Coupled Cells: Wreath Products and Direct Products

Martin Golubitsky
Department of Mathematics
University of Houston
Houston, TX 77204-3476
USA

Ian Stewart
Mathematics Institute
University of Warwick
Coventry CV4 7AL
UK

Benoit Dionne
Department of Mathematics
University of Ottawa
Ottawa, Ontario
K1N 6N5
CANADA

October 18, 1993

1 Introduction

In this note we discuss the structure of systems of coupled cells (which we view as systems of ordinary differential equations) where symmetries of the system are obtained through the group $\mathcal{G}$ of global permutations of the cells and the group $\mathcal{L}$ of local internal symmetries of the dynamics in each cell. We show that even when the cells are assumed to be identical with identical coupling, the way that $\mathcal{G}$ and $\mathcal{L}$ combine to form the total symmetry group of the system Γ depends on properties of the coupling. We illustrate this point by showing how the combination of $\mathcal{L}$ with $\mathcal{G}$ can lead to a symmetry group Γ that is either a direct product or a wreath product. The symmetry group has strong implications for the dynamics of the system of cells, and the distinction between the two cases is substantial. This has important implications for the modeling of systems by coupled cells.

Several authors have studied abstractly systems of coupled cells *cf.* Alexander [1]. It has been noted previously that the form of the coupling can seriously affect the dynamics in systems of coupled cells [2]. More recently, Dangelmayr *et. al.* [6] have studied coupled cell systems with specific internal and global symmetries where the coupling produces direct product symmetry groups. Here we emphasize the point that the type of coupling does influence the total symmetry group and describe a few general bifurcation results for two natural types of coupling.

In the next section we discuss the form of the differential equations describing coupled cells. Section 3 develops the properties of the coupling that lead to direct and wreath products and Section 4 presents a number of examples of each type of coupling. In Section 5 we discuss the types of bifurcating branches that may occur in steady-state bifurcations for wreath product systems and in the last section, Section 6, we consider these bifurcations for direct product symmetries. These sections preview work that will appear in [8].

2 Identical Coupled Cells

In this section we discuss the form taken by systems of differential equations that model systems of identical cells with identical coupling. Imagine an array of N coupled cells — by which we mean a set of N cells with arrows connecting cell i to cell j when the output of cell i is coupled to cell j. Define the $N \times N$ *connection matrix* C by setting

$$C(i,j) = \begin{cases} 1 & \text{if cell } i \text{ is coupled to cell } j \\ 0 & \text{otherwise.} \end{cases}$$

Note that C need not be a symmetric matrix (the coupling may be directed).

Let $X_j \in \mathbf{R}^{k_j}$ be the state variables of cell j and let $X = (X_1, \ldots, X_N)$ be the state variables of the entire system of cells. Suppose that the dynamics of the coupled cell system is modeled by a differential equation

$$\dot{X} = F(X). \tag{2.1}$$

The structure of coupled cells allows us to write (2.1) in the form

$$\dot{X}_j = f_j(X_j) + h_j(X)$$

where f_j models the internal dynamics of the j^{th} cell and h_j represents the coupling of all other cells to the j^{th} cell.

For simplicity of exposition, assume that the total coupling to cell j is the *sum* of couplings from all cells i to cell j. (Similar conclusions may also be derived for more general types of coupling.) This assumption may be stated as

$$h_j(X) = \sum_{\{i:C(i,j)=1\}} h_{ij}(X_j, X_i)$$

for appropriate functions h_{ij}.

Now assume that all cells are identical. Then $k_j = k$ and $f_j = f$ for $j = 1, \ldots, N$; that is, the internal dynamics of each cell is governed by the same set of equations. The assumption that all couplings between cells are identical leads to the identity $h_{ij} = h$ for all i, j.

To summarize: the system of ODEs (2.1) has the form

$$\dot{X}_j = f(X_j) + \sum_{i=1}^{N} C(i,j) h(X_j, X_i) \qquad (2.2)$$

where $X_j \in \mathbf{R}^k$ for $j = 1, \ldots, N$.

3 Symmetries in Coupled Cells

As mentioned in the introduction, the symmetries of coupled systems appear in two ways: through global permutation symmetries and through local internal symmetries. Studies of the effects of the global symmetries have been made by many authors, but studies of the effects of internal symmetries have been less frequent.

Global Permutation Symmetries $\mathcal{G}$

In coupled cell systems with identical cells and identical coupling the *global symmetries* are permutations, determined by patterns in the couplings themselves. There are three especially popular patterns in which the cells form rings, directed rings, or simplexes ('all-to-all' coupling). In these cases the permutation symmetries $\mathcal{G}$ are the dihedral groups $\mathbf{D}_N$, the cyclic group $\mathbf{Z}_N$ and the permutation group $\mathbf{S}_N$, respectively. A permutation $\sigma \in \mathbf{S}_N$ acts on state space by

$$\sigma \cdot X = (X_{\sigma^{-1}(1)}, \ldots, X_{\sigma^{-1}(N)}).$$

The permutation σ is a symmetry of the coupled cell system if

$$F(\sigma \cdot X) = \sigma \cdot F(X)$$

which happens precisely when

$$\sigma C \sigma^{-1} = C, \tag{3.3}$$

where σ is viewed as a permutation matrix.

Local Internal Symmetries $\mathcal{L}$

A $k \times k$ matrix ρ is an *internal symmetry* if it is a symmetry of the internal dynamics of each cell, that is,

$$f(\rho X_j) = \rho f(X_j).$$

When do the internal symmetries lead to symmetries of the entire coupled cell array? The answer depends on the type of coupling h. We discuss two types of coupling.

Direct Products

Suppose that the coupling is equivariant with respect to the internal symmetries. That is,

$$h(\rho X_j, \rho X_i) = \rho h(X_j, X_i)$$

for all $\rho \in \mathcal{L}$. If we let $\mathcal{L}$ act on state space by

$$\rho \cdot X = (\rho X_1, \ldots, \rho X_N),$$

then $\mathcal{L} \times \mathcal{G}$ is a group of symmetries of the full system of ODEs (2.1). An example is diagonal linear coupling

$$h(X_j, X_i) = \lambda(X_i - X_j),$$

where $\lambda \in \mathbf{R}$ is the coupling strength.

If h satisfies no other invariance or equivariance conditions, then the symmetry group of (2.2) is the direct product of the groups of local and global symmetries. When the coupling leads to a direct product, the internal symmetries are symmetries of the whole cell system only when they act 'diagonally' — that is, in the same way on each cell.

Wreath Products

In the second type of coupling *any* internal symmetry acting on any individual cell is a symmetry of the entire system (2.2). In this case the coupling of cell i to cell j must not 'feel' the effect of an internal symmetry applied to cell i alone. This invariance may be formalized as:

$$\begin{aligned} h(\rho X_j, \rho X_i) &= \rho h(X_j, X_i) \\ h(X_j, \rho X_i) &= h(X_j, X_i) \end{aligned}$$

for all $\rho \in \mathcal{L}$. Equivalently

$$h(\rho X_j, \sigma X_i) = \rho h(X_j, X_i)$$

for all $\rho, \sigma \in \mathcal{L}$. (That is, the coupling is invariant in X_i, equivariant in X_j.)

With such coupling the symmetries of (2.1) include the group $\mathcal{L}^N$ acting by

$$(\rho_1, \ldots, \rho_N) \cdot X = (\rho_1 X_1, \ldots, \rho_N X_N).$$

The *wreath product* of $\mathcal{L}$ with the permutation group $\mathcal{G}$, denoted by $\mathcal{L} \wr \mathcal{G}$, is the smallest group generated by $\mathcal{L}^N$ and $\mathcal{G}$ in the given actions of $\mathcal{L}^N$ and $\mathcal{G}$ on state space. See Robinson [19] p.18 or Scott [20] p.215 for general information on wreath products. The wreath product contains the direct product (identify $\mathcal{L}$ with the diagonal subgroup of $\mathcal{L}^n$) but it is huge in comparison. If the coupling satisfies no further group-theoretic constraints, then the wreath product will be the full symmetry group of these coupled cell systems. We note that an example of wreath product coupling is:

$$h(X_j, X_i) = |X_i|^2 X_j.$$

4 Examples

At first sight the above abstract considerations may appear rather artificial, especially as regards the wreath product. However, examples of both kinds of coupled cell system are widespread — and in several respects the wreath product is the most natural and the most interesting. In this section we discuss a number of such examples.

Wreath Product Examples

(a) *Coupled arrays of Josephson junctions.*

This example was suggested by Kurt Wiesenfeld. There is an extensive literature studying arrays of identical coupled Josephson junctions [17, 3]. Indeed, such arrays are prototypical examples of systems exhibiting all-to-all coupling, since the coupling is electrical and is felt equally by all junctions. Thus an array having k junctions is modeled by a system of differential equations with $\mathbf{S}_k$ symmetry. Josephson junction arrays are usually posed as an example of a system of coupled cells with no internal symmetry. However, we shall consider each Josephson junction array to be a single cell, with $\mathbf{S}_k$ as its internal symmetry group.

Josephson junction arrays are often used to model certain kinds of computer chip. From this point of view, it is natural to consider an array of N chips, also electrically coupled. Thus the global symmetry group is $\mathbf{S}_N$, since the system of chips may be modeled as having all-to-all coupling. When the resistances in the individual chip and in the array of chips are different — a reasonable modeling assumption — then this system of coupled chips has $\mathbf{S}_k \wr \mathbf{S}_N$ symmetry (rather than $\mathbf{S}_{kN}$ symmetry).

(b) *Discretizations of PDEs with gauge symmetry.*

In systems of PDEs with local gauge symmetry, the gauge group acts independently at each point in space. For example, in systems with an abelian gauge, such as the original complex Ginzburg-Landau equation modeling superconductivity, the local gauge symmetry is a phase shift and (except for smoothness considerations) the phase shift operates independently at each point in space. It is well known that when discretizations of systems of PDE are made, the resulting system of ODEs has the structure of a coupled system of cells with each cell representing the dynamics of the PDE at one point or in one small region of space. From this point of view it is natural for the gauge symmetries to act independently in each cell. Should the system of PDEs be posed on a symmetric domain, then the total symmetry group of the discrete system will be the wreath product of the local gauge symmetry with the global (permutation) symmetry of the domain.

(c) *Molecular dynamics.*

As suggested by John Guckenheimer, another example of wreath product symmetry should occur in molecular dynamics. Molecules are made up of atoms (the cells) and have permutation symmetries that depend on the type of atoms and the bonds (coupling) between the atoms. On the other hand, atoms themselves have internal symmetries and the application of one of these symmetries to one atom should have no effect on the bonds between that atom and another. If this description is valid, then symmetries of models for the dynamics of molecules that include internal variables from the individual atoms will have a wreath product symmetry. If it is merely an approximation, then the system can be considered as a symmetry-breaking perturbation of one with wreath product symmetry.

(d) *Heteroclinic cycles.*

Perhaps the best known example of a structurally stable heteroclinic cycle in a symmetric system is the one abstracted by Guckenheimer and Holmes [15] from a model by Busse and Heikes [4] on rotating convection. In the experiment the dynamics of the convection system passes near three rolls patterns — each rotated by 120° from the previous one. Guckenheimer and Holmes observed that the model in [4] can be abstracted using a certain 24 element symmetry group; this symmetry group is just $\mathbf{Z}_2 \wr \mathbf{Z}_3$. The system of ODEs has the form of a system of three coupled cells with one internal state variable ($k = 1$) and one nontrivial internal symmetry ($\mathbf{Z}_2$). Due to the rotation in the model, the coupling from cell i to cell j is not equal to the coupling from cell j to cell i. Thus the symmetry in this system is that of a directed ring system. One wonders whether the existence of heteroclinic cycles is related to the coupling pattern. Examples of Field and Richardson [9] on symmetry groups $\mathbf{Z}_2 \wr \mathbf{Z}_N$ substantiate this point of view. The 'instant chaos' scenario of Guckenheimer and Worfolk [16] involves a subgroup of index two in $\mathbf{Z}_2 \wr \mathbf{Z}_4$. The symmetry group of the cube is the wreath product $\mathbf{Z}_2 \wr \mathbf{D}_3$.

Direct Product Examples

(a) *Neural networks.*

Wegelin *et. al.* [21, 6, 7] study coupled systems of three cells where each cell is itself a system of three identical cells. In order to study patterns of oscillation they consider direct product couplings. In particular they consider the types of Hopf bifurcation that occur with this symmetry. They find eleven different patterns of oscillation, as well as states with more complicated dynamics, and they discuss the stability of the periodic solutions that they find.

(b) *Discretization of PDEs with range symmetries.*

Suppose that a system of PDEs in k functions u is posed on a domain with a symmetry group $\mathcal{G}$ and a group of range symmetries $\mathcal{L}$ acting on $\mathbf{R}^k$. For example, consider the reaction-diffusion system on the interval $[0, 1]$ satisfying Dirichlet boundary conditions

$$u_t = \Delta u + f(u),$$

where $f(-u) = -u$. In this case the nontrivial domain symmetry is $x \mapsto 1 - x$ and the nontrivial range symmetry is $u \mapsto -u$. Other examples include elastic buckling of rods and plates with various symmetric geometries and appropriate boundary conditions — see for instance Buzano *et al.* [5] who study rods whose cross-sections are regular polygons, leading to $\mathbf{Z}_2 \times \mathbf{D}_n$ symmetry.

Discretizations of such PDEs will lead to coupled cell systems with the direct product of the domain and range symmetries as the group of symmetries. Here the range symmetries must act identically on all cells.

(c) *Direct products that occur by themselves.*

In a number of applications direct products occur in the standard models. For example, the Couette-Taylor system is posed on a circular cylinder with periodic boundary conditions in the axial direction. The group of symmetries for this model is $\mathbf{O(2)} \times \mathbf{SO(2)}$. Euclidean-invariant PDEs on rectangular domains with periodic boundary conditions have $\mathbf{O(2)} \times \mathbf{O(2)}$ symmetry, see Gomes and Stewart [13, 14] which also give connections with Neumann and Dirichlet boundary conditions.

5 Wreath Product Bifurcations

Wreath product bifurcations lead to some rather remarkable states, in which some cells are active while the remainder are quiescent. This kind of spatial differentiation has been found only rarely in bifurcation analyses, but appears to be natural in coupled cells with wreath product symmetry. In this section we summarize our knowledge of steady-state bifurcation in systems of coupled cells with wreath product symmetries. Details may be found in [8] along with a description of the corresponding Hopf bifurcation. To simplify the discussion here we assume that the global permutation symmetries $\mathcal{G}$ act transitively on the N cells.

Generically, steady-state bifurcations in systems of ODEs occur when the linearization at an equilibrium has zero eigenvalues and the kernel of the linearization is an absolutely irreducible representation of the group of symmetries of that equilibrium. In the context of coupled cells with wreath product $\mathcal{L} \wr \mathcal{G}$ symmetry we consider steady-state bifurcation from a group invariant equilibrium. The corresponding irreducible representations have local symmetries acting either trivially or nontrivially. When the local symmetries act trivially, the types of bifurcation reduce to the types considered in coupled cell systems with no internal symmetry. Here we only consider those types of bifurcation in which the internal symmetry group $\mathcal{L}$ acts nontrivially. In the case of a nontrivial action of $\mathcal{L}$ the irreducible spaces have the form

$$W \oplus \cdots \oplus W,$$

where W is an irreducible representation of $\mathcal{L}$. Here we use the assumption that $\mathcal{G}$ acts transitively on the N cells. The irreducible representation on the kernel is absolutely irreducible precisely when the representation of $\mathcal{L}$ on W is absolutely irreducible, which we henceforth assume.

Recall that the equivariant branching lemma [12] guarantees that generically there exists a branch of bifurcating equilibria for each isotropy subgroup with a one-dimensional fixed-point subspace. We call an isotropy subgroup *axial* if it has a one-dimensional fixed-point subspace; we also call the corresponding bifurcating solutions axial. It turns out that axial subgroups of wreath product symmetry groups are relatively easy to describe.

Definition 5.1 *A subset $J \subset \{1, \ldots, N\}$ is a* block *if there exists a subgroup $Q \subset \mathcal{G}$ which leaves J invariant and acts transitively on J. Let Q_J be the largest subgroup of $\mathcal{G}$ that leaves J invariant.*

We now show how to form axial subgroups of $\mathcal{L} \wr \mathcal{G}$ from an axial subgroup A of $\mathcal{L}$ acting on W and a block J. Let

$$\Sigma(A, J) = (H_1, \ldots, H_N) \dot{+} Q_J,$$

where $H_j = A$ if $j \in J$ and $H_j = \mathcal{L}$ otherwise.

Theorem 5.2 *The subgroup $\Sigma \subset \mathcal{L} \wr \mathcal{G}$ is axial if and only if Σ is conjugate to $\Sigma(A, J)$ for some axial subgroup $A \subset \mathcal{L}$ and some block J.*

Note that solutions $X = (X_1, \ldots, X_N)$ corresponding to the subgroup $\Sigma(A, J)$ have the property that $X_j = 0$ for all $j \notin J$ and $X_j \neq 0$ for all $j \in J$. Thus the cells j are quiescent when $j \notin J$ and active when $j \in J$. We note that similar results hold for Hopf bifurcation in the presence of wreath product symmetry.

6 Direct Product Bifurcations

Steady-state Bifurcation.

As in the previous section we begin our discussion with the irreducible representations of $\mathcal{L} \times \mathcal{G}$. Over $\mathbf{C}$ the irreducible representations of the direct product are just tensor products of irreducible representations of $\mathcal{L}$ and $\mathcal{G}$. This is not always true over $\mathbf{R}$ — but it is often true. (The precise description depends upon the commuting linear maps of irreducible representations, see [8].) In this note we consider only absolutely irreducible representations that are tensor products of absolutely irreducible representations. So assume that $\mathcal{L} \times \mathcal{G}$ acts absolutely irreducibly on the real tensor product $U \otimes V$. The following theorem identifies a class of axial solutions of systems with direct product symmetry.

Theorem 6.1 *Let A be axial for $\mathcal{L}$ acting on U and let B be axial for $\mathcal{G}$ acting on V. Then $A \times B$ is either axial or a subgroup of index two in an axial subgroup.*

Hopf Bifurcations.

Similar ideas apply to Hopf bifurcation. The equivariant Hopf bifurcation theorem [12] allows us to find branches of time periodic solutions for each isotropy subgroup $\Sigma \subset \Gamma \times \mathbf{S}^1$ which has a two-dimensional fixed-point subspace (in an appropriate representation). We say that Σ is $\mathbf{C}$-*axial* if its fixed-point subspace is two-dimensional.

In general, isotropy subgroups $\Sigma \subset \Gamma \times \mathbf{S}^1$ have the form of a *twisted* subgroup, that is, there is a subgroup $A \subset \Gamma$ and a homomorphism $\varphi : A \to \mathbf{S}^1$ such that $\Sigma = A^\varphi$ where

$$A^\varphi = \{(a, \varphi(a) \in \Gamma \times \mathbf{S}^1 : a \in A\}.$$

In order to introduce the $\mathbf{C}$-axial subgroups of $\Gamma = \mathcal{L} \times \mathcal{G}$ we need one additional definition. There is a natural way to take the product of two twisted subgroups. Recall that the twisting reflects the fact that the general symmetry of a periodic solution is a mixture of a space symmetry a with a phase shift $\varphi(a)$. When taking the product of two twisted groups one must add the phase shifts. More precisely, let A^φ and B^ψ be twisted subgroups. Then define

$$A^\varphi \dot{\times} B^\psi = \{(a, b, \varphi(a) + \psi(b)) \in \mathcal{L} \times \mathcal{G} \times \mathbf{S}^1\}.$$

The following theorem shows that there are many periodic solutions whose isotropy subgroups are products.

Theorem 6.2 *If $A^\varphi \subset \mathcal{L} \times \mathbf{S}^1$ and $B^\psi \subset \mathcal{G} \times \mathbf{S}^1$ are $\mathbf{C}$-axial in the representations on U and V, then $A^\varphi \dot{\times} B^\psi$ is $\mathbf{C}$-axial in $\mathcal{L} \times \mathcal{G} \times \mathbf{S}^1$ in the representation on $U \otimes V$.*

For example, it is known that there are two $\mathbf{C}$-axial subgroups of $\mathbf{O(2)}$ acting on $U = \mathbf{C} \otimes \mathbf{C}$ corresponding to rotating waves and standing waves (see [12]). It is also known that there are three $\mathbf{C}$-axial subgroups of $\mathbf{D}_3$ acting on $V = U$ corresponding to a discrete rotating wave (ponies on a merry-go-round) and two standing waves (see [12]). Thus in the action of $\mathcal{L} \times \mathcal{G} = \mathbf{O(2)} \times \mathbf{D_3}$ on $U \otimes V$, there are at least six $\mathbf{C}$-axial subgroups. This Hopf bifurcation problem is considered by Wegelin [21] when analyzing a system of three coupled lasers and these periodic solutions are there found by explicit computation. Wegelin also finds one additional $\mathbf{C}$-axial subgroup in this bifurcation. Similarly, in the corresponding Hopf bifurcation for $\mathbf{D}_3 \times \mathbf{D}_3$ symmetry Theorem 6.2 determines nine $\mathbf{C}$-axial product subgroups. Wegelin *et al.* [6, 21] find these solutions in their study of neural nets with *macro* and *micro* symmetry. They also find two additional $\mathbf{C}$-axial subgroups in this representation. It is possible to use representation theoretic ideas to compute the additional (non-product) $\mathbf{C}$-axial subgroups. The details may be found in [8].

Acknowledgment: We thank Gerhard Dangelmayr and Michael Wegelin for helpful discussions concerning the nature of the symmetries in coupled cells. This research was supported in part by NSF Grant DMS-9101836 (MG), the Texas Advanced Research Program (003652037) (MG), a grant from the Science and Engineering Research Council of the UK (INS), and a European Community Laboratory Twinning grant (INS). The research of INS was carried out under the auspices of the European Bifurcation Theory Group. All three authors were supported by the Fields Institute for Research in the Mathematical Sciences, and are grateful for the hospitality extended to them.

References

[1] J.C. Alexander. Patterns at primary Hopf bifurcations of a plexus of identical oscillators, *SIAM J. Appl. Math.* **46**(2) (1986) 199-221.

[2] J.C. Alexander and B. Fiedler. Global decoupling of coupled symmetric oscillators. In: *Differential Equations* (C.M. Dafermos, G. Ladas and G. Papanicolaou, eds.), Lecture Notes in Pure and Applied Mathematics **118**, Marcel Dekker, Inc. New York, 1989, 7-16.

[3] D.G. Aronson, M. Golubitsky and M. Krupa. Large arrays of Josephson junctions and iterates of maps with S_n symmetry. *Nonlinearity* **4** (1991) 861-902.

[4] F.H. Busse and Heikes. Convection in a rotating layer: a simple case of turbulence. *Science* **208** (1980) 173-175.

[5] E.Buzano, G.Geymonat, and T.Poston. Post-buckling behavior of a nonlinearly hyperelastic thin rod with cross section invariant under the dihedral group $\mathbf{D}_n$, *Arch. Rational Mech. Anal.* **89** (1985) 307-388.

[6] G. Dangelmayr, W. Güttinger and M. Wegelin. Hopf bifurcation with $\mathbf{D}_3 \times \mathbf{D}_3$-symmetry, *ZAMP* **44** (1993) 595-638.

[7] G. Dangelmayr, W. Güttinger, J. Oppenländer, J. Tomes, and M. Wegelin. Coupled neural oscillators with $\mathbf{D}_3 \times \mathbf{D}_3$-symmetry. Preprint, 1992.

[8] B. Dionne, M. Golubitsky and I.N. Stewart. Arrays of oscillators with internal and global symmetries. In preparation.

[9] M. Field and R. Richardson. Symmetry breaking and the maximal isotropy subgroup conjecture for reflection groups. *Arch. Rational Mech. & Anal.* **105**, No. 1 (1989) 61-94.

[10] M.J. Field and R.W. Richardson. Symmetry breaking and branching patterns in equivariant bifurcation theory II, *Arch. Rational Mech. Anal.* **120** (1992) 147-190.

[11] M.J. Field and J.W. Swift. Stationary bifurcation to limit cycles and heteroclinic cycles, *Nonlinearity* **4** (1992) 1001-1043.

[12] M. Golubitsky, I.N. Stewart and D.G. Schaeffer. *Singularities and Groups in Bifurcation Theory* vol.II, Appl. Math. Sci. **69**, Springer-Verlag, New York 1988.

[13] M.G.M. Gomes and I.N.Stewart, Steady PDEs on generalized rectangles: a change of genericity in mode interactions, *Nonlinearity.* To appear.

[14] M.G.M. Gomes and I.N.Stewart, Hopf bifurcations on generalized rectangles with Neumann boundary conditions, in *Dynamics, Bifurcations, Symmetry*, Cargese 1993. To appear.

[15] J. Guckenheimer and P. Holmes. Structurally stable heteroclinic cycles, *Math. Proc. Camb. Phil. Soc.* **103** (1988) 189-192.

[16] J.Guckenheimer and P.Worfolk, Instant chaos, *Nonlinearity* **5** (1992) 1211-1222.

[17] P. Hadley, M.R. Beasley and K. Wiesenfeld. Phase locking of Josephson-junction series arrays, *Phys. Rev. B* **38** (1988) 8712-8719.

[18] J. Oppenländer. *Zur Dynamik hierarchischer Oszillatorennetze*. Diplomarbeit, Universität Tübingen, Institut für Informationsverarbeitung, Fakultät für Physik, 1992.

[19] D.J.S.Robinson, *Finiteness Conditions and Generalized Soluble Groups* vol.2, *Ergebnisse der Math.* **63**, Springer-Verlag, New York 1972.

[20] W.R.Scott, *Group Theory.* Prentice-Hall., Englewood Cliffs NJ 1964.

[21] M. Wegelin. *Nichtlineare Dynamik raumzeitlicher Muster in hierarchischen Systemen.* Dissertation, Universität Tübingen, Fakultät für Physik, Institut für Informationsverarbeitung, 1993.