SIAM J. Applied Dynamical Systems
Vol. 5, No. 2, pp. 205–251

Nilpotent Hopf Bifurcations in Coupled Cell Systems*

Toby Elmhirst† and Martin Golubitsky‡

Abstract. Network architecture can lead to robust synchrony in coupled systems and, surprisingly, to codimension one bifurcations from synchronous equilibria at which the associated Jacobian is nilpotent. We prove three theorems concerning nilpotent Hopf bifurcations from synchronous equilibria to periodic solutions, where the critical eigenvalues have algebraic multiplicity two and geometric multiplicity one, and discuss these results in the context of three different networks in which the bifurcations occur generically. Phenomena stemming from these bifurcations include multiple periodic solutions, solutions that grow at a rate faster than the standard $\lambda^{\frac{1}{2}}$, and solutions that grow slower than the standard $\lambda^{\frac{1}{2}}$. These different bifurcations depend on the network architecture and, in particular, on the flow-invariant subspaces that are forced to exist by the architecture.

Key words. Hopf bifurcation, coupled cells, nonsemisimple normal form

AMS subject classifications. 34C23, 34C25, 37G05

DOI. 10.1137/050635559

1. Introduction.

Stewart, Golubitsky, Pivato, and Török (see [9, 5]) formalized the concept of a *coupled cell network*, where a *cell* is a system of ordinary differential equations (ODEs) and a *coupled cell system* consists of cells whose equations are coupled. These researchers defined the architecture of coupled cell networks and developed a theory that shows how network architecture leads to synchrony. The architecture of a coupled cell network is a graph that indicates which cells have the same phase space, which cells are coupled to which, and which couplings are the same. Coupled cell systems with a given architecture are called *admissible*. In this theory, local network symmetries (which form a groupoid; see [9] for details) generalize symmetry as a way of organizing network dynamics, and synchrony-breaking bifurcations replace symmetry-breaking bifurcations as a basic way in which transitions to complicated dynamics occur.

This paper is concerned with *homogeneous networks*. These are networks in which there is only one type of cell and one type of coupling. In particular, the differential equations defining the time evolution of each cell in any admissible system are identical. Thus these networks have the property that the diagonal subspace Δ, formed by setting the coordinates in all cells equal, is flow-invariant for all admissible coupled cell systems. It is therefore expected that branches of synchronous equilibria can exist in Δ and that synchrony-breaking bifurcations from these equilibria (bifurcations in which critical eigenvectors of the Jacobian J at the equilibrium are not in Δ) can occur naturally as one parameter in the differential equations is varied.

*Received by the editors August 2, 2005; accepted for publication (in revised form) by T. Kaper January 24, 2006; published electronically April 21, 2006. This work was supported in part by NSF grant DMS-0244529.
http://www.siam.org/journals/siads/5-2/63555.html
†Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Z2 (toby@math.uh.edu).
‡Department of Mathematics, University of Houston, Houston, TX 77204-3008 (mg@math.uh.edu).

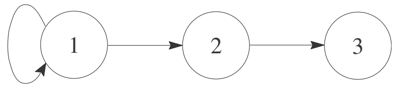

Figure 1. *The three-cell feed-forward chain.*

Golubitsky, Nicol, and Stewart [2] observed that a certain three-cell feed-forward network (see Figure 1) has codimension one synchrony-breaking bifurcations where J restricted to the center subspace is nonsemisimple. The corresponding Hopf bifurcation, which we call a *nilpotent Hopf bifurcation*, leads to periodic solutions whose amplitudes grow with the surprising growth rate of $1/6$, rather than the expected growth rate from Hopf bifurcation of $1/2$. Leite [6] and Leite and Golubitsky [7] showed that there are 34 different types of homogeneous three-cell networks where the number of inputs to each cell is either one or two, and that several of these, in addition to the feed-forward network, lead to nilpotent bifurcations.

In this paper we develop an approach to nilpotent Hopf bifurcation theory which enables us to complete the work in [2] by showing that the $1/6$ power growth rate is generic in the codimension one nilpotent Hopf bifurcations of the feed-forward network, and to classify the periodic solutions that can emanate from codimension one nilpotent Hopf bifurcations in certain other homogeneous cell networks.

Nilpotent (or 1:1 resonant nonsemisimple) Hopf bifurcations have been considered previously in a generic setting in [10, 1, 8]. Without the coupled cell framework, such bifurcations occur in codimension three. However, the structure imposed on admissible vector fields at the linear level by certain network architectures implies that nilpotent Hopf bifurcations can occur in these systems at codimension one. Moreover, these same architectures can also put restrictions on the higher order terms of admissible vector fields, which force surprising branching of the solutions.

When investigating structured systems, one fundamental question is "How does the architecture of the system affect the dynamics," and already we see unexpected, complex behavior in simple-looking systems. In addition to aiding our understanding of natural systems, we also expect applications of a more synthetic nature. In particular, we believe that the "amplification" seen in the $1/6$ growth rate in the network of Figure 1 could have interesting engineering consequences.

We begin with a brief review of ordinary Hopf bifurcation and a summary of our main results.

The standard Hopf theorems. Hopf bifurcation occurs at an equilibrium x_0 and at a parameter value λ_0 of

$$\dot{x} = F(x, \lambda), \qquad x \in \mathbf{R}^n, \lambda \in \mathbf{R}, \tag{1.1}$$

when the linearization of $(\mathrm{d}F)_{x_0,\lambda_0}$ has a pair of purely imaginary eigenvalues. Generically, the critical eigenvalues are simple, and no other eigenvalues lie on the imaginary axis. Under these assumptions we may assume, after a change of coordinates and a rescaling of time, that $x_0 = 0$, $\lambda_0 = 0$, the critical eigenvalues of $(\mathrm{d}F)_{0,0}$ are $\pm i$, and locally $F(0, \lambda) = 0$.

Let $\sigma(\lambda) + i\omega(\lambda)$, where $\sigma(0) = 0$ and $\omega(0) = 1$, be the eigenvalue extension in $(\mathrm{d}F)_{0,\lambda}$ of

i. Assume that the *eigenvalue crossing condition* holds, that is,

$$\sigma'(0) \neq 0. \tag{1.2}$$

The two standard Hopf bifurcation theorems [4] then state the following:

1. There is a unique branch of small amplitude periodic solutions to (1.1) with period near 2π.
2. Generically, the amplitude of the periodic solutions on this branch grows at a rate of order $\lambda^{1/2}$; that is, the bifurcation is of pitchfork type.

The main results. We present three theorems concerning nilpotent Hopf bifurcations in coupled cell systems. Specifically, in this paper we shall use the term *nilpotent Hopf bifurcation* to indicate that there are critical eigenvalues $\pm\omega i$ of $(\mathrm{d}F)_{0,0}$ at the equilibrium $(0, 0)$, where $\omega > 0$, that are each double, but with only one (complex conjugate) pair of corresponding eigenvectors. Typically we will also rescale time so that $\omega = 1$. We note that network architectures can easily be found that lead to codimension one bifurcations in admissible vector fields in which the critical eigenvalues have algebraic multiplicity greater than two and geometric multiplicity one. For example, the n-cell feed-forward chain, obtained by attaching additional cells to the end of the network in Figure 1, has eigenvalues of algebraic multiplicity $n - 1$ and geometric multiplicity one.

We show below that nilpotent Hopf bifurcations can occur generically in codimension one bifurcations from a synchronous equilibrium in coupled cell networks. This point was noted previously in [2, 6]. Our focus here is on the nonlinear theory, in which we show that network architecture can lead generically to multiple periodic solutions whose amplitude growth rate is greater than, equal to, or less than $1/2$.

This variety in nilpotent Hopf bifurcations is due to the type of nonlinear degeneracies forced by different network architectures on their admissible vector fields. In our approach we study classes of networks whose architectures force, in codimension one, a particular type of nonlinear degeneracy in the Liapunov–Schmidt reduced equation. Given this assumption on architecture, we classify the branches of periodic solutions that occur generically in codimension one bifurcations. Each network architecture can, in principle, lead to different codimension one bifurcations, just as each symmetry group can lead to different equivariant bifurcations.

We illustrate our results by discussing the following three specific networks:

(a) the three-cell feed-forward network in Figure 1, whose nilpotent Hopf bifurcation generically leads to two branches of periodic solutions with amplitude growth at rates of $1/6$ and $1/2$. The existence of these solutions in a restricted class of coupled cell systems is shown in [2].

(b) the three-cell network in Figure 2, whose nilpotent Hopf bifurcation generically leads either to two or four branches of periodic solutions with amplitude growth at the standard rate of $\lambda^{1/2}$.

(c) the five-cell network in Figure 3, whose nilpotent Hopf bifurcation generically leads to two branches of periodic solutions with amplitude growth at rate λ.

Coupled cell networks and nilpotent normal forms. A general theory for the differential equations associated with coupled cell networks is outlined in [9, 5]. In particular, an algo-

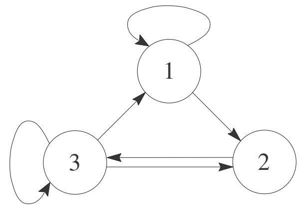

Figure 2. *A three-cell network with nilpotent linear part.*

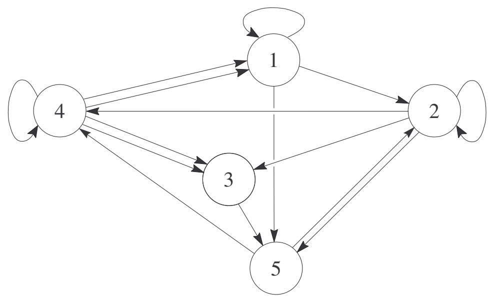

Figure 3. *A five-cell network with nilpotent linear part.*

rithmic way of identifying a class of systems of differential equations with a directed graph is given. The identification is reasonably intuitive, so we do not describe the general setup here. Rather, we just list the results for the three networks of Figures 1–3.

(a) Following [2], the coupled cell systems corresponding to the three-cell feed-forward network in Figure 1 have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, x_1), \\ \dot{x}_2 &= f(x_2, x_1), \\ \dot{x}_3 &= f(x_3, x_2), \end{aligned} \tag{1.3}$$

where $x_1, x_2, x_3 \in \mathbf{R}^k$ and $f : \mathbf{R}^k \times \mathbf{R}^k \to \mathbf{R}^k$ is arbitrary. Note that the synchrony subspace $x_1 = x_2 = x_3$ is flow-invariant for such systems, and the existence of a synchronous equilibrium (satisfying $f(a, a) = 0$) is to be expected. Without loss of generality we may assume that the synchronous equilibrium is at the origin. The Jacobian of (1.3) at the origin has the form

$$J = \begin{pmatrix} A+B & 0 & 0 \\ B & A & 0 \\ 0 & B & A \end{pmatrix},$$

where $A = f_1(\mathbf{0})$ is the linearized internal dynamics and $B = f_2(\mathbf{0})$ is the linearized coupling. The $3k$ eigenvalues and eigenvectors of J are

Eigenvector	Eigenvalues	Algebraic multiplicity	Geometric multiplicity
$(0, 0, u)^t$	A	2	1
$(v, v, v)^t$	$A + B$	1	1

where u is an eigenvector of A and v is an eigenvector of $A+B$. It follows that when $k \geq 2$, (1.3) can have a codimension one nilpotent Hopf bifurcation.

(b) The coupled cell systems corresponding to the three-cell network in Figure 2 have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_1, x_3}), \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_3}), \\ \dot{x}_3 &= f(x_3, \overline{x_2, x_3}), \end{aligned} \tag{1.4}$$

where $x_1, x_2, x_3 \in \mathbf{R}^k$ and the overbar indicates that $f : \mathbf{R}^k \times \mathbf{R}^{2k} \to \mathbf{R}^k$ satisfies $f(a,b,c) = f(a,c,b)$. The synchrony subspace $x_1 = x_2 = x_3$ is still flow-invariant for such systems, and the existence of a synchronous equilibrium, which without loss of generality we may assume is at the origin, is to be expected. The Jacobian of (1.4) at the origin has the form

$$J = \begin{pmatrix} A+B & 0 & B \\ B & A & B \\ 0 & B & A+B \end{pmatrix}, \tag{1.5}$$

where $A = f_1(\mathbf{0})$ is the linearized internal dynamics and $B = f_2(\mathbf{0}) = f_3(\mathbf{0})$ is the linearized coupling. The $3k$ eigenvalues and eigenvectors of J are

Eigenvector	Eigenvalues	Algebraic multiplicity	Geometric multiplicity
$(u,u,-u)^t$	A	2	1
$(v,v,v)^t$	$A+2B$	1	1

where u is an eigenvector of A and v is an eigenvector of $A+2B$. Thus, if $k \geq 2$, codimension one nilpotent Hopf bifurcations occur in this network as well.

(c) The coupled cell systems corresponding to the five-cell network in Figure 3 have the form

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_1, x_4, x_4}), \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_2, x_5}), \\ \dot{x}_3 &= f(x_3, \overline{x_2, x_4, x_4}), \\ \dot{x}_4 &= f(x_4, \overline{x_2, x_4, x_5}), \\ \dot{x}_5 &= f(x_5, \overline{x_1, x_2, x_3}), \end{aligned} \tag{1.6}$$

where $x_j \in \mathbf{R}^k$, $f : \mathbf{R}^k \times \mathbf{R}^{3k} \to \mathbf{R}^k$, and the overbar indicates that $f(a,b,c,d)$ is invariant under permutation of b,c,d. The synchrony subspace $x_1 = x_2 = x_3 = x_4 = x_5$ is flow-invariant, and a synchronous equilibrium, which again we may assume is at the origin, is therefore to be expected. The Jacobian of (1.6) at the origin is

$$J = \begin{pmatrix} A+B & 0 & 0 & 2B & 0 \\ B & A+B & 0 & 0 & B \\ 0 & B & A & 2B & 0 \\ 0 & B & 0 & A+B & B \\ B & B & B & 0 & A \end{pmatrix},$$

where $A = f_1(\mathbf{0})$ is the linearized internal dynamics and $B = f_2(\mathbf{0}) = f_3(\mathbf{0}) = f_4(\mathbf{0})$ is the linearized coupling. The $5k$ eigenvalues of J are

Eigenvalues	Algebraic multiplicity	Geometric multiplicity
$A \pm iB$	2	1
$A + 3B$	1	1

Assuming that (1.6) depends on a parameter λ, we can arrange for a codimension one nilpotent Hopf bifurcation at $\lambda = 0$ by taking $k = 1$, $A(\lambda) = \lambda$, and $B(\lambda) \equiv -1$.

Review of the Liapunov–Schmidt reduction proof of Hopf bifurcation. We use the standard procedure of Liapunov–Schmidt reduction for finding periodic solutions through Hopf bifurcation (see [3]), but nilpotence dramatically changes this analysis.

Since periodic solutions to (1.1) will not in general have period 2π, rescale time in the usual way by letting $s = (1+\tau)t$ so that (1.1) becomes

$$(1+\tau)\frac{\mathrm{d}z}{\mathrm{d}s} = F(z, \lambda). \tag{1.7}$$

Fixing the period allows us to define the operator $\Phi : \mathcal{C}^1_{2\pi} \times \mathbf{R} \times \mathbf{R} \longrightarrow \mathcal{C}_{2\pi}$ by

$$\Phi(x, \lambda, \tau) = (1+\tau)\frac{\mathrm{d}x}{\mathrm{d}s} - F(x, \lambda), \tag{1.8}$$

where $\mathcal{C}_{2\pi}$ and $\mathcal{C}^1_{2\pi}$ are respectively the Banach spaces of continuous and continuously differentiable 2π-periodic functions $x : \mathbf{S}^1 \longrightarrow \mathbf{R}^n$. Note that

(a) the solutions to $\Phi(x, \lambda, \tau) = 0$ correspond to near 2π-periodic solutions of (1.1),
(b) $\Phi(0, \lambda, \tau) \equiv 0$ since $F(0, \lambda) \equiv 0$,
(c) Φ is $\mathbf{S}^1$-equivariant, where $\theta \in \mathbf{S}^1$ acts on $u \in \mathcal{C}_{2\pi}$ by

$$(\theta \cdot u)(s) = u(s - \theta).$$

In standard Hopf bifurcation, the kernel and cokernel of the Frechet derivative $(\mathrm{d}\Phi)_{\mathbf{0}}$ are two-dimensional, and these spaces may be identified with $\mathbf{C}$. Liapunov–Schmidt reduction implies the existence of a mapping $\phi : \mathbf{C}\times\mathbf{R}\times\mathbf{R} \to \mathbf{C}$, whose zeros near the origin parameterize the small amplitude periodic solutions of $\Phi = 0$. Moreover, this reduction can be performed to preserve symmetry; that is, we can assume that

$$\phi(e^{i\theta}z, \lambda, \tau) = e^{i\theta}\phi(z, \lambda, \tau). \tag{1.9}$$

It follows that

$$\phi(z, \lambda, \tau) = p(|z|^2, \lambda, \tau)z + q(|z|^2, \lambda, \tau)iz, \tag{1.10}$$

where p, q are real-valued smooth functions satisfying $p(\mathbf{0}) = q(\mathbf{0}) = 0$. Using (1.9), we need only look for solutions where $z = x \in \mathbf{R}$. Hence solutions to $\phi = 0$ are of two types: $x = 0$ (the trivial equilibrium) and solutions to the system $p = q = 0$ (the desired small amplitude periodic solutions).

In standard Hopf bifurcation, a calculation shows that $q_\tau(\mathbf{0}) = -1$. Hence, the equation $q = 0$ can be solved by the implicit function theorem for $\tau = \tau(x^2, \lambda)$, where $\tau(\mathbf{0}) = 0$, and small amplitude periodic solutions to (1.1) are found by solving

$$r(x^2, \lambda) \equiv p(x^2, \lambda, \tau(x^2, \lambda)) = 0. \tag{1.11}$$

Another (more complicated) calculation shows that

$$r_\lambda(\mathbf{0}) = \sigma'(0).$$

It follows from the eigenvalue crossing condition that $r = 0$ can be solved by another application of the implicit function theorem for $\lambda = \lambda(x^2)$, where $\lambda(0) = 0$, and the first Hopf theorem is proved. Setting $u = x^2$, the second Hopf theorem (the square root growth of amplitude) is proved by making the genericity assumption $r_u(\mathbf{0}) \neq 0$. The calculation of $r_u(\mathbf{0})$ in terms of (1.1) is the most difficult of the calculations.

Statements of the main theorems. In a nilpotent Hopf bifurcation the kernel and cokernel of the Frechet derivative $(\mathrm{d}\Phi)_\mathbf{0}$ are still two-dimensional—just like ordinary Hopf bifurcation. It follows that the Liapunov–Schmidt reduced equation $\phi = 0$ has the same $\mathbf{S}^1$-equivariance as in standard Hopf bifurcation and hence has the form of (1.10).

We show in section 2.2 that a nilpotent Hopf bifurcation leads to the following result.

Proposition 1.1. *Let p and q be as in* (1.10). *Then*

$$p_\tau(\mathbf{0}) = 0, \qquad q_\tau(\mathbf{0}) = 0, \tag{1.12}$$

$$p_\lambda(\mathbf{0}) = 0, \qquad q_\lambda(\mathbf{0}) = 0. \tag{1.13}$$

It follows from Proposition 1.1 that we cannot employ the implicit function theorem in the same way as it is used in the standard Hopf theorem; higher derivatives are necessary to understand the bifurcation. Note that Proposition 1.1 does not require any assumptions about network architecture. Indeed, it follows simply from the fact that $(\mathrm{d}F)_\mathbf{0}$ is nilpotent.

All is not lost, however. In section 2.3 we assume that F is a homogeneous coupled cell system with nilpotent linearization, and we obtain the following generalization of the eigenvalue crossing condition (see [10] for a version of this proposition for generic vector fields).

Proposition 1.2. *If F is a homogeneous coupled cell system, then*

$$p_{\tau\tau}(\mathbf{0}) = -2, \qquad q_{\tau\tau}(\mathbf{0}) = 0, \tag{1.14}$$

$$p_{\lambda\lambda}(\mathbf{0}) = 2(\sigma'(0)^2 - \omega'(0)^2), \qquad q_{\lambda\lambda}(\mathbf{0}) = -4\sigma'(0)\omega'(0), \tag{1.15}$$

$$p_{\lambda\tau}(\mathbf{0}) = 2\omega'(0), \qquad q_{\lambda\tau}(\mathbf{0}) = 2\sigma'(0). \tag{1.16}$$

These explicit calculations enable us to proceed with the calculation of solutions to $p = q = 0$. The three theorems mentioned previously can now be stated (in reverse order) and the proofs sketched.

We begin by stating the following theorem, which is proved in section 3.

Theorem 1.3. *Assume*

$$q_u(\mathbf{0}) \neq 0 \tag{1.17}$$

and the eigenvalue crossing condition (1.2)*. Then there exist two branches of near 2π-periodic solutions, each one growing at order λ. Moreover, one branch is subcritical, and the other is supercritical.*

This theorem is proved by a straightforward application of the implicit function theorem. We also show that the network in Figure 3 generically satisfies (1.17).

Observe that Theorem 1.3 refers to the class of networks whose architecture does not force restrictions on the nonlinear terms of the reduced equation. The linear structure alone forces a different branching pattern than for generic Hopf bifurcation. It does not follow, however, that this situation is somehow the "generic" case for coupled cell networks. Indeed, there are network architectures, such as the feed-forward network, that do force $p_u = q_u = 0$.

In section 4 we prove the next theorem.

Theorem 1.4. *Assume*

$$p_u(\mathbf{0}) = q_u(\mathbf{0}) = 0, \tag{1.18}$$

$$\begin{aligned} &p_{uu}(\mathbf{0}) \neq -\tfrac{1}{2}p_{u\tau}(\mathbf{0})^2, && p_{u\lambda}(\mathbf{0}) \neq -\omega'(0)p_{u\tau}(\mathbf{0}),\\ &q_{uu}(\mathbf{0}) \neq -p_{u\tau}(\mathbf{0})q_{u\tau}(\mathbf{0}), && q_{u\tau}(\mathbf{0}) \neq 0,\\ &q_{u\lambda}(\mathbf{0}) \neq -\big(\sigma'(0)p_{u\tau}(\mathbf{0}) + \omega'(0)q_{u\tau}(\mathbf{0})\big), \end{aligned} \tag{1.19}$$

$$p_{uu}(\mathbf{0}) \neq \frac{q_{uu}(\mathbf{0})}{q_{u\tau}(\mathbf{0})}\left(p_{u\tau}(\mathbf{0}) + \frac{q_{uu}(\mathbf{0})}{2q_{u\tau}(\mathbf{0})}\right), \tag{1.20}$$

and the eigenvalue crossing condition (1.2)*. Then there exist either two or four branches of near 2π-periodic solutions to* (1.1)*, with the number of branches depending on F, and each branch grows as $\lambda^{\frac{1}{2}}$.*

The constraint (1.18) means that the implicit function theorem cannot be used to solve $p = q = 0$, so the proof of Theorem 1.4 takes a form different from that for Theorem 1.3. We show that there are either two or four branches of solutions to $p = q = 0$ (depending on the uu, $u\tau$, and $u\lambda$ derivatives of p and q). Each of these solution branches is defined by $\lambda = O(u)$, and the growth rate of the amplitude is the standard $1/2$ power. The proof of this theorem, given in section 4, is based on showing that generically solution branches to $p = q = 0$ are determined at quadratic order in u, λ, and τ. Then in section 4.3 we show that the network in Figure 2 generically satisfies (1.18) and (1.19).

In section 5 we prove a third result.

Theorem 1.5. *Generic nilpotent Hopf bifurcation in the feed-forward chain yields two branches of near 2π-periodic solutions, the amplitude of one growing as $\lambda^{\frac{1}{2}}$ and the amplitude of the other growing as $\lambda^{\frac{1}{6}}$.*

We show, using Poincaré–Birkhoff normal form techniques, that the feed-forward network has two branches of solutions: one with growth rate $1/2$ and the other with growth rate $1/6$. This step builds on the results in [2]. Then we show that the existence of these branches of solutions implies that

$$p_u(\mathbf{0}) = p_{uu}(\mathbf{0}) = p_{uuu}(\mathbf{0}) = q_u(\mathbf{0}) = q_{uu}(\mathbf{0}) = q_{uuu}(\mathbf{0}) = 0. \tag{1.21}$$

Furthermore, using the Liapunov–Schmidt reduction, we show that

$$p_{uuuu}(\mathbf{0}) \neq 0 \quad \text{and} \quad q_{uuuu}(\mathbf{0}) \neq 0. \tag{1.22}$$

It follows that there are no other branches of solutions to $p = q = 0$, or else these fourth derivatives would also vanish.

In all three theorems and their corresponding examples, flow-invariant subspaces play a vital role since they force the derivatives of p and q to vanish in (1.18) and (1.21). The five-cell network of Figure 3 has no nontrivial invariant subspaces, and thus the derivatives are not constrained. On the other hand, the feed-forward chain in Figure 1 and the three-cell network of Figure 2 both possess a synchrony subspace $S = \{(u, u, v) : u, v \in \mathbf{R}^k\}$. Furthermore, at nilpotent Hopf bifurcations both networks satisfy the following:

(1.23) The network has a flow-invariant subspace, S, that contains the critical eigenspace but does not contain the corresponding generalized eigenspace.

An immediate consequence of (1.23) is the following.

Proposition 1.6. *Suppose that a coupled cell system satisfies* (1.23)*. Then at a nilpotent Hopf bifurcation there exists a branch of solutions that grows at* $O(\lambda^{\frac{1}{2}})$*, and* (1.18) *holds.*

Proof. Note that (1.23) implies that we can restrict the vector field to S, and because S does not contain the generalized eigenvectors we can apply the standard Hopf theorem to deduce that there is a standard Hopf bifurcation in S. Thus there is at least one branch that grows as $O(\lambda^{\frac{1}{2}})$. This branch can be parameterized by u, so that

$$p(u, \lambda(u), \tau(u)) \equiv 0 \quad \text{and} \quad q(u, \lambda(u), \tau(u)) \equiv 0.$$

Differentiating both expressions with respect to u and evaluating at the origin yields

$$p_u(\mathbf{0}) + p_\lambda(\mathbf{0})\lambda_u(0) + p_\tau(\mathbf{0})\tau_u(0) = 0 \quad \text{and} \quad q_u(\mathbf{0}) + q_\lambda(\mathbf{0})\lambda_u(0) + q_\tau(\mathbf{0})\tau_u(0) = 0.$$

So by Proposition 1.1, $p_u(\mathbf{0}) = q_u(\mathbf{0}) = 0$. ■

Observe that the feed-forward chain has an additional flow-invariant subspace $\hat{S} = \{(0, 0, v) : v \in \mathbf{R}^k\}$, and it turns out that $\hat{S}$ also satisfies (1.23). However, $\hat{S}$ is not a synchrony subspace since it is not a polydiagonal, and this implies stronger consequences than in Proposition 1.6. It is this that underlies the additional degeneracies in (1.21).

Theorems 1.3, 1.4, and 1.5 are concerned only with the existence of solutions branches, and we do not consider the stability of the solutions or other dynamical features of the systems. We would expect such an analysis to turn up some interesting features.

In section 6 we give three more examples of three-cell networks that can undergo nilpotent Hopf bifurcation in codimension one. One of these falls into the same category as the network in Figure 2, in that it has two or four branches of solutions given by Theorem 1.4. The other two are similar to the feed-forward chain of Figure 1, in that they have two branches of solutions, one growing at $O(\lambda^{\frac{1}{2}})$ and the other at $O(\lambda^{\frac{1}{6}})$.

Finally, we have placed the (lengthy) expressions for the derivatives of p and q into the appendix, so as not to distract from the flow of the calculations.

2. Reduction with nilpotent normal form. In this section we derive information about the λ and τ derivatives of the Liapunov–Schmidt reduced mapping. Throughout we make the standard assumptions that there exists a trivial (synchronous) equilibrium $F(0,\lambda) = 0$, that $J(\lambda) = (\mathrm{d}F)_{0,\lambda}$ has a complex conjugate pair of eigenvalues $\sigma(\lambda) \pm i\omega(\lambda)$ with $\sigma(0) = 0$ and $\omega(0) = 1$, and that the eigenvalue crosses the imaginary axis with nonzero speed, $\sigma'(0) \neq 0$. In addition we assume that the critical eigenvalues of $(\mathrm{d}F)_{0,0}$ have algebraic multiplicity 2 and geometric multiplicity 1. We also assume throughout that F is a coupled cell system, since nilpotent Hopf bifurcations are nongeneric in systems without a coupled cell structure.

In section 2.1 we set up the generalities of the Liapunov–Schmidt reduction for a nilpotent Hopf bifurcation and obtain a reduced bifurcation problem of the form (1.10). Then in section 2.2 we prove Proposition 1.1. Finally, in section 2.3 we make the additional assumption that F is a *homogeneous* coupled cell system and prove Proposition 1.2.

2.1. The Liapunov–Schmidt reduction. We study the Liapunov–Schmidt reduction of (1.8) onto the kernel of the linearization of Φ at the origin. The linearization of Φ at the origin is given by

$$\mathcal{L}u \equiv (\mathrm{d}\Phi)_{\mathbf{0}}u = \frac{\mathrm{d}u}{\mathrm{d}s} - Ju, \tag{2.1}$$

where $J = (\mathrm{d}F)_{\mathbf{0}}$. Then $\mathcal{K} = \ker \mathcal{L}$ consists of the 2π-periodic solutions to the linear system

$$\frac{\mathrm{d}u}{\mathrm{d}s} = Ju. \tag{2.2}$$

The Liapunov–Schmidt reduction requires an invariant splitting,

$$\begin{aligned} \mathcal{C}^1_{2\pi} &= \mathcal{K} \oplus \mathcal{M}, \\ \mathcal{C}_{2\pi} &= \mathcal{N} \oplus \mathcal{R}, \end{aligned}$$

where $\mathcal{R} = \operatorname{range} \mathcal{L}$, such that $\mathcal{L}|_{\mathcal{M}} : \mathcal{M} \to \mathcal{R}$ is invertible. As we now show, we can take $\mathcal{M} = \mathcal{K}^{\perp}$ and $\mathcal{N} = \mathcal{K}^*$, where $\mathcal{K}^*$ is the kernel of

$$\mathcal{L}^*u = -\frac{\mathrm{d}u}{\mathrm{d}s} - J^t u, \tag{2.3}$$

which is the adjoint of $\mathcal{L}$ with respect to the inner product

$$\langle u, v\rangle = \frac{1}{2\pi}\int_0^{2\pi} \overline{v(s)^t} u(s)\, \mathrm{d}s. \tag{2.4}$$

Proposition 2.1. *Assume that J has eigenvalues $\pm i$ with algebraic multiplicity 2 and geometric multiplicity 1 and no other eigenvalues on the imaginary axis. Then*

1. $\dim \mathcal{K} = \dim \mathcal{K}^* = 2$;
2. *there are bases $\{v_1, v_2\}$ for $\mathcal{K}$ and $\{v_1^*, v_2^*\}$ for $\mathcal{K}^*$ such that $\mathcal{K}$ and $\mathcal{K}^*$ can be identified with $\mathbf{R}^2$ so that $\mathbf{S}^1$ acts on both spaces by counterclockwise rotation;*
3. *there are invariant splittings of $C^1_{2\pi}$ and $\mathcal{C}_{2\pi}$:*

$$\begin{aligned} &\text{(a)} \quad \mathcal{C}_{2\pi} = \mathcal{K}^* \oplus \mathcal{R}, \\ &\text{(b)} \quad \mathcal{C}^1_{2\pi} = \mathcal{K} \oplus \mathcal{K}^{\perp}. \end{aligned} \tag{2.5}$$

Proof. We show that $\mathcal{K}$ and $\mathcal{K}^*$ are two-dimensional by constructing bases. Note that if $n > 4$, then J has eigenvalues off the imaginary axis, so solutions not lying in the space spanned by the eigenvectors of $\pm i$ will not be periodic.

Let $c \in \mathbf{C}^n$ be an eigenvector of J with eigenvalue i. Then setting

$$v_1(s) = \mathrm{Re}\{e^{is}c\}, \qquad v_2(s) = \mathrm{Im}\{e^{is}c\} \tag{2.6}$$

forms a basis for $\mathcal{K}$, and in particular, $\dim \mathcal{K} = 2$.

Since J^t has the same eigenvalues as J, and in particular has double eigenvalues $\pm i$ and no other eigenvalues on the imaginary axis, we can construct a basis for $\mathcal{K}^*$ in a similar way. Let $d \in \mathbf{C}^n$ be an eigenvector $J^t d = -id$. Then

$$v_1^*(s) = \mathrm{Re}\{e^{is}d\} \quad \text{and} \quad v_2^*(s) = \mathrm{Im}\{e^{is}d\} \tag{2.7}$$

forms a basis for $\mathcal{K}^*$, and $\dim \mathcal{K}^* = 2$.

We can identify $\mathcal{K}$ and $\mathcal{K}^*$ with $\mathbf{R}^2$ via the mappings

$$(x, y) \mapsto xv_1 + yv_2 \quad \text{and} \quad (x, y) \mapsto xv_1^* + yv_2^*. \tag{2.8}$$

Observe that $\mathbf{S}^1$ acts on $\mathcal{K}$ as

$$\theta \cdot v_1(s) = v_1(s - \theta) = \mathrm{Re}\{e^{-i\theta}e^{is}c\} = \cos\theta\, v_1(s) + \sin\theta\, v_2(s),$$

and similarly

$$\theta \cdot v_2(s) = -\sin\theta\, v_1(s) + \cos\theta\, v_2(s).$$

Therefore, the action of $\mathbf{S}^1$ on $\mathcal{K}$, coordinatized by (2.8), is given by

$$\theta \cdot \begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix}. \tag{2.9}$$

That is, $\theta \in \mathbf{S}^1$ acts by counterclockwise rotation through θ. By the same argument applied to the identification given for $\mathcal{K}^*$ in (2.8), $\mathbf{S}^1$ also acts on $\mathcal{K}^*$ by counterclockwise rotation as in (2.9).

The invariant splittings given in (2.5a,b) follow from the fact that $\mathcal{L}$ is Fredholm of index zero. Specifically, the Fredholm alternative states that

$$\mathcal{R}^\perp = \mathcal{K}^*, \tag{2.10}$$

which gives the decomposition in (2.5a). ■

An important point of departure of this case from the standard Hopf bifurcation is that here we have

$$\mathcal{K}^* \perp \mathcal{K}, \tag{2.11}$$

as is shown in Lemma 2.2 below. This is essentially the reason why the first λ and τ derivatives of p and q vanish in Proposition 1.1. The derivatives in Proposition 1.2 come from the fact that the generalized eigenvectors of J with eigenvalues $\pm i$ are not orthogonal to $\mathcal{K}^*$. Notice also that $\mathcal{K} \perp \mathcal{K}^* = \mathcal{R}^\perp$ implies that $\mathcal{K} \subset \mathcal{R}$.

Let b be a generalized eigenvector of J such that

$$Jb = ib + c, \tag{2.12}$$

and choose b so that

$$b^t \bar{c} = 0.$$

Then define

$$u_1 = \mathrm{Re}\{e^{is}b\} \quad \text{and} \quad u_2 = \mathrm{Im}\{e^{is}b\} \tag{2.13}$$

by analogy with v_j and v_j^*. Note that

$$\mathcal{L}u_j = -v_j.$$

The following lemma summarizes the relations between b, c, and d, and can be contrasted with [3, Chapter VIII, Lemma 2.4].

Lemma 2.2. *Let F be any vector field such that $(\mathrm{d}F)_{0,0}$ is nilpotent, and let b, c, and d be defined as above. Then*

$$c^t \bar{d} = 0 \tag{2.14}$$

and the eigenvector d can be scaled so that

$$b^t \bar{d} = 2. \tag{2.15}$$

Proof. Observe that

$$ib^t\bar{d} = b^t[J^t\bar{d}] = [Jb]^t\bar{d} = (ib + c)^t\bar{d} = ib^t\bar{d} + c^t\bar{d}.$$

Therefore, $c^t\bar{d} = 0$.

Similarly, let v be any eigenvector of J with eigenvalue μ. Then

$$\mu v^t\bar{d} = [Jv]^t\bar{d} = v^t J^t \bar{d} = iv^t\bar{d}. \tag{2.16}$$

Thus every eigenvector of J with eigenvalue different from i is orthogonal to $\bar{d}$. But $c^t\bar{d} = 0$, so $b^t\bar{d}$ must be nonzero or $\bar{d}$ is orthogonal to every eigenvector of J, which is a contradiction. Therefore $b^t\bar{d} \neq 0$, and we can scale d so that $b^t\bar{d} = 2$. ■

Before continuing with the reduction, we give the following useful formulas concerning v_j, v_j^*, and u_j.

Lemma 2.3.

$$\frac{\mathrm{d}v_1}{\mathrm{d}s} = -v_2, \qquad \frac{\mathrm{d}v_1}{\mathrm{d}s} = -v_2, \qquad \frac{\mathrm{d}u_1}{\mathrm{d}s} = -u_2, \qquad \frac{\mathrm{d}u_2}{\mathrm{d}s} = u_1, \tag{2.17}$$

$$\langle v_j^*, v_k \rangle = 0, \qquad \langle v_j^*, u_k \rangle = \delta_{jk}, \tag{2.18}$$

where $\langle \cdot, \cdot \rangle$ *is the inner product defined in* (2.4).

Proof. These formulas are straightforward calculations from the definitions of v_j, v_j^*, and u_j in (2.6), (2.7), and (2.13). We give two examples of these calculations; the others follow very similar lines.

Observe that (2.6) implies

$$\frac{\mathrm{d}v_1}{\mathrm{d}s} = \frac{\mathrm{d}}{\mathrm{d}s}\mathrm{Re}\{e^{is}c\} = \mathrm{Re}\{ie^{is}c\} = -\mathrm{Im}\{e^{is}c\} = -v_2.$$

Similar calculations yield the other derivatives in (2.17).

We verify only that $\langle v_j^*, u_k \rangle = \delta_{jk}$. Since

$$v_1^* = \mathrm{Re}\{e^{is}d\} = \tfrac{1}{2}(e^{is}d + e^{-is}\bar{d}), \qquad v_2^* = \mathrm{Im}\{e^{is}d\} = -\tfrac{i}{2}(e^{is}d - e^{-is}\bar{d}),$$
$$u_1 = \mathrm{Re}\{e^{is}d\} = \tfrac{1}{2}(e^{is}b + e^{-is}\bar{b}), \qquad u_2 = \mathrm{Im}\{e^{is}d\} = -\tfrac{i}{2}(e^{is}b - e^{-is}\bar{b}),$$

we can write

$$v_j^* = \frac{(-i)^{j-1}}{2}(e^{is}d + (-1)^{j-1}e^{-is}\bar{d}) \quad \text{and} \quad u_k = \frac{(-i)^{k-1}}{2}(e^{is}b + (-1)^{k-1}e^{-is}\bar{b}).$$

Then using (2.4) and the fact that $b^t\bar{d} = 2$ by (2.15),

$$\begin{aligned}
\langle v_j^*, u_k \rangle &= \frac{i^{k-1}(-i)^{j-1}}{8\pi}\int_0^{2\pi}(e^{-is}\bar{b} + (-1)^{k-1}e^{is}b)^t(e^{is}d + (-1)^{j-1}e^{-is}\bar{d})\mathrm{d}s \\
&= \frac{i^{k-1}(-i)^{j-1}}{8\pi}\int_0^{2\pi}\big((-1)^{k-1}e^{2is}b^td + (-1)^{j+k}b^t\bar{d} \\
&\qquad\qquad + \bar{b}^td + (-1)^{j-1}e^{-2is}\bar{b}^t\bar{d}^t\big)\mathrm{d}s \\
&= \frac{1}{2}i^{k-1}(-i)^{j-1}(1 + (-1)^{j+k}) \\
&= \delta_{jk}. \qquad \blacksquare
\end{aligned}$$

To continue the reduction, let $E : \mathcal{C}_{2\pi} \to \mathcal{R}$ be the projection with $\ker E = \mathcal{K}^*$, and write $x \in \mathcal{C}_{2\pi}^1$ as $x = v + w$, where $v \in \mathcal{K}$ and $w \in \mathcal{K}^\perp$. Then $\Phi(v + w, \lambda, \tau) = 0$ if and only if

$$\text{(2.19)} \qquad \begin{aligned} &\text{(a)} \quad E\Phi(v + w, \lambda, \tau) = 0, \\ &\text{(b)} \quad (I - E)\Phi(v + w, \lambda, \tau) = 0, \end{aligned}$$

where $I - E$ is the complementary projection of $\mathcal{C}_{2\pi}$ onto $\mathcal{K}^*$ with kernel $\mathcal{R}$. The differential of $E\Phi(v + w, \lambda, \tau)$ with respect to w at the origin is just $\mathcal{L}|_{\mathcal{K}^\perp}$, and $\mathcal{L}|_{\mathcal{K}^\perp} : \mathcal{K}^\perp \to \mathcal{R}$ is invertible because $\mathcal{L}$ is Fredholm of index zero. Thus the implicit function theorem implies that (2.19a) can be solved for $w = W(v, \lambda, \tau)$, where $W : \mathcal{K} \times \mathbf{R} \times \mathbf{R} \to \mathcal{K}^\perp$ is such that $W(\mathbf{0}) = 0$ and

$$\text{(2.20)} \qquad E\Phi(xv_0 + W(xv_0, \lambda, \tau), \lambda, \tau) \equiv 0$$

for any $v_0 \in \mathcal{K}$ and $x \in \mathbf{R}$. Solving (2.19b) is therefore equivalent to solving $\Phi(u, \lambda, \tau) = 0$, and hence, to finding the near 2π-periodic solutions to (1.1).

The following lemma greatly simplifies a number of formulas later on.

Lemma 2.4.

$$W_x(\mathbf{0}) = W_\alpha(\mathbf{0}) = W_{\alpha\beta}(\mathbf{0}) = 0, \tag{2.21}$$

where α and β are placeholders for λ and τ.

Proof. Differentiating (2.20) with respect to x and evaluating all derivatives at the origin yields

$$\mathrm{d}\Phi(v_0 + W_x(\mathbf{0})) = 0$$

and therefore

$$0 = \mathcal{L}v_0 + \mathcal{L}W_x(\mathbf{0}) = \mathcal{L}W_x(\mathbf{0}).$$

However, $W_x(\mathbf{0}) \in \mathcal{K}^\perp$, and thus $W_x(\mathbf{0}) = 0$.

Observe that setting $v = 0$, $w = W(0, \lambda, \tau)$ in (2.19a) yields

$$E\left((1+\tau)\frac{\mathrm{d}}{\mathrm{d}s}W(0,\lambda,\tau) - F(W(0,\lambda,\tau),\lambda)\right) \equiv 0.$$

This is solved for $W(0, \lambda, \tau) \equiv 0$ since $F(0, \lambda) \equiv 0$, and this must be the only solution since the implicit function theorem guarantees that $W(0, \lambda, \tau)$ is the unique solution to (2.19a). This implies that all λ and τ derivatives of W evaluated at the origin vanish. ∎

Define the reduced mapping $\phi : \mathcal{K} \times \mathbf{R} \times \mathbf{R} \to \mathcal{K}^*$ by

$$\phi(v, \lambda, \tau) = (I - E)\Phi(v + W(v, \lambda, \tau), \lambda, \tau). \tag{2.22}$$

Then for $j = 1, 2$

$$\phi_j(x, y, \lambda, \tau) = \left\langle v_j^*, \phi(xv_1 + yv_2, \lambda, \tau)\right\rangle.$$

Since $\mathcal{K}$ and $\mathcal{K}^*$ are invariant subspaces under the action of $\mathbf{S}^1$ as specified in (2.9), it follows from Proposition 3.3 of Chapter VII in [3] that ϕ commutes with this action of $\mathbf{S}^1$ on $\mathcal{K}$ and $\mathcal{K}^*$. Therefore, by Proposition 2.3 of Chapter VIII in [3], the reduced mapping has the form in (1.10), where x and y come from the identification of $\mathcal{K}$ with $\mathbf{C}$ given by (2.8).

Solutions to (1.7) that are 2π-periodic correspond to solutions to $\phi(x, y, \lambda, \tau) = 0$, and by (1.10) these solutions are given by $x = y = 0$ or $p = q = 0$. The former correspond to the trivial steady-state solutions $z = 0$ to (1.7), whereas the latter (with $x^2 + y^2 > 0$) correspond to nonconstant 2π-periodic solutions to (1.7).

Because ϕ is $\mathbf{S}^1$-equivariant, solutions to $\phi(x, y, \lambda, \tau) = 0$ come in group orbits, and we can therefore rotate the plane so that $y = 0$ and

$$x \geq 0. \tag{2.23}$$

In particular,

$$\begin{aligned}
\phi_1(x, 0, \lambda, \tau) &= \langle v_1^*, \phi(xv_1, \lambda, \tau)\rangle = p(x^2, \lambda, \tau)x,\\
\phi_2(x, 0, \lambda, \tau) &= \langle v_2^*, \phi(xv_1, \lambda, \tau)\rangle = q(x^2, \lambda, \tau)x.
\end{aligned}$$

2.2. Linear terms in the reduced equation. We now consider the λ and τ derivatives of p and q. The first x derivatives of ϕ,

$$\frac{\partial \phi_1}{\partial x}(\mathbf{0}) = p(\mathbf{0}) \quad \text{and} \quad \frac{\partial \phi_2}{\partial x}(\mathbf{0}) = q(\mathbf{0}),$$

are both zero because linear terms vanish in the Liapunov–Schmidt reduction. Indeed, suppose that linear terms remained in $\phi = (I - E)\Phi$. Then we could have used the implicit function theorem to solve for these terms as we did for (2.19a). Since we could not do this, there can be no linear terms in ϕ. The second x derivatives,

$$\frac{\partial^2 \phi_1}{\partial x^2}(\mathbf{0}) = p_x(\mathbf{0}) \quad \text{and} \quad \frac{\partial^2 \phi_2}{\partial x^2}(\mathbf{0}) = q_x(\mathbf{0}),$$

are also clearly zero because p and q are quadratic in x.

Proof of Proposition 1.1. For any parameter α, the general formula for $\phi_{\alpha x}$ is

$$\phi_{\alpha x} = (I - E)\left(\mathrm{d}\Phi_\alpha(v_1 + W_x) + \mathrm{d}\Phi(W_{\alpha x}) + \mathrm{d}^2\Phi(v_1 + W_x, W_\alpha)\right), \tag{2.24}$$

from (A.5), but we can simplify this by using (2.21). With these results, and bearing in mind that

(a) $\mathrm{d}^2\Phi(\cdot,\cdot)$ is bilinear, so any terms of the form $\mathrm{d}\Phi_\alpha(\cdot, 0)$ vanish, and

(b) range $\mathrm{d}\Phi = \ker(I - E)$, so any terms in the range of $\mathrm{d}\Phi$ also vanish,

formula (2.24) becomes

$$\phi_{\alpha x} = (I - E)\left(\mathrm{d}\Phi_\alpha(v_1)\right). \tag{2.25}$$

To verify (1.12), (2.25) implies

$$\phi_{\tau x} = (I - E)\left(\mathrm{d}\Phi_\tau(v_1)\right).$$

Observe that

$$\mathrm{d}\Phi_\tau(v_1) = \frac{\mathrm{d}v_1}{\mathrm{d}s} = -v_2,$$

since $\mathrm{d}\Phi_\tau = \frac{\mathrm{d}}{\mathrm{d}s}$ and by (2.17). Since $v_2 \in \mathcal{K} \subset \mathcal{R} = \ker(I - E)$, it follows that $\phi_{\tau x} = 0$ and therefore that $p_\tau(0, 0, 0) = q_\tau(0, 0, 0) = 0$. This verifies (1.12).

To verify (1.13), (2.25) implies

$$\phi_{\lambda x} = (I - E)\left(\mathrm{d}\Phi_\lambda(v_1)\right).$$

Observe that

$$\Phi_\lambda(x, \lambda, \tau) = -F_\lambda(x, \lambda) \tag{2.26}$$

and that $F(x, \lambda)$ can be written as

$$F(x, \lambda) = J(\lambda)x + h.o.t.$$

Thus $F_\lambda(x,\lambda) = J'(\lambda)x + h.o.t.$, and therefore, evaluating at the origin,

$$\mathrm{d}\Phi_\lambda = -\mathrm{d}F_\lambda(x,\lambda)|_{\mathbf{0}} = -J'(0). \tag{2.27}$$

In (2.28) in Lemma 2.5, below, we prove that

$$J'(0)c = (\sigma'(0) + i\omega'(0))c - (J - iI_n)c'(0),$$

from which it follows that

$$\begin{aligned} J'(0)v_1 &= \mathrm{Re}\{e^{is}J'(0)c\} \\ &= \sigma'(0)\mathrm{Re}\{e^{is}c\} + \omega'(0)\mathrm{Re}\{ie^{is}c\} - \mathrm{Re}\{(J - iI_n)c'(0)\} \\ &= \sigma'(0)v_1 - \omega'(0)v_2 - \mathrm{Re}\{e^{is}(J - iI_n)c'(0)\}. \end{aligned}$$

So, recalling from (2.18) that $\langle v_i^*, v_j\rangle = 0$,

$$\begin{aligned} \left\langle v_j^*, \mathrm{d}\Phi_\lambda(v_1)\right\rangle &= \left\langle v_j^*, \sigma'(0)v_1 - \omega'(0)v_2\right\rangle - \left\langle v_j^*, \mathrm{Re}\{e^{is}(J - iI_n)c'(0)\}\right\rangle \\ &= \left\langle v_j^*, \mathrm{Re}\{e^{is}(J - iI_n)c'(0)\}\right\rangle. \end{aligned}$$

Observe that

$$[(J - iI_n)c'(0)]^t\bar{d} = c'(0)^t(J - iI_n)^t\bar{d} = 0,$$

since $\bar{d}$ is an eigenvector of J^t with eigenvalue i. Therefore,

$$\begin{aligned} \left\langle v_1^*, \mathrm{Re}\{e^{is}(J - iI_n)c'(0)\}\right\rangle &= \frac{1}{2}\mathrm{Re}\{[(J - iI_n)c'(0)]^t\bar{d}\} = 0, \\ \left\langle v_2^*, \mathrm{Re}\{e^{is}(J - iI_n)c'(0)\}\right\rangle &= -\frac{1}{2}\mathrm{Im}\{[(J - iI_n)c'(0)]^t\bar{d}\} = 0, \end{aligned}$$

which verifies (1.13). ■

The following calculations are needed.

Lemma 2.5. *Let $\mu(\lambda) = \sigma(\lambda) + i\omega(\lambda)$ be the eigenvalue of $J(\lambda)$ such that $\mu(0) = i$ with eigenvector $c(\lambda)$ such that $c(0) = c$. Let $b(\lambda)$ be the corresponding generalized eigenvector such that $J(\lambda)b(\lambda) = \mu(\lambda)b(\lambda) + c(\lambda)$. Then*

$$J'(0)c = \mu'(0)c - (J - iI_n)c'(0), \tag{2.28}$$

$$J''(0)c = \mu''(0)c - 2(J'(0) - \mu'(0)I_n)c'(0) - (J - iI_n)c''(0), \tag{2.29}$$

$$J'(0)b = \mu'(0)b - (J - iI_n)b'(0) + c'(0). \tag{2.30}$$

Proof. Since $c(\lambda)$ is an eigenvector of $J(\lambda)$ with eigenvalue $\mu(\lambda)$,

$$J(\lambda)c(\lambda) = \mu(\lambda)c(\lambda). \tag{2.31}$$

Differentiating (2.31) with respect to λ and evaluating at $\lambda = 0$ gives

$$J'(0)c(0) + Jc'(0) = \mu'(0)c(0) + \mu(0)c'(0) = \mu'(0)c(0) + ic'(0),$$

since $\mu(0) = i$, and this rearranges to give (2.28).

Differentiating (2.31) twice with respect to λ, we obtain

$$J''(0)c(0) + 2J'(0)c'(0) + Jc''(0) = \mu''(0)c(0) + 2\mu'(0)c'(0) + ic''(0),$$

which rearranges to give (2.29).

Similarly, for the generalized eigenvector we have

$$J(\lambda)b(\lambda) = \mu(\lambda)b(\lambda) + c(\lambda),$$

and differentiating this with respect to λ and evaluating at the origin, we obtain

$$J'(0)b(0) + Jb'(0) = \mu'(0)b(0) + \mu(0)b'(0) + c'(0),$$

which rearranges to give (2.30). ■

2.3. Quadratic terms in the reduced equation. To prove Proposition 1.2 we require Lemmas 2.6 and 2.7, which we prove at the end of the section.

Lemma 2.6. *For any vector field with nilpotent linearization*

$$W_{\tau x} = -u_2, \tag{2.32}$$

$$W_{\lambda x} = -\sigma'(0)u_1 + \omega'(0)u_2 + \mathrm{Re}\{e^{is}(c'(0) - \zeta c)\}, \tag{2.33}$$

where

$$\zeta = \frac{1}{\|c\|^2} c'(0)^t \bar{c} \tag{2.34}$$

so that ζc is the projection of $c'(0)$ onto the critical eigenspace of J.

Lemma 2.7. *Suppose that F is a coupled cell system such that $(\mathrm{d}F)_{\mathbf{0}}$ is nilpotent and that*

(a) *each cell in the network has identical linearized internal dynamics,*

(b) *the linearized coupling between any two cells takes the form $mB(\lambda)$, where $B(\lambda)$ is a $k \times k$ matrix and $m \in \mathbf{R}$.*

Then

$$c^{(l)}(0)^t \bar{d} = 0 \qquad \text{for all } l \geq 0, \tag{2.35}$$

$$[J^{(k)}(0)c^{(l)}(0)]^t \bar{d} = 0 \qquad \text{for all } k, l \geq 0, \tag{2.36}$$

where

$$J^{(k)}(0) = \left.\frac{\partial^k J(\lambda)}{\partial \lambda^k}\right|_{\lambda=0} \qquad \text{and} \qquad c^{(l)}(0) = \left.\frac{\partial^l c(\lambda)}{\partial \lambda^l}\right|_{\lambda=0}.$$

Remark 2.8. Assumptions (a) and (b) in Lemma 2.7 are instant if the network is homogeneous.

Proof of Proposition 1.2. The general formula for $\phi_{\alpha\beta x}$ is given by (A.6), but the same arguments that we used to derive (2.25) can be used to obtain

$$\phi_{\alpha\beta x} = (I - E)\big(\mathrm{d}\Phi_{\alpha\beta}(v_1) + \mathrm{d}\Phi_\alpha(W_{\beta x}) + \mathrm{d}\Phi_\beta(W_{\alpha x})\big). \tag{2.37}$$

To verify (1.14), we have from (2.37)

$$\phi_{\tau\tau x} = (I-E)\big(\mathrm{d}\Phi_{\tau\tau}(v_1) + 2\mathrm{d}\Phi_\tau(W_{\tau x})\big) = (I-E)\left(2\frac{\mathrm{d}}{\mathrm{d}s}W_{\tau x}\right),$$

where the second equality follows because $\mathrm{d}\Phi_\tau = \frac{\mathrm{d}}{\mathrm{d}s}$ and $\Phi_{\tau\tau} = 0$. By the formula for $W_{\tau x}$ in (2.32) and the derivative given in (2.17),

$$\frac{\mathrm{d}}{\mathrm{d}s}W_{\tau x} = -\frac{\mathrm{d}}{\mathrm{d}s}u_2 = -u_1.$$

So, by (2.18), the jth component of $\phi_{\tau\tau x}$ is

$$\phi_{\tau\tau x,j} = -2\left\langle v_j^*, u_1\right\rangle = -2\delta_{j1}, \qquad j = 1,2.$$

For the $\lambda\lambda$ derivative in (1.15), we have from (2.37)

$$\phi_{\lambda\lambda x} = (I-E)\big(\mathrm{d}\Phi_{\lambda\lambda}(v_1) + 2\mathrm{d}\Phi_\lambda(W_{\lambda x})\big). \tag{2.38}$$

Consider the first term $(I-E)(\mathrm{d}\Phi_{\lambda\lambda}(v_1))$, and note that (2.27) can be extended to the second λ derivative, to yield

$$\mathrm{d}\Phi_{\lambda\lambda}(v_1) = -\mathrm{d}F_{\lambda\lambda}(v_1) = -J''(0)v_1 = -\mathrm{Re}\{e^{is}J''(0)c\}.$$

Thus, since $[J''(0)c]^t\bar{d} = 0$ by (2.36), we have

$$\phi_{\lambda\lambda x,j} = \left\langle v_j^*, \mathrm{d}\Phi_{\lambda\lambda}(v_1)\right\rangle = -\left\langle v_j^*, \mathrm{Re}\{e^{is}J''(0)c\}\right\rangle = 0.$$

Therefore by (2.27), equation (2.38) becomes

$$\phi_{\lambda\lambda x} = 2(I-E)\big(\mathrm{d}\Phi_\lambda(W_{\lambda x}\big) = -2(I-E)\big(J'(0)W_{\lambda x}\big),$$

and by formula (2.33) for $W_{\lambda x}$

$$J'(0)W_{\lambda x} = -\sigma'(0)\mathrm{Re}\{e^{is}J'(0)b\} + \omega'(0)\mathrm{Im}\{e^{is}J'(0)b\} + \mathrm{Re}\{e^{is}J'(0)(c'(0)-\zeta c)\}.$$

Using (2.30) for $J'(0)b$ and rearranging, this becomes

$$J'(0)W_{\lambda x} = -\mathrm{Re}\big\{e^{is}\big(\mu'(0)^2 b - \mu'(0)(J-iI_n)b'(0) + \mu'(0)c'(0) - J'(0)(c'(0)-\zeta c)\big)\big\},$$

so that

$$\begin{aligned}\phi_{\lambda\lambda x,1} &= -2\left\langle v_1^*, J'(0)W_{\lambda x}\right\rangle \\ &= \mathrm{Re}\big\{\mu'(0)^2 b^t\bar{d} - \mu'(0)[(J-iI_n)b'(0)]^t\bar{d} \\ &\qquad + \mu'(0)c'(0)^t\bar{d} - [J'(0)c'(0)]^t\bar{d} + [\zeta J'(0)c]^t\bar{d}\big\},\end{aligned} \tag{2.39}$$

with $\phi_{\lambda\lambda x,2}$ being minus the imaginary part of the same expression.

Observe that the last three terms in (2.39) vanish by Lemma 2.7 and that the second term vanishes because

$$[(J - iI_n)b'(0)]^t\bar{d} = b'(0)^t[(J - iI_n)^t\bar{d}] = 0,$$

since $\bar{d}$ is an eigenvector of J^t with eigenvalue i. Hence,

$$\begin{aligned}\phi_{\lambda\lambda x,1} &= -2\left\langle v_1^*, J'(0)W_{\lambda x}\right\rangle = \mathrm{Re}\{\mu'(0)^2 b^t\bar{d}\},\\ \phi_{\lambda\lambda x,2} &= -2\left\langle v_2^*, J'(0)W_{\lambda x}\right\rangle = -\mathrm{Im}\{\mu'(0)^2 b^t\bar{d}\},\end{aligned}$$

which, given that $b^t\bar{d} = 2$ by (2.15), proves (1.15).

Turning now to (1.16), we have from (2.37)

$$\phi_{\lambda\tau x} = (I - E)\big(\mathrm{d}\Phi_{\lambda\tau}(v_1) + \mathrm{d}\Phi_\lambda(W_{\tau x}) + \mathrm{d}\Phi_\tau(W_{\lambda x})\big). \tag{2.40}$$

First note that $\mathrm{d}\Phi_{\lambda\tau} = 0$. For the second term in (2.40), (2.27) and (2.32) imply that

$$\mathrm{d}\Phi_\lambda(W_{\tau x}) = J'(0)u_2 = \mathrm{Im}(e^{is}J'(0)b).$$

Expanding $J'(0)b$ by (2.30), we obtain

$$\begin{aligned}\mathrm{d}\Phi_\lambda(W_{\tau x}) &= \mathrm{Im}\{e^{is}(\mu'(0)b - (J - iI_n)b'(0) + c'(0))\}\\ &= \omega'(0)u_1 + \sigma'(0)u_2 - \mathrm{Im}\{e^{is}[(J - iI_n)b'(0) + c'(0)]\}.\end{aligned} \tag{2.41}$$

For the third term in (2.40), we have from (2.33) and the fact that $\mathrm{d}\Phi_\tau = \frac{\mathrm{d}}{\mathrm{d}s}$ that

$$\begin{aligned}\mathrm{d}\Phi_\tau(W_{\lambda x}) &= -\sigma'(0)\dot{u}_1 + \omega'(0)\dot{u}_2 + \mathrm{Re}\{ie^{is}(c'(0) - \zeta c)\}\\ &= \omega'(0)u_1 + \sigma'(0)u_2 - \mathrm{Im}\{e^{is}(c'(0) - \zeta c)\}.\end{aligned}$$

Putting this together with (2.41) and taking inner products yields

$$\begin{aligned}\left\langle v_j^*, \mathrm{d}\Phi_\lambda(W_{\tau x}) + \mathrm{d}\Phi_\tau(W_{\lambda x})\right\rangle &= \langle v_j^*, 2\omega'(0)u_1 + 2\sigma'(0)u_2\\ &\qquad - \mathrm{Im}\{e^{is}[(J - iI_n)b'(0) + c'(0) + c'(0) - \zeta c]\}\rangle\\ &= 2\omega'(0)\left\langle v_j^*, u_1\right\rangle + 2\sigma'(0)\left\langle v_j^*, u_2\right\rangle,\end{aligned}$$

where the second line follows because $c'(0)^t\bar{d} = 0 = c^t\bar{d}$ and $[(J - iI_n)b'(0)]^t\bar{d} = b'(0)^t(J - iI_n)^t\bar{d} = 0$. The formulas in (1.16) then follow from Lemma 2.3. ■

Proof of Lemma 2.6. To show (2.32), we have from formula (A.14) and from the facts that $W_x = W_\tau = 0$, $\mathrm{d}\Phi_\tau = \frac{\mathrm{d}}{\mathrm{d}s}$, and $\mathcal{L}u_2 = -v_2$ that

$$W_{\tau x} = -\mathcal{L}^{-1}E(\mathrm{d}\Phi_\tau(v_1)) = -\mathcal{L}^{-1}E\left(\frac{\mathrm{d}v_1}{\mathrm{d}s}\right) = \mathcal{L}^{-1}E(v_2) = -u_2,$$

as required.

To prove (2.33), we use (A.14) and (2.27) to obtain

$$W_{\lambda x} = -\mathcal{L}^{-1}E(\mathrm{d}\Phi_\lambda(v_1)) = \mathcal{L}^{-1}E(J'(0)v_1).$$

So, using formula (2.28) for $J'(0)v_1 = \mathrm{Re}\{e^{is}J'(0)c\}$, $W_{\lambda x}$ is the solution to the differential equation

$$\mathcal{L}W_{\lambda x} = \sigma'(0)v_1 - \omega'(0)v_2 - \mathrm{Re}\{e^{is}(J - iI_n)c'(0)\}. \tag{2.42}$$

Write $c'(0) = c_0 + c_1$, where $c_0 \in \ker(J - iI_n)$ and $c_1 \in \ker(J - iI_n)^{\perp}$. Then

$$c_0 = \frac{c'(0)^t\bar{c}}{\|c\|^2}c$$

and

$$\mathrm{Re}\{e^{is}(J - iI_n)c'(0)\} = \mathrm{Re}\left\{e^{is}(J - iI_n)\left(c'(0) - \frac{c'(0)^t\bar{c}}{\|c\|}c\right)\right\}.$$

With this substitution it is straightforward to check that (2.42) is solved for $W_{\lambda x}$ as in (2.33). ■

Proof of Lemma 2.7. For any two matrices M and N we define $[N]_M$ to be the matrix obtained by replacing every entry m_{ij} in M with the block $m_{ij}N$. Using this notation, the Jacobian of a coupled cell system satisfying (a) and (b) has the form

$$\begin{aligned} J(\lambda) &= \begin{pmatrix} A(\lambda) & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & A(\lambda) \end{pmatrix} + \begin{pmatrix} m_{11}B(\lambda) & \cdots & m_{1n}B(\lambda) \\ \vdots & \ddots & \vdots \\ m_{n1}B(\lambda) & \cdots & m_{nn}B(\lambda) \end{pmatrix} \\ &= [A(\lambda)]_{I_n} + [B(\lambda)]_M, \end{aligned} \tag{2.43}$$

where $A(\lambda)$ is the linearized internal dynamics, $B(\lambda)$ is the linearized coupling, and $m_{ij} \in \mathbf{R}$.

As shown in [6], the eigenvalues of a homogeneous network are the eigenvalues of the $k \times k$ matrices $A(\lambda) + \mu_j B(\lambda)$, where $\mu_1, \ldots, \mu_n$ are the eigenvalues of M. Fix μ_c such that $A(0) + \mu_c B(0)$ has a critical eigenvalue, and observe that μ_c is independent of λ. Note that μ_c has algebraic multiplicity 2 and geometric multiplicity 1. Then $A(0)^t + \mu_c B(0)^t$ also has a critical eigenvalue.

The eigenvectors $u(\lambda)$ of $J(\lambda)$ have the form

$$u(\lambda) = \begin{pmatrix} v_1 w(\lambda) \\ \vdots \\ v_n w(\lambda) \end{pmatrix} = [w(\lambda)]_v,$$

where $w(\lambda) \in \mathbf{C}^k$ is an eigenvector of $A(\lambda) + \mu_j B(\lambda)$ and $v \in \mathbf{C}^n$ is an eigenvector of M. Note that v does not depend on λ since the matrix M does not depend on λ. Thus we can write

$$c(\lambda) = [\beta(\lambda)]_\alpha \quad \text{and} \quad d(\lambda) = [\delta(\lambda)]_\gamma,$$

where α and γ are respectively the appropriate eigenvectors of M and M^t with eigenvalues μ_c, and $\beta(\lambda)$ and $\delta(\lambda)$ are respectively the eigenvectors of $A(\lambda) + \mu_c B(\lambda)$ and $A(\lambda)^t + \mu_c B(\lambda)^t$ with eigenvalues $\sigma(\lambda) + i\omega(\lambda)$. Then

$$c(\lambda)^t \bar{d}(\lambda) = ([\beta(\lambda)]_\alpha)^t [\bar{\delta}(\lambda)]_{\bar{\gamma}} = (\alpha^t \bar{\gamma})(\beta(\lambda)^t \bar{\delta}(\lambda)). \tag{2.44}$$

Observe that $\beta(0)^t\bar{\delta}(0) \neq 0$ since $\beta(0)$ and $\bar{\delta}(0)$ are respectively the eigenvectors of $A(0) + \mu_c B(0)$ and $(A(0) + \mu_c B(0))^t$ with the same simple eigenvalue i, and recall from Lemma 2.2 that $c^t\bar{d} = 0$ since $(\mathrm{d}F)_{\mathbf{0}}$ is nilpotent. Then (2.44) implies $\alpha^t\bar{\gamma} = 0$.

Hence,

$$c^{(l)}(0)^t\bar{d}(0) = ([\beta^{(l)}(0)]_\alpha)^t[\bar{\delta}(0)]_{\bar{\gamma}} = (\alpha^t\bar{\gamma})(\beta^{(l)}(0)^t\bar{\delta}(0)) = 0,$$

which proves (2.35). To show (2.36), use (2.43) to calculate

$$\begin{aligned}[J^{(k)}(0)c^{(l)}(0)]^t\bar{d}(0) &= \left(J^{(k)}(0)[\beta^{(l)}(0)]_\alpha\right)^t[\bar{\delta}(0)]_{\bar{\gamma}} \\ &= (\alpha^t\bar{\gamma})\left((A^{(k)}(0) + \mu_c B^{(k)}(0))\beta^{(l)}(0)\right)^t\bar{\delta}(0) \\ &= 0\end{aligned}$$

since $\alpha^t\bar{\gamma} = 0$. ■

3. Hopf bifurcation with linear u terms.

In this section we consider nilpotent Hopf bifurcations in coupled cell systems that satisfy (1.17). Note that we are making two assumptions in (1.17); that the network architecture does not force $q_u(\mathbf{0})$ to vanish, and that the bifurcation is generic for that network. Theorem 1.3 is proved below, and the corresponding bifurcation diagram is shown in Figure 4. We then show that there exists a vector field on the five-cell network of Figure 3 such that $q_u(\mathbf{0}) \neq 0$. Hence, by Theorem 1.3, this system has two branches of solutions that grow linearly with λ.

Proof of Theorem 1.3. By Propositions 1.1 and 1.2, we can write

$$p(u,\lambda,\tau) = p_u(\mathbf{0})u - \tau^2 + (\sigma'(0)^2 - \omega'(0)^2)\lambda^2 + 2\omega'(0)\lambda\tau + \cdots, \tag{3.1}$$

$$q(u,\lambda,\tau) = q_u(\mathbf{0})u - 2\sigma'(0)\omega'(0)\lambda^2 + 2\sigma'(0)\lambda\tau + \cdots, \tag{3.2}$$

and we require solutions to $p = q = 0$. Assuming (1.17) and applying the implicit function theorem, we can solve $q = 0$ near the origin for

$$u = \frac{2\sigma'(0)\lambda}{q_u(\mathbf{0})}(\omega'(0)\lambda - \tau) + \cdots .$$

Substituting this into (3.1) and setting $p = 0$ yields the following equation in τ and λ:

$$\tau^2 + 2\left(\frac{p_u(\mathbf{0})}{q_u(\mathbf{0})}\sigma'(0) - \omega'(0)\right)\lambda\tau + \left(\omega'(0)^2 - \frac{2p_u(\mathbf{0})}{p_u(\mathbf{0})}\sigma'(0)\omega'(0) - \sigma'(0)^2\right)\lambda^2 + \cdots = 0,$$

where quadratic terms are solved for

$$\tau = \left[\omega'(0) - \frac{\sigma'(0)}{q_u(\mathbf{0})}\left(p_u(\mathbf{0}) \pm \sqrt{p_u(\mathbf{0})^2 + q_u(\mathbf{0})^2}\right)\right]\lambda. \tag{3.3}$$

Since these solutions are real and distinct we can apply the recognition problem for simple bifurcation using τ as the state variable and λ as the bifurcation parameter to prove that higher order terms are unimportant. See [3, Proposition 9.3, p. 95].

Substitute (3.3) into $q = 0$, for q as in (3.2), and rearrange to obtain

$$u = \frac{2\sigma'(0)^2}{q_u(\mathbf{0})^2}\left(p_u(\mathbf{0}) \pm \sqrt{p_u(\mathbf{0})^2 + q_u(\mathbf{0})^2}\right)\lambda^2.$$

Note that only the $+$ sign is relevant, since we require $u = x^2 > 0$. Thus, since $u = x^2$ grows as $O(\lambda^2)$, x grows as $O(\lambda)$, and we obtain the bifurcation diagram in Figure 4. ■

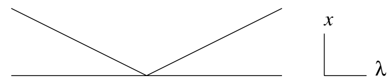

Figure 4. *Bifurcation diagram for nilpotent Hopf bifurcation with* $q_u(\mathbf{0}) \neq 0$.

The five-cell network of Figure 3. To illustrate the results of this section we consider the five-cell system shown in Figure 3 and defined by (1.6) with the vector field defined by

$$f(x_i; x_j, x_k, x_l) = \lambda x_i - x_j - x_k - x_l - x_i^3. \tag{3.4}$$

We compute $p_u(\mathbf{0})$ and $q_u(\mathbf{0})$ for this system and show that $p_u(\mathbf{0}) \neq 0 \neq q_u(\mathbf{0})$.

Recall that

$$p_u(\mathbf{0}) = \frac{1}{2} p_{xx}(\mathbf{0}) = \frac{1}{2}\frac{\partial^3 \phi_1}{\partial x^3}(\mathbf{0}) \quad \text{and} \quad q_u(\mathbf{0}) = \frac{1}{2} q_{xx}(\mathbf{0}) = \frac{1}{2}\frac{\partial^3 \phi_2}{\partial x^3}(\mathbf{0}).$$

By (A.1) we have

$$\phi_{xxx} = (I - E)(\mathrm{d}^3\Phi(v_1, v_1, v_1) + 3\mathrm{d}^2\Phi(v_1, W_{xx})) \tag{3.5}$$

since $\mathrm{d}\Phi(W_{xxx}) \in \mathcal{R} = \ker(I - E)$.

As a reminder of the higher order differentials of Φ, let $u_1, \ldots, u_m$ be any functions in $\mathcal{C}_{2\pi}$, and let $u_{i,j}$ denote the jth component of u_i. By definition, evaluating all derivatives at the origin, we have

$$\begin{aligned}(\mathrm{d}^m\Phi)_{0,0,0}(u_1, \ldots, u_m) &= \frac{\partial}{\partial t_1} \cdots \frac{\partial}{\partial t_m} \Phi(t_1 u_1 + \cdots + t_m u_m, 0, 0)\bigg|_{t=0} \\ &= -\sum_{i_1,\ldots,i_m=1}^{n} \frac{\partial^m F}{\partial x_{i_1} \ldots \partial x_{i_m}}\bigg|_{0,0} u_{1,i_1} \cdots u_{m,i_m} \\ &= -(\mathrm{d}^m F)_{0,0,0}(u_1, \ldots, u_m).\end{aligned}$$

See [3, pp. 31–32]. Letting $v_{1,j}$ denote the jth component of v_1 gives

$$\begin{aligned}\mathrm{d}^3\Phi(v_1, v_1, v_1) &= -\sum_{i,j,k=1}^{n} \frac{\partial^3 F}{\partial x_i \partial x_j \partial x_k}\bigg|_0 v_{1,i} v_{1,j} v_{1,k} \\ &= -\frac{1}{4}\mathrm{Re}\{e^{3is}\mathrm{d}^3F(c, c, c) + 3e^{is}\mathrm{d}^3F(c, c, \bar{c})\}\end{aligned} \tag{3.6}$$

since

$$\begin{aligned}v_{1,i} v_{1,j} v_{1,k} &= \frac{1}{8}(e^{is}c_i + e^{-is}\bar{c}_i)(e^{is}c_j + e^{-is}\bar{c}_j)(e^{is}c_k + e^{-is}\bar{c}_k) \\ &= \frac{1}{4}\mathrm{Re}\{e^{3is}c_i c_j c_k + e^{is}(c_i c_j \bar{c}_k + c_i \bar{c}_j c_k + \bar{c}_i c_j c_k)\}.\end{aligned}$$

Therefore, since second derivatives of F in (3.4) vanish, formula (3.5) becomes

$$\phi_{xxx} = (I - E)\mathrm{d}^3\Phi(v_1, v_1, v_1) = -\frac{1}{4}(I - E)\mathrm{Re}\big\{e^{3is}\mathrm{d}^3F(c, c, c) + 3e^{is}\mathrm{d}^3F(c, c, \bar{c})\big\}.$$

Thus the components of ϕ_{xxx} on $\mathcal{K}^*$ are

$$p_u(\mathbf{0}) = -\frac{1}{8}\left\langle v_1^*, \mathrm{Re}\{e^{3is}\mathrm{d}^3F(c, c, c) + e^{is}3\mathrm{d}^3F(c, c, \bar{c})\}\right\rangle = -\frac{3}{16}\mathrm{Re}\{\mathrm{d}^3F(c, c, \bar{c})^t\bar{d}\},$$
$$q_u(\mathbf{0}) = -\frac{1}{8}\left\langle v_2^*, \mathrm{Re}\{e^{3is}\mathrm{d}^3F(c, c, c) + e^{is}3\mathrm{d}^3F(c, c, \bar{c})\}\right\rangle = \frac{3}{16}\mathrm{Im}\{\mathrm{d}^3F(c, c, \bar{c})^t\bar{d}\}.$$

To compute $\mathrm{d}^3F(c, c, \bar{c})$, rewrite the equations in the form $\dot{x}_i = f_i(x)$ for $i = 1, \dots, 5$ and observe that

$$\frac{\partial^3 f_i}{\partial x_i^3} = -6 \quad \text{for} \quad i = 1, \dots, 5 \qquad \text{and} \qquad \frac{\partial^3 f_i}{\partial x_j^3} = 0 \quad \text{if} \quad i \neq j.$$

Hence

$$\mathrm{d}^3F(c, c, \bar{c}) = \sum_{i,j,k=1}^{5} \left.\frac{\partial^3 F}{\partial x_i^3}\right|_0 c_i c_j \bar{c}_k = -6 \begin{pmatrix} c_1c_1\bar{c}_1 \\ c_2c_2\bar{c}_2 \\ c_3c_3\bar{c}_3 \\ c_4c_4\bar{c}_4 \\ c_5c_5\bar{c}_5 \end{pmatrix}.$$

The critical eigenspace is spanned by the real and imaginary parts of

$$c = (2, -2 + 2i, -4i, -1 - i, 2)^t,$$

and therefore

$$\mathrm{d}^3F(c, c, \bar{c}) = -12(4, -8 + 8i, -32i, -1 - i, 4)^t.$$

The generalized eigenvector orthogonal to c is

$$b = \frac{1}{17}(-45 - 27i, 4 - 18i, 14 - 12i, -8 + 36i, 57 + 7i),$$

and the eigenvector of J^t with eigenvalue $-i$ is

$$d = \frac{1}{20}(-3 - i, 1 - 3i, 1 - 3i, -2 + 6i, 3 + i)^t,$$

which is chosen so that $b^t\bar{d} = 2$, as in (2.15). Hence

$$\mathrm{d}^3F(c, c, \bar{c})^t\bar{d} = -36 + 24i,$$

and therefore, since $p_u(\mathbf{0}) = \frac{1}{2}p_{xx}(\mathbf{0})$ and $q_u(\mathbf{0}) = \frac{1}{2}q_{xx}(\mathbf{0})$,

$$p_u(\mathbf{0}) = -\frac{3}{16}\mathrm{Re}\{-36 + 24i\} = \frac{27}{4} \qquad \text{and} \qquad q_u(\mathbf{0}) = \frac{3}{16}\mathrm{Im}\{-36 + 24i\} = \frac{9}{2}.$$

Since $q_u(\mathbf{0}) \neq 0$, Theorem 1.3 guarantees linear growth near to a bifurcation.

4. Hopf bifurcation with quadratic u terms. In this section we consider nilpotent Hopf bifurcations in networks that satisfy (1.18) and (1.19). In section 4.1 we prove Theorem 1.4. Then in section 4.2 we discuss the implications of (1.23) for the Liapunov–Schmidt reduction and derive expressions for the partial derivatives of the reduced mapping. Finally, in section 4.3 we show that the network in Figure 2 satisfies (1.23) and, using the formulas derived in section 4.2, that admissible vector fields can be chosen to give either two or four branches, as stated in Theorem 1.4, with any combination of super- and subcritical branches.

The number of branches is determined by the number of points in the intersection of two quadratics in the (u,τ)-plane. These quadratics are not arbitrary because, as we show in Lemma 4.2, the null intersection is not possible. However, in the two or four branch cases any configuration of super- and subcritical branches may be obtained by admissible vector fields for the network in Figure 2.

4.1. Proof of Theorem 1.4. Proposition 1.2 and (1.18) imply that p and q can be written as

$$\begin{aligned} p(u,\lambda,\tau) &= \tfrac{1}{2}p_{uu}(\mathbf{0})u^2 + p_{u\tau}(\mathbf{0})u\tau + p_{u\lambda}(\mathbf{0})u\lambda - \tau^2 \\ &\qquad + (\sigma'(0)^2 - \omega'(0)^2)\lambda^2 + 2\omega'(0)\lambda\tau + \cdots\,, \\ q(u,\lambda,\tau) &= \tfrac{1}{2}q_{uu}(\mathbf{0})u^2 + q_{u\tau}(\mathbf{0})u\tau + q_{u\lambda}(\mathbf{0})u\lambda \\ &\qquad - 2\sigma'(0)\omega'(0)\lambda^2 + 2\sigma'(0)\lambda\tau + \cdots\,. \end{aligned} \tag{4.1}$$

We seek solutions to $p(u,\lambda,\tau) = q(u,\lambda,\tau) = 0$.

The first step in finding solutions to (4.1) is to introduce changes of coordinates that simplify the equations but do not affect the branching behavior at the bifurcation.

Lemma 4.1. *The mapping (p,q) in (4.1) is strongly equivalent to*

$$\begin{aligned} p(u,\lambda,\tau) &= \alpha u^2 + \gamma u\lambda - \tau^2 + \sigma'(0)^2\lambda^2 + \cdots\,, \\ q(u,\lambda,\tau) &= a u^2 + b u\tau + c u\lambda + 2\sigma'(0)\lambda\tau + \cdots\,, \end{aligned} \tag{4.2}$$

where

$$\begin{aligned} \alpha &= \frac{1}{4}\big(2p_{uu}(\mathbf{0}) + p_{u\tau}(\mathbf{0})^2\big), & a &= \frac{1}{2}\big(q_{uu}(\mathbf{0}) + p_{u\tau}(\mathbf{0})q_{u\tau}(\mathbf{0})\big), \\ & & b &= q_{u\tau}(\mathbf{0}), \\ \gamma &= p_{u\lambda}(\mathbf{0}) + \omega'(0)p_{u\tau}(\mathbf{0}), & c &= q_{u\lambda}(\mathbf{0}) + \sigma'(0)p_{u\tau}(\mathbf{0}) + \omega'(0)q_{u\tau}(\mathbf{0}). \end{aligned} \tag{4.3}$$

Furthermore, the nondegeneracy conditions in (1.19) imply that all coefficients in (4.2) are nonzero, and condition (1.20) implies that when $\lambda = 0$ the only solution to $p = q = 0$ near the origin is the origin itself.

We assume that the coefficients in (4.2) are independent and arbitrary. In section 4.3 we use the results derived in section 4.2 to show that this assumption is valid for the network shown in Figure 2.

The second step in finding solutions to (4.1) is to truncate p and q at quadratic order and to introduce similarity variables

$$\hat{u} = \lambda u \quad \text{and} \quad \hat{\tau} = \tau\lambda. \tag{4.4}$$

We then prove that generically there are two or four branches of solutions to $p = q = 0$ in two stages. Consider the equations

$$\begin{aligned} \hat{p}(\hat{u},\hat{\tau}) &\equiv \alpha\hat{u}^2 + \gamma\hat{u} - \hat{\tau}^2 + \sigma'(0)^2 &= 0, \\ \hat{q}(\hat{u},\hat{\tau}) &\equiv a\hat{u}^2 + b\hat{u}\hat{\tau} + c\hat{u} + 2\sigma'(0)\tau &= 0. \end{aligned} \tag{4.5}$$

Note that solutions to (4.5) correspond to lines of solutions (parametrized by λ) in the zeros for the quadratic truncations of p and q in (4.2). We prove the following.

Lemma 4.2. *Assume* (1.19). *Then, generically, solutions to* (4.5) *consist of either two or four points, with the precise number depending on the coefficients in* (4.3).

Recall that the only solutions to (4.2) of interest are those with $u = x^2 > 0$, since we require real solutions for x. However, any solution $(\hat{u}_0, \hat{\tau}_0)$ to (4.5) corresponds to a ray of solutions to the quadratic truncations of p and q. If $\hat{u}_0 > 0$, the ray consists of points $u_0 = \hat{u}_0\lambda$ and $\tau_0 = \hat{\tau}_0\lambda$, where $\lambda \geq 0$. On the other hand, if $\hat{u}_0 < 0$, the ray consists of (u_0, τ_0), where $\lambda < 0$. Each ray of solutions in (u, τ) space corresponds to a parabola of solutions in (x, τ, λ) space, where the solutions are supercritical if $\hat{u}_0 > 0$ and subcritical if $\hat{u}_0 < 0$.

Finally, we use hyperbolicity to justify truncating (4.2) at quadratic order.

Lemma 4.3. *Generically in the coefficients* (4.3)*, all solutions to* (4.5) *are hyperbolic.*

Lemma 4.3 implies that higher order perturbations of the truncated equations merely move the branches of solutions in (u, λ, τ) space and do not affect the existence of the branches. The proof of Theorem 1.4 follows from Lemmas 4.1, 4.2, and 4.3.

Figures 5, 6, and 7 show that two or four branches with all choices of super- and subcritical branches are indeed possible, given that the coefficients in (4.2) are arbitrary. Recall that negative $\hat{u}$ solutions to (4.5) correspond to subcritical branches of solutions to $p = q = 0$ and that positive solutions correspond to supercritical branches.

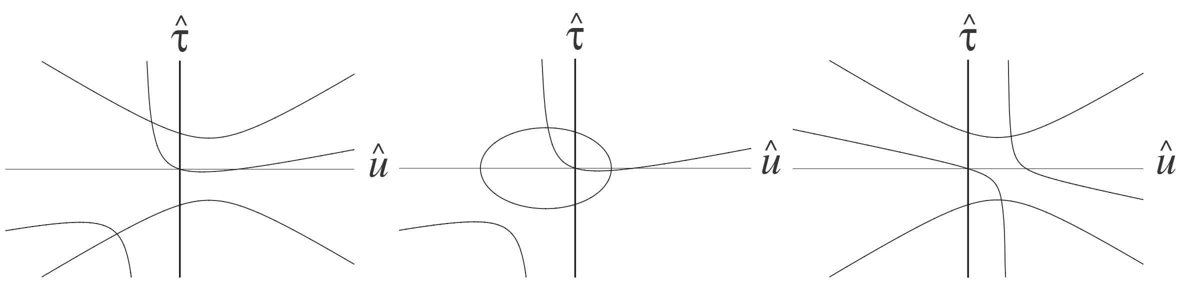

Figure 5. *Two solutions to* (4.5) *with* $\gamma = -1$, $a = -1$, $c = 1$, $\sigma'(0) = 1$. *Left: Two solutions with* $\hat{u} < 0$; $\alpha = 1$, $b = 3$. *Center: One solution with* $\hat{u} < 0$ *and one with* $\hat{u} > 0$; $\alpha = -1$, $b = 3$. *Right: Two solutions with* $\hat{u} > 0$; $\alpha = 1$, $b = -3$.

Proof of Lemma 4.1. For strong equivalence (see [4]) we may make changes of coordinates of the form

$$g(x,\tau,\lambda) = S(x,\tau,\lambda)\phi(X(x,\tau,\lambda), T(x,\tau,\lambda), \lambda).$$

Thus we can transform τ by

$$\tau \mapsto \tau + \frac{p_{u\tau}(\mathbf{0})}{2}u + \omega'(0)\lambda,$$

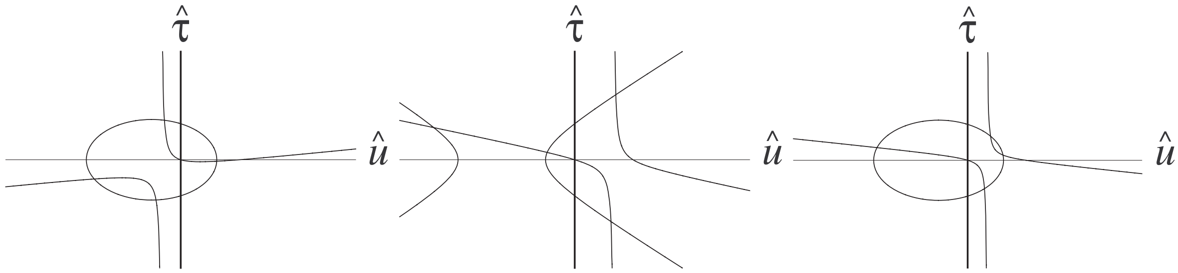

Figure 6. *Four solutions to (4.5) with $a = -1$, $c = 1$, $\sigma'(0) = 1$. Left: Three solutions with $\hat{u} < 0$ and one with $\hat{u} > 0$; $\alpha = -1$, $\gamma = -1$, $b = 6$. Center: Two solutions with $\hat{u} < 0$ and two with $\hat{u} > 0$; $\alpha = 1$, $\gamma = 2.5$, $b = -3$. Right: One solution with $\hat{u} < 0$ and three with $\hat{u} > 0$; $\alpha = -1$, $\gamma = -1$, $b = -6$.*

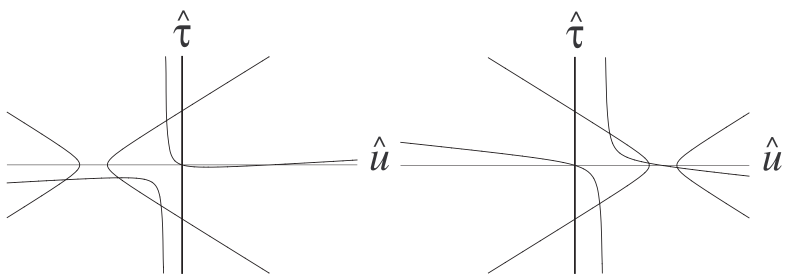

Figure 7. *Four solutions to (4.5) with $\alpha = 1$, $a = -1$, $c = 1$, $\sigma'(0) = 1$. Left: Four solutions with $\hat{u} < 0$; $\gamma = 2.025$, $b = 10$. Right: Four solutions with $\hat{u} > 0$; $\gamma = -2.025$, $b = -6$.*

which yields

$$
\begin{aligned}
p(u,\lambda,\tau) &= \tfrac{1}{4}\big(2p_{uu}(\mathbf{0}) + p_{u\tau}(\mathbf{0})^2\big)u^2 + \big(p_{u\lambda}(\mathbf{0}) + \omega'(0)p_{u\tau}(\mathbf{0})\big)u\lambda \\
&\qquad - \tau^2 + \sigma'(0)^2\lambda^2 + \cdots, \\
q(u,\lambda,\tau) &= \tfrac{1}{2}\big(q_{uu}(\mathbf{0}) + p_{u\tau}(\mathbf{0})q_{u\tau}(\mathbf{0})\big)u^2 + q_{u\tau}(\mathbf{0})u\tau \\
&\qquad + \big(q_{u\lambda}(\mathbf{0}) + \omega'(0)q_{u\tau}(\mathbf{0}) + \sigma'(0)p_{u\tau}(\mathbf{0})\big)u\lambda + 2\sigma'(0)\lambda\tau + \cdots.
\end{aligned} \tag{4.6}
$$

This gives (4.2) and the coefficients in (4.3). Moreover, (1.19) implies that all coefficients are nonzero.

Note that if $\lambda = 0$, then

$$p(u,0,\tau) = \alpha u^2 - \tau^2 + \cdots \quad \text{and} \quad q(u,0,\tau) = au^2 + bu\tau + \cdots.$$

Use the implicit function theorem to solve $q = 0$ to quadratic order for $\tau = -\frac{a}{b}u + \cdots$, and substitute into $p = 0$, giving

$$\left(\alpha - \frac{a^2}{b^2}\right)u^2 + \cdots = 0,$$

which has nonzero solutions for u only if $\alpha b^2 - a^2 = 0$. By (4.3), $\alpha b^2 - a^2 \neq 0$ is equivalent to (1.20). So the only solution for $\lambda = 0$ is the origin. ■

Proof of Lemma 4.2. We show that (1.19) implies that there always exists at least one solution. Therefore, generically, there must be either two or four solutions.

Observe that $\hat{q} = 0$ can be solved for

$$\hat{\tau} = \hat{\tau}(\hat{u}) = -\frac{a\hat{u}^2 + c\hat{u}}{b\hat{u} + 2\sigma'(0)}. \tag{4.7}$$

Substituting this into $\hat{p}$ in (4.5) yields

$$\hat{p}(\hat{u}, \hat{\tau}(\hat{u})) = \frac{1}{(b\hat{u} + 2\sigma'(0))^2}\big[(\alpha\hat{u}^2 + \gamma\hat{u} + \sigma'(0)^2)(b\hat{u} + 2\sigma'(0))^2 - (a\hat{u}^2 + c\hat{u})^2\big].$$

Thus $\hat{p} = 0$ only when

$$h(\hat{u}) \equiv (\alpha\hat{u}^2 + \gamma\hat{u} + \sigma'(0)^2)(b\hat{u} + 2\sigma'(0))^2 - (a\hat{u}^2 + c\hat{u})^2 = 0. \tag{4.8}$$

Observe that

$$h(0) = 4\sigma'(0)^4 > 0 \quad \text{and} \quad h\left(-\frac{2\sigma'(0)}{b}\right) = -(a\hat{u}^2 + c\hat{u})^2 \leq 0.$$

Hence, by the mean value theorem, there exists a $\hat{u}_0$ between 0 and $-2\sigma'(0)/b$ such that $h(\hat{u}_0) = 0$. Therefore, for $\hat{\tau}_0 = \hat{\tau}(\hat{u}_0)$ as given in (4.7), $\hat{p}(\hat{u}_0, \hat{\tau}_0) = 0$. Since there is at least one solution to (4.5), generically there must be either two or four solutions. ■

Proof of Lemma 4.3. We show that the Jacobian of the mapping

$$R(\hat{u}, \hat{\tau}) = (\hat{p}(\hat{u}, \hat{\tau}), \hat{q}(\hat{u}, \hat{\tau}))$$

is singular at a solution to (4.5) only if $\hat{u}$ satisfies a quartic equation that is different from $h = 0$. Hence, generically, a point $\hat{u}_0$ will not solve both equations simultaneously.

Observe that the Jacobian of R is

$$\mathrm{d}_{\hat{u},\hat{\tau}}R = \begin{pmatrix} \frac{\partial\hat{p}}{\partial\hat{u}} & \frac{\partial\hat{p}}{\partial\hat{\tau}} \\ \frac{\partial\hat{q}}{\partial\hat{u}} & \frac{\partial\hat{q}}{\partial\hat{\tau}} \end{pmatrix} = \begin{pmatrix} 2\alpha\hat{u} + \gamma & -2\hat{\tau} \\ 2a\hat{u} + b\hat{\tau} + c & b\hat{u} + 2\sigma'(0) \end{pmatrix}.$$

Thus $\mathrm{d}R$ is singular only if its determinant

$$(2\alpha\hat{u} + \gamma)(b\hat{u} + 2\sigma'(0)) + 2\hat{\tau}(2a\hat{u} + b\hat{\tau} + c) = 0.$$

Evaluating this on the manifold of solutions $\hat{\tau} = \hat{\tau}(\hat{u})$ to $\hat{q} = 0$ given by (4.7) yields a second quartic equation:

$$2(a\hat{u} + c)\big(b(a\hat{u} + c)\hat{u} - (2a\hat{u} + c)(b\hat{u} + 2\sigma'(0))\big)\hat{u} + (2\alpha\hat{u} + \gamma)(b\hat{u} + 2\sigma'(0))^3 = 0. \tag{4.9}$$

Generically in the coefficients, the two quartics (4.8) and (4.9) will not have any simultaneous roots. Therefore, solutions will generically be hyperbolic. ■

4.2. Liapunov–Schmidt reduction with a flow-invariant synchrony subspace. In this section we discuss the effect that (1.23) has on the derivatives of the reduced mapping ϕ. The existence of a flow-invariant synchrony subspace S forces certain derivatives of the reduced equation ϕ to vanish because trajectories in S are trapped in S for all time. The space of 2π-periodic solutions that lie inside S forms a subspace of $\mathcal{C}_{2\pi}$, and so we define

$$\mathcal{S}_{2\pi} = \{u \in \mathcal{C}_{2\pi} \; : \; u(s) \in S \text{ for all } s\}\,. \tag{4.10}$$

The essential point is that if $u_1, \ldots, u_k \in \mathcal{S}_{2\pi}$, then $\mathrm{d}^k\Phi(u_1, \ldots, u_k) \in \mathcal{R}$ and thus vanishes under the projection $(I - E)$ onto $\mathcal{K}^*$. This and related results are given by the following lemma.

Lemma 4.4. *Suppose that a coupled cell network has a synchrony subspace S satisfying* (1.23)*. Then*

(a) *if $w \in S$, then*

$$w^t \bar{d} = 0; \tag{4.11}$$

(b) *if $c_1, \ldots, c_k \in S$, then*

$$\mathrm{d}^k F(c_1, \ldots, c_k) \in S; \tag{4.12}$$

(c) *if $u_1, \ldots, u_k \in \mathcal{S}_{2\pi}$, then*

$$\mathrm{d}^k\Phi(u_1, \ldots, u_k) \in \mathcal{S}_{2\pi}; \tag{4.13}$$

(d)

$$\mathcal{S}_{2\pi} \subseteq \mathcal{R}. \tag{4.14}$$

Proof. To prove (4.11), note that any synchrony subspace, being invariant under J, must be the direct sum of eigenspaces of J. In the proof of Lemma 2.2 it was shown that the only eigenvector to which $\bar{d}$ is not orthogonal is the generalized eigenvector b, which is not contained in S by hypothesis (1.23). Therefore $\bar{d}$ is orthogonal to every vector in S.

The statement in (4.12) follows simply because S is invariant for F and hence for all differentials of F. Similarly for (4.13), $\mathcal{S}_{2\pi}$ is invariant for Φ and hence for all differentials of Φ.

Finally, to prove (4.14) note that, by the definitions of v_1^* and v_2^* in (2.7), and by (4.11), trajectories in $\mathcal{S}_{2\pi}$ are orthogonal to v_1^* and v_2^*. So $\mathcal{S}_{2\pi} \subseteq (\mathcal{K}^*)^\perp = \mathcal{R}$ by (2.11). ■

With these results in mind we compute the second partial derivatives of p and q. To simplify these calculations we assume that F is odd, since it turns out that this is sufficient for our purposes in section 4.3. To begin, we require the following lemma.

Lemma 4.5. *Assume that a coupled cell system satisfies* (1.23) *with flow-invariant subspace S and that F is odd. Then*

$$W_{xx}(\mathbf{0}) = W_{xxxx}(\mathbf{0}) = W_{\tau xx}(\mathbf{0}) = W_{\lambda xx}(\mathbf{0}) = 0 \tag{4.15}$$

and

$$W_{xxx}(\mathbf{0}) = -\frac{1}{4}\mathrm{Re}\{e^{3is}\eta_3 + e^{is}\eta_1\}, \tag{4.16}$$

where η_1 *and* η_3 *are such that*

$$(J - iI_n)\eta_1 = 3\mathrm{d}^3F(c, c, \bar{c}) \quad \text{and} \quad (J - 3iI_n)\eta_3 = \mathrm{d}^3F(c, c, c). \tag{4.17}$$

Proof. The statements in (4.15) follow immediately from the assumption that F is odd. This assumption also implies that formula (A.11) in the appendix becomes

$$W_{xxx} = -\mathcal{L}^{-1}(\mathrm{d}^3\Phi(v_1, v_1, v_1)).$$

By (3.6) it follows that W_{xxx} is the solution to the differential equation

$$\mathcal{L}W_{xxx} = \frac{1}{4}\mathrm{Re}\big\{e^{3is}\mathrm{d}^3F(c, c, c) + 3e^{is}\mathrm{d}^3F(c, c, \bar{c})\big\}.$$

It is straightforward to verify that this solution is given by (4.16). ■

Then we have the following.

Proposition 4.6. *Assume the same hypotheses as in Lemma 4.5. Then*

$$p_{uu}(\mathbf{0}) = \frac{5}{16}\mathrm{Re}\{\xi_1^t\bar{d}\}, \qquad q_{uu}(\mathbf{0}) = -\frac{5}{16}\mathrm{Im}\{\xi_1^t\bar{d}\}, \tag{4.18}$$

$$p_{u\tau}(\mathbf{0}) = -\frac{1}{8}\mathrm{Im}\{\xi_2^t\bar{d}\}, \qquad q_{u\tau}(\mathbf{0}) = -\frac{1}{8}\mathrm{Re}\{\xi_2^t\bar{d}\}, \tag{4.19}$$

$$p_{u\lambda}(\mathbf{0}) = \frac{1}{8}\mathrm{Re}\{\xi_3^t\bar{d}\}, \qquad q_{u\lambda}(\mathbf{0}) = -\frac{1}{8}\mathrm{Im}\{\xi_3^t\bar{d}\}, \tag{4.20}$$

where

$$\xi_1 = 2\mathrm{d}^3F(c, \bar{c}, \eta_1) + \mathrm{d}^3F(c, c, \bar{\eta}_1), \tag{4.21}$$

$$\xi_2 = 6\mathrm{d}^3F(c, \bar{c}, b) - 3\mathrm{d}^3F(c, c, \bar{b}) + \eta_1, \tag{4.22}$$

$$\xi_3 = 6\mu'(0)\mathrm{d}^3F(c, \bar{c}, b) + 3\bar{\mu}'(0)\mathrm{d}^3F(c, c, \bar{b}) + J'(0)\eta_1, \tag{4.23}$$

where $\mu'(0) = \sigma'(0) + i\omega'(0)$.

Proof. Applying Lemmas 4.4 and 4.5 to formula (A.2) yields

$$\phi_{xxxxx} = 10(I - E)\mathrm{d}^3\Phi(v_1, v_1, W_{xxx}). \tag{4.24}$$

Using (4.16) and the linearity of $\mathrm{d}^3\Phi$, we compute

$$\begin{aligned}\mathrm{d}^3\Phi(v_1, v_1, W_{xxx}) = \tfrac{1}{16}\mathrm{Re}\big\{&e^{5is}\mathrm{d}^3F(c, c, \eta_3) \\ &+ e^{3is}\big(\mathrm{d}^3F(c, c, \eta_1) + 2\mathrm{d}^3F(c, \bar{c}, \eta_3)\big) \\ &+ e^{is}\big(2\mathrm{d}^3F(c, \bar{c}, \eta_1) + \mathrm{d}^3F(c, c, \bar{\eta}_1) + \mathrm{d}^3F(\bar{c}, \bar{c}, \eta_3)\big)\big\}.\end{aligned} \tag{4.25}$$

Note that, by (4.17), $\eta_3 \in S$ since $\mathrm{d}^3F(c,c,c) \in S$ and S is invariant under $J-3iI_n$. Therefore $e^{is}\mathrm{d}^3F(\bar{c},\bar{c},\eta_3) \in \mathcal{R}$ by Lemma 4.4. Since only terms involving e^{is} have nonzero projection onto $\mathcal{K}^*$, we obtain

$$\phi_{xxxxx,j} = 10\left\langle v_j^*, \mathrm{d}^3\Phi(v_1,v_1,W_{xxx})\right\rangle = \frac{5}{8}\left\langle v_j^*, \mathrm{Re}\left\{e^{is}\left(2\mathrm{d}^3F(c,\bar{c},\eta_1) + \mathrm{d}^3F(c,c,\bar{\eta}_1)\right)\right\}\right\rangle.$$

From this it is straightforward to verify that

$$\phi_{xxxxx,1} = \frac{5}{16}\mathrm{Re}\{\xi_1^t\bar{d}\} \quad \text{and} \quad \phi_{xxxxx,2} = -\frac{5}{16}\mathrm{Im}\{\xi_1^t\bar{d}\},$$

where ξ_1 is as defined in (4.21).

Now consider the formulas in (4.19) and (4.20). Using formula (A.8) and recalling from (2.21) and (4.15) that $W_\tau = W_\lambda = W_{xx} = 0$, we obtain

$$\phi_{\alpha xxx} = (I-E)\left(3\mathrm{d}^3\Phi(v_1,v_1,W_{\alpha x}) + \mathrm{d}^3\Phi_\alpha(v_1,v_1,v_1) + \mathrm{d}\Phi_\alpha(W_{xxx})\right), \tag{4.26}$$

where α is either τ or λ.

Consider the case where $\alpha = \tau$. By the fact that $\mathrm{d}^k\Phi_\tau = \frac{\mathrm{d}}{\mathrm{d}s}$ if $k=1$ and 0 if $k>1$, we obtain

$$\mathrm{d}^3\Phi_\tau(v_1,v_1,v_1) = 0 \quad \text{and} \quad \mathrm{d}\Phi_\tau(W_{xxx}) = \frac{1}{4}\mathrm{Im}\{3e^{3is}\eta_3 + e^{is}\eta_1\}. \tag{4.27}$$

Since $W_{\tau x} = -u_2$ by (2.32) we compute

$$\mathrm{d}^3\Phi(v_1,v_1,W_{\tau x}) = \frac{1}{4}\mathrm{Im}\left\{e^{3is}\mathrm{d}^3F(c,c,b) + e^{is}(2\mathrm{d}^3F(c,\bar{c},b) - \mathrm{d}^3F(c,c,\bar{b}))\right\}. \tag{4.28}$$

Substituting (4.27) and (4.28) into (4.26), we obtain

$$\phi_{\tau xxx,j} = \frac{1}{4}\left\langle v_j^*, \mathrm{Im}\left\{e^{is}\left(6\mathrm{d}^3F(c,\bar{c},b) - 3\mathrm{d}^3F(c,c,\bar{b}) + \eta_1\right)\right\}\right\rangle$$

and hence

$$\phi_{\tau xxx,1} = -\frac{1}{8}\mathrm{Im}\{\xi_2^t\bar{d}\} \quad \text{and} \quad \phi_{\tau xxx,2} = -\frac{1}{8}\mathrm{Re}\{\xi_2^t\bar{d}\},$$

where ξ_2 is defined in (4.22).

In the case when $\alpha = \lambda$ we have from (4.26)

$$\phi_{\lambda xxx} = (I-E)\left(3\mathrm{d}^3\Phi(v_1,v_1,W_{\lambda x}) + \mathrm{d}^3\Phi_\lambda(v_1,v_1,v_1) + \mathrm{d}\Phi_\lambda(W_{xxx})\right). \tag{4.29}$$

Observe that $\mathrm{d}^3\Phi_\lambda(v_1,v_1,v_1) = -\mathrm{d}^3F_\lambda(v_1,v_1,v_1)$ and note that F_λ is a vector field on the same network as F and therefore has the same flow-invariant subspaces. Hence, by Lemma 4.4, $(I-E)\mathrm{d}^3\Phi(v_1,v_1,v_1) = 0$ since $v_1 \in \mathcal{S}_{2\pi}$. Then using formulas (2.33) for $W_{\lambda x}$ and (4.16) for W_{xxx} we have

$$\begin{aligned}
\mathrm{d}^3\Phi(v_1,v_1,W_{\lambda x}) = \frac{1}{4}\mathrm{Re}\Big\{&e^{3is}\left(\mu'(0)\mathrm{d}^3F(c,c,b) - \mathrm{d}^3F(c,c,c'(0)-\zeta c)\right) \\
&+ e^{is}\big(2\mu'(0)\mathrm{d}^3F(c,\bar{c},b) + \bar{\mu}'(0)\mathrm{d}^3F(c,c,\bar{b}) \\
&\qquad - 2\mathrm{d}^3F(c,\bar{c},c'(0)-\zeta c) - \mathrm{d}^3F(c,c,\bar{c}'(0)-\bar{\zeta}\bar{c})\big)\Big\}, \\
\mathrm{d}\Phi_\lambda(W_{xxx}) = \frac{1}{4}\mathrm{Re}\{&e^{3is}J'(0)\eta_3 + e^{is}J'(0)\eta_1\},
\end{aligned}$$

where ζ is given by (2.34) and $c(\lambda)$ is the continuation of the critical eigenvector for $J(\lambda)$ so that $c(0) = c$. Note that $c(\lambda) \in S$ for all λ. Therefore, $c'(0) \in S$, and $\mathrm{d}^3F(c, c, c'(0) - \zeta c) \in S$ by (4.12). Thus we obtain

$$\phi_{\lambda xxx,j} = \frac{1}{4}\left\langle v_j^*, \mathrm{Re}\left\{e^{is}\left(6\mu'(0)\mathrm{d}^3F(c,\bar{c},b) + 3\bar{\mu}'(0)\mathrm{d}^3F(c,c,\bar{b}) + J'(0)\eta_1\right)\right\}\right\rangle.$$

It is now straightforward to verify that

$$\phi_{\lambda xxx,1} = \frac{1}{8}\mathrm{Re}\{\xi_3^t\bar{d}\} \quad \text{and} \quad \phi_{\lambda xxx,2} = -\frac{1}{8}\mathrm{Im}\{\xi_3^t\bar{d}\},$$

where ξ_3 is as in (4.23). ■

4.3. The three-cell network of Figure 2. In this section we show that the network in Figure 2 can have bifurcations with either two or four branches of solutions, depending on the specific vector field, as stated in Theorem 1.4. Recall that $S = \{(u, u, v) : u, v \in \mathbf{R}^n\}$ is a flow-invariant synchrony subspace for the network, and that the critical eigenvector has the form $c = (a, a, -a)$, where $Aa = ia$. The corresponding generalized eigenvector has the form

$$b = \zeta \begin{pmatrix} a \\ -a \\ 0 \end{pmatrix} + \begin{pmatrix} w \\ w \\ -w \end{pmatrix},$$

where ζ and w are chosen so that $(A - iI_k)w = -(\zeta B - I_k)u$. Hence, S contains the critical eigenspace but not the generalized eigenspace, since $(u, -u, 0)$ breaks synchrony. Therefore the network satisfies (1.23), and thus Proposition 1.6 applies and (1.18) holds. In Proposition 4.7 we use the formulas in Proposition 4.6 to show that the second partial derivatives of p and q are independent and arbitrary, and hence that (1.19) and (1.20) also hold generically. Therefore, in this network the coefficients of (4.2) can be varied to obtain either two or four branches of solutions, as stated in Theorem 1.4.

Proposition 4.7. *The partial derivatives $p_{uu}(\mathbf{0})$, $p_{u\lambda}(\mathbf{0})$, $p_{u\tau}(\mathbf{0})$, $q_{uu}(\mathbf{0})$, $q_{u\lambda}(\mathbf{0})$, and $q_{u\tau}(\mathbf{0})$ are arbitrary and can be varied independently for the network defined by* (1.4).

Proof. We show that the real and imaginary parts of $\xi_j^t\bar{d}$ are arbitrary and independent, for ξ_j defined in (4.21) to (4.23). In fact, we can do this in a very restricted setting to simplify the calculations. Showing that the derivatives of p and q are arbitrary and independent under restrictive assumptions clearly implies that the result holds for generic vector fields on this network.

The first of these assumptions is that each cell has two-dimensional internal dynamics so that

$$z_1 = \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad z_2 = \begin{pmatrix} x_3 \\ x_4 \end{pmatrix}, \quad \text{and} \quad z_3 = \begin{pmatrix} x_5 \\ x_6 \end{pmatrix}.$$

We also assume that

$$A(0) = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad B(0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrix $A'(0)$ is left arbitrary.

Let $a = (i, 1)^t$ be the eigenvector of A with eigenvalue i, and set

$$c = (a, a, -a)^t. \tag{4.30}$$

Then since $B = I_2$, the corresponding generalized eigenvector is

$$b = (a, -a, 0)^t. \tag{4.31}$$

Finally, observe that $a = (i, 1)^t$ is also an eigenvector of A^t such that $A^t a = -ia$, so

$$d = (a, -a, 0)^t$$

is an eigenvector of J^t with eigenvalue $-i$.

We also make a number of assumptions about the higher derivatives of F. As in Lemma 4.5 and Proposition 4.6, we assume that F is odd. Since $n = 2$ we can write

$$f(u, \overline{v, w}, \lambda) = \begin{pmatrix} g(u, \overline{v, w}, \lambda) \\ h(u, \overline{v, w}, \lambda) \end{pmatrix},$$

where $g, h : \mathbf{R}^6 \times \mathbf{R} \longrightarrow \mathbf{R}$, with u being the internal variable and v and w being the input variables. Now set all third derivatives equal to zero except for $f_{u_1u_1u_1}$, which we fix at

$$f_{u_1u_1u_1} = \begin{pmatrix} g_{u_1u_1u_1} \\ h_{u_1u_1u_1} \end{pmatrix} = \begin{pmatrix} 2 \\ 0 \end{pmatrix}, \tag{4.32}$$

and

$$f_{u_2u_2v_2} = \begin{pmatrix} g_{u_2u_2v_2} \\ h_{u_2u_2v_2} \end{pmatrix} \quad \text{and} \quad f_{v_1v_1w_2} = \begin{pmatrix} g_{v_1v_1w_2} \\ h_{v_1v_1w_2} \end{pmatrix},$$

which we leave arbitrary. Note that $f_{u_2u_2v_2} = f_{u_2u_2w_2}$ and $f_{v_1v_1w_2} = f_{w_1w_1v_2}$ by the invariance of f.

First we compute η_1 and show that $[J'(0)\eta_1]^t \bar{d} \neq 0$. Then it follows that $\xi_3^t \bar{d}$ is arbitrary and independent of $\xi_1^t \bar{d}$ and $\xi_2^t \bar{d}$ due to the occurrence of the term $J'(0)\eta_1$ in (4.23). Finally we show that under the above assumptions about F,

$$\begin{pmatrix} \xi_1^t \bar{d} \\ \xi_2^t \bar{d} \end{pmatrix} = 3 \begin{pmatrix} -6i & 6 & -4i & 4 \\ -i & 1 & 2i & -2 \end{pmatrix} \begin{pmatrix} g_{u_2u_2v_2} \\ h_{u_2u_2v_2} \\ g_{v_1v_1w_2} \\ h_{v_1v_1w_2} \end{pmatrix} + \begin{pmatrix} 72 \\ 24 \end{pmatrix}. \tag{4.33}$$

Because the matrix in (4.33) has full rank, and because $g_{u_2u_2v_2}$, $h_{u_2u_2v_2}$, $g_{v_1v_1w_2}$, and $h_{v_1v_1w_2}$ are arbitrary, the real and imaginary parts of $\xi_1^t \bar{d}$ and $\xi_2^t \bar{d}$ can be manipulated arbitrarily from just these four derivatives of f. Therefore, the coefficients in (4.2) are arbitrary and can be varied independently for this network.

Recall from (4.17) that η_1 is such that $(J - iI_n)\eta_1 = 3\mathrm{d}^3F(c, c, \bar{c})$. Using (4.32), we compute

$$\mathrm{d}^3F(c, c, \bar{c}) = \begin{pmatrix} if_{u_1u_1u_1} \\ if_{u_1u_1u_1} \\ -if_{u_1u_1u_1} \end{pmatrix} = \begin{pmatrix} 2i \\ 0 \\ 2i \\ 0 \\ -2i \\ 0 \end{pmatrix}.$$

Then it is easy to verify that

$$\eta_1 = 3b - \frac{3i}{2}\bar{c} \tag{4.34}$$

by using the definitions of c and b in (4.30) and (4.31). Hence

$$[J'(0)\eta_1]^t\bar{d} = 3[J'(0)b]^t\bar{d} - \frac{3i}{2}[J'(0)\bar{c}]^t\bar{d}.$$

Observe that $J'(0)\bar{c} \in S$ since $\bar{c} \in S$ and S is invariant for $J'(0)$. So by (4.11), $[J'(0)\bar{c}]^t\bar{d} = 0$. Also observe that

$$J'(0)b = \begin{pmatrix} A'(0)a \\ -A'(0)a \\ 0 \end{pmatrix} + \begin{pmatrix} B'(0)a \\ B'(0)a \\ -B'(0)a \end{pmatrix},$$

so that

$$[J'(0)\eta_1]^t\bar{d} = 6[A'(0)a]^t\bar{a}$$

since $(B'(0)a, B'(0)a, -B'(0)a)^t \in S$ is orthogonal to $\bar{d}$. Since the entries in $A'(0)$ are arbitrary, $[J'(0)\eta_1]^t\bar{d}$ is an arbitrary complex number, and therefore $p_{u\lambda}(\mathbf{0})$ and $q_{u\lambda}(\mathbf{0})$ can be varied independently from each other. Furthermore, since only $\xi_3^t\bar{d}$ depends on this term, $p_{u\lambda}(\mathbf{0})$ and $q_{u\lambda}(\mathbf{0})$ are independent of the other derivatives. We will therefore consider only $\xi_1^t\bar{d}$ and $\xi_2^t\bar{d}$ from here on.

By plugging (4.34) into the expressions for ξ_1 and ξ_2 in (4.21) and (4.22), using the linearity of d^3F, and observing that $\bar{c}^t\bar{d} = c^t\bar{d} = 0$, we obtain

$$\begin{aligned} \xi_1^t\bar{d} &= 12\mathrm{d}^3F(c, \bar{c}, b)^t\bar{d} + 6\mathrm{d}^3F(c, c, \bar{b})^t\bar{d}, \\ \xi_2^t\bar{d} &= 6\mathrm{d}^3F(c, \bar{c}, b)^t\bar{d} - 3\mathrm{d}^3F(c, c, \bar{b})^t\bar{d} + 12. \end{aligned} \tag{4.35}$$

Using the definitions of $c = (a, a, -a)$ and $b = (a, -a, 0)$, we obtain by direct calculation

$$\mathrm{d}^3F(c, \bar{c}, b) = \begin{pmatrix} if_{u_1u_1u_1} + f_{v_1v_1w_2} + 2f_{u_2u_2v_2} \\ -if_{u_1u_1u_1} + f_{v_1v_1w_2} + f_{u_2u_2v_2} \\ -3f_{v_1v_1w_2} - f_{u_2u_2v_2} \end{pmatrix},$$

$$\mathrm{d}^3F(c, c, \bar{b}) = \begin{pmatrix} if_{u_1u_1u_1} - f_{v_1v_1w_2} + 2f_{u_2u_2v_2} \\ -if_{u_1u_1u_1} - 3f_{v_1v_1w_2} + f_{u_2u_2v_2} \\ 3f_{v_1v_1w_2} - f_{u_2u_2v_2} \end{pmatrix},$$

and thus we compute (4.35) as

$$\begin{aligned}\xi_1^t \bar{d} &= 6[6if_{u_1u_1u_1} + 3f_{u_2u_2v_2} + 2f_{v_1v_1w_2}]^t \bar{a}\\ &= -18(ig_{u_2u_2v_2} - h_{u_2u_2v_2}) - 12(ig_{v_1v_1w_2} - h_{v_1v_1w_2}) + 72,\\ \xi_2^t \bar{d} &= 3[2if_{u_1u_1u_1} + f_{u_2u_2v_2} - 2f_{v_1v_1w_2}]^t \bar{a} + 12\\ &= -3(ig_{u_2u_2v_2} - h_{u_2u_2v_2}) + 6(ig_{v_1v_1w_2} - h_{v_1v_1w_2}) + 24.\end{aligned}$$

It is a straightforward matter to show that this can be written as (4.33). ■

5. Hopf bifurcation in the feed-forward chain.

The proof of Theorem 1.5 divides into two parts. First, in section 5.1 we prove the following.

Proposition 5.1. *At a nilpotent Hopf bifurcation in the feed-forward chain there exist two branches of near 2π-periodic solutions, one growing as $\lambda^{\frac{1}{2}}$ and the other growing as $\lambda^{\frac{1}{6}}$.*

Then in section 5.2 we prove the following.

Proposition 5.2. *The branches given by Proposition 5.1 are generically the only branches.*

Before we delve into the proofs of Propositions 5.1 and 5.2 we need to consider the various invariant subspaces that play a role later on. Recall that the critical eigenvector of J is $c = (0, 0, a)^t$, where a is an eigenvector of A with eigenvalue i. The corresponding generalized eigenvector is

$$b = (0, \zeta a, w), \tag{5.1}$$

where $\zeta \in \mathbf{C}$ and $w \in \mathbf{C}^k$ are chosen so that $w^t \bar{a} = 0$ and $(A - iI_n)w = -(\zeta B - I_n)a$, and hence $\bar{c}^t b = 0$.

Observe that

$$\begin{aligned} S &= \left\{(u, u, v) \;:\; u, v \in \mathbf{R}^k\right\},\\ \hat{S} &= \left\{(0, 0, u) \;:\; u \in \mathbf{R}^k\right\}\end{aligned} \tag{5.2}$$

are flow-invariant subspaces for (1.3), which both contain the critical eigenvector c but not the generalized eigenvector given by (5.1). The feed-forward chain thus satisfies (1.23) for both S and $\hat{S}$. Thus Proposition 1.6 implies that there exists a branch of solutions in $\hat{S}$ that grows as $O(\lambda^{\frac{1}{2}})$.

By analogy with (4.10), we define

$$\hat{\mathcal{S}}_{2\pi} = \left\{u \in \mathcal{C}_{2\pi} \;:\; u(s) \in \hat{S} \text{ for all } s\right\}.$$

Recall from Lemma 4.4 that $\mathcal{S}_{2\pi} \subset \mathcal{R}$. Thus we have

$$\hat{\mathcal{S}}_{2\pi} \subset \mathcal{S}_{2\pi} \subset \mathcal{R}. \tag{5.3}$$

Suppose that (1.3) depends on a bifurcation parameter λ and undergoes a nilpotent Hopf bifurcation at $\lambda = 0$. Assume that the eigenvalues of $B = B(0)$ have negative real part and that z_1 has reached its asymptotic state $z_1 = 0$, so that we can restrict our attention to the subsystem

$$\begin{aligned} \dot{y} &= f(y, 0, \lambda) = g(y, \lambda), \\ \dot{z} &= f(z, y, \lambda) = h(z, y, \lambda) \end{aligned} \tag{5.4}$$

with linearization

$$J(\lambda) = (\mathrm{d}F)_{0,\lambda} = \begin{pmatrix} A(\lambda) & 0 \\ B(\lambda) & A(\lambda) \end{pmatrix}.$$

Let $\mu(\lambda) = \sigma(\lambda) + i\omega(\lambda)$ such that $\sigma(0) = 0$ and $\omega(\lambda) = 1$ be the continuation of the critical eigenvalue of $A(0)$.

Proposition 5.1 is proved in [2, Lemma 6.1] with the assumption that the normal form of the vector field on the center manifold is $\mathbf{S}^1$-equivariant, where $\mathbf{S}^1$ acts as

$$f(e^{i\theta} y, e^{i\theta} z) = e^{i\theta} f(y, z). \tag{5.5}$$

In section 5.1 we show that this assumption is satisfied generically, and so [2, Lemma 6.1] holds in full generality. Specifically, we prove the following.

Proposition 5.3. *Up to third order, the normal form of the subsystem* (5.4) *is*

$$\begin{aligned} \dot{y} &= \mu(\lambda) y + c_3(\lambda) y^2 \bar{y} + O(5), \\ \dot{z} &= \mu(\lambda) z + y + c_3(\lambda) z^2 \bar{z} + \alpha \bar{y} z^2 + \beta y z \bar{z} + O(4), \end{aligned} \tag{5.6}$$

which is $\mathbf{S}^1$*-equivariant under the action given by* (5.5).

Because of the work in [2], Proposition 5.3 suffices to prove Proposition 5.1 for the truncated equations without higher order terms. For the truncated equations the $\lambda^{1/6}$ branch is also shown in [2] to consist of asymptotically stable solutions. Hence, a scaling argument may be used to prove the result for (5.6).

The following two lemmas prove Proposition 5.2 and hence Theorem 1.5.

Lemma 5.4. *At a nilpotent Hopf bifurcation in the feed-forward chain the reduced equation* (1.10) *satisfies* (1.21).

Lemma 5.5. *The feed-forward chain generically satisfies* (1.22).

These lemmas are proved in section 5.2 and use Liapunov–Schmidt reduction instead of the normal form from (5.6). It may be possible to prove Theorem 1.5 using only normal form methods, but this requires proving that (5.6) is equivariant to all orders. While we fully expect this to be the case, a proof remains elusive.

5.1. Proof of Proposition 5.3.

By [2, Lemma 6.2] there exists a center manifold $\mathcal{M}$ for (5.4) such that the vector field on $\mathcal{M}$ is in skew-product form. On this center manifold we can therefore change coordinates to put $(\mathrm{d}F)_{0,\lambda}$ in complex Jordan form:

$$(\mathrm{d}F)_{0,\lambda} = \begin{pmatrix} A(\lambda) & 0 \\ I_2 & A(\lambda) \end{pmatrix},$$

where

$$A(\lambda) = |\mu(\lambda)| \begin{pmatrix} e^{i\theta(\lambda)} & 0 \\ 0 & e^{-i\theta(\lambda)} \end{pmatrix}.$$

Thus we consider

$$\begin{aligned} \dot{y} &= |\mu(\lambda)|e^{i\theta(\lambda)}y + G(y, \bar{y}, \lambda), \\ \dot{z} &= |\mu(\lambda)|e^{i\theta(\lambda)}z + y + H(y, \bar{y}, z, \bar{z}, \lambda), \end{aligned} \tag{5.7}$$

where G and H are $O(2)$.

First we make identical changes on y and z,

$$y \mapsto y + \phi(y, \bar{y}) \quad \text{and} \quad z \mapsto z + \phi(z, \bar{z}), \tag{5.8}$$

to put $\dot{y} = g(y, \lambda)$ in standard normal form for Hopf bifurcation. Since the changes are the same on both variables we have

$$\begin{aligned} &\text{(a)} \quad \dot{y} = \mu(\lambda)y + c(|y|^2, \lambda)y, \\ &\text{(b)} \quad \dot{z} = \mu(\lambda)z + c(|z|^2, \lambda)z + y + H_2 + H_3 + \cdots, \end{aligned} \tag{5.9}$$

where H_j is order j in y, $\bar{y}$, z, and $\bar{z}$, and

$$H_j(0, 0, z, \bar{z}, \lambda) = 0 \tag{5.10}$$

since all the terms depending only on z and $\bar{z}$ appear in $c(|z|^2, \lambda)z$.

Next we make changes of the form

$$y \mapsto y \quad \text{and} \quad z \mapsto \phi(y, \bar{y}, z, \bar{z}), \tag{5.11}$$

where ϕ is order 2. Substituting this into (5.9b), we obtain

$$\dot{z} = \mu z + y + \mu\phi - \mu\phi_y y - \bar{\mu}\phi_{\bar{y}}\bar{y} - \mu\phi_z z - \phi_z y - \bar{\mu}\phi_{\bar{z}}\bar{z} - \phi_{\bar{z}}\bar{y} + H_2 + O(3),$$

and so second order terms can be eliminated if we can choose ϕ so that

$$\mu\phi - \mu\phi_y y - \bar{\mu}\phi_{\bar{y}}\bar{y} - \mu\phi_z z - \phi_z y - \bar{\mu}\phi_{\bar{z}}\bar{z} - \phi_{\bar{z}}\bar{y} + H_2 = 0.$$

Let M_k denote the space of order k monomials in y, $\bar{y}$, z, and $\bar{z}$, and define

$$\tilde{M}_k = M_{k-1}y + M_{k-1}\bar{y}.$$

Then

$$\tilde{M}_2 = \text{span}\left\{y^2, y\bar{y}, yz, y\bar{z}, \bar{y}^2, \bar{y}z, \bar{y}\bar{z}\right\}, \tag{5.12}$$

and H_2 is a linear combination of elements of $\tilde{M}_2$ by (5.10).

Define the map $\Psi_2 : \tilde{M}_2 \longrightarrow \tilde{M}_2$ by

$$\Psi_2(\phi) = \mu\phi - \mu\phi_y y - \bar{\mu}\phi_{\bar{y}}\bar{y} - \mu\phi_z z - \phi_z y - \bar{\mu}\phi_{\bar{z}}\bar{z} - \phi_{\bar{z}}\bar{y}.$$

Then we seek solutions to the linear equation

$$\Psi_2(\phi) + H_2 = 0.$$

The action of Ψ_2 on the basis elements in (5.12) yields

$$\begin{aligned} y^2 &\mapsto -\mu y^2, & y\bar{y} &\mapsto -\bar{\mu} y\bar{y}, \\ yz &\mapsto -\mu yz - y^2, & y\bar{z} &\mapsto -\bar{\mu} y\bar{z} - y\bar{y}, \\ \bar{y}^2 &\mapsto (\mu - 2\bar{\mu})\bar{y}^2, & \bar{y}z &\mapsto -\bar{\mu}\bar{y}z - y\bar{y}. \\ \bar{y}\bar{z} &\mapsto (\mu - 2\bar{\mu})\bar{y}\bar{z} - \bar{y}^2, \end{aligned}$$

Thus, with respect to this basis, Ψ_2 can be written as

	y^2	$y\bar{y}$	yz	$y\bar{z}$	$\bar{y}^2$	$\bar{y}z$	$\bar{y}\bar{z}$
y^2	$-\mu$	$\cdot$	-1	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$y\bar{y}$	$\cdot$	$-\bar{\mu}$	$\cdot$	-1	$\cdot$	-1	$\cdot$
yz	$\cdot$	$\cdot$	$-\mu$	$\cdot$	$\cdot$	$\cdot$	$\cdot$
$y\bar{z}$	$\cdot$	$\cdot$	$\cdot$	$-\bar{\mu}$	$\cdot$	$\cdot$	$\cdot$
$\bar{y}^2$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\mu - 2\bar{\mu}$	$\cdot$	-1
$\bar{y}z$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$-\bar{\mu}$	$\cdot$
$\bar{y}\bar{z}$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\cdot$	$\mu - 2\bar{\mu}$

Note that $\mu(0) \neq 0$ and $\mu(0) = -\bar{\mu}(0)$, so for sufficiently small λ, $\mu(\lambda) \neq 0$ and $\mu - 2\bar{\mu} \neq 0$. Therefore, for sufficiently small λ, range $\Psi_2 = \tilde{M}_2$, so all quadratics may be eliminated.

Moving on to cubic terms, we again make changes of the form (5.11), but with ϕ being order 3. Making this substitution into (5.9b), we obtain

$$\dot{z} = \mu z + y + \mu\phi - \mu\phi_y y - \bar{\mu}\phi_{\bar{y}}\bar{y} - \mu\phi_z z - \phi_z y - \bar{\mu}\phi_{\bar{z}}\bar{z} - \phi_{\bar{z}}\bar{y} + c_3 z^2\bar{z} + H_3 + O(4),$$

and so third order terms can be eliminated if we solve the linear equation $\Psi_3(\phi) + H_3 = 0$, where $\Psi_3 : \tilde{M}_3 \longrightarrow \tilde{M}_3$ is defined by

$$\Psi_3(\phi) = \mu\phi - \mu\phi_y y - \bar{\mu}\phi_{\bar{y}}\bar{y} - \mu\phi_z z - \phi_z y - \bar{\mu}\phi_{\bar{z}}\bar{z} - \phi_{\bar{z}}\bar{y}.$$

Observe that

$$\tilde{M}_3 = \operatorname{span}\{y^3, y^2\bar{y}, y\bar{y}^2, y^2z, y^2\bar{z}, y\bar{y}z, y\bar{y}\bar{z}, yz^2, yz\bar{z}, y\bar{z}^2, \bar{y}^3, \bar{y}^2z, \bar{y}^2\bar{z}, \bar{y}z^2, \bar{y}z\bar{z}, \bar{y}\bar{z}^2\}. \tag{5.13}$$

Table 1

Matrix representation of the mapping $\Psi_3 : \tilde{M}_3 \longrightarrow \tilde{M}_3$ *with respect to the basis in* (5.13), *where* $\alpha = -(\mu + \bar{\mu})$ *and* $\beta = \mu - 3\bar{\mu}$.

	y^3	$y^2\bar{y}$	$y\bar{y}^2$	y^2z	$y^2\bar{z}$	$y\bar{y}z$	$y\bar{y}\bar{z}$	yz^2	$yz\bar{z}$	$y\bar{z}^2$	$\bar{y}^3$	$\bar{y}^2z$	$\bar{y}^2\bar{z}$	$\bar{y}z^2$	$\bar{y}z\bar{z}$	$\bar{y}\bar{z}^2$
y^3	-2μ	·	·	-1	·	·	·	·	·	·	·	·	·	·	·	·
$y^2\bar{y}$	·	α	·	·	-1	-1	·	·	·	·	·	·	·	·	·	·
$y\bar{y}^2$	·	·	$-2\bar{\mu}$	·	·	·	-1	·	·	·	·	-1	·	·	·	·
y^2z	·	·	·	-2μ	·	·	·	-2	·	·	·	·	·	·	·	·
$y^2\bar{z}$	·	·	·	·	α	·	·	·	-1	·	·	·	·	·	·	·
$y\bar{y}z$	·	·	·	·	·	α	·	·	-1	·	·	·	·	-2	·	·
$y\bar{y}\bar{z}$	·	·	·	·	·	·	$-2\bar{\mu}$	·	·	-2	·	·	·	·	-1	·
yz^2	·	·	·	·	·	·	·	-2μ	·	·	·	·	·	·	·	·
$yz\bar{z}$	·	·	·	·	·	·	·	·	α	·	·	·	·	·	·	·
$y\bar{z}^2$	·	·	·	·	·	·	·	·	·	$-2\bar{\mu}$	·	·	·	·	·	·
$\bar{y}^3$	·	·	·	·	·	·	·	·	·	·	β	·	-1	·	·	·
$\bar{y}^2z$	·	·	·	·	·	·	·	·	·	·	·	$-2\bar{\mu}$	·	·	-1	·
$\bar{y}^2\bar{z}$	·	·	·	·	·	·	·	·	·	·	·	·	β	·	·	-2
$\bar{y}z^2$	·	·	·	·	·	·	·	·	·	·	·	·	·	α	·	·
$\bar{y}z\bar{z}$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	$-2\bar{\mu}$	·
$\bar{y}\bar{z}^2$	·	·	·	·	·	·	·	·	·	·	·	·	·	·	·	β

Then Ψ_3 applied to each of the basis elements in (5.13) yields

$$
\begin{aligned}
y^3 &\mapsto -2\mu y^3, & y^2\bar{y} &\mapsto -(\mu+\bar{\mu})y^2\bar{y},\\
y\bar{y}^2 &\mapsto -2\bar{\mu}y\bar{y}^2, & y^2z &\mapsto -2\mu y^2 z - y^3,\\
y^2\bar{z} &\mapsto -(\mu+\bar{\mu})y^2\bar{z} - y^2\bar{y}, & y\bar{y}z &\mapsto -(\mu+\bar{\mu})y\bar{y}z - y^2\bar{y},\\
y\bar{y}\bar{z} &\mapsto -2\bar{\mu}y\bar{y}\bar{z} - y\bar{y}^2, & yz^2 &\mapsto -2\mu yz^2 - 2y^2z,\\
yz\bar{z} &\mapsto -(\mu+\bar{\mu})yz\bar{z} - y^2\bar{z} - y\bar{y}z, & y\bar{z}^2 &\mapsto -2\bar{\mu}y\bar{z}^2 - 2y\bar{y}\bar{z},\\
\bar{y}^3 &\mapsto (\mu - 3\bar{\mu})\bar{y}^3, & \bar{y}^2z &\mapsto -2\bar{\mu}\bar{y}^2z - y\bar{y}^2,\\
\bar{y}^2\bar{z} &\mapsto \mu\bar{y}^2\bar{z} - 2\bar{\mu}\bar{y}^2\bar{z} - \bar{y}^3, & \bar{y}z^2 &\mapsto -(\mu+\bar{\mu})\bar{y}z^2 - 2y\bar{y}z,\\
\bar{y}z\bar{z} &\mapsto -2\bar{\mu}\bar{y}z\bar{z} - y\bar{y}\bar{z} - \bar{y}^2z, & \bar{y}\bar{z}^2 &\mapsto -(\mu+\bar{\mu})\bar{y}\bar{z}^2 - 2\bar{y}^2\bar{z}.
\end{aligned}
$$

Thus Ψ_3 can be represented by the matrix shown in Table 1.

Observe that

$$-2\mu(0) = -2i, \quad -2\bar{\mu}(0) = 2i, \quad \mu(0) - 3\bar{\mu}(0) = 4i, \quad \text{and} \quad \mu(0) + \bar{\mu}(0) = 0.$$

Thus for λ sufficiently close to 0

$$-2\mu(\lambda) \neq 0, \quad -2\bar{\mu}(\lambda) \neq 0, \quad \text{and} \quad \mu(\lambda) - 3\bar{\mu}(\lambda) \neq 0.$$

It is straightforward to check that

$$\ker \Psi|_{\lambda=0} = \left\{\bar{y}z^2, yz\bar{z}\right\}.$$

Thus for λ sufficiently small, third order terms other than $\bar{y}z^2$ or $yz\bar{z}$ can be eliminated. Therefore the normal form up to third order is as in (5.6).

It is straightforward to verify that $h(y,z) = \alpha\bar{y}z^2 + \beta yz\bar{z}$ is $\mathbf{S}^1$-equivariant under the action of (5.5). Thus it follows that (5.6) is also $\mathbf{S}^1$-equivariant. ■

5.2. Proof of Lemmas 5.4 and 5.5.

Proof of Lemma 5.4. Observe that Proposition 1.6 implies that $p_u(\mathbf{0}) = q_u(\mathbf{0}) = 0$ since the feed-forward chain satisfies (1.23) for S and $\hat{S}$. Then by Propositions 1.1 and 1.2 the general form of the Liapunov–Schmidt reduced equation is

$$\begin{aligned} 0 &= p(u,\lambda,\tau) = u\hat{p}(u,\lambda,\tau) - \tau^2 + \lambda^2 + O(|\tau,\lambda|^3), \\ 0 &= q(u,\lambda,\tau) = u\hat{q}(u,\lambda,\tau) + 2\tau\lambda + O(|\tau,\lambda|^3). \end{aligned} \tag{5.14}$$

Consider the branch of solutions on which $u = x^2$ grows at $O(\lambda^{\frac{1}{3}})$, and introduce a scaling parameter s such that $\lambda = s^3$. Then since τ scales linearly with λ we have

$$u = sv(s) \quad \text{and} \quad \tau = s^3\tilde{\tau}(s),$$

where $v(0) \neq 0$ and $\tilde{\tau}(0) \neq 0$. Then (5.14) becomes

$$\begin{aligned} 0 &= sv\hat{p}(sv, s^3, s^3\tilde{\tau}) - s^6\tilde{\tau}^2 + s^6 + O(s^9), \\ 0 &= sv\hat{q}(sv, s^3, s^3\tilde{\tau}) + 2s^6\tilde{\tau} + O(s^9). \end{aligned} \tag{5.15}$$

Expanding $\hat{p}$ and $\hat{q}$ in powers of s, we obtain

$$\begin{aligned} \hat{p}(sv, s^3, s^3\tilde{\tau}) &= s\hat{p}_u v + s^2\hat{p}_{uu}v + O(s^3), \\ \hat{q}(sv, s^3, s^3\tilde{\tau}) &= s\hat{q}_u v + s^2\hat{q}_{uu}v + O(s^3), \end{aligned}$$

and so (5.15) becomes

$$\begin{aligned} 0 &= s^2\hat{p}_u v^2 + s^3\hat{p}_{uu}v^2 + O(s^4), \\ 0 &= s^2\hat{q}_u v^2 + s^3\hat{q}_{uu}v^2 + O(s^4). \end{aligned}$$

Equating powers of s, we obtain

$$\hat{p}_u = \hat{p}_{uu} = \hat{q}_u = \hat{q}_{uu} = 0,$$

which implies the result, by definition of $\hat{p}$ and $\hat{q}$ in (5.14). ∎

Proof of Lemma 5.5. Since the feed-forward chain satisfies (1.23) for S and $\hat{S}$, Lemma 4.4 applies. However, the skew-product form of the feed-forward chain and the fact that $\hat{S}$ is not polydiagonal lead to the following stronger form of Lemma 4.4.

Lemma 5.6. *If one of the arguments $c_1, \ldots, c_m$ lies in $\hat{S}$, then*

$$\mathrm{d}^m F(c_1, \ldots, c_m) \in \hat{S}. \tag{5.16}$$

If one of the arguments $u_1, \ldots, u_m$ lies in $\hat{\mathcal{S}}_{2\pi}$, then

$$\mathrm{d}^m \Phi(u_1, \ldots, u_m) \in \hat{\mathcal{S}}_{2\pi}.$$

Proof. Suppose that one of the arguments $c_1, \dots, c_m$ lies in $\hat{S}$. Since $\mathrm{d}^m F$ is symmetric in $c_1, \dots, c_m$ we can assume without loss of generality that this is the first component. Then $c_{1,j} = 0$ if $j \le 2k$, so

$$\mathrm{d}^m F(c_1, \dots, c_m) = \sum_{\substack{i_1 = 2k+1,\dots,3k \\ i_2,\dots,i_m = 1,\dots,3k}} \left. \frac{\partial^m F}{\partial x_{i_1} \cdots \partial x_{i_m}} \right|_0 c_{1,i_1} \cdots c_{m,i_m}, \tag{5.17}$$

since all terms with $i_1 \le 2k$ vanish.

Observe that for $i_1 = 2k+1, \dots, 3k$,

$$\left. \frac{\partial^m F}{\partial x_{i_1} \cdots \partial x_{i_m}} \right|_0 = \begin{pmatrix} \frac{\partial^m f(z_1, z_1)}{\partial x_{i_1} \cdots \partial x_{i_m}} \\ \frac{\partial^m f(z_2, z_1)}{\partial x_{i_1} \cdots \partial x_{i_m}} \\ \frac{\partial^m f(z_3, z_2)}{\partial x_{i_1} \cdots \partial x_{i_m}} \end{pmatrix}_0 = \begin{pmatrix} 0 \\ 0 \\ \frac{\partial^m f(z_3, z_2)}{\partial x_{i_1} \cdots \partial x_{i_m}} \end{pmatrix}_0,$$

and so (5.17) becomes

$$\mathrm{d}^m F(c_1, \dots, c_m) = \sum_{\substack{i_1 = 2k+1,\dots,3k \\ i_2,\dots,i_m = 1,\dots,2k}} \begin{pmatrix} 0 \\ 0 \\ \frac{\partial^m f(z_3, z_2)}{\partial x_{i_1} \cdots \partial x_{i_m}} \end{pmatrix}_0 c_{1,i_1} \cdots c_{m,i_m},$$

which lies in $\hat{S}$ by the definition in (5.2).

Similarly, suppose that $u_1 \in \hat{\mathcal{S}}_{2\pi}$. Then each u_1 is a linear combination of terms of the form $e^{lis} c_1$ for some $c_1 \in \hat{S}$, $l \in \mathbf{Z}$. So $\mathrm{d}^m \Phi(u_1, \dots, u_m)$ is a linear combination of terms of the form

$$e^{lis} \mathrm{d}^m F(c_1, \dots, c_m),$$

which lie in $\hat{\mathcal{S}}_{2\pi}$ since $\mathrm{d}^m F(c_1, \dots, c_m) \in \hat{S}$ by (5.16). ■

In line with previous calculations, we assume that F is odd: if the result is true in this restricted case, then it will certainly be true generically. To simplify notation, let $\phi_{(k)}$ and $W_{(k)}$ denote the kth x derivatives of ϕ and W. Then using the formula for $\phi_{(9)}$ given in (A.4) along with Lemmas 4.5 and 5.6 and the fact that $\mathrm{d}\Phi(W_{(9)}) \in \mathcal{R}$ by definition, we obtain

$$\phi_{(9)} = 280(I - E)\mathrm{d}^3 \Phi(W_{(3)}, W_{(3)}, W_{(3)}). \tag{5.18}$$

We claim that

$$\left\langle v_j^*, \mathrm{d}^3 \Phi(W_{(3)}, W_{(3)}, W_{(3)}) \right\rangle = \left\langle v_j^*, \mathrm{Re}\{e^{is} \mathrm{d}^3 F(\eta_1, \eta_1, \bar{\eta}_1)\} \right\rangle. \tag{5.19}$$

To see this, observe that $c = (0, 0, a) \in \hat{S}$, and thus $\mathrm{d}^3 F(c, c, c)$ and $\mathrm{d}^3 F(c, c, \bar{c})$ both lie in $\hat{S}$ by (5.16). Now consider the occurrence of η_3 in the formula for $W_{(3)}$ in Lemma 4.5 and observe that $\eta_3 = (J - 3iI_n)^{-1} \mathrm{d}^3 F(c, c, c)$ also lies in $\hat{S}$ because $\hat{S}$ is invariant for $(J - 3iI_n)^{-1}$. Thus any terms in the expansion of $\mathrm{d}^3 \Phi(W_{(3)}, W_{(3)}, W_{(3)})$ of the form $e^{\pm mis} \mathrm{d}^3 F(\eta_3, \cdot, \cdot)$ will lie in $\hat{\mathcal{S}}_{2\pi}$

by Lemma 5.6 and vanish in the projection onto $\mathcal{K}^*$ by (5.3). Furthermore, $e^{\pm 3is}\mathrm{d}^3F(\eta_1, \eta_1, \eta_1)$ also vanishes in the projection, and we are left with (5.19). Therefore, if we show that

$$\mathrm{d}^3F(\eta_1, \eta_1, \bar{\eta}_1)^t\bar{d} \neq 0, \tag{5.20}$$

then the result will follow.

Recall from (4.17) that

$$(J - iI_n)\eta_1 = 3\mathrm{d}^3F(c, c, \bar{c}).$$

Since $\mathrm{d}^3F(c, c, \bar{c}) \in \hat{S}$, and since there are no other constraints on $\mathrm{d}^3F(c, c, \bar{c})$, the projection onto the critical eigenspace E_i will generically be nonzero. Note also that $(J - iI_n)$ is not invertible. The kernel of $(J - iI_n)$ is E_i, and the preimage of E_i under $(J - iI_n)$ is the generalized eigenspace G_i. Hence, the projection of η_1 onto G_i will also be generically nonzero in order to pick up the component of $\mathrm{d}^3F(c, c, \bar{c})$ in E_i. Thus we can write

$$\eta_1 = \alpha b + w,$$

where $\alpha \in \mathbf{C}$ and $w \in \hat{S} - E_i$. Therefore $\eta_1 \notin S$ since $b \notin S$.

Thus, using the linearity of d^3F, we have

$$\mathrm{d}^3F(\eta_1, \eta_1, \bar{\eta}_1)^t\bar{d} = \alpha^3\mathrm{d}^3F(b, b, \bar{b})^t\bar{d}$$

since any terms $\mathrm{d}^3F(w, \cdot, \cdot)$ lie in $\hat{S}$ by (5.16) and are therefore orthogonal to d by (4.11). Generically $\alpha^3\mathrm{d}^3F(b, b, \bar{b})^t\bar{d} \neq 0$, because $b^t\bar{d} \neq 0$ by (2.15). This proves (5.20), and hence Proposition 5.5. ■

6. Further examples of nilpotent Hopf bifurcation. In this section we consider three additional examples of three-cell networks, shown in Figures 8, 9, and 10, that can have nilpotent Hopf bifurcations.

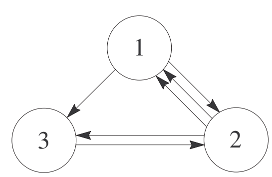

Figure 8. *Another three-cell network with two or four branches at a nilpotent Hopf bifurcation.*

6.1. Another network with multiple $O(\lambda^{\frac{1}{2}})$ branches. The network in Figure 8 is defined by

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_2, x_2}), \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_3}), \\ \dot{x}_2 &= f(x_3, \overline{x_1, x_2}) \end{aligned} \tag{6.1}$$

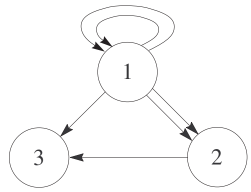

Figure 9. *A feed-forward-like three-cell network.*

and has Jacobian

$$J = \begin{pmatrix} A & 2B & 0 \\ B & A & B \\ B & B & A \end{pmatrix}.$$

The $3k$ eigenvalues and eigenvectors of J are

Eigenvector	Eigenvalues	Algebraic multiplicity	Geometric multiplicity
$(-2u, u, u)^t$	$A - B$	2	1
$(v, v, v)^t$	$A + 2B$	1	1

where u is an eigenvector of $A - B$ and v is an eigenvector of $A + 2B$. It follows that when $k \geq 2$, (6.1) can have a codimension one nilpotent Hopf bifurcation if $A - B$ has a purely imaginary pair of eigenvalues.

Suppose that a is the critical eigenvector of $A - B$. Then the critical eigenvector of J and the corresponding generalized eigenvector are

$$c = \begin{pmatrix} -2a \\ a \\ a \end{pmatrix} \quad \text{and} \quad b = \zeta \begin{pmatrix} 2a \\ -7a \\ 11a \end{pmatrix} + \begin{pmatrix} -2w \\ w \\ w \end{pmatrix},$$

where $\zeta \in \mathbf{C}$ and $w \in \mathbf{C}^k$ are chosen so that $\bar{w}^t a = 0$ and $(A - B - iI_k)w = -(6\zeta B - I_k)a$.

Observe that $S = \{(u, v, v) \ : \ u, v \in \mathbf{R}^k\}$ is a synchrony subspace for this network and that S contains the critical eigenspace but not the generalized eigenspace. Hence this network satisfies (1.23). Therefore, Proposition 1.6 implies that there exists a branch of solutions that grows at $O(\lambda^{\frac{1}{2}})$, and that (1.18) holds. Thus, in the absence of any further constraints that force $p_{uu} = q_{uu} = 0$, it follows from Theorem 1.4 that there exist two or four solutions, each growing at $O(\lambda^{\frac{1}{2}})$. We do not verify the absence of such constraints here, but we note that the absence of any other flow-invariant subspaces suggests, by analogy with the networks studied previously, that the second derivatives of p and q are indeed unconstrained.

6.2. Two networks with branches that grow at $O(\lambda^{\frac{1}{6}})$. In addition to the three-cell feed-forward chain of section 5, there are two other three-cell networks, shown in Figures 9 and 10, that can have branches of solutions that grow at $O(\lambda^{\frac{1}{6}})$.

The network in Figure 9 is defined by

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_1, x_1}), \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_1}), \\ \dot{x}_3 &= f(x_3, \overline{x_1, x_2}) \end{aligned} \tag{6.2}$$

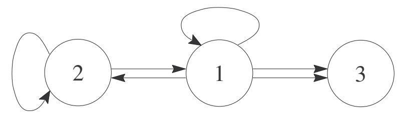

Figure 10. *Another feed-forward-like three-cell network.*

and has Jacobian

$$J = \begin{pmatrix} A+2B & 0 & 0 \\ 2B & A & 0 \\ B & B & A \end{pmatrix}.$$

The network in Figure 10 is defined by

$$\begin{aligned} \dot{x}_1 &= f(x_1, \overline{x_1, x_2}), \\ \dot{x}_2 &= f(x_2, \overline{x_1, x_2}), \\ \dot{x}_3 &= f(x_3, \overline{x_1, x_1}) \end{aligned} \tag{6.3}$$

and has Jacobian

$$J = \begin{pmatrix} A+B & B & 0 \\ B & A+B & 0 \\ 2B & 0 & A \end{pmatrix}.$$

In both cases, the $3k$ eigenvalues and eigenvectors of J are

Eigenvector	Eigenvalues	Algebraic multiplicity	Geometric multiplicity
$(0,0,u)^t$	A	2	1
$(v,v,v)^t$	$A+2B$	1	1

where u is an eigenvector of A and v is an eigenvector of $A+2B$. It follows that when $k \geq 2$, (6.2) and (6.3) can have a codimension one nilpotent Hopf bifurcation if A has a purely imaginary pair of eigenvalues.

Consider first the network shown in Figure 9 and defined by (6.2). Suppose that a is the critical eigenvector of A. Then the critical eigenvector of J and the corresponding generalized eigenvector are

$$c = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} 0 \\ \zeta a \\ w \end{pmatrix},$$

where $\zeta \in \mathbf{C}$ and $w \in \mathbf{C}^k$ are chosen so that $\bar{w}^t a = 0$ and $(A - iI_k)w = -(\zeta B - I_k)a$.

Observe that $S = \{(u,u,v) \; : \; u,v \in \mathbf{R}^k\}$ and $\hat{S} = \{(0,0,v) \; : \; v \in \mathbf{R}^k\}$ are invariant subspaces for this network, which both contain the critical eigenspace but not the generalized eigenspace. Hence this network satisfies (1.23), and Proposition 1.6 implies that there exists a branch of solutions that grows at $O(\lambda^{\frac{1}{2}})$ and that (1.18) holds. This branch is obtained by restricting the system to $\hat{S}$ so that $x_1 = x_2 = 0$ and observing that cell 3 undergoes a standard Hopf bifurcation.

This branch of solutions is unstable because the origin is unstable for cell 2. However, the same argument used in [2, Lemma 6.1] can be employed to show that there exists an additional branch of solutions that grows at $O(\lambda^{\frac{1}{6}})$. Suppose that the eigenvalues of B are negative so that the origin in the first cell is stable for $\dot{x}_1 = f(x_1, \overline{x_1, x_1}, \lambda)$ if λ is sufficiently small. Thus we may assume that $x_1 = 0$.

$$\begin{aligned}\dot{x}_2 &= f(x_2, \overline{0, 0}, \lambda) = g(x_2, \lambda),\\ \dot{x}_3 &= f(x_3, \overline{0, x_2}, \lambda) = h(x_3, x_2, \lambda),\end{aligned}$$

which is precisely the form of the reduced feedforward network in (5.4). Thus the same $\mathbf{S}^1$-equivariant normal form can be obtained as in section 5.1, and hence the arguments of [2, Lemma 6.1] are applicable.

Now consider the network shown in Figure 10 and defined by (6.3). Suppose that a is the critical eigenvector of A. Then the critical eigenvector of J and the corresponding generalized eigenvector are

$$c = \begin{pmatrix} 0 \\ 0 \\ a \end{pmatrix} \quad \text{and} \quad b = \begin{pmatrix} \zeta a \\ -\zeta a \\ w \end{pmatrix},$$

where $\zeta \in \mathbf{C}$ and $w \in \mathbf{C}^k$ are chosen so that $\bar{w}^t a = 0$ and $(A - iI_k)w = -(2\zeta B - I_k)a$.

Again, observe that $S = \{(u, u, v) \,:\, u, v \in \mathbf{R}^k\}$ and $\hat{S} = \{(0, 0, v) \,:\, v \in \mathbf{R}^k\}$ are invariant subspaces that satisfy (1.23), so that Proposition 1.6 implies a branch of solutions that grows at $O(\lambda^{\frac{1}{2}})$ and that (1.18) holds.

Observe that cells 1 and 2 form a $\mathbb{Z}_2$-equivariant subsystem that is not influenced by cell 3. A synchrony-breaking Hopf bifurcation in this subsystem yields a branch of periodic solutions that grows as $\lambda^{\frac{1}{2}}$ and satisfies

$$x_1(t) = x_2\left(t + \tfrac{1}{2}\right).$$

Thus the bifurcation in cell 3 is forced by $x_1(t)$ in exactly the same way as it is forced by cell 2 in the feed-forward chain. Assuming that the normal form on the center manifold is $\mathbf{S}^1$-equivariant under the action in (5.5), it follows from the proof of [2, Lemma 6.1] that there exists a branch that grows at $O(\lambda^{\frac{1}{6}})$.

Appendix. Derivatives of the reduced mapping. The following is a collection of all the derivatives of the reduced mapping ϕ. These are derived by definition; see [3, pp. 31–33]. For higher derivatives we use the notation $\phi_{(k)}$ and $W_{(k)}$ to signify the kth x derivative of ϕ and W. In order to keep these formulas as readable as possible we have used the fact that $W_x(0, 0, 0) = 0$, but it should be remembered that in deriving $\phi_{(k+1)}$ from $\phi_{(k)}$, an argument of v_1 should be read as $v_1 + W_x$.

$$\phi_{xxx} = (I - E)(\mathrm{d}^3\Phi(v_1, v_1, v_1) + 3\mathrm{d}^2\Phi(v_1, W_{xx}) + \mathrm{d}\Phi(W_{xxx})) \tag{A.1}$$

$$\begin{aligned}\phi_{(5)} = (I-E)\big(&\mathrm{d}^5\Phi(v_1,v_1,v_1,v_1,v_1) + 10\mathrm{d}^4\Phi(v_1,v_1,v_1,W_{xx})\\ &+ 15\mathrm{d}^3\Phi(v_1,W_{xx},W_{xx}) + 10\mathrm{d}^3\Phi(v_1,v_1,W_{xxx})\\ &+ 10\mathrm{d}^2\Phi(W_{xx},W_{xxx}) + 5\mathrm{d}^2\Phi(v_1,W_{(4)}) + \mathrm{d}\Phi(W_{(5)})\big)\end{aligned} \tag{A.2}$$

$$\begin{aligned}\phi_{(7)} = (I-E)\big(&\mathrm{d}^7\Phi(v_1,v_1,v_1,v_1,v_1,v_1,v_1) + 21\mathrm{d}^6\Phi(v_1,v_1,v_1,v_1,v_1,W_{xx})\\ &+ 105\mathrm{d}^5\Phi(v_1,v_1,v_1,W_{xx},W_{xx}) + 35\mathrm{d}^5\Phi(v_1,v_1,v_1,v_1,W_{xxx})\\ &+ 105\mathrm{d}^4\Phi(v_1,W_{xx},W_{xx},W_{xx}) + 210\mathrm{d}^4\Phi(v_1,v_1,W_{xx},W_{xxx})\\ &+ 35\mathrm{d}^4\Phi(v_1,v_1,v_1,W_{(4)}) + 105\mathrm{d}^3\Phi(W_{xx},W_{xx},W_{xxx})\\ &+ 70\mathrm{d}^3\Phi(v_1,W_{xxx},W_{xxx}) + 105\mathrm{d}^3\Phi(v_1,W_{xx},W_{(4)})\\ &+ 21\mathrm{d}^3\Phi(v_1,v_1,W_{(5)}) + 35\mathrm{d}^2\Phi(W_{xxx},W_{(4)})\\ &+ 21\mathrm{d}^2\Phi(W_{xx},W_{(5)}) + 7\mathrm{d}^2\Phi(v_1,W_{(6)}) + \mathrm{d}\Phi(W_{(7)})\big)\end{aligned} \tag{A.3}$$

$$\begin{aligned}\phi_{(9)} = (I-E)\big(&\mathrm{d}^9\Phi(v_1,v_1,v_1,v_1,v_1,v_1,v_1,v_1,v_1) + 36\mathrm{d}^8\Phi(v_1,v_1,v_1,v_1,v_1,v_1,v_1,W_{xx})\\ &+ 378\mathrm{d}^7\Phi(v_1,v_1,v_1,v_1,v_1,W_{xx},W_{xx}) + 84\mathrm{d}^7\Phi(v_1,v_1,v_1,v_1,v_1,v_1,W_{xxx})\\ &+ 1260\mathrm{d}^6\Phi(v_1,v_1,v_1,W_{xx},W_{xx},W_{xx}) + 1260\mathrm{d}^6\Phi(v_1,v_1,v_1,v_1,W_{xx},W_{xxx})\\ &+ 126\mathrm{d}^6\Phi(v_1,v_1,v_1,v_1,v_1,W_{(4)}) + 945\mathrm{d}^5\Phi(v_1,W_{xx},W_{xx},W_{xx},W_{xx})\\ &+ 3780\mathrm{d}^5\Phi(v_1,v_1,W_{xx},W_{xx},W_{xxx}) + 840\mathrm{d}^5\Phi(v_1,v_1,v_1,W_{xxx},W_{xxx})\\ &+ 1260\mathrm{d}^5\Phi(v_1,v_1,v_1,W_{xx},W_{(4)}) + 126\mathrm{d}^5\Phi(v_1,v_1,v_1,v_1,W_{(5)})\\ &+ 1260\mathrm{d}^4\Phi(W_{xx},W_{xx},W_{xx},W_{xxx}) + 2520\mathrm{d}^4\Phi(v_1,W_{xx},W_{xxx},W_{xxx})\\ &+ 1890\mathrm{d}^4\Phi(v_1,W_{xx},W_{xx},W_{(4)}) + 1260\mathrm{d}^4\Phi(v_1,v_1,W_{xxx},W_{(4)})\\ &+ 756\mathrm{d}^4\Phi(v_1,v_1,W_{xx},W_{(5)}) + 84\mathrm{d}^4\Phi(v_1,v_1,v_1,W_{(6)})\\ &+ 280\mathrm{d}^3\Phi(W_{xxx},W_{xxx},W_{xxx}) + 1260\mathrm{d}^3\Phi(W_{xx},W_{xxx},W_{(4)})\\ &+ 378\mathrm{d}^3\Phi(W_{xx},W_{xx},W_{(5)}) + 315\mathrm{d}^3\Phi(v_1,W_{(4)},W_{(4)})\\ &+ 504\mathrm{d}^3\Phi(v_1,W_{xxx},W_{(5)}) + 252\mathrm{d}^3\Phi(v_1,W_{xx},W_{(6)})\\ &+ 36\mathrm{d}^3\Phi(v_1,v_1,W_{(7)}) + 126\mathrm{d}^2\Phi(W_{(4)},W_{(5)}) + 84\mathrm{d}^2\Phi(W_{xxx},W_{(6)})\\ &+ 36\mathrm{d}^2\Phi(W_{xx},W_{(7)}) + 9\mathrm{d}^2\Phi(v_1,W_{(8)}) + \mathrm{d}\Phi(W_{(9)})\big)\end{aligned} \tag{A.4}$$

The following are differentials involving parameters α and β:

$$\phi_{\alpha x} = (I-E)\left(\mathrm{d}\Phi_\alpha(v_1) + \mathrm{d}\Phi(W_{\alpha x}) + \mathrm{d}^2\Phi(v_1,W_\alpha)\right), \tag{A.5}$$

$$\begin{aligned}\phi_{\alpha\beta x} = (I-E)\big(&\mathrm{d}\Phi_{\alpha\beta}(v_1) + \mathrm{d}\Phi_\alpha(W_{\beta x}) + \mathrm{d}\Phi_\beta(W_{\alpha x}) + \mathrm{d}\Phi(W_{\alpha\beta x})\\ &+ \mathrm{d}^2\Phi_\alpha(v_1,W_\beta) + \mathrm{d}^2\Phi_\beta(v_1,W_\alpha) + \mathrm{d}^2\Phi(W_{\alpha x},W_\beta)\\ &+ \mathrm{d}^2\Phi(W_{\beta x},W_\alpha) + \mathrm{d}^2\Phi(v_1,W_{\alpha\beta}) + \mathrm{d}^3\Phi(v_1,W_\alpha,W_\beta)\big),\end{aligned} \tag{A.6}$$

$$\begin{aligned}\phi_{\alpha xx} = (I-E)\big(&\mathrm{d}^3\Phi(v_1,v_1,W_\alpha) + 2\mathrm{d}^2\Phi(v_1,W_{\alpha x}) + \mathrm{d}^2\Phi(W_{xx},W_\alpha)\\ &+ \mathrm{d}\Phi(W_{\alpha xx}) + \mathrm{d}^2\Phi_\alpha(v_1,v_1) + \mathrm{d}\Phi_\alpha(W_{xx})\big),\end{aligned} \tag{A.7}$$

$$\begin{aligned}\phi_{\alpha xxx} = (I-E)\big(&\mathrm{d}^4\Phi(v_1,v_1,v_1,W_\alpha) + 3\mathrm{d}^3\Phi(v_1,v_1,W_{\alpha x})\\ &+ 3\mathrm{d}^3\Phi(v_1,W_{xx},W_\alpha) + 3\mathrm{d}^2\Phi(v_1,W_{\alpha xx})\\ &+ 3\mathrm{d}^2\Phi(W_{xx},W_{\alpha x}) + \mathrm{d}^2\Phi(W_{xxx},W_\alpha) + \mathrm{d}\Phi(W_{\alpha xxx})\\ &+ \mathrm{d}^3\Phi_\alpha(v_1,v_1,v_1) + 3\mathrm{d}^2\Phi_\alpha(v_1,W_{xx}) + \mathrm{d}\Phi_\alpha(W_{xxx})\big).\end{aligned} \tag{A.8}$$

The following formulas for the $W_{(k)}$ are obtained by differentiating

$$E\Phi(xv_1 + W(xv_1,\lambda,\tau),\lambda,\tau) \equiv 0 \tag{A.9}$$

k times with respect to x. This yields an expression of the form

$$E(\cdots) + E\mathrm{d}\Phi(W_{(k)}) = 0,$$

which can be rearranged to give

$$\mathrm{d}\Phi(W_{(k)}) = -E(\cdots)$$

since $\mathrm{d}\Phi(W_{(k)}) \in \mathcal{R}$ on which E acts as the identity, and hence

$$W_{(k)} = -\mathcal{L}^{-1}E(\cdots).$$

In this way, we obtain

$$W_{xx} = -\mathcal{L}^{-1}E\big(\mathrm{d}^2\Phi(v_1,v_1)\big), \tag{A.10}$$

$$W_{xxx} = -\mathcal{L}^{-1}E\big(\mathrm{d}^3\Phi(v_1,v_1,v_1) + 3\mathrm{d}^2\Phi(v_1,W_{xx})\big), \tag{A.11}$$

$$\begin{aligned}W_{xxxx} = -\mathcal{L}^{-1}E\big(&\mathrm{d}^4\Phi(v_1,v_1,v_1,v_1) + 6\mathrm{d}^3\Phi(v_1,v_1,W_{xx})\\ &+ 3\mathrm{d}^2\Phi(W_{xx},W_{xx}) + 4\mathrm{d}^2\Phi(v_1,W_{xxx})\big).\end{aligned} \tag{A.12}$$

Similarly, we obtain the following expressions for $W_{\alpha x}$ by differentiating (A.9) with respect to α and x to obtain

$$E\left(\mathrm{d}^2\Phi(v_1+W_x,W_\alpha) + \mathrm{d}\Phi(W_{\alpha x}) + \mathrm{d}\Phi_\alpha(v_1+W_x)\right) = 0, \tag{A.13}$$

which rearranges to give

$$W_{\alpha x} = -\mathcal{L}^{-1}E\left(\mathrm{d}^2\Phi(v_1,W_\alpha) + \mathrm{d}\Phi_\alpha(v_1)\right). \tag{A.14}$$

By further differentiation of (A.13) with respect to x and rearranging, we obtain

$$\begin{aligned}W_{\alpha xx} = -\mathcal{L}^{-1}E\big(&\mathrm{d}^3\Phi(v_1,v_1,W_\alpha) + \mathrm{d}^2\Phi_\alpha(v_1,v_1)\\ &+ \mathrm{d}^2\Phi(W_{xx},W_\alpha) + 2\mathrm{d}^2\Phi(v_1,W_{\alpha x}) + \mathrm{d}\Phi_\alpha(W_{xx})\big),\end{aligned} \tag{A.15}$$

$$\begin{aligned}W_{\alpha xxx} = -\mathcal{L}^{-1}E\big(&\mathrm{d}^4\Phi(v_1,v_1,v_1,W_\alpha) + 3\mathrm{d}^3\Phi(v_1,v_1,W_{\alpha x})\\ &+ 3\mathrm{d}^3\Phi(v_1,W_{xx},W_\alpha) + \mathrm{d}^3\Phi_\alpha(v_1,v_1,v_1)\\ &+ 3\mathrm{d}^2\Phi(W_{xx},W_{\alpha x}) + \mathrm{d}^2\Phi(W_{xxx},W_\alpha) + 3\mathrm{d}^2\Phi(v_1,W_{\alpha xx})\\ &+ 3\mathrm{d}^2\Phi_\alpha(v_1,W_{xx}) + \mathrm{d}^2\Phi_\alpha(W_{xxx})\big).\end{aligned} \tag{A.16}$$

Acknowledgments. We thank Yunjiao Wang and Maria Leite for helpful conversations.

REFERENCES

[1] S. van Gils, M. Krupa, and W. Langford, *Hopf bifurcation with non-semisimple* 1:1 *resonance*, Nonlinearity, 3 (1990), pp. 825–850.

[2] M. Golubitsky, M. Nicol, and I. Stewart, *Some curious phenomena in coupled cell networks*, J. Nonlinear Sci., 14 (2004), pp. 119–236.

[3] M. Golubitsky and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory*, Vol. I, Appl. Math. Sci. 51, Springer-Verlag, New York, 1984.

[4] M. Golubitsky, I. N. Stewart, and D. G. Schaeffer, *Singularities and Groups in Bifurcation Theory*, Vol. II, Appl. Math. Sci. 69, Springer-Verlag, New York, 1988.

[5] M. Golubitsky, I. Stewart, and A. Török, *Patterns of synchrony in coupled cell networks with multiple arrows*, SIAM J. Appl. Dyn. Syst., 4 (2005), pp. 78–100.

[6] M. Leite, *Homogeneous Three-Cell Networks*, Ph.D. thesis, Department of Mathematics, University of Houston, Houston, TX, 2005.

[7] M. Leite and M. Golubitsky, *Homogeneous three-cell networks*, Networks, submitted.

[8] N. S. Namachchivaya, M. Doyle, W. Langford, and N. Evans, *Normal form for generalized Hopf bifurcation with non-semisimple* 1:1 *resonance*, Z. Angew. Math. Phys., 45 (1994), pp. 312–335.

[9] I. Stewart, M. Golubitsky, and M. Pivato, *Symmetry groupoids and patterns of synchrony in coupled cell networks*, SIAM J. Appl. Dyn. Syst., 2 (2003), pp. 609–646.

[10] A. Vanderbauwhede, *Hopf bifurcation at non-semisimple eigenvalues*, in Multiparameter Bifurcation Theory, M. Golubitsky and J. Guckenheimer, eds., Contemp. Math. 56, AMS, Providence, RI, 1986, pp. 343–353.