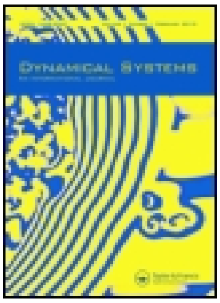

Dynamics and Stability of Systems
Publication details, including instructions for authors and subscription information:
http://www.tandfonline.com/loi/cdss19

Modulated rotating waves in O(2) mode interactions

John David Crawford [a] , Martin Golubitsky [b] & William F. Langford [c]

[a] Institute for Nonlinear Science, University of California—San Diego, La Jolla, California, 92093, USA

[b] Department of Mathematics, University of Houston, Houston, Texas, 77004, USA

[c] Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario, NIG 2W1, Canada

Version of record first published: 21 Mar 2007.

To cite this article: John David Crawford, Martin Golubitsky & William F. Langford (1988): Modulated rotating waves in O(2) mode interactions, Dynamics and Stability of Systems, 3:3-4, 159-175

To link to this article: http://dx.doi.org/10.1080/02681118808806054

Dynamics and Stability of Systems Vol. 3 Nos. 3 & 4 1988

Modulated rotating waves in O(2) mode interactions

John David Crawford
Institute for Nonlinear Science, University of California—San Diego, La Jolla, California 92093, *USA*

Martin Golubitsky
Department of Mathematics, University of Houston, Houston, Texas 77004, *USA*

and William F. Langford
Department of Mathematics and Statistics, University of Guelph, Guelph, Ontario, Canada N1G 2W1

Abstract

The interaction of steady-state and Hopf bifurcations in the presence of O(2) symmetry generically gives a secondary Hopf bifurcation to a family of 2-tori, from the primary rotating wave branch. We present explicit formulas for the coefficients which determine the direction of bifurcation and the stability of the 2-tori. These formulas show that the tori are determined by third-degree terms in the normal-form equations, evaluated at the origin. The flow on the torus near criticality has a small second frequency, and is close to linear flow, without resonances. Existence of an additional SO(2) symmetry, as in the Taylor–Couette problem, forces the flow to be exactly linear; however, the tori are unstable at bifurcation in the Taylor–Couette case. More generally, these tori may reveal themselves physically as slowly modulated rotating waves, for example in reaction–diffusion problems.

1. Introduction

Bifurcation in systems of differential equations with O(2) symmetry has been the subject of much recent study (cf. (Golubitsky, Stewart and Schaeffer, 1988)). In systems with one parameter both steady-state and Hopf bifurcations from an invariant state may be expected, while in systems with two parameters various kinds of mode interactions may be unavoidable. As we explain below, we study one particular secondary bifurcation to an invariant 2-torus which occurs in a Hopf–steady-state mode interaction.

Generically in systems of differential equations with symmetry group Γ the eigenspaces of the linearization of the differential equations at a Γ-invariant steady state will be irreducible under the action of Γ. Thus the existence of O(2) symmetry implies that the eigenvalues of this linearization will be either simple or double, since irreducible representations of O(2) are either one- or two-dimensional. Whether these eigenvalues are simple or double may be determined as follows. In either steady-state or Hopf bifurcation if a bifurcating branch of solutions consists of states which break the rotational SO(2) symmetry, then the critical eigenvalue must be double.

A number of authors have studied Hopf bifurcation with O(2) symmetry when the critical eigenvalue is double (cf. (Ruelle, 1973; van Gils 1984; Golubitsky and Stewart, 1985)). The basic observation resulting from these studies is that generically there exist two types of periodic solutions, a family of standing waves and a pair of rotating waves.

In this paper we consider steady-state–Hopf mode interaction with O(2) symmetry where the critical eigenvalues are all double. This interaction leads to a six-dimensional centre manifold and to an interesting variety of steady and periodic solution branches, arising as primary and secondary bifurcations. In addition, there are *secondary* and *tertiary* branches of tori; see (Golubitsky and Stewart, 1986). There is, however, precisely one secondary branch of 2-tori which yields a quasiperiodic state and it is this branch that we analyse here. We show that under certain non-degeneracy conditions the direction of branching, the stability and the frequencies for this family of 2-tori are determined by the third-order terms in the normal-form equations.

Our analysis is motivated by the Taylor–Couette system, where the 2-tori correspond to a modulated spiral flow. Unfortunately, using the formula derived here, it can be shown that these 2-tori are unstable (see (Golubitsky and Langford, 1988)) and section 5 below. Our results, however, may be applicable to a number of other models where O(2) symmetry is present. We mention reaction–diffusion equations on a disk and fluid flow through an articulated hosepipe.

The branch of 2-tori we study bifurcates from the primary branch of rotating waves. Since, near the bicritical point where mode interaction occurs, both the second frequency and the amplitude of the torus flow are small when compared to those of the rotating wave, we refer to these torus solutions as *modulated rotating waves*. We note that the coupling between the primary and secondary oscillations is weak; in fact, the flow on the torus is topologically conjugate to a linear flow with two independent frequencies. In section 4 we describe more precisely the nature of the flow on the torus, using results of Rand (1982) and Renardy (1982).

Earlier studies of steady-state–Hopf mode interactions without O(2) symmetry (cf. (Langford, 1979; Holmes, 1980; Guckenheimer, 1981; Langford and Iooss, 1980)) also revealed bifurcation of 2-tori. These cases differ from the present situation however, in that these 2-tori arise via bifurcations which are degenerate (or nearly so), are tertiary rather than secondary bifurcations, and occur only in restricted cases. The 2-torus described here exists generically in the context of O(2) mode interactions.

After a preliminary discussion of the hypotheses and normal forms in section 2, the analysis begins in section 3 with the truncated Poincaré–Birkhoff normal form. Symmetry properties are exploited to reduce the question of bifurcation of the torus from the rotating wave to a question of Hopf bifurcation for an autonomous complex ordinary differential equation. Because the formula in the standard Hopf theorem does not apply directly to this case, a modified Hopf bifurcation formula is given in the Appendix. It is presented in greater generality than is required here, to facilitate application to other systems of complex differential equations.

In section 4, truncation of the tail of the normal form is shown not to change the qualitative nature of the flow on the torus of modulated rotating waves, nor the existence of the torus itself. In applications with an additional SO(2) symmetry, such as the Taylor–Couette problem, the formal normal form may in fact be exact, and then the results of section 3 apply directly. See section 5.

The method of averaging provides an alternative to the approach presented here for the calculation of the bifurcation coefficients. In fact, one of the authors has independently derived the same bifurcation formulas using the method of averaging. These averaging calculations are not included in this paper.

2. Hypotheses and notation

Consider a 2-parameter family of differential equations

$$u' = f(u, \mu), \tag{2.1}$$

where f is smooth (C^∞) and $f: X \times \mathbb{R}^2 \to X$. Here we shall consider $X = \mathbb{R}^n$ with $n \geq 6$, but, with suitable hypotheses on f, X could be infinite-dimensional; for example (2.1) could be the Navier–Stokes equations (see (Golubitsky and Langford, 1987; Iooss, 1984; Renardy, 1982)).

Assume that a linear action of O(2) is defined on X, and that (2.1) is equivariant with respect to this action; that is, for all $(u, \mu) \in X \times \mathbb{R}^2$ and $\gamma \in \mathrm{O}(2)$,

$$f(\gamma u, \mu) = \gamma f(u, \mu). \tag{2.2}$$

Assume that (2.1) has an O(2)-invariant equilibrium solution which without loss of generality we may take to be $u = 0$. Thus

$$f(0, \mu) = 0. \tag{2.3}$$

Moreover, assume that at $\mu = 0 \in \mathbb{R}^2$, the Frechet derivative

$$A \equiv (d_u f)_{(0,0)} \tag{2.4}$$

has eigenvalues 0 and $\pm i\omega_0$ ($\omega_0 \neq 0$), and no others on the imaginary axis. It follows from differentiation of (2.2) that

$$A\gamma = \gamma A \quad \text{for all } \gamma \in \mathrm{O}(2), \tag{2.5}$$

so that eigenvalues of A are generically either simple or double and semisimple, see (Golubitsky *et al.*, 1988). Because we assume here that both the steady-state and Hopf bifurcations break symmetry, we require that the eigenvalues 0 and $\pm i\omega_0$ are double. It follows that these eigenvalues have continuous extensions as double eigenvalues, for $\mu \neq 0$. Furthermore, (2.3) is in fact generic in this setting.

By the centre manifold theorem, equation (2.1) has an invariant 6-dimensional centre manifold, tangent to the direct sum of the eigenspaces of the eigenvalues 0 and $\pm i\omega$. If the remaining eigenvalues of A have negative real parts, this centre manifold is attracting near 0. Thus we can restrict our attention to this centre manifold. Furthermore, the equivariance (2.2) is preserved in the reduction to the centre manifold, so we may assume that (2.1) is a 6-dimensional system, satisfying (2.2) to (2.5).

It is possible to further simplify f by transforming the vector field to Poincaré–Birkhoff normal form. This is a recursive procedure, which eliminates 'non-resonant' terms in f up to any finite order k. In this process, the tail becomes more complicated and in general blows up as $k \to \infty$. We shall assume in this section that this calculation has been performed to order k and that the tail has been truncated. We call the resulting polynomial vector field the *formal normal form* of f. (In section 4 we consider the effects of the tail on solutions of (2.1).) The formal normal form of (2.1) has an additional phase-shift symmetry, which we denote by S^1, associated to the purely imaginary eigenvalues. The full group of symmetries of the formal normal form is then

$$O(2) \times S^1. \tag{2.6}$$

After factoring out the kernel of the $O(2) \times S^1$ action on $\mathbb{R}^6 \cong \mathbb{C}^3$, we may choose complex coordinates $z = (z_0, z_1, z_2)$ such that

$$\left.\begin{array}{ll} \text{(a)} & \theta \cdot z = (e^{ki\theta} z_0, e^{li\theta} z_1, e^{-li\theta} z_2) \quad \text{for all } \theta \in SO(2), \\ \text{(b)} & \kappa \cdot z = (\bar{z}_0, z_2, z_1), \ \kappa \in O(2) \sim SO(2), \ \kappa^2 = \text{id}, \\ \text{(c)} & \phi \cdot z = (z_0, e^{i\phi} z_1, e^{i\phi} z_2) \quad \text{for all } \phi \in S^1, \end{array}\right\} \tag{2.7}$$

where k and l are relatively prime. Observe that the z_0-coordinate corresponds to steady-state bifurcation and the z_1, z_2-coordinates correspond to Hopf bifurcation. See (Golubitsky, Stewart and Schaeffer, 1988) for further details.

Recall that a two-dimensional irreducible representation of O(2) may be identified with an integer m where $\theta \in SO(2)$ acts on $z \in \mathbb{C}$ by $e^{mi\theta} z$. The kernel of this action of O(2) is $\mathbb{Z}_m$. When mode interaction does not occur, we may divide out by $\mathbb{Z}_m$ and obtain an effective action of O(2) where $m = 1$. When mode interaction does occur this reduction is, in general, not possible and formula (2.7a) results. As we indicated in section 1, our interest in this secondary torus bifurcation is motivated by analyses of the Taylor–Couette system. In this application

$$k = l = 1, \tag{2.8}$$

which we henceforth assume. Analyses of this bifurcating 2-torus should be possible for general k and l, though we do not pursue that issue here.

The general formal normal form on $\mathbb{C}^3$, with $O(2) \times S^1$ equivariance, has been calculated in (Golubitsky and Stewart, 1986); see also (Golubitsky *et al.*, 1988; Langford, 1986). It can be written as

$$\begin{aligned} g(z, \bar{z}, \mu) = {} & (c^1 + i\,\delta c^2)\begin{bmatrix} z_0 \\ 0 \\ 0 \end{bmatrix} + (c^3 + i\,\delta c^4)\begin{bmatrix} z_0 z_1 \bar{z}_2 \\ 0 \\ 0 \end{bmatrix} \\ & + (p^1 + iq^1)\begin{bmatrix} 0 \\ z_1 \\ z_2 \end{bmatrix} + (p^2 + iq^2)\,\delta\begin{bmatrix} 0 \\ z_1 \\ -z_2 \end{bmatrix} \\ & + (p^3 + iq^3)\begin{bmatrix} 0 \\ z_0^2 z_2 \\ \bar{z}_0^2 z_1 \end{bmatrix} + (p^4 + iq^4)\,\delta\begin{bmatrix} 0 \\ z_0^2 z_2 \\ -\bar{z}_0^2 z_1 \end{bmatrix}, \end{aligned} \tag{2.9}$$

where $\delta = z_2\bar{z}_2 - z_1\bar{z}_1$, and c^j, p^j and q^j, $j = 1, \ldots, 4$ are real-valued functions of μ and the five invariant polynomials

$$\left.\begin{array}{lll} \rho = z_0\bar{z}_0, & N = z_1\bar{z}_1 + z_2\bar{z}_2, & \\ \Delta = \delta^2, & \Phi = \mathrm{Re}\,(z_0^2\bar{z}_1 z_2), & \Psi = \mathrm{Im}\,(z_0^2\bar{z}_1 z_2)\,\delta. \end{array}\right\} \tag{2.10}$$

When it is necessary to list these arguments we shall use the notation

$$c^1 = c^1(\rho, N, \Delta, \Phi, \Psi, \mu).$$

The assumptions on the eigenvalues of A imply that

$$c^1(0) = 0, \qquad p^1(0) = 0, \qquad q^1(0) = \omega_0. \tag{2.11}$$

Solutions of the formal normal form equations, corresponding to rotating waves and modulated rotating waves, are analysed in section 3. The effects of the higher-order tail, which breaks the S^1 symmetry, on these solutions are considered in section 4.

3. Bifurcation analysis for the normal form

Here we consider bifurcations of the complex differential equation

$$dz/dt = g(z, \bar{z}, \mu) \tag{3.1}$$

in formal normal form; that is to say, with g satisfying (2.9) to (2.11). The steady-state and periodic solution branches of (3.1) are described in (Golubitsky and Langford, 1987; Golubitsky and Stewart, 1986; Golubitsky *et al.*, 1988) and need not be presented in detail here. These solution branches lie in invariant fixed-point subspaces of $\mathbb{C}^3$, corresponding to isotropy subgroups of $O(2) \times S^1$, as summarized in Table 1. Equation (3.1), when restricted to each of the fixed-point subspaces, decouples into amplitude and phase equations, which may be solved for steady-state ($a = 0$) and periodic ($a > 0$) solutions. There are in fact two rotating-wave solutions; the second obtained by applying κ to the one in Table 1.

Table 1. Steady-state and periodic solutions

Name	Isotropy subgroup	Fixed-point subspace	Subspace equations
Trivial	$O(2) \times S^1$	$\{(0, 0, 0)\}$	0
Steady state	$\{\kappa, 1\} \times S^1$	$\{(x, 0, 0)\}, x \in \mathbb{R}$	$x' = xc^1$
Rotating wave	$\{(\phi, -\phi)\}$ $\cong \widetilde{SO}(2)$	$\{(0, z, 0)\}$ $z = ae^{i\Omega} \in \mathbb{C}$	$a' = a[p^1 - a^2p^2]$ $\Omega' = q^1 - a^2q^2$
Standing wave	$\{(\kappa, 0), (\pi, \pi)\}$	$\{(0, z, z)\}$ $z = ae^{i\Omega} \in \mathbb{C}$	$a' = ap^1$ $\Omega' = q^1$
Secondary$_1$	$\mathbb{Z}_2(\kappa, \pi)$	$\{(x, z, -z)\}$ $x \in \mathbb{R}$ $z = ae^{i\Omega} \in \mathbb{C}$	$x' = x[c^1 - a^2c^3]$ $a' = a[p^1 - x^2p^3]$ $\Omega' = q^1 - x^2q^3$
Secondary$_2$	$\mathbb{Z}_2(\kappa)$	$\{(x, z, z)\}, x \in \mathbb{R}$ $z = ae^{i\Omega} \in \mathbb{C}$	$x' = x[c^1 + a^2c^3]$ $a' = a[p^1 + x^2p^3]$ $\Omega' = q^1 + x^2q^3$

The nomenclature for the rotating-wave solutions is natural, since this solution is invariant whenever a spatial rotation through ϕ is combined with a temporal phase shift through $-\phi$ (or ϕ for the conjugate rotating wave).

Of the solutions listed in Table 1, only the rotating-wave branch is considered in this paper. The amplitude of the rotating wave is a root of the equation

$$p^1(0, a^2, a^4, 0, 0, \mu) - a^2p^2(0, a^2, a^4, 0, 0, \mu) = 0, \tag{3.2}$$

and for each such root the rotating wave is given by

$$\mathbf{z}(t) = (0, z_1^r(t), 0), \qquad z_1^r(t) = ae^{i\Omega^r(t)} \tag{3.3}$$

and the phase Ω^r is given by

$$\Omega^r(t) = t[q^1(0, a^2, a^4, 0, 0, \mu) - a^2q^2(0, a^2, a^4, 0, 0, \mu)] + \Omega_0. \tag{3.4}$$

Locally, equation (3.2) may be solved by the implicit function theorem. The Taylor expansion of (3.2) gives (recalling that $p^1(0) = 0$)

$$p_\mu^1 \cdot \mu + (p_N^1 - p^2)a^2 + \dots = 0, \tag{3.5}$$

so that, assuming that $p_N^1 - p^2 \neq 0$, we have

$$a^2 = -(p_\mu^1 \cdot \mu)/(p_N^1 - p^2) + \dots . \tag{3.6}$$

In (3.5), (3.6) all coefficients and derivatives are evaluated at the origin, subscripts denote partial derivatives, and $p_\mu^1 \cdot \mu$ is a directional derivative in parameter space. Bifurcation of rotating waves from the trivial solution occurs on the curve in the μ-plane defined by $a^2 = 0$ in (3.6), see Fig. 1.

The search for solutions bifurcating from the rotating-wave branch is faciliated by a translation of the origin of coordinates to the rotating wave, together with a transformation to a rotating frame which moves with the rotating wave, and a rescaling to the amplitude of the rotating wave. Therefore, we introduce new

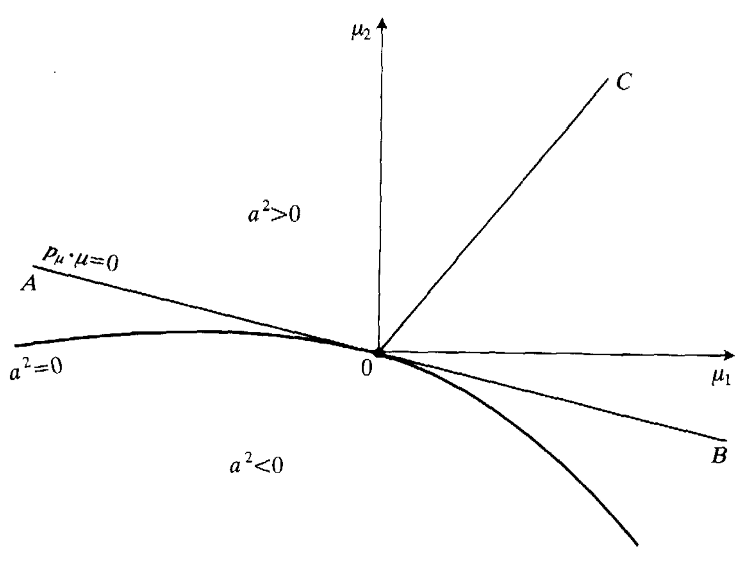

Fig. 1

coordinates (v_0, r_1, v_2) in $\mathbb{C} \times \mathbb{R} \times \mathbb{C}$, defined by

$$z = \begin{bmatrix} z_0 \\ z_1 \\ z_2 \end{bmatrix} = \begin{bmatrix} av_0 \\ a(1+r_1)e^{i\Omega(t)} \\ av_2 e^{i\Omega(t)} \end{bmatrix}. \tag{3.7}$$

Note that with $(v_0, r_1, v_2) = (0, 0, 0)$ this reduces to (3.3). We have chosen to rotate the z_2-coordinate as well as the z_1-coordinate in (3.7), in order to exploit the S^1 symmetry of the system. It is understood in (3.7) that the amplitude a is defined by (3.5), (3.6); however, the phase $\Omega(t)$ is no longer given exactly by (3.4).

The transformed equations, obtained by substitution of (3.7) into (3.1), are

$$\left.\begin{aligned} v_0' &= (c^1 - i\,\delta c^2)v_0 + a^2(c^3 + i\,\delta c^4)\bar{v}_0(1+r_1)\bar{v}_2, \\ r_1' &= (p^1 + \delta p^2)(1+r_1) + a^2(p^3 + \delta p^4)\operatorname{Re}(v_0^2 v_2) - a^2(q^3 + \delta q^4)\operatorname{Im}(v_0^2, v_2), \\ \Omega' &= (q^1 + \delta q^2) + \frac{a^2}{1+r_1}[(p^3 + \delta p^4)\operatorname{Im}(v_0^2 v_2) + (q^3 + \delta q^4)\operatorname{Re}(v_0^2 v_2)], \\ v_2' &= (p^1 - \delta p^2 - 2i\,\delta q^2)v_2 + a^2(p^3 - \delta p^4)\bar{v}_0^2(1+r_1) + ia^2(q^3 - \delta q^4)\bar{v}_0^2(1+r_1) \\ &\quad - i\frac{a^2}{1+r_1}[(p^3 + \delta p^4)\operatorname{Im}(v_0^2 v_2) + (q^3 + \delta q^4)\operatorname{Re}(v_0^2 v_2)]v_2. \end{aligned}\right\} \tag{3.8}$$

By grace of the S^1-invariance, the phase variable Ω has disappeared completely from the right-hand side of (3.8). This means that the (v_0, r_1, v_2)-equations decouple and can be solved independently of Ω. If such a solution $(v_0(t), r_1(t), v_2(t))$ is found, then the Ω-equation in (3.8) can be solved by direct integration:

$$\Omega(t) = \int_{t_0}^{t} \{q^1 + \delta q^2 + \dots\}\,dt + \Omega_0. \tag{3.9}$$

Furthermore, since $q^1(0) = \omega_0 \neq 0$, if $(v_0(t), r_1(t), v_2(t))$ is bounded and sufficiently small (for example a small periodic solution) then $\Omega(t)$ in (3.9) is a strictly monotone function of t. Note that, given one such solution, there must be a circle of such solutions, distinguished by the initial phase Ω_0 in (3.9), but all are conjugate under the S^1 symmetry. This implies that, if a Hopf bifurcation to a periodic solution occurs in the (v_0, r_1, v_2)-equations, the corresponding torus of solutions in the full $O(2) \times S^1$-equivariant equations (3.1) has 'linear flow', in the sense that every solution is S^1-conjugate to every other solution. The solutions may be periodic or quasiperiodic, but there is no resonance.

The first step in establishing a Hopf bifurcation in the (v_0, r_1, v_2)-equations is a linear stability analysis. The Jacobian matrix at $(v_0, r_1, v_2) = (0, 0, 0)$ is

$$\begin{bmatrix} [c^1 - ia^2c^2]^r & 0 & 0 \\ 0 & 2a^2[(p_N^1 - p^2)^r + O(a^2)] & 0 \\ 0 & 0 & 2a^2[p^2 + iq^2]^r \end{bmatrix}. \tag{3.10}$$

The superscripts r indicate that the expressions are evaluated on the rotating-wave branch (3.2) to (3.6). Local bifurcation can occur only at non-hyperbolic points; that is, where (3.10) has an eigenvalue with zero real part. Generically,

$$[p^1_N(0) - p^2(0)] \neq 0 \neq p^2(0), \tag{3.11}$$

and by continuity these coefficients will remain non-zero in a neighbourhood of the origin. Furthermore, inequalities (3.11) are required for non-degeneracy of the primary bifurcation of rotating waves and standing waves, respectively. Hence the only remaining possibility for a local bifurcation from rotating waves is

$$[c^1]^r = 0. \tag{3.12}$$

Expanding (3.12) at 0, recalling that $c^1(0) = 0$, gives

$$c^1_\mu \cdot \mu + c^1_N a^2 + \dots = 0. \tag{3.13}$$

This is to be solved concurrently with equation (3.5) for the rotating-wave branch, which can be done uniquely, provided that c^1_μ and p^1_μ satisfy the transversality condition

$$\det\left[\frac{\partial[c^1, p^1]}{\partial[\mu_1, \mu_2]}\right] \neq 0. \tag{3.14}$$

Henceforth we assume that (3.14) holds. This is a natural generalization of the Hopf eigenvalue crossing condition for classical Hopf bifurcation.

Now we can eliminate a^2 between (3.5) and (3.13) and obtain, to leading order,

$$[p^1_N - p^2]c^1_\mu \cdot \mu - c^1_N p^1_\mu \cdot \mu + \dots = 0. \tag{3.15}$$

This gives the tangent to a curve through the origin in the μ-plane on which (3.10) is non-hyperbolic. This tangent line is restricted to a ray by the condition that $a^2 > 0$, which to first order from (3.6) is

$$[p^1_\mu \cdot \mu]/[p^1_N - p^2] < 0; \tag{3.16}$$

see the ray $0C$ in Fig. 1.

Now suppose that as μ varies near $(0, 0)$ in the parameter plane, μ crosses the arc (3.15) (which is tangent to the ray $0C$ in Fig. 1). Then the Jacobian (3.10) has a complex eigenvalue crossing the imaginary axis, and experience leads us to expect a Hopf bifurcation in the (v_0, r_1, v_2)-equations. However, this is not quite the standard Hopf bifurcation, because for example (3.10) has no corresponding complex-conjugate eigenvalue. What is required here is a version of the Hopf bifurcation theorem for a special class of complex ordinary differential equations. Such a theorem is presented in the Appendix.

A second difficulty is the fact that the rotating-wave branch has not been calculated exactly, but only asymptotically for small a^2. It is clear from (3.5) and (3.13) that, for the parameter values of interest, $\mu = O(a^2)$. Therefore we rescale μ to make this explicit, and define

$$\mu = (\mu_1, \mu_2) = a^2\lambda = a^2(\lambda_1, \lambda_2). \tag{3.17}$$

Now all the terms in (3.10) are explicitly $O(a^2)$, and in fact the differential equations for $v \equiv (v_0, r_1, v_2)$ take the form

$$dv/dt = a^2 f^0(v, \bar{v}, \lambda) + a^4 f^1(v, \bar{v}, \lambda, a^2), \tag{3.18}$$

where f^0 is independent of a and f^1 is bounded as $a \to 0$. Now introduce a slow time variable

$$\tau = a^2 t \tag{3.19}$$

and let $a \to 0$ in (3.18), to obtain the blown-up equation

$$dv/d\tau = f^0(v, \bar{v}, \lambda). \tag{3.20}$$

The explicit form of f^0 is given below in (3.23). First we remark that if there exists a hyperbolic periodic orbit of (3.20) (for example from a Hopf bifurcation), then by standard transversality arguments, for sufficiently small a^2 there is a nearby periodic orbit of

$$dv/d\rho = f^0 + a^2 f^1 \tag{3.21}$$

and hence for (3.18), and orbital stability is preserved. To calculate f^0, we recall that

$$\left.\begin{aligned} \rho &= a^2 v_0 \bar{v}_0, \\ N &= a^2[(1+r_1)^2 + v_2\bar{v}_2], \\ \delta &= -a^2 + a^2[v_2\bar{v}_2 - (2+r_1)r_1], \end{aligned}\right\} \tag{3.22}$$

and make use of the fact that a^2 satisfies the rotating-wave branching equation (3.2). In (3.23), all coefficients and derivatives are evaluated at the origin:

$$\left.\begin{aligned} dv_0/d\tau = f_0^0 &= [c_\mu^0 \cdot \lambda + c_N^1 - ic^2]v_0 + c_\rho^1 v_0\bar{v}_0 v_0 + [c_N^1 + ic^2]v_2\bar{v}_2 v_0 \\ &\quad + [c_N^1 - ic^2][2+r_1]r_1 v_0 + c^3[1+r_1]\bar{v}_0\bar{v}_2, \\ dr_1/d\tau = f_1^0 &= [p_N^1 - p^2][2+r_1][1+r_1]r_1 + [p_N^1 + p^2][1+r_1]v_2\bar{v}_2 \\ &\quad + p_\rho^1 v_0\bar{v}_0[1+r_1] + \mathrm{Re}\,[[p^3 + iq^3][v_0^2 v_2]], \\ dv_2/d\tau = f_2^0 &= 2[p^2 + iq^2]v_2 + [p_N^1 + p^2 + 2iq^2][2+r_1]r_1 v_2 \\ &\quad + [p_N^1 - p^2 - 2iq^2]v_2^2\bar{v}_2 + p_\rho^1 v_0\bar{v}_0 v_2 \\ &\quad + [p^3 + iq^3][1+r_1]\bar{v}_0^2 + [\text{4th-degree terms in } v]. \end{aligned}\right\} \tag{3.23}$$

In f_2^0, there are fourth-degree terms in v_0, r_1, v_2, which come from the corresponding terms in (3.8), but they have not been written out explicitly because they have no effect on the Hopf bifurcation.

In order to apply the bifurcation formula in the Appendix, we require the first three derivatives of f^0 evaluated at criticality. These are

$$A_0 \equiv [d_v f^0]_0 = \begin{bmatrix} -ic^2 & 0 & 0 \\ 0 & 2[p_N^1 - p^2] & 0 \\ 0 & 0 & 2[p^2 + iq^2] \end{bmatrix}, \tag{3.24}$$

$$\begin{aligned} [d_v^2 f^0]_0[u, \bar{u}][v, \bar{v}] &\equiv \frac{d}{dt_1}\frac{d}{dt_2} f^0[t_1 u + t_2 v, t_1\bar{u} + t_2\bar{v}, \lambda]\,\big|_{t_j=0} \\ &= \begin{bmatrix} 2[c_N^1 - ic^2](u_1 v_0 + v_1 u_0) + c^3(\bar{u}_0\bar{v}_2 + \bar{v}_0\bar{u}_2) \\ 6[p_N^1 - p^2]u_1 v_1 + [p_N^1 + p^2](u_2\bar{v}_2 + v_2\bar{u}_2) + p_\rho^1(u_0\bar{v}_0 + \bar{u}_0 v_0) \\ 2[p_N^1 + p^2 + 2iq^2](u_1 v_2 + v_1 u_2) + 2[p^3 + iq^3]\bar{u}_0\bar{v}_0 \end{bmatrix}. \end{aligned} \tag{3.25}$$

Anticipating the fact that only the first component of the third derivative of f^0

will be required we calculate

$$
\begin{aligned}
&[d^3f_0^0]_0[u, \bar{u}][v, \bar{v}][w, \bar{w}] \\
&\quad = [c_N^1 + ic^2][u_2\bar{v}_2w_0 + u_2\bar{w}_2v_0 + \bar{u}_2v_2w_0 + \bar{u}_2w_2v_0 + u_0v_2\bar{w}_2 + u_0w_2\bar{v}_2] \\
&\quad + 2[c_N^1 - ic^2][u_0v_1w_1 + v_0u_1w_1 + w_0u_1v_1] \\
&\quad + c^3[u_1\bar{v}_0\bar{w}_2 + u_1\bar{w}_0\bar{v}_2 + \bar{u}_0v_1\bar{w}_2 + \bar{u}_0\bar{v}_2w_1 + \bar{u}_2v_1\bar{w}_0 + \bar{u}_2w_1\bar{v}_0] \\
&\quad + 2c_\rho^1[u_0v_0\bar{w}_0 + u_0w_0\bar{v}_0 + v_0w_0\bar{u}_0]. \qquad (3.26)
\end{aligned}
$$

Now equation (3.20) has the required form for application of the complex Hopf bifurcation theorem and formula in the Appendix. In particular, the linear part A operates only on v and not on $\bar{v}$. The special fact that the r_1-equation in (3.20) is real causes no difficulty; the real v_1-axis is invariant under the flow.

Applying the formulas of the Appendix, we have

$$[d_v^2 f^0]_0[u, \bar{u}][v, \bar{v}] = h_{zz}[u, v] + h_{z\bar{z}}[u, \bar{v}] + h_{\bar{z}z}[\bar{u}, v] + h_{\bar{z}\bar{z}}[\bar{u}, \bar{v}], \qquad (3.27)$$

$$\omega_0 = -c^2(0), \qquad \mathbf{c} = \mathbf{d} = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \qquad (3.28)$$

$$b_0 = \frac{1}{2}\left[\frac{-p_\rho^1}{p_N^1 - p^2}\right]\begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \qquad b_2 = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \qquad b_{-2} = \frac{-(p^3 + iq^3)}{2[p^2 - i(c^2 - q^2)]}\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, \qquad (3.29)$$

$$w_2(s) = b_0 + e^{-2is}b_{-2}, \qquad (3.30)$$

and the Hopf coefficient H is

$$H = c_\rho^1 - \frac{p_\rho^1[c_N^1 - ic^2]}{p_N^1 - p^2} - \frac{c^3[p^3 - iq^3]}{2[p^2 + i(c^2 - q^2)]}. \qquad (3.31)$$

The direction of bifurcation, relative to a given choice of path in the parameter plane (transverse to the ray $0C$) is determined by the sign of

$$-\mathrm{Re}\,(H)/\sigma'(0), \qquad (3.32)$$

where $\sigma'(0)$ is the directional derivative of c^1 at criticality, along the given path, evaluated on the rotating-wave branch. We have a supercritical branch if (3.32) is positive and a subcritical branch if it is negative. It follows from the transversality hypothesis (3.14) that generically $\sigma'(0) \neq 0$. The bifurcating torus is asymptotically stable if the following quantities are negative:

$$\mathrm{Re}\,(H), \qquad p_N^1 - p^2, \qquad p^2. \qquad (3.33)$$

We conclude that, for a generic parameter path crossing the ray $0C$, equation (3.20) has a Hopf bifurcation of a branch of periodic orbits, with direction and stability determined by formulas (3.31) to (3.33). According to the remarks following (3.20), this implies existence of a branch of periodic orbits of (3.18) with frequency asymptotic to $c^2(0)a^2$, for small a^2.

Finally, for the reasons given after (3.9), this implies the bifurcation of an invariant torus of solutions of (3.8) and hence of (3.1). On this torus, we know that the second frequency is $O(a^2)$ and furthermore the flow is linear; that is, all solutions on the torus are obtained from the S^1 phase shift in (3.9). In fact, normal hyperbolicity of the bifurcating torus guarantees that the torus is invariant under the action of $SO(2) \times S^1$ and hence any group orbit. It then follows more generally that the vector field on the torus must be topologically conjugate to linear flow (see (Rand, 1982) and section 4).

4. The higher-order tail

In section 3 we investigated the bifurcation of an invariant 2-torus of solutions, under the assumption that the vector field (3.1) is in formal normal form, that is to say, has exact $O(2) \times S^1$ symmetry. However, in general the vector field has a remainder or tail of order $k+1$ which has only the O(2) symmetry and not the S^1 symmetry. In this section we analyse how the tail modifies the conclusions of section 3.

To begin, we reconsider the $O(2) \times S^1$-equivariant normal form (2.9). Instead of (2.9), we now have a vector field which is a sum

$$g(z, \bar{z}; \mu) = g^k(z, \bar{z}; \mu) + h^k(z, \bar{z}; \mu). \tag{4.1}$$

Here g^k is a vector-valued polynomial of degree k in $(z, \bar{z})$ whereas h^k is a remainder of order $k+1$ in $(z, \bar{z})$; further, g^k has the form (2.9), where h^k is O(2)-equivariant but not S^1-equivariant. We shall show that the torus bifurcation found in section 3 persists under the assumption that

$$k \geqslant 3. \tag{4.2}$$

The transformation to a rotating coordinate system centred on the rotating-wave solution proceeds as before, using (3.7). However, now the transformed equations are not exactly as in (3.8). Instead, the right-hand side is a sum of a polynomial part of the same form as (3.8) and a remainder which is $O(a^k)$ and is not S^1-equivariant. The failure of S^1-equivariance is manifested in the presence of factors $\exp(i\Omega)$ in the $O(a^k)$ part, so that the vector field is now 2π-periodic in Ω. It remains true that the vector field is autonomous, and that for sufficiently small a and μ, and bounded v, Ω is a strictly monotone function of t. We use this fact to change the independent variable in (3.8) from t to Ω;

$$\frac{dv}{d\Omega} = \left[\frac{dv}{dt}\right] \Big/ \left[\frac{d\Omega}{dt}\right]. \tag{4.3}$$

Let us also rescale the parameter μ as in (3.17). The result is a new system of the form

$$dv/d\Omega = a^2 P(v, \bar{v}, \lambda, a^2) + a^k R(v, \bar{v}; \lambda, a, \Omega), \tag{4.4}$$

where P is a vector-valued polynomial in $(v, \bar{v})$ independent of Ω, and the remainder R is 2π-periodic in Ω. Further consideration of P to first order as $a \to 0$ shows that

$$P(v, \bar{v}, \lambda, 0) = \left[\frac{1}{\omega_0}\right] f^0(v, \bar{v}, \lambda), \tag{4.5}$$

where f^0 is as in (3.18). We now apply first-order averaging to the equation (4.4). Since P is independent of Ω, the averaging is trivial, and the first-order averaged equation is

$$dv/d\Omega = a^2P(v, \bar{v}, \lambda, 0). \tag{4.6}$$

But, by (4.5), this is identical to equation (3.20), for which there is generically a Hopf bifurcation to a branch of hyperbolic periodic solutions (that is, if $\mathrm{Re}\,(H) \neq 0$). Now by a theorem of Hale (1969), the existence of hyperbolic periodic orbit for the averaged equation (4.6) implies the existence of a hyperbolic invariant torus for the periodically forced equation (4.4). Since (4.4) is equivalent to the original system near criticality, the local persistence of the torus to the perturbations of the tail is established.

Now the loss of S^1-equivariance means that the solutions of (4.4) are not all S^1 conjugates. Nevertheless, the O(2) symmetry is sufficient to preserve the qualitative nature of the flow on the torus, as we now show. The $\mathbb{Z}_2$ part of the O(2) symmetry has been broken by the primary rotating-wave bifurcation; there are two isolated $\mathbb{Z}_2$-conjugate rotating waves, each of which is SO(2)-invariant (since each is an isolated periodic solution). The secondary Hopf bifurcation from the rotating wave yields an isolated 2-torus which is normally hyperbolic, provided $H \neq 0$. Therefore this 2-torus is also SO(2)-invariant. Now a theorem of Rand (1982) describes flow on a 2-torus that is both flow-invariant and invariant with respect to the action of SO(2). It is required that SO(2) is not the isotropy group of a solution in the torus (that is, there exists a non-axisymmetric solution); this is clearly satisfied, as seen from (2.7a) and (3.7). The conclusion is that any trajectory on the torus is either asymptotic to a rotating wave in the torus, or is a modulated wave and topologically conjugate to a linear flow. But a rotating wave, by definition, coincides with a group orbit of SO(2). In the present case, (2.8) implies that the rotation number of an SO(2) group orbit is equal to 1, while the rotation number of the flow is asymptotically proportional to the second frequency calculated in section 3, which is $O(a^2)$. Thus, on the torus the group orbits and flow orbits are transverse, and rotating waves are excluded. Thus we conclude that every solution on the 2-torus is a modulated wave (in the sense of Rand (1982)), topologically conjugate to a linear flow with two independent frequencies depending continuously on the parameter μ. In particular, there is no resonance (or frequency entrainment).

For the case in which the modulated wave arises via Hopf bifurcation from a rotating wave, there are 'selection rules' which constrain the possible spatio-temporal patterns that can appear. These rules apply to the modulated waves obtained on the 2-torus; for more details see (Rand, 1982; Renardy 1982).

Recall, however, that the normal form is valid only locally, and although we can perform the calculation to make the order k of the tail as high as we please, the neighbourhood of validity of the formal normal form in general shrinks to zero as k increases. This implies that at onset the torus is smooth, but as the torus grows it may rapidly lose differentiability. Thus the possibility of more complicated or even 'chaotic' dynamics away from criticality remains open.

5. Application to the Taylor–Couette problem

In the Taylor–Couette system, two concentric cylinders are rotated independently, and the space between them is filled with fluid. Since the work of Taylor

(1923), an outstanding problem has been to explain the variety of patterns of fluid motion observed in the experiments; see the review article of DiPrima and Swinney (1981). A standard mathematical idealization is to assume that the fluid motion is periodic in the axial coordinate z. This corresponds well to experimental observations away from the ends of long cyclinders. Together with the reflectional symmetry $z \to -z$ along the axis of the cylinder, this imposes a symmetry group O(2) on the problem. The rotation speeds of the two cylinders (or the two corresponding non-dimensional Reynolds numbers) provide two independent parameters in the problem. In the counterrotating case, there exists a bicritical point for these two parameters, at which the linearization of the Navier–Stokes equations simultaneously has a zero eigenvalue and a conjugate pair of purely imaginary eigenvalues. The O(2) symmetry forces all three of these eigenvalues to be double. Thus this bicritical point falls into context of the present work.

There is an additional SO(2) symmetry in the Taylor–Couette experiment, corresponding to azimuthal rotations of the cylinders. Mathematically, in the normal form this SO(2) symmetry plays the same role as the S^1 phase-shift symmetry, except that it is exact to all orders. Thus the non-symmetric tail considered in section 4 does not exist, and the results of section 3 apply directly to the Taylor–Couette problem.

We have computed the bifurcation coefficients, given in this paper, from the Navier–Stokes equations for the Taylor–Couette problem over the range of radius ratio 0.45 to 0.98; see (Golubitsky and Langford, 1987). In this context, the branch of rotating waves is usually called 'spirals', therefore we refer to the modulated rotating waves as modulated spirals. Our conclusion is that, whenever spirals are stable, there exists a bifurcation to modulated spirals, but the modulated spirals are unstable at bifurcation. Thus modulated spirals are not likely to be observed in experiments near the bicritical point.

Acknowledgement

This research was supported in part by the AMCP program of DARPA. In addition the second author received support from NASA-Ames grant NAG-279 and NSF grant DMS-8402604, and the third author from NSERC, Canada.

Bibliography

Chossat, P. (1986) Bifurcation secondaire de solutions quasi-periodiques dans un probleme de bifurcation de Hopf invariant par symmetrie O(2). *Comptes Rendus Hebdomadaires des Séances de l'Académie des Sciences* **302**, 539–541.

DiPrima, R. C. and Swinney, H. L. (1981) Instabilities and transition in flow between concentric rotating cylinders. *Hydrodynamic Instabilities and the Transition to Turbulence,* Topics in Applied Physics **45** (eds H. L. Swinney and J. P. Gollub; Springer, Berlin) 139–180.

Golubitsky, M. and Langford, W. F. (1981) Classification and unfoldings of degenerate Hopf bifurcations. *Journal of Differential Equations* **41**, 375–415.

Golubitsky, M. and Langford, W. F. (1988) Pattern formation and bistability in flow between counterrotating cylinders. *Physica* **D** to appear.

Golubitsky, M. and Schaeffer, D. G. (1985) *Singularities and Groups in Bifurcation Theory,* Vol. I, Applied Mathematical Sciences **51** (Springer, New York).

Golubitsky, M. and Stewart, I. (1985) Hopf bifurcation in the presence of symmetry. *Archive for Rational Mechanics and Analysis* **87**, 107–165.

Golubitsky, M. and Stewart, I. (1986) Symmetry and stability in Taylor–Couette flow. *SIAM Journal on Mathematical Analysis* **17**, 249–288.

Golubitsky, M., Stewart, I. and Schaeffer, D. G. (1988) *Singularities and Groups in Bifurcation Theory,* Vol. II, Applied Mathematical Sciences **69** (Springer, New York).

Guckenheimer, J. (1981) On a codimension two bifurcation. *Dynamical Systems and Turbulence,* Warwick 1980, Lecture Notes in Mathematics **898** (eds D. A. Rand and L.-S. Young; Springer, Berlin) 99–142.

Guckenheimer, J. and Holmes, P. (1983) *Nonlinear Oscillations, Dynamical Systems and Bifurcation of Vector Fields,* Applied Mathematical Sciences **42** (Springer, Berlin).

Hale, J. (1969) *Ordinary Differential Equations* (Wiley, New York).

Hassard, B. D., Kazarinoff, N. D. and Wan, Y.-H. (1981) *Theory and Applications of Hopf Bifurcation,* London Mathematical Society Lecture Notes **41** (University Press, Cambridge).

Holmes, P. (1980) Unfolding a degenerate nonlinear oscillator: a codimension two bifurcation. *Nonlinear Dynamics,* Annals of the New York Academy of Science **357** (ed. R. H. S. Helleman) 473–488.

Hopf, E. (1942) Abzweigung einer periodischen Losung von einer stationaren Losung eines Differentialsystems. *Berichten der Mathematisch-Physischen Klasse der Sächsischen Akademie der Wissenschaften zu Leipzig* **94**, 1–22.

Iooss, G. (1984) Bifurcation and transition to turbulence in hydrodynamics. *CIME Session on Bifurcation Theory and Applications,* Lecture Notes in Mathematics **1057** (ed. L. Salvadori; Springer, New York) 152–201.

Langford, W. F. (1979) Periodic and steady-state mode interactions lead to tori. *SIAM Journal on Applied Mathematics* **37**, 22–48.

Langford, W. F. (1986) Equivariant normal forms for the Taylor–Couette problem. *Multiparameter Bifurcation Theory,* Contemporary Mathematics **56** (eds M. Golubitsky and J. Guckenheimer; American Mathematical Society, Providence) 363–366.

Langford, W. F. and Iooss, G. (1980) Interactions of Hopf and pitchfork bifurcations. *Bifurcation Problems and Their Numerical solution,* International Series on Numerical Mathematics **54** (eds H. D. Mitteleman and H. Weber; Birkhauser, Basel) 103–134.

Langford, W. F., Tagg, R., Kostelich, E., Swinney, H. L. and Golubitsky, M. (1988) Primary instabilities and bicriticality in flow between counterrotating cylinders. *Physics of Fluids* **31**, 776–785.

Marsden, J. and McCracken, M. (1976) *Hopf Bifurcation and Its Applications,* Applied Mathematical Sciences **19** (Springer, Berlin).

Rand, D. (1982) Dynamics and symmetry: predictions for modulated waves in rotating fluids. *Archive for Rational Mechanics and Analysis* **79**, 1–38.

Renardy, M. (1982) Bifurcation from rotating waves. *Archive for Rational Mechanics and Analysis* **79**, 49–84.

Ruelle, D. (1973) Bifurcation in the presence of a symmetry group. *Archive for Rational Mechanics and Analysis* **51**, 136–152.

Taylor, G. I. (1923) Stability of viscous liquid contained between two rotating cylinders. *Philosophical Transactions of the Royal Society* **A 223**, 289–343.

Vanderbauwhede, A. (1982) *Local Bifurcation and Symmetry,* Research Notes in Mathematics **75** (Pitman, London).

van Gils, S. A. (1984) Some studies in dynamical systems theory. Thesis, University of Amsterdam.

(Received May 1988)

Appendix

Complex Hopf bifurcation

The standard Hopf bifurcation theorem applies to a real system of differential equations

$$dy/dt = f(y, \mu), \tag{A.1}$$

where $f: \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$; see for example Hopf (1942) and the books of Marsden and McCracken (1976), Hassard, Kazarinoff and Wan (1981), Vanderbauwhede (1982), and Golubitsky and Schaeffer (1985). All of these references give explicit formulas for the coefficients which determine the direction of bifurcation, stability, and the period of the periodic solution, whenever the Hopf bifurcation hypotheses are satisfied.

In the study of normal forms of differential equations which possess symmetry, it is often convenient to use complex coordinates, and one is lead naturally to a complex generalization of (A.1). Such systems may also undergo a Hopf bifurcation but the standard real formulas do not directly apply to the complex case. It is the purpose of this Appendix to present explicit formulas for the Hopf bifurcation coefficients for a class of complex differential equations.

The complex differential equations considered here are assumed to be of the form

$$dz/dt = A(\mu)z + h(z, \bar{z}; \mu), \tag{A.2}$$

where $z \in C^n$, $\bar{z}$ is the complex conjugate of z, $\mu \in \mathbb{R}$ (a real parameter), $A(\mu): \mathbb{C}^n \to \mathbb{C}^n$ is linear for each μ, and $h: \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{R} \to \mathbb{C}^n$. We assume that both A and h are sufficiently smooth in their arguments (three continuous derivatives is sufficient), and h is 'higher order' in the sense that

$$h(0, 0; \mu) = h_z(0, 0; \mu) = h_{\bar{z}}(0, 0; \mu) = 0. \tag{A.3}$$

Note that a major restriction implied by the form of (A.2) is that the linear terms must not contain $\bar{z}$. This situation frequently occurs in normal form differential equations with symmetry. Of course, such symmetries will also restrict the form of h, but for generality we place no further symmetry restrictions on h.

For a Hopf bifurcation, assume that $A(0) \equiv A_0$ has an algebraically simple purely imaginary eigenvalue $i\omega_0$, $\omega_0 \neq 0$, and no others with zero real part. Then for μ sufficiently near 0, $A(\mu)$ has a simple eigenvalue

$$\sigma(\mu) + i\omega(\mu), \qquad \sigma(0) = 0, \qquad \omega(0) = \omega_0, \tag{A.4}$$

depending smoothly on μ. We assume that the classical crossing condition holds:

$$\sigma'(0) \neq 0. \tag{A.5}$$

We remark that it is not necessary for A to have a conjugate pair of eigenvalues in (A.4), as in the real case. However, the system (A.2) embeds naturally in $\mathbb{C}^{2n}$ as the system

$$\begin{bmatrix} z \\ \bar{z} \end{bmatrix}' = \begin{bmatrix} A & 0 \\ 0 & \bar{A} \end{bmatrix} \begin{bmatrix} z \\ \bar{z} \end{bmatrix} + \begin{bmatrix} h(z, \bar{z}; \mu) \\ \bar{h}(z, \bar{z}; \mu) \end{bmatrix}, \tag{A.6}$$

and obviously for (A.6), complex eigenvalues occur in conjugate pairs.

The method of analysis is the Liapunov–Schmidt procedure, as in (Golubitsky and Langford, 1981) but generalized to the complex setting. We introduce a rescaled time variable

$$s = (\omega_0 + \tau)t \tag{A.7}$$

and seek solutions which are 2π-periodic in s, that is, $2\pi/(\omega_0 + \tau)$-periodic in t. The starting point is the linearized system

$$\omega_0 \frac{dz}{ds} = A_0 z \tag{A.8}$$

with $\mu = \tau = 0$. Clearly (A.8) has 2π-periodic solutions

$$\Phi(s) = e^{is}c, \tag{A.9}$$

where c is an eigenvector of A_0 satisfying

$$A_0 c = i\omega_0 c, \qquad c^*c = 1. \tag{A.10}$$

Here * denotes conjugate-transpose. We shall have need of the adjoint eigenvector and eigenfunction defined by

$$A^*d = -i\omega_0 d, \qquad d^*c = 1, \qquad \Psi(s) = de^{is}. \tag{A.11}$$

Define the inner product on $\mathbb{C}^n$-valued 2π-periodic functions

$$\langle u, v\rangle = \frac{1}{2\pi}\int_0^{2\pi} v^*(s)u(s)\, ds, \tag{A.12}$$

and it is clear that $\langle \Phi, \Psi\rangle = 1$. Now the same type of analysis as for the real case in (Golubitsky and Langford, 1981) leads to the following.

Existence Theorem *Under the assumptions* (A.3) *to* (A.5) *above; equation* (A.2) *has a one-parameter branch of periodic solutions, with* (z, μ) *near* $(0, 0)$ *and with period near* $2\pi/\omega_0$ *in* t, *of the form*

$$\left.\begin{aligned} z &= \varepsilon\Phi(s) + \varepsilon^2 W_2(s) + O(\varepsilon^3),\\ \mu &= \varepsilon^2\mu_2 + O(\varepsilon^3),\\ \tau &= \varepsilon^2\tau_2 + O(\varepsilon^3), \end{aligned}\right\} \tag{A.13}$$

where ε *and* W_2 *satisfy*

$$\langle z, \Psi\rangle = \varepsilon, \qquad \langle W_2, \Psi\rangle = 0. \tag{A.14}$$

The solution (A.13) is unique up to phase shift, in the sense that any neighbouring solution with period near $2\pi/\omega_0$ can be expressed in the form (A.13) for a suitable choice of the eigenvector c in (A.9), or equivalently by a shift in time t.

Our goal is to present explicit formulas for W_2, μ_2 and τ_2 in (A.13), in terms of derivatives of the given vector field h. Let us use subscripts z and $\bar{z}$ to denote partial Frechet derivatives of h with respect to these arguments. Then, for example, if $(d_z h)_0$ denotes the total Frechet derivative of $h(z, \bar{z}; \mu)$ with respect to z at the origin, we write

$$(d_z h)_0(v, \bar{v}) = h_z v + h_{\bar{z}} \bar{v}. \tag{A.15}$$

Here, and henceforth, all derivatives are evaluated at $(z, \bar{z}; \mu) = (0, 0; 0)$.

Define complex vectors b_0, b_2, b_{-2} by

$$\left.\begin{aligned} A_0 b_0 &= -h_{z\bar{z}} c\bar{c}, \\ [A_0 - 2i\omega_0 I] b_2 &= -\tfrac{1}{2} h_{zz} cc, \\ [A_0 + 2i\omega_0 I] b_{-2} &= -\tfrac{1}{2} h_{\bar{z}\bar{z}} \bar{c}\bar{c}. \end{aligned}\right\} \tag{A.16}$$

Then W_2, μ_2 and τ_2 in (A.13) are given by the formulas

$$\left.\begin{aligned} W_2(s) &= e^{2is} b_2 + b_0 + e^{-2is} b_{-2}, \\ \mu_2 &= -\mathrm{Re}\,(H)/\sigma'(0), \\ \tau_2 &= \mathrm{Im}\,(H) - \mathrm{Re}\,(H)\omega'(0)/\sigma'(0), \end{aligned}\right\} \tag{A.17}$$

where we define

$$H = \tfrac{1}{2} d^* h_{zz\bar{z}} cc\bar{c} + d^*[h_{zz} c b_0 + h_{z\bar{z}} c\bar{b}_2 + h_{\bar{z}\bar{z}} \bar{c}\bar{b}_{-2}]. \tag{A.18}$$

In (A.17), $\sigma'(0)$ and $\omega'(0)$ are the derivatives of the eigenvalue in (A.4), (A.5), and may be calculated from the identity

$$d^* A'(0) c = \sigma'(0) + i\omega'(0). \tag{A.19}$$

Formulas (A.16) to (A.18) are the generalization, to this complex setting, of the corresponding formulas in (Golubitsky and Langford, 1981) for real vector fields.