of interest to a wide variety of specialists in applied mathematics and engineering and should be on the bookshelf of anyone interested in ill posed problems. There are numerous examples and illustrations. The translator has taken pains to insure that the English reads smoothly.

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Catastrophe theory and its applications, by Tim Poston and Ian Stewart, Surveys and Reference Works in Mathematics, Pitman, London, 1978, xviii + 491 pp., $50.00.

What mathematical discovery has provoked recent articles in *Scientific American*, *Nature*, *Newsweek*, *Science*, *The New York Times*, *The Times Higher Education Supplement*, *L'Express* and the *New York Review of Books*? What current theory now ranks only behind the weather and old movies as a subject of cocktail conversation between mathematicians and nonmathematicians? Is there any content to this theory which has been described in *Science* as an emperor without clothes? Has all the notoriety been public relations–beginning with the creator's brilliant choice of name? This, for instance, has led *The New York Times* to blunder on its front page article with the headline "Experts Debate the Prediction of Disasters." In short, is this theory really–as *Newsweek* described it–the most important mathematical advance since Newton's invention of the calculus?

The answer to the last question is simply no; however, catastrophe theory does have merit both in mathematics and in applications. How, then does one find out about its successes and why is there a controversy? The answers to these questions are related, but before discussing them one point should be made. To my knowledge no one has suggested that the mathematics behind catastrophe theory is anything less than superb.

The mathematics–in its *narrowest* interpretation–is the study of the qualitative nature of isolated singularities of C^∞ functions from R^n to R defined on some (small) neighborhood of the origin. By "qualitative" one means up to a change of coordinates in the domain and the addition of a scalar in the range. The major accomplishments are the unfolding theorem which allows the qualitative enumeration of all small perturbations of the original function and the classification theorem which gives a first step–through the notion of codimension–towards the determination of just how complicated a given singularity is. This body of mathematics is now called *elementary catastrophe theory*. The mathematics is profound as befits the mathematicians mainly responsible for the theory: Morse, Whitney, Thom, Malgrange, Mather and Arnold, to name just a few.

It is not difficult to understand why the "real world" press has been interested in catastrophe theory. The list of topics to which elementary catastrophe theory has been applied–with no judgement made on the quality of these applications–is truly amazing: stock markets, prison riots, Darwin's theory, anorexia nervosa, love, the twinkling of stars, the design of bridges, nerve impulses and the division of cells. Although mathematical discussion in the press is rare, it would be strange indeed not to report on a mathematical theory which gave unity to the above phenomena. The controversy is–in part–whether or not the applications have any merit.

Another aspect of the controversy over the usefulness of catastrophe theory is what should be called catastrophe theory. The critics maintain that catastrophe theory should be considered as the mathematics and applications of elementary catastrophe theory restricted to the seven simplest singularities. Some critics seem to argue further that this term should only be applied to those applications in the social sciences and biology **[12]**. On the other hand, some of the supporters, including Thom himself, view the mathematical extent of catastrophe theory in such generalities that they go beyond what is proven rigorously into areas which are vague and–as some suggest–mystical.

This dichotomy is unfortunate and leads to an unstable situation. The critics have a tendency to label applications in the social sciences and biology as "applied catastrophe theory" while reserving the term "singularity theory" for those applications which are based on sound mathematical models and are usually found in physics and engineering. Another result is that criticisms of applications to the "soft sciences" are interpreted to mean criticisms of all uses of catastrophe theory, which is both misleading and unfair to those who have made "solid" use of catastrophe theory methods.

One reason for this divergence of opinion has been the lack of availability of information about the hard applications of catastrophe theory. So we return to the question, "How does one find out about catastrophe theory's successes?" An obvious suggestion is to read some of the numerous expository articles–for example **[20]**, **[4]**, **[13]**, **[3]**–describing the mathematics and some of the applications. Even the most visible critic **[14]** as well as this reviewer **[5]** have written such articles. Unfortunately surveys are constrained by a lack of space to describe the uses of catastrophe theory superficially or to describe superficial uses of catastrophe theory. Another suggestion is to read Thom's seminal work **[17]** or the recently published collection of

Zeeman's papers [21]. However, the first book is about the problems that Thom hoped would be described by catastrophe theory and not about what has actually been accomplished while controversy swirls about the second. Zeeman deals mainly with applications to the social sciences and biology–at least these are the applications which are most often quoted–areas which have persistently defied sophisticated mathematical techniques. Thus, a diligent person interested in catastrophe theory should read Zeeman and his critics [15], but to do only this would lead one to miss entirely what are perhaps the most interesting examples of catastrophes.

It seems to this reviewer that to resolve the question of whether catastrophe theory has useful applications, it is not necessary to approve or disparage applications of elementary catastrophe theory to the social and biological sciences. Instead we may focus on areas where the use of mathematics has a successful history and where this new theory claims to have made significant progress rather than on areas where even the use of mathematics is in doubt. There have been many of this kind of application of catastrophe theory to problems in physics and engineering. That most readers of Newsweek should not appreciate this fact is not surprising; that most professional mathematicians should be similarly ignorant calls for some explanation. As we have discussed above, the reasons are twofold. First, there is disagreement as to what should be called catastrophe theory. Second, it was until now difficult to find a coherent, self-contained reader on catastrophe theory and hard applications. To be such a reader is the purpose of Poston and Stewart's book. It is a book whose time has come, which may serve as a basis for redirecting the catastrophe theory debate, and if successful will be outdated–in the best engineering sense–within a few years.

Before discussing Poston and Stewart's book I would like to return to the question of what is catastrophe theory. Of all the ideas associated with elementary catastrophe theory it is the notion of universal unfolding (or equivalently, the stability of a parametrized family) which uses new mathematics. In particular, it is here that one needs the Malgrange Preparation Theorem, easily the deepest and most technical analytic result associated with catastrophe theory. Thus, I would like to define *catastrophe theory* to be the study and use of classes of germs of mappings of R^n into R^m considered under some equivalence relation (usually given by changes of coordinates) for which the unfolding theorem is valid. One should observe that all of the theorems which are known in this more general setting are proved in a fashion similar to the unfolding theorem of elementary catastrophe theory and there is substantial mathematical precedent for grouping similar theories under a single title. Finally, for those who find the term "catastrophe theory" anathema, I suggest the use of "singularity theory."

Catastrophe theory and its applications divides naturally into two parts: theory and practice. The mathematics is presented in a leisurely manner rather than in the more usual and terse theorem-proof format. The basic theorems are described, but their proofs are mostly just replaced by references to the literature. As there is now extant an extensive and detailed literature this seems a reasonable approach. The authors propose, in the first part of their book, to give the reader a working knowledge of elementary catastrophe theory, a knowledge sufficient to understand the subsequent examples. It is indeed a curious, though hardly surprising, pedagogical fact that the theory is defined and described by those very examples.

The list of applications fall under the following chapter titles: ship design, fluids, optics, elastic structures, thermodynamics, lasers, biology, and social modelling. Disregarding the last two subjects–the most controversial–still leaves one with a rather ambitious list. The remainder of this review will be confined to fluids and elastic structures. These applications should give a flavor for the type considered by Poston and Stewart; they also represent different kinds of applications, the first showing how the theory may help in the analysis of a specific model while the second describes a field where catastrophe theory may help to organize the extensive existing knowledge. My own prejudice is that work on elastic structures and, more generally, the relation between singularity theory and classical bifurcation theory will strengthen both theories as well as mute the criticisms that catastrophe theory is a theory dealing only with real-valued functions. On the other hand, this relationship with bifurcation theory points out the valid criticism that to date catastrophe theory is an essentially static theory. There have been attempts by Arnold **[1]** and Takens **[16]** to use singularity theory for the study of dynamical systems but the tangible results *so far* are not very encouraging.

The application of elementary catastrophe theory described by Poston and Stewart under the title of fluids is work done by Berry and Mackley **[2]** on experimentally finding a measure for how much a given polymer deviates from being Newtonian. A non-Newtonian liquid is one where the viscous forces at a point in the liquid depend on the history of the liquid at that point. The size and geometry of the polymer chains contribute to making a given fluid non-Newtonian. Berry and Mackley use this data to study "such points of physical interest as molecular relaxation times." It turns out that this application depends on a detailed understanding of the geometry of the elliptic umbilic catastrophe.

The models under consideration are two dimensional, time independent (or *steady*) flow; that is, it is assumed that there is a vector field $v(x,y)$ which gives the velocity of the fluid particle at (x,y). These assumptions are, of course, physically unrealistic in many situations but they are assumptions made classically and they give a good approximation to actual flows in certain cases. The experiments of Berry and Mackley seem to provide such an instance.

For steady, two-dimensional flow there always exists a stream function ϕ such that $v = (\phi_y, -\phi_x)$. Note that it is through the stream function that elementary catastrophe theory enters the analysis. In general, the Navier-Stokes equations which govern fluid flow are only invariant under changes of coordinates which are volume preserving; whereas the theorems of elementary catastrophe theory demand equivalences which are given by arbitrary C^∞ coordinate changes. However, differential invariants of the flow–for example, the number and type of critical points (or *stagnation points*) of ϕ–are *a fortiori* invariants of volume preserving coordinate changes.

Consider, as an example, "pure shear" flow; that is, $\phi = xy$. Geometrically this flow is a standard hyperbolic flow about a saddle point. G. I. Taylor suggested in 1934 an experimental apparatus known as the *four roll mill*, represented in Figure 1, for generating this flow. Using symmetry and some elementary Morse theory one can see why this apparatus should generate pure shear flow. By symmetry there is a unique stagnation point at the origin and $\phi = axy + \ldots$ where a may be shown to be nonzero. Morse theory states that ϕ is C^∞ equivalent to pure shear flow so that the hyperbolic character of the flow is a necessity. Moreover, the stability of nondegenerate critical points under small perturbations indicates that the flow should be experimentally realizable and it is.

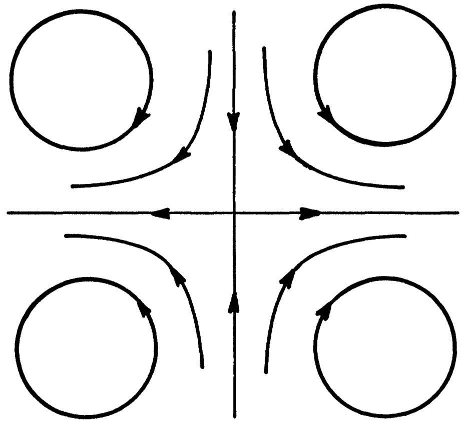

FIGURE 1. All roller speeds are equal

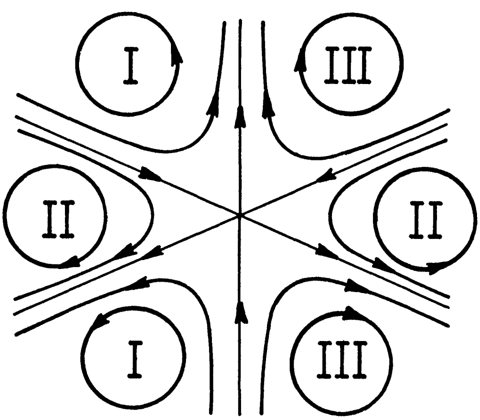

FIGURE 2. All roller speeds are equal

The experiment considered by Berry and Mackley is the *six roll mill* pictured in Figure 2. Using symmetry and some simple elementary catastrophe theory techniques one can see that the flow for the idealized six roll mill is the monkey saddle flow presented in Figure 2. In particular the ϕ for this experiment is C^∞ equivalent to $x^3 - 3xy^2$, the elliptic umbilic. Here now is a difference with the four roll mill; this ϕ is not stable under small perturbations. In fact, the unfolding theorem states that three parameters are necessary to describe all small perturbations of ϕ up to C^∞ equivalence.

These perturbations may be represented by $\Phi = \phi + w(x^2 + y^2) + uy + zx$. This simple form for the pertrubation terms allows one to catalogue–using the well understood geometry of the elliptic umbilic–all of the way the monkey saddle flow can change under small perturbations of ϕ. This is done graphically by Poston and Stewart and appears on the dust cover of their book.

Berry and Mackley ask how the unfolding parameters u, z and w can be realized experimentally. First, they let W_I, W_{II} and W_{III} be independent roller speeds for the pairs of rollers I, II and III in Figure 2. As only the ratios of these speeds matter, one obtains two independent parameters. The third unfolding parameter turns out to measure the deviation of a given fluid from being Newtonian. Of course, for a given fluid this parameter is fixed. However, Poston and Stewart note that by observing the patterns of the flow experimentally as the roller speeds are changed one can use the catalogue of possible flow patterns to actually determine the value of this third unfolding parameter, thus generating the desired information for the given fluid.

The chapter on elastic structures shows how to use elementary catastrophe theory to analyze a number of (idealized) mechanical structures. Except for some work on the von Kármán equations describing the buckling of plates, there is no claim that the results are new, just that there is a new perspective in which to view these problems. To understand the novelty in this approach one must understand some of the rudiments of steady-state bifurcation theory.

The basic idea is that one is given an equation (either algebraic, differential or integral) depending on a parameter λ and denoted symbolically by

$$G(x, \lambda) = 0. \tag{1}$$

(Usually G is given as a mapping between Banach spaces but may, for many problems, be reduced by some method like that of Lyapunov and Schmidt to a finite dimensional problem.) One wants to study how the *solution set* varies with λ. Those values of λ where two different solution branches $x_1(\lambda)$ and $x_2(\lambda)$ cross are called *bifurcation points*. A standard classical example is the buckling of an object under an applied load λ; the values of λ where buckling occurs are bifurcation points.

Elementary catastrophe theory does not apply to all such problems, only those for which $G = \partial V/\partial x$ for some potential function V. However, this is a rich class known as *conservative* problems so that such a restriction is a reasonable one.

Much of classical bifurcation theory rests on knowing a particular solution branch, called the *trivial solution*, and following along that branch until bifurcation occurs, thus mimicking a real experiment. A necessary condition for branching is that $d_x G$–the Jacobian of G with respect to the x-variables–be singular. The novelty of the singularity theory approach is to find those singular points first and then build the solution set from the knowledge of the type of singularity which is present. There are two advantages; a trivial solution may not be easily discernible and many singular phenomena do not actually include branching but are still in need of description **[6]**.

The technical ways which Poston and Stewart suggest that elementary catastrophe theory may be helpful are as follows:

(1) All too often bifurcation analysis has only considered the varying of one parameter. Already in the buckling of an Euler column several parameters are necessary to attain all nearby states. The suggestion here is to use the unfolding theorem to identify *all* of the relevant perturbation terms. Care must be exercised in this analysis for the answer that one obtains depends crucially on the exact nature of the question that is asked. I will say more about this point in *B* below.

(2) The refined algebraic nature of catastrophe theory calculations may be used to analyze problems where there are several x variables (multiple eigenvalue problems). This may well be one of the major technical contributions of singularity theory to bifurcation theory; however, even these refined techniques do not make the calculations easy! The application to the von Kármán equations alluded to above is a case in point.

(3) Thompson and Hunt [**18**] have developed a theory similar to elementary catastrophe theory to study imperfection sensitivity–the study of how the value of λ where buckling occurs varies with some assumed imperfection. It is assumed that by applying catastrophe theory techniques one should be able to carry out the Thompson-Hunt program for more complicated situations (as well as to provide mathematical proofs for their techniques).

There are several criticisms that one could make of these applications of elementary catastrophe theory.

(A) Even though a given problem may be conservative there is no guarantee that perturbations of that problem need be conservative (the wind may be blowing). Hence elementary catastrophe theory cannot classify all possible perturbations.

(B) There is a very good reason for considering only one parameter bifurcation problems; namely, when performing an experiment one must follow a one-dimensional path parametrized by time. If this theory is to be predictive, then it must yield information which can be compared to experiment. Thus λ should be considered separately from the other perturbation parameters, a situation not permitted in elementary catastrophe theory. When one does this the number of parameters needed for a universal unfolding increases, sometimes quite substantially.

(C) For many physical problems there is a built-in symmetry. For many of these problems one is only interested in the symmetry preserving perturbations.

Poston and Stewart are aware of these problems and through various examples they show explicitly why such considerations are important. The reader should be warned that their suggestion for how to take care of criticism (B), namely, by the use of $r - s$ stability theory, is wrong. (One should see [**7**] for a more detailed explanation of why this theory is inappropriate for the problem at hand.) Poénaru [**9**] has an unfolding theory which includes symmetry properties of the potential function; however, no systematic application of this theory has been attempted to date.

Since Poston and Stewart wrote their book the criticisms listed above, including the necessity of analyzing the potential function V rather than G directly, have been confronted by use of yet another of the standard unfolding theorems from singularity theory–Mather's theorem on contact equivalence. Now there exists a solid relationship between classical bifurcation theory and singularity theory [7], [8], [10], [11], [5].

I should like to end this review with some comments on the tone of *Catastrophe theory and its applications*. The authors are enthusiastic and there is no muting their enthusiasm. However, unlike in some other works on the subject, there is a welcome lack of pretension and an honest effort to explain how and why elementary catastrophe theory is useful. Great pains are taken to justify the catastrophe theory formalism and this is most desirable, given the general controversy. Finally, there is a lively wit which is sprinkled throughout; one of my favorites being:

> "As these examples illustrate, the distinction is between elastic and plastic *behavior* and not between *materials*. Moreover it is far from absolute: there are materials that can be deformed and left lying, apparently changed in shape, but a day later (like the mind of a bureaucrat you thought you had convinced of something) appear as if they had never been disturbed, having reverted slowly but inexorably to the original position."

Catastrophe theory suffered through a period of initial euphoria when mathematicians were curious and nonmathematicians overly impressed by what was claimed. More recently, there has been a backlash against those extravagances and doubt has been cast on every conceivable use of this theory. It is my hope that the collective wisdom regarding catastrophe theory will prove elastic.

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Innovation processes, by Yuriy A. Rozanov, John Wiley & Sons, New York, Toronto, London, and Sydney, 1977, vii + 136 pp., $14.50.

Complex valued random variables ξ_s, $s \in I$, with finite second moments being simply L_2-functions on a probability space, problems involving only the second moments are naturally set in the corresponding Hilbert space, the expectation $E[\xi_s\bar{\xi}_t] \equiv B(s, t)$ serving as the inner product. Throughout this review all random variables will be assumed to have second moments, and also, for convenience only, the first moment will be taken to be zero. The parameter set I will be taken to be the interval (α, β) on the line, where α or β may be infinite. $B(s, t)$ is the covariance function. Then $(\xi_s, s \in I)$ is a stochastic process; or a curve in Hilbert space. For $t \in I$, let $H_t(\xi)$ be the closed linear hull of $\{\xi_s: \alpha < s \leqslant t\}$, let $H(\xi)$ be the closure of the union of the $H_t(\xi)$, $t \in I$, and let P_t be the operator of orthogonal projection onto $H_t(\xi)$. The theme of Rozanov's book is the temporal evolution of the family of nondecreasing subspaces $(H_t(\xi), t \in I)$. This leads to questions in the geometry of Hilbert space naturally motivated by probabilistic considerations: $\{\xi_s: \alpha < s \leqslant t\}$ represents the observations available up to time t, and for $t < u < \beta$, $P_t\xi_u$ is the best linear predictor (in the sense of mean square error) of ξ_u in terms of the past up to time t. The process $(\xi_t, t \in I)$, will be assumed left-continuous, and this implies the same property for the family $(H_t, t \in I)$ and also the separability of $H(\xi)$. For simplicity ξ_t is taken to be complex-valued, but much recent work in the area has been devoted to vector space valued cases, and this is also the setting of Rozanov's book.

The notion of an innovations process associated with $(\xi_t, t \in I)$ is due to Cramèr [1], [2], [3]; for a related development see Hida [7]. It can be shown that there exists a finite or infinite sequence $\zeta^{(i)}$ of elements of $H(\xi)$ so that on putting $\zeta_t^{(i)} = P_t\zeta^{(i)}$ the following conditions hold: (i) $H_t(\zeta^{(i)}) \perp H_t(\zeta^{(j)})$ for $i \neq j$; (ii) setting $F_t^{(i)} = E[|\zeta_t^{(i)}|^2]$, $F^{(j)}$ is absolutely continuous with respect to $F^{(i)}$ for $j > i$; $H_t(\xi) = \Sigma_i \oplus H_t(\zeta_i)$. Of course each $(\zeta_t^{(i)}, t \in I)$ is a process with orthogonal increments. The length of the sequence $\zeta^{(i)}$ is called

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