F(x,y) = (r(u)x, s(u,v)y)$ as

$$\begin{array}{rcl} r(u) & = & \big(pu + n(u)\big)x \\ s(u,v)y & = & \big(qu + tv + m(u,v)\big)y, \end{array}$$

where $n \in \mathcal{M}_u^2$, $m \in \mathcal{M}_{u,v}^2$ are higher order maps. By Lemma 11.1,

$$\begin{array}{c} RT(F) = \left\langle \begin{bmatrix} x(pu+n) \\ 0 \end{bmatrix}, \begin{bmatrix} x(p+n_u)u \\ y(q+m_u)u \end{bmatrix} \right\rangle_{\{u\}} \oplus \\ \left\langle \begin{bmatrix} 0 \\ y(qu+tv+m) \end{bmatrix}, \begin{bmatrix} 0 \\ y(pu+n)u \end{bmatrix}, \begin{bmatrix} 0 \\ y\left(t+m_v\right)v \end{bmatrix} \right\rangle_{\{x,v\}}. \end{array} \tag{11.6}$$

By Lemma 11.2, each generator of $RT(F)$ in (11.6) is in $RT(\bar{F})$, so $RT(F) \subseteq RT(\bar{F})$. Next we apply Nakayama's Lemma to prove $RT(\bar{F}) \subseteq RT(F)$, for which we need to show $RT(\bar{F}) \subseteq RT(F) + (\mathcal{M}_u, \mathcal{M}_{u,v})RT(\bar{F})$. The set of generators of $RT(\bar{F})$ over the system of rings $(\mathcal{E}_u, \mathcal{E}_{u,v})$ is

$$\left\{ \begin{bmatrix} ux \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ vy \end{bmatrix}, \begin{bmatrix} 0 \\ uy \end{bmatrix} \right\}.$$

Therefore

$$(\mathcal{M}_u, \mathcal{M}_{u,v})RT(\bar{F}) = \left\langle \begin{bmatrix} u^2x \\ 0 \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ uvy \end{bmatrix}, \begin{bmatrix} 0 \\ v^2y \end{bmatrix}, \begin{bmatrix} 0 \\ u^2y \end{bmatrix} \right\rangle_{\{x,v\}} = \begin{bmatrix} \mathcal{M}_u^2 \langle x \rangle \\ \mathcal{M}_{u,v}^2 \langle y \rangle \end{bmatrix}.$$

Now we must show that

$$\left\langle \begin{bmatrix} ux \\ 0 \end{bmatrix} \right\rangle_{\{u\}} \oplus \left\langle \begin{bmatrix} 0 \\ vy \end{bmatrix}, \begin{bmatrix} 0 \\ uy \end{bmatrix} \right\rangle_{\{u,v\}} \subseteq RT(F) + \begin{bmatrix} \mathcal{M}_u^2 \langle x \rangle \\ \mathcal{M}_{u,v}^2 \langle y \rangle \end{bmatrix}.$$

This follows from

$$\begin{aligned} \begin{bmatrix} ux \\ 0 \end{bmatrix} &= \tfrac{1}{p} \begin{bmatrix} x(pu+n) \\ 0 \end{bmatrix} - \tfrac{1}{p} \begin{bmatrix} n \\ 0 \end{bmatrix} \in RT(F) + \begin{bmatrix} \mathcal{M}_u^2 \langle x \rangle \\ 0 \end{bmatrix} \\ \begin{bmatrix} 0 \\ vy \end{bmatrix} &= \tfrac{1}{t} \begin{bmatrix} 0 \\ y\left(t+m_v\right)v \end{bmatrix} - \tfrac{1}{t} \begin{bmatrix} 0 \\ yvm_v \end{bmatrix} \in RT(F) + \begin{bmatrix} 0 \\ \mathcal{M}_{xv}^2 \langle y \rangle \end{bmatrix} \\ \begin{bmatrix} 0 \\ uy \end{bmatrix} &= \tfrac{1}{q} \left(\begin{bmatrix} 0 \\ y(qu+tv+m) \end{bmatrix} - \begin{bmatrix} 0 \\ y\left(t+m_v\right)v \end{bmatrix} \right) - \tfrac{1}{q} \begin{bmatrix} 0 \\ ym - yvm_v \end{bmatrix} \\ &\in RT(F) + \begin{bmatrix} 0 \\ \mathcal{M}_{u,v}^2 \langle y \rangle \end{bmatrix}. \end{aligned}$$

Therefore the restricted tangent space of F is equal to the restricted tangent space of $\bar{F}$, so the two are equivalent. Moreover the transformation $r \to r/|p|$, $s \to s/|q|$ and $v \to |q|v/|t|$ takes $\bar{F}$ to the normal form (5.24). □

Remark 11.3. The restricted tangent space has finite codimension. A complement of the restricted tangent space of the normal form (5.24) in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,v} \cdot \{y\})$ is

$$\mathbb{R} \left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\}.$$

The codimension of $RT(\hat{F})$ is 2.

11.4 Tangent Space for H/H Mode Interaction

We find a universal unfolding in terms of the complement of the tangent space, which is forced to be identical to the restricted tangent space by the $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-symmetry.

Lemma 11.4. *The tangent space $T(F)$ of $F = (r(u)x, s(u,v)y)$ in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,v} \cdot \{y\})$ is equal to $RT(F)$.*

Proof. Computing $T(F)$ is similar to computing $RT(F)$ as in Lemma 11.1, except that now we do not require the origin to be fixed by the coordinate transformation $\Phi(x,y)$. However, $\mathbf{Z}_2 \oplus \mathbf{Z}_2$-symmetry forces $\Phi(0) = 0$, so $T(F) = RT(F)$. □

We can now compute a universal unfolding of the normal form (5.24) using the analog of Theorem 7.3.

Proof of the Universal Unfolding Theorem 5.25: By Lemma 11.4 and Remark 11.3, a complement to $T(F)$ in $(\mathcal{E}_u \cdot \{x\}, \mathcal{E}_{u,v} \cdot \{y\})$ is

$$\mathbb{R}\left\{ \begin{bmatrix} x \\ 0 \end{bmatrix}, \begin{bmatrix} 0 \\ y \end{bmatrix} \right\},$$

giving the universal unfolding (5.25).

Appendix A: Reduction Procedure for Networks with Higher-Dimensional Nodes

We describe a construction that converts any network $\mathcal{G}$ with higher-dimensional node phase spaces P_c into a network $\mathcal{G}^\dagger$ with 1-dimensional node phase spaces, without changing the space of admissible maps, when variables are suitably identified. We call $\mathcal{G}^\dagger$ the *expansion* of $\mathcal{G}$. Here we describe the construction only for fully inhomogeneous networks $\mathcal{G}$, but there is a straightforward generalisation to any network in the multiarrow formalism of Golubitsky *et al.* (2005).

Let $\mathcal{G}$ be fully inhomogeneous with nodes $\mathcal{C} = \{1, \ldots, n\}$. Assume that $\dim P_c = \delta(c) \geq 1$ for $c \in \mathcal{C}$. Each arrow $e \in \mathcal{E}$ can be identified with the pair $(\mathcal{H}(e), \mathcal{T}(e))$, and distinct arrows give distinct pairs. In the single-arrow formalism of Stewart *et al.* (2003), which applies to the fully inhomogeneous case, $c \notin I(c)$, so the pair (c,c) does not appear as an arrow. Moreover, the input set $I(c)$ can be identified with the set of tail nodes $\{\mathcal{T}(e) : e \in I(c)\}$ in this case.

Definition 11.5. Given $\mathcal{G}$, fully inhomogeneous, we define the *expansion* $\mathcal{G}^\dagger$ as follows.

Nodes and arrows are defined by

$$\begin{aligned} \mathcal{C}^\dagger &= \{[c,k] : 1 \leq k \leq \delta(c)\} \\ \mathcal{E}^\dagger &= \{([c,k],[d,l]) : (c,d) \in \mathcal{E}, 1 \leq k \leq \delta(c), 1 \leq l \leq \delta(d)\} \\ &\qquad \cup \{([c,k],[c,m]) : 1 \leq k, m \leq \delta(c), k \neq m\}, \end{aligned}$$

where for clarity, ordered pairs are denoted by square brackets $[c, k]$. Each node c is expanded to a clump of nodes $[c, k]$ where $1 \leq k \leq \delta(c)$. Each input arrow $e = (c, d)$ of c is expanded to a bundle of arrows between nodes in the corresponding clumps, one for each pair of heads and tails $[c, k]$ and $[d, l]$. Moreover, there are arrows between all distinct $[c, k], [c, m]$; that is, each clump is 'internally' all-to-all connected.

Heads and tails in $\mathcal{G}^\dagger$ are defined by the pairs of nodes. All node types and arrow types are distinct. $\diamond$

The simplest way to describe this construction is in terms of the *adjacency matrix* $A = A(\mathcal{G})$, defined by

$$A_{ij} = 1 \iff (i, j) \in \mathcal{E} \text{ or } i = j$$

Then $A^\dagger = A(\mathcal{G}^\dagger)$ is obtained from A be replacing every entry A_{ij} by a block matrix B^{ij} of size $\delta(i) \times \delta(j)$, whose entries are all the same as A_{ij}. That is, all 0s or all 1s.

Observe that $\mathcal{G}^\dagger$ is fully inhomogeneous since by definition all node types and arrow types are distinct.

Example 11.6. The 3-node network $\mathcal{G}$ of Figure 7 (left) gives the expansion $\mathcal{G}^\dagger$ of Figure 7 (right) when $\dim\, P_1 = 3, \dim\, P_2 = \dim\, P_3 = 2$.

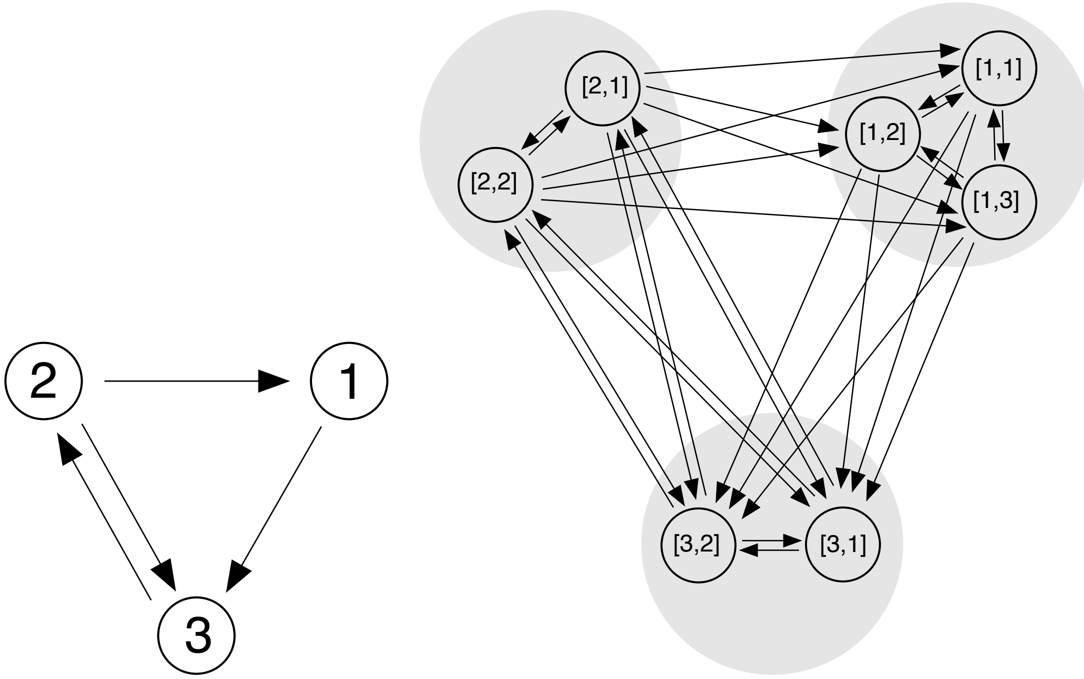

Figure 7: *Left*: A fully inhomogeneous network $\mathcal{G}$. *Right*: The expansion $\mathcal{G}^\dagger$ when $\dim\, P_1 = 3, \dim\, P_2 = \dim\, P_3 = 2$. (All arrows and nodes have distinct types.)

The adjacency matrix for $\mathcal{G}$ is

$$A(\mathcal{G}) = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$$

and that of $\mathcal{G}^\dagger$ is

$$A(\mathcal{G}^\dagger) = \left[\begin{array}{ccc|cc|cc} 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ 1 & 1 & 1 & 1 & 1 & 0 & 0 \\ \hline 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ \hline 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 1 & 1 & 1 & 1 & 1 & 1 \end{array}\right]$$

For both networks, the diagonal entries 1 come from the nodes, whereas the off-diagonal entries 1 in any diagonal block of $A(\mathcal{G}^\dagger)$ come from the all-to-all connections within that clump.

Clearly, for any i the input set $I([c,i])$ in $\mathcal{G}^\dagger$ is

$$I([c,i]) = (I(c))^\dagger = \{[d,j] : d \in I(c)\}$$

Next we define how to interpret an admissible map f for $\mathcal{G}$ as an admissible map $f^\dagger$ for $\mathcal{G}^\dagger$. For each node $[c,i]$ of $\mathcal{G}^\dagger$ let

$$P_{[c,i]} = \mathbb{R}$$

Then we identify

$$\mathbb{R}^{\delta(c)} = P_c = \bigoplus_i P_{[c,i]}$$

so that the $x_{[c,i]}$ are coordinates on P_c.

Now a map $f : P \to P$ with components $f_c : P_c \to P$ can be split into finer components

$$f_{[c,i]}(x) = (f(x))_{[c,i]}$$

where $x \in P$ can be identified with its component representations $(x_c)_{c \in \mathcal{C}} \in \oplus P_c$ and $(x_{[c,i]})_{[c,i] \in \mathcal{C}^\dagger} \in \oplus P_{[c,i]}$. We then have:

Theorem 11.7. *Let $\mathcal{G}$ be fully inhomogeneous. A map f is $\mathcal{G}$-admissible and only if, when represented in the natural manner using coordinates indexed by the $[c,k]$, it is $\mathcal{G}^\dagger$-admissible.*

Proof. In a fully inhomogeneous network, the only constraint on an admissible map is the domain condition. The adjacency matrices show that this holds for $f^\dagger$ if and only if it holds for f. That is, f_c depends only on the x_j for $j \in I(c)$ if and only if all $f_{[c,i]}$ depend only on the $x_{[j,k]}$ for $[j,k] \in I([c,i]) = (I(c))^\dagger$. All components are otherwise arbitrary. $\square$

Remark 11.8. (a) Expansion preserves path components in the sense that the union of the clumps of a path component in $\mathcal{G}$ is the corresponding path component in $\mathcal{G}^\dagger$.

(b) Expansion also preserves 'upstream/downstream', that is, the natural feedforward ordering between path components.

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