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PRIMITIVE SUBALGEBRAS OF EXCEPTIONAL LIE ALGEBRAS

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The object of this paper is to classify (up to inner automorphism) the primitive, maximal rank, reductive subalgebras of the (complex) exceptional Lie algebras. By primitive we mean that the subalgebras correspond to (possibly disconnected) maximal Lie subgroups. In [3], the corresponding classification for the (complex) classical Lie algebras was completed, as was the classification for the non-reductive, maximal rank, subalgebras of all the simple Lie algebras.

Using case by case techniques and some more general results given later, we prove the following theorem:

THEOREM 0. *The primitive, maximal rank, reductive subalgebras of the exceptional (complex, simple) Lie algebras are listed (up to conjugacy by an inner automorphism) in the table below. Further, all subalgebras isomorphic to one of these are conjugate by an inner automorphism.*

Algebra	*Primitive subalgebras*
E_8	$A_1 \oplus E_7,\ A_1^8,\ A_2 \oplus E_6,\ A_2^4,\ A_4^2,\ D_4^2,\ D_8,\ A_8,\ T^8$
E_7	$A_1 \oplus D_6,\ A_1^3 \oplus D_4,\ A_1^7,\ A_2 \oplus D_5,\ A_2^3 \oplus T^1,\ A_7;\ E_6 \oplus T^1,\ T^7$
E_6	$A_1 \oplus A_5,\ A_2^3;\ D_4 \oplus T^2,\ D_5 \oplus T^1,\ T^6$
F_4	$A_1 \oplus C_3,\ A_2^2,\ B_4,\ D_4$
G_2	$A_1^2,\ A_2$

T^k denotes the center of the subalgebra, where k is the dimension of that center. The other superscripts refer to the number of summands of the corresponding algebra.

We note that Theorem 5.5 (p. 148) in the reductive case of Dynkin [2] is incorrect. In particular $A_3 \oplus D_5$, $A_5 \oplus A_2 \oplus A_1$, $A_7 \oplus A_1$ in E_8, $A_3^2 \oplus A_1$ in E_7 and $A_3 \oplus A_1$ in F_4 are not maximal subalgebras. (See [2, Table 12, p. 155].)

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We now present some basic notation and a characterization of

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primitivity from [3].

Let p be a maximal rank subalgebra of a simple Lie algebra g. By maximal rank, we mean that there exists a Cartan subalgebra h of g which is contained in p. We fix h.

Let W be the Weyl group relative to h. p is then decomposed by h into

$$p = h \oplus \sum_{\phi} e_{\phi}$$

where the ϕ's are roots in g determined by h, and the e_{ϕ}'s are the corresponding one-dimensional root spaces. p is then uniquely determined by the roots ϕ for which $e_{\phi} \subset p$. Let $K_p = \{\phi \text{ a root of } g \text{ relative to } h \mid e_{\phi} \subset p\}$. Define $W_p = \{\alpha \in W \mid \alpha(K_p) = K_p\}$.

PROPOSITION 1.1. *Let p be a maximal rank subalgebra of g. Then p is primitive iff the following holds: If l is a subalgebra of g such that $p \subset l$, and $W_p \subset W_l$, then $p = l$ or $l = g$.*

PROOF. This is Proposition 3.2 of [3].

We will use this as our definition of primitivity.

Now let p be reductive, i.e. p is given uniquely as the direct sum of simple algebras and its center. Thus there exist nonisomorphic simple Lie subalgebras of $g: X_1, X_2, \cdots, X_r$ such that $p = X_1^{k_1} \oplus \cdots \oplus X_r^{k_r} \oplus T$ where T is the center of p and where $X_i^{k_i}$ denotes the direct sum of all ideals of p isomorphic to X_i, and k_i is the number of such ideals. Note that since $X_i = (h \cap X_i) \oplus \sum_{\phi} e_{\phi}$, where the ϕ are unique, K_{X_i} and W_{X_i} make sense.

LEMMA 1.2. *Let $q_i = X_i^{k_i}$. Then $W_p \subset W_{q_i}$.*

Let z be a subalgebra of g with Cartan subalgebra $h_z \subset h$. Assume that z is regular (in the sense of Dynkin [2]), i.e. let $K_z = \{\phi \in K_g \mid e_{\phi} \subset z\}$, then $z = h_z \oplus \sum_{\phi \in K_z} e_{\phi}$. Denote by $K_z^{\perp}$ all of the roots in K_g orthogonal to the set K_z. Let $h_z^{\perp}$ be the subspace of h orthogonal to h_z.

LEMMA 1.3. *Let ${}_z^{\perp} = h_z^{\perp} \oplus \sum_{\phi \in K_z^{\perp}} e_{\phi}$. Then $z^{\perp}$ is a subalgebra of g, and $K_z^{\perp} = K_z^{\perp}$.*

THEOREM 1.4. *Let p be a maximal rank, reductive subalgebra. Let $p = X_1^{k_1} \oplus \cdots \oplus X_r^{k_r} \oplus T$ (as described above). Let $q_i = X_i^{k_i}$. If p is primitive, then either $q_1^{\perp} = q_2 \oplus \cdots \oplus q_r$ or the subalgebra, generated by the vector subspace $q_1 + q_1^{\perp} + h$ is g.*

COROLLARY 1.5. *Let g be a simple Lie algebra all of whose roots have the same length (in particular E_6, E_7, and E_8). Then if p is a primitive subalgebra, $Y = X_2^{k_2} \oplus \cdots \oplus X_r^{k_r}$.*

To compute the primitive subalgebras we use a case by case technique. For each exceptional algebra we determine which simple algebras could be ideals in a maximal rank, reductive subalgebra. The possibilities are restricted by rank and the length of the roots. For instance, the only possibilities for X_1 in E_8 are A_l $(1 \leqq l \leqq 8)$, D_l $(4 \leqq l \leqq 8)$, E_6 and E_7. Then using 1.4 and 1.5, we know what the possibilities are for the subalgebra. For example, in E_8, given $X_1^{k_1}$, the only possibility for a primitive subalgebra is $X_1^{k_1} \oplus (X_1^{k_1})^\perp$ by 1.5.

The remaining task is to determine whether or not the given subalgebra is primitive. For this we use Proposition 1.1. Unfortunately (particularly for E_s, $s=6, 7, 8$) this is a nontrivial task. We need to know more about the Weyl group of E_s. We present here a theorem which we found extremely useful in computing with these Weyl groups.

We first describe the roots of the algebras E_6, E_7, and E_8 (see [1]). Let $z_1, z_2, \cdots, z_8$ be the standard orthonormal basis for the dual space to a fixed Cartan subalgebra of E_8. With respect to this basis, the roots of E_8 are given by

$$(\mathrm{I})_8 = \{\pm z_i \pm z_j \mid 1 \leqq i < j \leqq 8\}$$

and

$$(\mathrm{II})_8 = \{\tfrac{1}{2}(\pm z_1 \pm z_2 \pm \cdots \pm z_8) \mid \text{the number of minus signs is even.}\}.$$

We shall refer to these as type I roots and type II roots respectively.

We take E_6 and E_7 to be regular subalgebras of E_8. The roots of E_7 are all of those roots of E_8 orthogonal to z_7+z_8. Thus

$$(\mathrm{I})_7 = (\mathrm{I}) \cap K_{E_7} = \{\pm z_i \pm z_j \mid 1 \leqq i < j \leqq 6\} \cup \{z_7 - z_8\},$$
$$(\mathrm{II})_7 = (\mathrm{II}) \cap K_{E_7} = \{\pm \tfrac{1}{2}(z_1 \pm z_2 \pm \cdots \pm z_6 + z_7 - z_8) \in (\mathrm{II})_8\}.$$

The roots of $E_6 \subseteq E_7$ are as follows:

$$(\mathrm{I})_6 = (\mathrm{I}) \cap K_{E_6} = \{\pm z_i \pm z_j \mid 1 \leqq j < j \leqq 5\},$$
$$\begin{aligned}(\mathrm{II})_6 &= (\mathrm{II}) \cap K_{E_6} \\ &= \{\pm \tfrac{1}{2}(z_1 \pm z_2 \pm \cdots \pm z_5 + z_6 + z_7 - z_8) \in (\mathrm{II})_7\}.\end{aligned}$$

We note first that the reflections about roots of type I are just determined by signed permutations of $z_1, \cdots, z_8$. For if α and β are roots, then

$$S_\alpha(\beta) = \beta - \frac{2(\alpha, \beta)}{(\alpha, \alpha)}\alpha = \beta - (\alpha, \beta)\alpha, \qquad \text{as } (\alpha, \alpha) = 2.$$

If $\alpha = z_i - z_j$, then

$$\begin{aligned} S_\alpha(z_k) &= z_i && \text{if } k = j, \\ &= z_j && \text{if } k = i, \\ &= z_k && \text{otherwise.} \end{aligned}$$

If $\alpha = z_i + z_j$, then

$$\begin{aligned} S_\alpha(z_k) &= -z_i && \text{if } k = j, \\ &= -z_j && \text{if } k = i, \\ &= z_k && \text{otherwise.} \end{aligned}$$

We can thus, for example, identify S_α with the transposition $\binom{1\,2\,3\ldots 8}{2\,1\,3\ldots 8}$ if $\alpha = z_1 - z_2$ and with $\binom{1\;\;\;2\;3\ldots 8}{-2\,-1\,3\ldots 8}$ if $\alpha = z_1 + z_2$. These reflections, then, generate signed permutations with an even number of sign changes.

Let N_s be the subgroup of the Weyl group of E_s consisting of the signed permutations. We have the following normal forms for elements of the Weyl group:

THEOREM 1.6. *Let g be a Weyl group element. Then one of the following three cases holds*:

(a) $g \in N_s$.

(b) $\exists f \in N_s,\ \alpha \in (\mathrm{II})_s \ni g = fS_\alpha$.

(c) $\exists f \in N_s,\ \alpha,\ \beta \in (\mathrm{II})_s \ni (\alpha, \beta) = 0$ *and* $g = fS_\beta S_\alpha$.

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