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Ponies on a merry-go-round in large arrays of Josephson junctions

D G Aronson† Martin Golubitsky‡ and John Mallet-Paret§

† School of Mathematics, University of Minnesota, Minneapolis, MN 55455, USA
‡ Department of Mathematics, University of Houston, Houston, TX 77204-3476, USA
§ Division of Applied Mathematics, Brown University, Providence, RI 02912, USA

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Abstract. Numerical simulation of periodic solutions in large arrays of Josephson junctions indicates the existence of periodic solutions where each junction oscillates with the same waveform, but with equal phase lags. These solutions are called ponies on a merry-go-round or POMs for short. In this paper we prove the existence of POMs in the equations modelling large arrays of Josephson junctions by using global bifurcation techniques. The basic idea is to view the period of the solution and the phase lag as independent parameters and to prove, using *a priori* estimates, that the synchronous solution (with phase lag set to zero) can be continued to a solution with phase lag equal to (1/N)th of the period, a POM.

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1. Introduction

We continue here the study of periodic solutions in large arrays of Josephson junctions begun in [AGK]. We demonstrate the existence of ponies on a merry-go-round (POMs) periodic solutions, by using a global existence theorem for differential delay equations coupled with symmetry. These POMS are observed numerically in [AGK] for a certain type of Josephson junction model, thus motivating this analysis.

We divide this paper into three short sections. The equations for coupled arrays of Josephson junctions are described in the following section. Further discussion of this system may be found in the papers of Beasley, Hadley and Wiensenfeld [H, BHW] (which stimulated our own interest in this area) and [AGK]. Ponies on a merry-go-round are defined in section 3, where our main theorem is stated. The proof is given in section 4.

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2. Arrays of Josephson junctions

Following [AGK, BHW] we write the equations for a coupled array of N Josephson junctions as follows. Let φ_j ($j = 1, \ldots, N$) denote the difference in the phases of the quasiclassical superconducting wavefunctions on the two sides of the kth junction, and let I_L denote the current flowing through the load. We assume that all individual junctions are identical and that each junction feels the load I_L identically. Thus the circuit equations describing arrays of Josephson junctions are S_N symmetric, where S_N is the group of permutations of the junctions.

The evolution of the φ_j and I_L is governed by the system of equations:

$$\beta\ddot{\varphi}_j + \dot{\varphi}_j + \sin(\varphi_j) + I_L = I_B \qquad (j = 1, \ldots, N) \tag{2.1a}$$

$$\sum_{k=1}^{N} \dot{\varphi}_k = \mathscr{F}(I_L) \tag{2.1b}$$

where β is a dimensionless measure of the capacitance of the junctions, I_B is the bias current applied to each junction, and $\mathscr{F}$ is an integro-differential operator which depends on the particular load considered.

[BHW] consider a variety of loads, two of which are singled out for study in [AGK]. These are pure capacitive load for which (with appropriate normalization)

$$I_L = \frac{3}{N}\sum_{k=1}^{N} \ddot{\varphi}_k \tag{2.2c}$$

and pure resistive load for which

$$I_L = \frac{1}{N}\sum_{k=1}^{N} \dot{\varphi}_k. \tag{2.2r}$$

In the resistive case we can eliminate I_L from the first N equations of (2.1) by simply substituting the right-hand side of (2.2r) for I_L. We can also eliminate I_L in the capacitive case with a little more algebraic effort. The resulting system is

$$\beta\ddot{\varphi}_j + \dot{\varphi}_j + \sin(\varphi_j) + \frac{A}{N}\sum_{k=1}^{N} \dot{\varphi}_k + \frac{B}{N}\sum_{k=1}^{N} \sin(\varphi_k) = CI_B \qquad (j = 1, \ldots, N) \tag{2.3}$$

where for a capactive load

$$A = B = -\frac{3}{3+\beta}, \qquad C = \frac{\beta}{3+\beta} \tag{2.4c}$$

and

$$A = C = 1 \qquad B = 0 \tag{2.4r}$$

for a resistive load.

A *running solution* to (2.3) is one for which there is a minimal $T > 0$ such that

$$\varphi_j(t+T) = \varphi_j(t) + 2\pi \qquad (j = 1, \ldots, N) \tag{2.5}$$

for all $t \in \mathbb{R}$. We call T the *period* of the running solution. A *symmetric* or *in-phase* running solution is one for which

$$\varphi_1 = \varphi_2 = \ldots = \varphi_N. \tag{2.6}$$

It is shown in [AGK] that any symmetric running solution is asymptotically stable in the subspace of the full phase space defined by (2.6). Hence there can exist at most one symmetric running solution to (2.3) for any particular choice of parameter values.

3. Ponies on a merry-go-round

A running solution to (2.3) is said to be a *discrete travelling wave* solution or, more picturesquely, a *ponies on a merry-go-round* solution (POM) if

$$\varphi_j(t) = \varphi_1(t-(j-1)T/N) \qquad (j=1,\ldots,N) \tag{3.1}$$

To simplify notation we will write φ instead of φ_1 when we are dealing with a POM. In view of the permutation symmetry of the system (2.3), the existence of a POM for one ordering of the junctions implies the existence of POMs for any other ordering. Moreover, when $N = KM$ then POMs can also be formed by grouping the junctions into M blocks of K synchronous junctions each with a delay of T/M between successive blocks.

The existence of POM solutions has been studied in rings of oscillators by Alexander and Auchmuty [AA] and, as part of a study of Hopf bifurcation with dihedral symmetry, by Golubitsky and Stewart [GS, GSS]. Indeed, using the same group theoretic techniques as in [GSS] it would be possible to find POMs in S_N symmetric systems via Hopf bifurcation. However, this is not the way in which POMs arise in the Josephson junction models. This is because in the Josephson junction equations, running solutions (and hence POMs) can never have small amplitude.

We apply global bifurcation techniques to prove the existence of POMs for (2.3). Substitution of (3.1) in (2.3) yields the delay differential equation

$$\beta\ddot{\varphi}(t) + \dot{\varphi}(t) + \sin\varphi(t) + \frac{A}{N}\sum_{k=0}^{N-1}\dot{\varphi}(t-kT/N) + \frac{B}{N}\sum_{k=0}^{N-1}\sin(\phi(t-kT/N)) = CI_{\mathrm{B}}.$$

This delay equation is special in that the delays kT/N are coupled to the period T of the running solution φ. To use global techniques we decouple the period T and the delay τ and consider the equation

$$\beta\ddot{\varphi}(t) + \dot{\varphi}(t) + \sin\varphi(t) + \frac{A}{N}\sum_{k=0}^{N-1}\dot{\varphi}(t-k\tau/N) + \frac{B}{N}\sum_{k=0}^{N-1}\sin(\varphi(t-k\tau/N)) = CI_{\mathrm{B}}. \tag{3.2}$$

Here we make the natural restriction that $0 \leq \tau < NT$. Note that when $\tau = 0$ the running solution to (3.2) corresponds to the in-phase running solution to (2.3).

Our method for proving the existence of POMs is to prove the existence of a global branch of running solutions to (3.2) emanating from the in-phase running solution in the (T,τ)-plane, and to use the specific form of the Josephson junction equations to show that this branch must cross the line $T = \tau$. This is done by finding *a priori* upper and lower bounds for the period T of any running solution φ of (3.2) and an upper bound for $|\dot{\varphi}|$ which are uniform in τ.

Theorem 3.1. POM solutions exist for the Josephson junction system (1.3) at least when $I_{\mathrm{B}} > 1$ and $\beta > 0$. Moreover, if K divides N, then POM solutions exist when the junctions are subdivided into N/K groups each consisting of K junctions.

Remark. The restriction $I_{\mathrm{B}} > 1$ is certainly too stringent, since POMs have been observed numerically for values of $I_{\mathrm{B}} \leq 1$ (at least for capacitive loads). Both theoretical considerations and numerical evidence suggest that the numerically observed POMs are born as heteroclinic orbits. We will discuss the question in more detail elsewhere.

4. The proof of theorem 3.1

We will prove only the first assertion of theorem 3.1. The proof of the assertion concerning blocks of junctions is almost idential except for notation.

Let φ be a running solution of (3.1) with period T. We derive a bound for T which is independent of τ and β. To this end, integrate both sides of (3.2) over $(0, T)$. Since $\dot{\varphi}$ is T-periodic and φ satisfies (2.5) we find

$$(1+A)2\pi + (1+B)\int_0^T \sin\varphi \, \mathrm{d}t = CI_{\mathrm{B}}T$$

Thus, using (2.4c) and (2.4r) we obtain the identity

$$2\pi p + \int_0^T \sin\varphi \, \mathrm{d}t = I_{\mathrm{B}}T$$

where $p = 1$ for a capacitive load and $p = 2$ for a resistive load. This equation may be rewritten as:

$$T = 2\pi p/(I_{\mathrm{B}} - \langle \sin\varphi \rangle).$$

where

$$\langle \sin\varphi \rangle \equiv \frac{1}{T}\int_0^T \sin\varphi \, \mathrm{d}t.$$

Bounding the sine by ± 1 yields the estimate

$$\frac{2\pi p}{I_{\mathrm{B}}+1} \leqslant T \leqslant \frac{2\pi p}{I_{\mathrm{B}}-1} \tag{4.1}$$

when $I_{\mathrm{B}} > 1$.

Next, we seek bounds for

$$M \equiv \max_{[0,T]} |\dot{\varphi}(t)|$$

which are independent of τ. We claim that for a capacitive load

$$M \leqslant I_{\mathrm{B}} + 1 + 6/\beta \tag{4.2c}$$

and

$$M \leqslant N(I_{\mathrm{B}} + 1)/2 \tag{4.2r}$$

for a resistive load.

To verify this claim, let $\bar{t}$ be a point in $[0, T]$ where $|\dot{\varphi}|$ achieves the value M. Note that $\ddot{\varphi}(\bar{t}) = 0$. For a resistive load, equation (3.2) at $t = \bar{t}$ becomes

$$\dot{\varphi}(\bar{t})\left(1 + \frac{1}{N}\right) + \frac{1}{N}\sum_{k=1}^{N-1} \dot{\varphi}(\bar{t} - k\tau/N) = -\sin\varphi(\bar{t}) + I_{\mathrm{B}}.$$

Thus

$$2\frac{M}{N} = M\left(1 + \frac{1}{N}\right) - M\frac{N-1}{N} \leqslant I_{\mathrm{B}} + 1$$

and (4.2r) follows. Similarly in the case of a capacitive load, at $t = \bar{t}$ equation (3.2) becomes

$$\dot\varphi(\tilde t)-\frac{3}{N(3+\beta)}\sum_{k=0}^{N-1}\dot\varphi(\tilde t-k\tau/N)$$
$$=\frac{\beta}{3+\beta}I_B-\sin\phi(\tilde t)+\frac{3}{N(3+\beta)}\sum_{k=0}^{N-1}\sin\varphi(\tilde t-k\tau/N).$$

Therefore

$$\frac{\beta}{3+\beta}M=M-\frac{3}{3+\beta}M\leqslant\frac{\beta}{3+\beta}I_B+1+\frac{3}{3+\beta}$$

which implies (4.2c).

We seek T-periodic running solutions of (3.2). We do this by treating the parameter τ as a homotopy parameter, and looking for a continuum of T-periodic running solutions for τ in the range $0\leqslant\tau\leqslant T$.

If $\varphi(t)$ is a T-periodic running solution of (3.2) then so are all of its time translations $\varphi(t+\theta)$. Let $s=t/T$ and define

$$\psi(s)\equiv\varphi(t+\theta)-2\pi s$$

where θ is uniquely determined by requiring that

$$\int_0^1\psi(s)\,\mathrm{d}s=0. \tag{4.3}$$

Note that since $\varphi(t+\theta)$ is a running solution,

$$\psi(s+1)=\psi(s). \tag{4.4}$$

The periodic function $\psi(s)$ satisfies the non-autonomous delay differential equation

$$\beta\psi''+T\psi'+T^2\sin(\psi+2\pi s)+\frac{AT}{N}\sum_{k=0}^{N-1}\psi'(s-k\sigma/N)$$
$$+\frac{BT^2}{N}\sum_{k=0}^{N-1}\sin[\psi(s-k\sigma/N)+2\pi(s-k\sigma/N)]$$
$$=CI_BT^2-2\pi T(1+A) \tag{4.5}$$

where $'=\mathrm{d}/\mathrm{d}s$ and $\sigma=\tau/T$ plays the role of the homotopy parameter.

To formulate (4.5) in a functinal analytic setting, defined the real Banach spaces

$$\mathscr{X}\equiv\{\psi\in C^2(\mathbb{R}):\psi\text{ satisfies (4.3) and (4.4)}\}$$

endowed with the C^2-norm, and

$$\mathscr{Y}\equiv\{\psi\in C(\mathbb{R}):\psi\text{ satisfies (4.4)}\}$$

with the supremum norm. Also define

$$\mathscr{L}:\mathscr{X}\times\mathbb{R}\to\mathscr{Y}$$

and

$$\mathscr{N}_\sigma:\mathscr{X}\times\mathbb{R}\to\mathscr{Y}$$

by

$$\mathscr{L}(\psi,T)\equiv\beta\psi''(s)+T$$

and

$$\mathscr{N}_\sigma(\psi,T)\equiv-T+T\psi'+T^2\sin(\psi+2\pi s)+\frac{AT}{N}\sum_{k=0}^{N-1}\psi'(s-k\sigma/N)$$
$$+\frac{BT^2}{N}\sum_{k=0}^{N-1}\sin\{\psi(s-\sigma k/n)+2\pi(s-\sigma k/N)\}-CI_BT^2+2\pi T(A+1)$$

for each $\sigma \in [0, 1]$. Then (4.5) becomes

$$\mathscr{L}\xi + \mathscr{N}_\sigma(\xi) = 0, \tag{4.6}$$

where $\xi = (\psi, T) \in \mathscr{X} \times \mathbb{R}$. Indeed, since $\mathscr{L}$ is a linear isomorphism from $\mathscr{X} \times \mathbb{R}$ onto $\mathscr{Y}$ we can write

$$\xi + \mathscr{L}^{-1}\mathscr{N}_\sigma(\xi) = 0. \tag{4.7}$$

Moreover, the nonlinear operator $\mathscr{L}^{-1}\mathscr{N}_\sigma$ is a compact operator, uniformly in $\sigma \in [0, 1]$. It is also jointly continuous in (σ, T) and smooth in ξ for each fixed σ. Degree theory is therefore the natural technique with which to solve (4.7).

We recall the uniform bounds (4.1) on the periods of running solutions to (3.2), which we write as

$$0 < T_1 < T < T_2. \tag{4.8}$$

In view of the estimates (4.2), we also have uniform C^2 bounds on running solutions to (3.2). These bounds together with (4.8) imply the uniform bound

$$\|\psi\|_{\mathscr{X}} < K$$

for solutions to (4.6). Therefore, all our solutions of (4.7) lie in the interior of the set

$$\Theta = \{(\psi, T) \in \mathscr{X} \times \mathbb{R} : \|\psi|_{\mathscr{X}} < K,\ T_1 < T < T_2\}$$

and none lies on $\partial\Theta$, for each $\sigma \in [0, 1]$. To prove existence of at least one such solution for each σ, we show that the Leray–Schauder degree at $\sigma = 0$ is non-zero:

$$\deg(I + \mathscr{L}^{-1}\mathscr{N}_0, \Theta, 0) \neq 0. \tag{4.9}$$

To verify (4.9), recall that it has been shown in [AGK] that (2.3) has a unique T_0-periodic symmetric running solution φ_0 in a parameter region which includes the quarter-space $I_B > 1$, $\beta > 0$. The linearization of (3.2), with $\tau = 0$, about φ_0 is

$$\beta\ddot{\zeta} + (1 + A)\dot{\zeta} + (1 + B)\zeta\cos\varphi_0 = 0$$

and $\zeta = \dot{\varphi}_0$ is a T_0-periodic solution to this equation corresponding to the Floquet multiplier 1. The remaining Floquet multiplier is $\exp\{-(1 + A)T_0/\beta\} \in (0, 1)$. The construction in [AGK] shows that

$$\min_{[0, T_0]} \dot{\varphi}_0(t) > 0.$$

Therefore another solution to the linearized equation which is linearly independent of $\dot{\varphi}_0$ is given by

$$\zeta(t) = \dot{\varphi}_0(t) \int_0^t e^{-(1+A)s/\beta} \dot{\varphi}_0^{-2}(s)\, ds.$$

Define

$$\psi_0(s) \equiv \varphi_0(t + \theta) - 2\pi s$$

where $s = t/T_0$ and θ is chosen so that $\psi_0 \in \mathscr{X}$. Without loss of generality we can set $\theta = 0$. The pair $\xi_0 \equiv (\psi_0, T_0) \in \mathscr{X} \times \mathbb{R}$ is the unique solution to (4.6) at $\sigma = 0$. To complete the proof we have to show that the Fréchet derivative $\mathscr{L} + D\mathscr{N}_0(\xi_0)$ is non-singular, i.e., that

$$\ker(\mathscr{L} + D\mathscr{N}_0(\xi_0)) = \{(0, 0)\}. \tag{4.10}$$

Suppose that $\zeta = (\eta, \Lambda) \in \mathscr{X} \times \mathbb{R}$ is an element of $\ker(\mathscr{L} + D\mathcal{N}_0(\xi_0))$. Then

$$\mathscr{L}\zeta + D\mathcal{N}_0(\xi_0)\zeta = 0$$

and it follows by a straightforward calculation that (η, Λ) must satisfy the linear inhomogeneous ordinary differential equation

$$\begin{aligned}\beta\eta'' + (1+A)T_0\eta' + (1+B)T_0^2\eta \cos\varphi_0(t)\\ = -\Lambda\{(1+A)T_0\dot{\varphi}_0(t) + 2(1+B)T_0 \sin\varphi_0(t) - 2CT_0I_{\mathrm{B}}\}.\end{aligned} \tag{4.11}$$

In view of the fact that φ_0 is a T_0-periodic symmetric running solution to (2.3), it is easy to verify that the general solution to (4.11) has the form

$$\eta(s) = \dot{\varphi}_0(T_0 s)W(s)$$

where

$$W(s) \equiv c_1 + c_2 \int_0^{T_0 s} \mathrm{e}^{-(1+A)\tau/\beta}\dot{\varphi}_0^{-2}(\tau)\,\mathrm{d}\tau + \Lambda s$$

with arbitrary constants c_1 and c_2. Since $\dot{\varphi}_0$ is T_0-periodic and $\eta \in \mathscr{X}$, it follows that $\eta(s)/\dot{\varphi}_0(T_0 s)$ is 1-periodic in s. On the other hand $W(s)$ can only be periodic if $c_2 = \Lambda = 0$. Thus

$$\eta(s) = c_1\dot{\varphi}_0(T_0 s).$$

However, $\eta \in \mathscr{X}$ and φ_0 being a T_0-periodic symmetric running solution to (2.3) imply

$$0 = \int_0^1 \eta(s)\,\mathrm{d}s = c_1 \int_0^1 \dot{\varphi}_0(T_0 s)\,\mathrm{d}s = 2\pi c_1/T_0.$$

Therefore we must also have $c_1 = 0$ so that (4.10) holds.

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Note added in proof. We have recently received a preprint from R E Mirolla [M] in which the existence of POMs is proved by means of the Lefschetz trace formula.

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