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An Example of Moduli for Singular Symplectic Forms

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In [3] Martinet shows that there are four generic types of singularities for germs of closed C^∞ 2-forms on 4-manifolds and then defines a notion of stability for these germs. The stability of the first singularity type is just the classical Darboux theorem for symplectic forms. Martinet proved the stability of the second type; while, more recently, Roussarie [6] has shown the stability of the third. In this paper we shall show that forms exhibiting this last type of singularity are unfortunately not stable. In fact, we show that near any generic $\Sigma_{2,2,1}$ singularity there is, at least, a one parameter family of moduli.

In §1 we briefly describe the various singularities. In §2 we will show how to reduce the problem of stability to one involving a contact structure on $\mathbb{R}^3$ at 0. Section 3 contains the proof of instability.

Note: we assume that all functions, forms, vector fields, etc. are C^∞.

§1. The Singularity Types

Let w be the germ of a closed 2-form on $\mathbb{R}^4$ at 0. Let Ω be a volume form on $\mathbb{R}^4$ and let $w \wedge w = f\Omega$.

(i) If $f(0) \neq 0$ then w is symplectic and Darboux's Theorem states that there are coordinates x, y, z, t on $\mathbb{R}^4$ at 0 such that

$$w = dx \wedge dy + dz \wedge dt.$$

Next assume that $f(0)=0$ while $(df)(0) \neq 0$. Let $\Sigma_2 = \{f=0\}$, and let $i: \Sigma_2 \to \mathbb{R}^4$ be the inclusion. Note that Σ_2 is a dimension 3 submanifold of $\mathbb{R}^4$.

(ii) If $i^* w(0) \neq 0$, then w has a $\Sigma_{2,0}$ singularity at 0. Martinet [3, p. 157] has shown that for $\Sigma_{2,0}$ singularities there are coordinates x, y, z, t on $\mathbb{R}^4$ such that

$$w = x\, dx \wedge dy + dz \wedge dt.$$

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Next, assume that $i^* w(0)=0$ and let $\Sigma_{2,2}=\{i^* w=0\}$. Consider the 2-plane field ker w on Σ_2. $\Sigma_{2,2}$ is precisely the points where the 2-planes are tangent to Σ_2. Generically this tangency occurs in codimension 2 in Σ_2. So generically $\Sigma_{2,2}$ is a dimension 1 submanifold of Σ_2, [3, p. 124].

(iii) w has a $\Sigma_{2,2,0}$ singularity at 0 if ker $w \not\pitchfork \Sigma_{2,2}$ at 0 in Σ_2. Roussarie [6] has shown that for $\Sigma_{2,2,0}$ singularities there are coordinates x, y, z, t on $\mathbb{R}^4$ at 0 such that

$$w=dx\wedge dy+z\,dy\wedge dz+d\left(xz+ty-\frac{z^3}{3}\right)\wedge dt \qquad \text{(elliptic } \Sigma_{2,2,0}\text{).}$$

or

$$w=dx\wedge dy+z\,dy\wedge dz+d\left(xz-ty-\frac{z^3}{3}\right)\wedge dt \qquad \text{(hyperbolic } \Sigma_{2,2,0}\text{).}$$

These two cases are distinguished as follows: let Ω' be a volume form on Σ_2, and X the vector field such that $\sigma=i^* w=X \lrcorner \Omega'$. Clearly $X=0$ on $\Sigma_{2,2}$ since $\sigma=0$ there. Thus at least one eigenvalue of the linear part of X at a point in $\Sigma_{2,2}$ is zero. The fact that σ is closed means that X is volume preserving. So the trace of the linear part of X is zero, and the other two eigenvalues are either both real or both imaginary. This property is an invariant of the singularity type. If the eigenvalues are real and non-zero then the singularity is a hyperbolic $\Sigma_{2,2,0}$; if they are imaginary, it is an ellipic $\Sigma_{2,2,0}$.

(iv) w has a $\Sigma_{2,2,1}$ singularity at 0 if $T_0\Sigma_{2,2}\subset \operatorname{Ker} w(0)$.

Definition. The closed 2-form germ w is *stable at* 0 in $\mathbb{R}^4$ if for every neighborhood U of O there is a neighborhood V of w (in the C^∞ topology on closed 2-forms) such that if w' is in V, then there is a point p' in U and a germ of diffeomorphism ϕ: $(\mathbb{R}^4, 0)\to(\mathbb{R}^4, p')$ such that $w=\phi^* w'$ at 0.

The work of Darboux, Martinet, and Roussarie shows that symplectic, $\Sigma_{2,0}$, and both types of $\Sigma_{2,2,0}$ singularities are stable. Martinet [3, p. 123] and [5] show that an open dense set of forms on a compact 4-manifold consists of forms which exhibit only the four singularity types listed above. We shall show that $\Sigma_{2,2,1}$ singularities are not stable.

§2. The Reduction to Σ_2

Let w have a $\Sigma_{2,2}$ singularity at 0 and let x: $\mathbb{R}^4\to\mathbb{R}$ define Σ_2. Let $\sigma=i^* w$.

(2.1) **Lemma.** *Let* π: $\mathbb{R}^4\to\Sigma_2$ *be a submersion which is the identity on* Σ_2. *Then there is a* 1-*form* v *on* Σ_2 *such that*

(i) $w=\pi^*\sigma+dx\wedge\pi^* v$ *on* Σ_2,

(ii) $v\wedge dv(0)\neq 0$ (v *is a contact form*),

and

(iii) $\sigma\wedge v\equiv 0$.

Proof. We can certainly write $w=\pi^*\sigma+dx\wedge\pi^* v+x\mu$ for some 1-form v on Σ_2 and 2-form μ on $\mathbb{R}^4$. Restricting to Σ_2 yields (i). Next compute

$$w\wedge w=2\pi^*\sigma\wedge dx\wedge\pi^* v+2x[\pi^*\sigma\wedge\mu+dx\wedge\pi^* v\wedge\mu]+x^2\mu\wedge\mu.$$

Now $w\wedge w=0=2\pi^*\sigma\wedge dx\wedge\pi^* v$ on Σ_2.

Thus $\sigma \wedge v \equiv 0$ on Σ_2. Finally the fact that Σ_2 is generic implies that the coefficient of x is not zero at the origin. So $dx \wedge \pi^* v \wedge \mu(0) \neq 0$. Since w is closed and $dw|_{\Sigma_2} = dx \wedge \pi^* dv + dx \wedge \mu$ we have that $\mu(0) = -\pi^* dv(0) + c\,dx$ for some constant c.

Thus $dx \wedge \pi^* v \wedge \pi^* dv(0) \neq 0$ and (ii) follows.

Now since $\sigma \wedge v = 0$ we can write $\sigma = v \wedge \alpha$.

Let w and $\bar{w}$ have $\Sigma_{2,2,1}$ singularities at 0 and the same Σ_2 sets. By the last lemma we have that on Σ_2

$$w = \pi^* \sigma + dx \wedge \pi^* v \quad \text{and} \quad \bar{w} = \pi^* \bar{\sigma} + dx \wedge \pi^* \bar{v}.$$

(2.2) **Lemma.** *Let* $\psi: (\mathbb{R}^4, 0) \to (\mathbb{R}^4, 0)$ *be a germ of a diffeomorphism such that* $\psi^* \bar{w} = w$. *Then there is a diffeomorphism* $\phi: (\Sigma_2, 0) \to (\Sigma_2, 0)$ *such that*

(a) $\phi^* \bar{\sigma} = \sigma$

and

(b) $\phi^* \bar{v} = k(v + f\alpha)$

for some functions k and f.

Proof. Certainly $\psi^*(\bar{w} \wedge \bar{w}) = w \wedge w$, so $\psi(\Sigma_2) = \Sigma_2$.

Let $\phi = \psi|_{\Sigma_2}$. Applying i^* to $\psi^* \bar{w} = w$ yields (a). Next note that

$$0 = \phi^*(\bar{v} \wedge \bar{\sigma}) = \phi^* \bar{v} \wedge \sigma.$$

Using Martinet [3, Lemma 3, p. 163] or direct computation we have that

$$\phi^* \bar{v} = k(v + f\alpha) \qquad \text{since } \sigma = v \wedge \alpha.$$

(2.3) **Corollary.** *If there exist arbitrarily small perturbations σ_s of σ with $\sigma_s \wedge v = 0$ for which there is not a diffeomorphism $\phi_s: (\Sigma_2, 0) \to (\Sigma_2, 0)$ satisfying $\phi_s^* \sigma_s = \sigma$ and $\phi_s^* v = k_s(v + f_s \alpha)$. Then w is not stable.*

Proof. This is obvious from the above by letting $w_s = w + \pi^*(\sigma_s - \sigma)$. Then $w_s = \pi^* \sigma_s + dx \wedge \pi^* v$ on Σ_2.

§3. The Proof of Instability

Let σ and v be as above. Since v is a contact form one can choose coordinates y, z, t on Σ_2 so that $v = dy + z\,dt$. We first show that we can put $\Sigma_{2,2}$ into a normal form while fixing v.

(3.1) **Lemma.** *If w has a generic $\Sigma_{2,2,1}$ singularity at 0, then there is a contact diffeomorphism $\phi: (\Sigma_2, 0) \to (\Sigma_2, 0)$ such that $\phi(\Sigma_{2,2}) = \{y = 0 = z - t\}$, and $\phi^* v = \pm v$.*

First we describe what we mean by genericity. Along $\Sigma_{2,2}$ the kernel field of w equals the kernel of v. (For on Σ_2, $w = \pi^* \sigma + dx \wedge \pi^* v$ and on $\Sigma_{2,2}$, $\sigma = 0$. So $w = dx \wedge \pi^* v$ on $\Sigma_{2,2}$.) Let A be a non-zero vector field tangent to $\Sigma_{2,2}$. Write

$$A = p \frac{\partial}{\partial z} + q \left(\frac{\partial}{\partial t} - z \frac{\partial}{\partial y} \right) + r \frac{\partial}{\partial y}$$

where the first two summands are in $\ker v$. Since 0 is a $\Sigma_{2,2,1}$ singularity for w, $r(0) = 0$. We say that 0 is a generic $\Sigma_{2,2,1}$ singularity if $dr(A) \neq 0$ at 0.

Since $\sigma \wedge v = 0$ there are functions a and b so that $\sigma = a\,dy \wedge dt + b\,dy \wedge dz + zb\,dt \wedge dz$. Now $\Sigma_{2,2} = \{\sigma = 0\} = \{a = b = 0\}$. The genericity of $\Sigma_{2,2}$ implies that da and db are linearly independent at 0. Since σ is closed we have

$$a_z - b_t + z b_y \equiv 0. \tag{1}$$

Proof of Lemma 3.1. First we put $\Sigma_{2,2}$ in the plane $y=0$. Now $\Sigma_{2,2}$ is transverse to either the z-axis or the t-axis. In the first case we write $\Sigma_{2,2} = \{(y(t), z(t), t)\}$. Note that $y(0) = 0 = z(0)$ and since $v = dy$ at 0, $y'(0) = 0$. Consider the change of coordinates

$$\bar{y} = y - y(t)$$
$$\bar{z} = z + y'(t)$$
$$\bar{t} = t.$$

This change of coordinates preserves v, sends 0 to 0, and satisfies $\bar{y}(\Sigma_{2,2}) = 0$.

In the second case $\Sigma_{2,2} = \{(y(z), z, t(z))\}$. Consider the change of coordinates

$$\bar{y} = y - y(z)$$
$$\bar{z} = z$$
$$\bar{t} = t + \int_0^z \frac{y'(s)}{s}\, ds.$$

As before $y(0) = 0$ and $y'(0) = 0$; so this change of coordinates makes sense, sends 0 to 0 and preserves v.

In either case $\Sigma_{2,2} \subset \{\bar{y} = 0\}$. Drop the bars. Since $\Sigma_{2,2} \subset \{y = 0\}$. A is tangent to $y = 0$ along $\Sigma_{2,2}$. Thus

$$A = p\frac{\partial}{\partial z} + q\left(\frac{\partial}{\partial t} - z\frac{\partial}{\partial y}\right) + zq\frac{\partial}{\partial y}$$

along $\Sigma_{2,2}$. Thus $dr(A) = A(zq) = p(0) \cdot q(0)$ at 0. So $p(0)$ and $q(0) \neq 0$. Now $\Sigma_{2,2}$ is transverse to the z-axis and $\Sigma_{2,2} = \{(0, z(t), t)\}$ where $z(0) = 0$ and $z'(0) \neq 0$. We can assume that $z'(0) > 0$. If not, use the contact change of coordinates

$$(y, z, t) \to (-y, -z, t).$$

Next note that any diffeomorphism of the form $\phi(y, z, t) = (y, z/k'(t), k(t))$ where $k(0) = 0$ and $k'(0) \neq 0$ is a contact transformation. Furthermore

$$\phi(\Sigma_{2,2}) = \phi(0, z(t), t) = (0, z(t)/k'(t), k(t)).$$

There obviously is a k such that $z(t) = k(t)k'(t)$. So we can assume that $\Sigma_{2,2} = \{y = z - t = 0\}$.

(3.2) **Proposition.** *There are no stable $\Sigma_{2,2,1}$ singularities.*

Clearly if w has a stable $\Sigma_{2,2,1}$ singularity at 0, then it has a generic one. This we assume. As above, $\sigma = a\,dy \wedge dt + b\,dy \wedge dz + zb\,dt \wedge dz$. Using Lemma 3.1, we can assume that

$$a = a_1 y + a_2(z-t) + \cdots$$

$$b = b_1 y + b_2(z-t) + \cdots$$

Equation (1) implies that $b_2 = -a_2$. Also note that $\alpha = a\,dt + b\,dz$. The proposition follows from the following:

(3.3) **Lemma.** *Let ϕ be a diffeomorphism of Σ_2 preserving* 0 *and $\Sigma_{2,2}$ for which $\phi^* v = k(v + f\alpha)$. Also assume $\phi^* \sigma \wedge v = 0$ (see* (2.1)*) which implies that*

$$\phi^* \sigma = \bar{a}\,dy \wedge dt + \bar{b}\,dy \wedge dz + z\bar{b}\,dt \wedge dz$$

where

$$\bar{a} = \bar{a}_1 y + \bar{a}_2(z-t) + \cdots$$

and

$$\bar{b} = \bar{b}_1 y - \bar{a}_2(z-t) + \cdots.$$

Then

$$\frac{(\bar{a}_1 + \bar{b}_1)^4}{\bar{a}_2^5} = \frac{(a_1 + b_1)^4}{a_2^5}.$$

This number is an invariant of a generic $\Sigma_{2,2,1}$ singularity which is easily perturbed. For example let $\sigma_s = (a + sy)\,dy \wedge dt + b\,dt \wedge dz + zb\,dt \wedge dz$. Apply Lemma 3.3 and Corollary 2.3 to see that w is not stable. In fact this gives the example of moduli promised in the introduction.

Proof of Lemma 3.3. We need only compute the 1-jet of $\phi^*\sigma$. Since $a(0) = b(0) = 0$, we need only know $(d\phi)(0)$ to do this. First we determine what restrictions the facts that $\phi^* v = K(v + f\alpha)$ and $\phi(\Sigma_{2,2}) = \Sigma_{2,2}$ put on $(d\phi)(0)$.

Let $\phi = (A, B, C)$. We use the following notation: A^{yt} denotes the coefficient of yt in the power series expansion of A at 0, etc. $\phi^* v = K(v + f\alpha)$ implies

$$A_z + BC_z = Kfb \tag{2}$$

$$A_t + BC_t = K[z + fa] \tag{3}$$

$$A_y + BC_y = K. \tag{4}$$

Evaluating (2) and (3) at 0 implies

$$A^z = A^t = 0. \tag{5}$$

$\phi(\Sigma_{2,2}) = \Sigma_{2,2} = \{y = 0 = z - t\}$ implies

$$A(0, z, z) = 0 \tag{6}$$

$$B(0, z, z) = C(0, z, z). \tag{7}$$

Thus

$$A^{tt} + A^{tz} + A^{zz} = 0 \qquad \text{from (6)} \tag{8}$$

and

$$B^z + B^t = C^z + C^t \qquad \text{from (7)} \tag{9}$$

$$A^{tz} + B^t C^z = K(0) f(0) a_2 \qquad \text{from (2)} \tag{10}$$

$$2A^{zz} + B^z C^z = -K(0) f(0) a_2 \qquad \text{from (2)} \tag{11}$$

$$2A^{tt} + B^t C^t = -K(0) f(0) a_2 \qquad \text{from (3)} \tag{12}$$

$$A^{tz} + B^z C^t = K(0)[f(0) a_2 + 1] \qquad \text{from (3).} \tag{13}$$

Now (10)+(11)+(12)+(13) imply that

$$(B^t + B^z)(C^t + C^z) = K(0) \qquad \text{using (8).} \tag{14}$$

Since $\phi^* v(0) = K(0) v(0) \neq 0$, we have

$$B^t + B^z \neq 0. \tag{15}$$

Next note that (14)−(13)+(10) yields

$$(B^t + B^z)(B^t + C^z) = 0 \qquad \text{using (9).} \tag{16}$$

Now (16) and (15) imply

$$B^t = -C^z \tag{17}$$

$$B^z = C^t + 2C^z \qquad \text{from (9) and (17).} \tag{18}$$

So our assumptions on ϕ imply that

$$(d\phi)(0) = \begin{pmatrix} A^y & 0 & 0 \\ B^y & C^t + 2C^z & -C^z \\ C^y & C^z & C^t \end{pmatrix}.$$

Next, by a long but straightforward calculation of $\phi^* v$, we have

$$\bar{a}_1 = A^y [a_1 A^y C^t - b_1 A^y C^z + a_2 (B^y - C^y)(C^t + C^z)]$$

$$\bar{b}_1 = A^y [a_1 A^y C^z + b_1 A^y (C^t + 2C^z) - a_2 (B^y - C^y)(C^t + C^z)]$$

$$\bar{a}_2 = a_2 A^y (C^t + C^z)^2.$$

Hence

$$\frac{(\bar{a}_1 + \bar{b}_1)^4}{\bar{a}_2^5} = \frac{(a_1 + b_1)^4}{a_2^5} \cdot \frac{(A^y)^3}{(C^t + C^z)^6}.$$

Finally,

$$A^y = K(0) \qquad \text{by (4)}$$

and

$$(C^t + C^z)^2 = K(0) \qquad \text{by (14) and (9).}$$

So

$$\frac{(\bar{a}_1 + \bar{b}_1)^4}{\bar{a}_2^5} = \frac{(a_1 + b_1)^4}{a_2^5}.$$

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