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The Structure of Symmetric Attractors

IAN MELBOURNE, MICHAEL DELLNITZ & MARTIN GOLUBITSKY

Abstract

We consider discrete equivariant dynamical systems and obtain results about the structure of attractors for such systems. We show, for example, that the symmetry of an attractor cannot, in general, be an arbitrary subgroup of the group of symmetries. In addition, there are group-theoretic restrictions on the symmetry of connected components of a symmetric attractor. The symmetry of attractors has implications for a new type of pattern formation mechanism by which patterns appear in the time-average of a chaotic dynamical system.

Our methods are topological in nature and exploit connectedness properties of the ambient space. In particular, we prove a general lemma about connected components of the complement of preimage sets and how they are permuted by the mapping.

These methods do not themselves depend on equivariance. For example, we use them to prove that the presence of periodic points in the dynamics limits the number of connected components of an attractor, and, for one-dimensional mappings, to prove results on sensitive dependence and the density of periodic points.

1. Introduction

Our goal in this paper is to describe mathematical properties of symmetric attractors that have been observed in the numerical simulations of equivariant discrete dynamical systems in [6, 11, 16]. These properties include connectedness, sensitive dependence on initial conditions and, indeed, the actual symmetry of the attractor. While pursuing this goal we have used techniques that are equally valid for systems possessing no symmetry, and these techniques lead to interesting results for asymmetric systems as well.

In contrast with much of the literature on discrete dynamical systems we make no assumptions on the mapping other than continuity, and our definition of attractor is fairly general (Definition 2.2 and Remark 2.3). We shall prove results of the following type.

(a) If an attractor contains a point of period k, then it has at most k connected components (Theorem 2.10). Symmetry often forces the origin to be fixed. So when an attractor for such a system contains the origin, it must be connected. (Such attractors also have full symmetry (Proposition 4.8).) In addition, topologically-mixing attractors are connected (Theorem 2.7).

(b) For continuous maps on the line, attractors (and, more generally, ω-limit sets) are contained in the closure of the set of periodic points (Theorem 3.1). This 'closing lemma' holds without the usual assumptions of genericity and differentiability. In addition, nonminimal attractors of continuous maps on the line display sensitive dependence on initial conditions (Theorem 3.2).

(c) There are representation-theoretic restrictions on the symmetry of attractors (Theorem 4.10).

(d) There are group-theoretic restrictions on the symmetry of connected components of symmetric attractors (see Theorem 4.6).

(e) Mappings of the plane having $\boldsymbol{D}_m$-symmetry (where $\boldsymbol{D}_m$ is the *dihedral* group of symmetries of the regular m-sided polygon in the plane) *cannot* have attractors with symmetry $\boldsymbol{D}_k$ where $2 < k < m$ (Theorem 5.3). Also, if a $\boldsymbol{D}_m$-equivariant mapping has an attractor with symmetry $\boldsymbol{D}_m$, then each of its components must have $\boldsymbol{D}_m$-symmetry (Theorem 5.8).

(f) Fully symmetric ω-limit sets of planar $\boldsymbol{D}_m$-equivariant mappings automatically have a form of sensitive dependence (Corollary 5.6).

We shall also show that the proofs of all of these theorems rely in part on a single topological lemma (Lemma 2.1) which uses connected components of the complement of a preimage set to cover invariant sets. This and related results are described in Section 2. Applications of Lemma 2.1 to one-dimensional mappings are given in Section 3. In Section 4 we discuss how symmetry is brought into the study of attractors of mappings and prove some general results. We apply the results in Section 4 to derive theorems about symmetric attractors for planar mappings with dihedral symmetry in Section 5.

It was observed in [6] from computer experimentation that symmetry-increasing bifurcations seem to be the rule in the discrete dynamics of maps in the plane with dihedral $\boldsymbol{D}_m$-symmetry. These bifurcations occur through the collision of conjugate attractors. In [10] we use the results about preimage sets, in particular Theorems 3.5 and 5.9, as the basis for a numerical algorithm for the computation of certain types of symmetry increasing bifurcations.

An important observation in the theory of equivariant steady-state bifurcations states is that there are restrictions on the possible symmetry of bifurcating equilibria. Our results in Sections 4 and 5 indicate that a similar remark holds for general attractors.

To conclude this Introduction we discuss briefly why the symmetry of attractors may be important in applications. We first consider systems of partial differential equations and then the Faraday experiment on parametrically excited surface waves.

Suppose that $u(x, t)$ is a (vector-valued) solution to a Γ-invariant system of partial differential equations where Γ is a (compact Lie) group of symmetries. Let A be the ω-limit set in phase space of the solution u. Numerical integration has suggested that even when the solution u is varying chaotically in time the set A can itself be invariant under a nontrivial subgroup Σ of symmetries in Γ. For example, see the work by PLATT, SIROVICH & FITZMAURICE [18] on Kolmogorov flow and our own work [10] on reaction-diffusion systems on the line (in particular, the Brusselator and the Ginzburg-Landau equation). There are two consequences of the existence of the symmetries Σ that we wish to discuss. The first is a technical comment and the second is, perhaps, of more consequence for applications.

1. A popular method for approximating the dynamics of a system of partial differential equations by a system of ordinary differential equations is by the use of the Karhunen-Loève (or proper orthogonal) decomposition. In [9] it is shown that this decomposition can be completed naturally in a way that the approximating system of ordinary differential equations is precisely Σ-equivariant. See also [3, 4].

2. The second consequence is difficult to prove rigorously in any given system of partial differential equations, but is likely to be valid in many systems. The statement is that the time-average of the solution is Σ-invariant – thus allowing for a distinctive regular pattern to appear in the time-average even though this pattern is never present at any instant of time. This comment suggests a new method for pattern formation based on chaotic dynamics rather than the standard kind of pattern formation based on stationary or time-periodic bifurcations. See [9, 10].

We now discuss this second point in a little more detail. Let

$$U(x) = \lim_{T\to\infty} \frac{1}{T} \int_0^T u(x, t)\, dt;$$

then $U(x)$ is the *time-average* of the solution $u(x, t)$. Our claim is that we expect

$$U(\sigma x) = U(x) \qquad \forall \sigma \in \Sigma.$$

To justify this conclusion, one has to presume the existence of a Sinai-Bowen-Ruelle measure on A which is both flow- and Σ-invariant. Then one can use the ergodic theorem to transform the time-average to an integral of u over A in phase space; and change of variables in this integration yields the result. This issue is discussed in [9, 10, 2]. What is illustrated numerically in [10] is that asymmetric chaotic solutions of the Brusselator and the Ginzburg-Landau equations can have spatially symmetric time-averages.

This last discussion is illustrated in recent experiments by GOLLUB and coworkers [12] on the Faraday surface wave experiment. In this experiment a fluid layer is vibrated at a fixed amplitude and at a fixed frequency. At small amplitudes of vibration the fluid layer remains flat while at larger amplitudes it deforms. Patterns are well known to appear in the deformed surface [21, 7] and these patterns can be related to the symmetry of the apparatus [8]. In the experiment the patterns are detected by viewing the focusing and diffusing of light transmitted through the fluid.

It is also well known that at even larger amplitudes of vibration the fluid surface appears to vary chaotically in time and to have no discernible pattern at a fixed instant of time. What GOLLUB and co-workers have shown, however, is that if the intensity of transmitted light is time-averaged at each point in space, then the resulting time-average can have a well-defined spatial pattern, a pattern consistent with the symmetry of the apparatus. We believe that the associated attractor has this symmetry and that the symmetry in the time-average stems from the symmetry of the attractor.

Finally, we note that if the symmetry group Σ of an attractor is important, then it is necessary to have a numerically efficient method for computing Σ. Algorithms for finding Σ when the group Γ is finite are presented in [2, 9] and it is shown that generically this algorithm can be expected to give the correct answer.

2. Topological dynamics using preimage sets

In this section we introduce the topological results that we use. The main observation (Lemma 2.1) states that ω-limit sets are either contained in the closure of a preimage set or are covered by a finite number of connected components of the complement of that preimage set. This result along with the definition of preimage sets is presented in Subsection (a). This observation has a number of applications which appear throughout this paper. In Subsections (b) and (c) we show how Lemma 2.1 can be used to prove the connectedness theorems promised in the Introduction and a general result concerning sensitive dependence.

(a) *Preimage sets*

Let X be a finite-dimensional Euclidean space and suppose that $f: X \to X$ is a continuous mapping. (In fact, we need only assume that X is a complete, locally connected, metric space for our results to be valid.) Recall that if $x \in X$, the *ω-limit set* of x is defined to be the set $\omega(x)$ consisting of points $y \in X$ for which there is an increasing sequence $\{n_k\}$ of positive integers such that $f^{n_k}(x) \to y$. Basic properties of $\omega(x)$ include

1. $\omega(x)$ is closed,
2. $\omega(f(x)) = \omega(x)$,
3. $f(\omega(x)) \subset \omega(x)$ with equality if $\omega(x)$ is compact.

We call a set A *topologically transitive* if $A = \omega(x)$ for some $x \in A$. Equivalently, A has a dense orbit. The set A is *topologically mixing* if for any open subsets $U, V \subset A$ there exists a positive integer N such that $f^{-n}(U) \cap V \neq \emptyset$ for all $n \geqq N$. If A is topologically mixing, then A is topologically transitive under f^k for all $k \geqq 1$ (cf. [17]).

Let S be a subset of X. We define

$$\mathscr{P}_S(f) = \bigcup_{n=0}^{\infty} (f^n)^{-1}(S)$$

to be the set consisting of S and all of the preimages of S under f. Usually, when the context is clear, we write $\mathscr{P}_S(f) = \mathscr{P}_S$. Observe that $f^{-1}(\mathscr{P}_S) \subset \mathscr{P}_S$. It follows that f induces a mapping

$$f : X - \mathscr{P}_S \to X - \mathscr{P}_S.$$

Since f is continuous, connected components of $X - \mathscr{P}_S$ are mapped into connected components.

The following topological lemma (and Corollary 2.5) are used frequently throughout the paper.

Lemma 2.1. *Let $x \in X$ and let S be a subset of X. Then either $\omega(x) \subset \overline{\mathscr{P}_S}$ or the following are valid.*

(a) *$\omega(x) - \mathscr{P}_S$ is covered by finitely many connected components $C_0, \ldots, C_{r-1}$ of $X - \mathscr{P}_S$.*

(b) *These components can be ordered so that $f(C_i) \subset C_{(i+1)\bmod r}$.*

(c) *$\omega(x) \subset \overline{C_0} \cup \cdots \cup \overline{C_{r-1}}$.*

Proof. We assume that $\omega(x) \not\subset \overline{\mathscr{P}_S}$. Choose $y \in \omega(x) - \overline{\mathscr{P}_S}$ and $\varepsilon > 0$ such that $B_\varepsilon(y) \subset X - \overline{\mathscr{P}_S}$ where $B_\varepsilon(y)$ is the open ball of radius ε centered at y. Since $B_\varepsilon(y)$ is connected, it lies inside a connected component C_0 of $X - \mathscr{P}_S$. Since $y \in \omega(x)$, there exist a smallest integer $k \geqq 0$ such that $f^k(x) \in B_\varepsilon(y)$. Also, there is a smallest integer $l > k$ such that $f^l(x) \in B_\varepsilon(y)$. If $r = l - k$, then $f^r(B_\varepsilon(y)) \cap B_\varepsilon(y) \neq \emptyset$ and it follows by continuity that $f^r(C_0) \subset C_0$.

Write $x' = f^k(x)$ and let C_i be the connected component of $X - \mathscr{P}_S$ containing $f^i(x')$ for $i = 0, \ldots, r-1$. It follows by continuity that $f(C_i) \subset C_{(i+1)\bmod r}$, and so

$$f^i(x') \in C_0 \cup \cdots \cup C_{r-1}, \quad i \geqq 0.$$

Hence,

$$\omega(x) = \omega(x') \subset \overline{C_0 \cup \cdots \cup C_{r-1}} = \overline{C_0} \cup \cdots \cup \overline{C_{r-1}}.$$

In addition, since there are only finitely many connected components, they have no limit points lying in another connected component of $X - \mathscr{P}_S$. Hence

$$\omega(x) \subset C_0 \cup \cdots \cup C_{r-1} \cup \mathscr{P}_S,$$

from which (a) follows. □

An f-invariant set A is *stable* if for any open neighborhood U of A there is a smaller open neighborhood V of A such that $f^n(V) \subset U$ for all $n \geqq 0$.

Definition 2.2. An *attractor* is a stable ω-limit set.

Remark 2.3. There are several definitions of 'attractor' in the literature. Since we do not require the existence of a dense orbit in A or that A has an open basin of attraction, our definition is reasonably general. For example, the 'Feigenbaum' limit set is an attractor by our definition even though it does not have an open basin of attraction.

Many of our results hold under a weaker definition of attractor where we only require stability at one point in the following sense. A set A is *stable at one point* if there is a point $y \in A$ such that for any neighborhood U of A there is a neighborhood V of y such that $f^n(V) \subset U$ for all $n \geqq 0$. An *attractor at one point* is an ω-limit set that is stable at one point. For example, the one-dimensional map $f(x) = 4x(1-x)$ has $[0, 1]$ as an invariant interval with a dense orbit. All points outside $[0, 1]$ iterate to $-\infty$ and so $[0, 1]$ is not stable and hence is not an attractor. However, $[0, 1]$ is stable at any point in its interior and so is an attractor at one point. In fact, our definition of attractor at one point is sufficiently general to include any set that has ever been called an 'attractor' for the logistic mapping $f(x) = \mu x(1-x)$.

Throughout this paper we shall state our results for attractors. However, the majority of results hold also for attractors at one point. In fact, the only results that require the stronger notion of attractor are those that make use of Proposition 4.8, that is, Proposition 4.9 and Theorem 4.10 in Section 4, and Proposition 5.2, Theorem 5.3 and Proposition 5.7 in Section 5.

Proposition 2.4. *Let S and A be closed sets and suppose that A is a stable f-invariant set. Then the following statements are equivalent*:

(a) $A \cap S = \emptyset$,

(b) $A \cap \overline{\mathscr{P}_S} = \emptyset$.

Proof. Since $S \subset \overline{\mathscr{P}_S}$, it is clear that (b) implies (a). Now suppose that (a) is valid. Since A and S are closed, there is an open set U containing A with $U \cap S = \emptyset$. Let V be a smaller neighborhood of A such that $\underline{f^n}(V) \subset U$ for $n = 0, 1, 2, \ldots$. It follows that $V \cap \mathscr{P}_S = \emptyset$ and so $A \cap \overline{\mathscr{P}_S} = \emptyset$ as required. □

Corollary 2.5. *Suppose that A is an attractor, S is closed and $A \cap S = \emptyset$. Then*

$$A \subset C_0 \cup \cdots \cup C_{r-1}.$$

Lemma 2.6. *Let M and S be closed subsets. Assume that*

(a) *A is an attractor and $A \cap S = \emptyset$.*

(b) *C is a connected component of $X - \overline{\mathscr{P}_S}$ and $A \cap C \neq \emptyset$.*

(c) *M is f-invariant and $A \cap M \neq \emptyset$.*

Then M intersects C.

Proof. By (b), A intersects C; hence C must be one of the connected components guaranteed by Lemma 2.1. These connected components are permuted cyclically by f; by (a) and Corollary 2.5 they cover the whole of A. By (c), M intersects at least one connected component; invariance implies that M intersects all the connected components. In particular, M intersects C. □

(b) *Connectedness results*

We can now prove two rather strong connectedness results for attractors.

Theorem 2.7. *Let A be a topologically-mixing attractor for a continuous mapping f. Then A is connected.*

Note that this theorem may be proved easily if A is assumed to have a finite number of connected components; the content stems from eliminating the possibility of an infinite number of components.

Proof. Suppose that A is not connected. Then we may write A as the disjoint union of two closed sets A_1 and A_2. Let S be a closed subset of X such that A_1 and A_2 lie inside distinct connected components of $X - S$. By Corollary 2.5, $A \subset C_0 \cup \cdots \cup C_{r-1}$, where $C_0, \ldots, C_{r-1}$ are connected components of $X - \mathscr{P}_S$ and are permuted cyclically by f. Also $r \geqq 2$ by construction. In particular, $A \cap C_0$ and $A \cap C_1$ are invariant under f^r, so that f^r has no attractor containing both of these subsets of A. Hence A is not an attractor for f^r, which contradicts the assumption that A is topologically mixing. □

Corollary 2.8. *If A is a topologically-mixing attractor topologically conjugate to a subshift, then A is a fixed point.*

Proof. It is well known that spaces topologically conjugate to a subshift are totally disconnected; cf. Proposition 11.9 in MAÑÉ [17]. Combining this with Theorem 2.7, we see that A is both connected and totally disconnected. □

Remark 2.9. The standard examples of nontrivial topologically-mixing spaces conjugate to subshifts are not attractors by any definition. On the other hand, it is not difficult to construct examples of nontrivial topologically-mixing attractors that are semiconjugate to subshifts and even conjugate to subshifts off a negligible subset.

Theorem 2.10. *Let A be an attractor for a continuous mapping f. Suppose A contains a periodic point of period k. Then A has at most k connected components.*

Proof. Suppose we can write A as a disjoint union of closed sets

$$A = A_1 \cup \cdots \cup A_{k+1}.$$

Choose S to be a closed set that separates the A_js and such that $S \cap A = \emptyset$. By Corollary 2.5,

$$A \subset C_0 \cup \cdots \cup C_{r-1}$$

where $C_0, \ldots, C_{r-1}$ are connected components of $X - \mathscr{P}_S$. Since S separates the A_js, at most one A_j can intersect a given C_i. It follows that

$$k + 1 \leqq r.$$

Now we let M be the periodic trajectory consisting of k points that is assumed to exist in A. Since M is f-invariant, we may apply Lemma 2.6 to conclude that $M \cap C_j \neq \emptyset$ for each C_j. It follows that

$$k \geqq r.$$

This contradicts the assumption that we can write A as a disjoint union of $k+1$ closed subsets. □

There are a number of consequences of Theorem 2.10 – we mention two.

Corollary 2.11. (a) *If an attractor contains a fixed point, then it is connected.*
(b) *If an attractor contains a periodic point, then it cannot be a Cantor set.*

(c) *Sensitive dependence*

We begin by recalling the notion of sensitive dependence on initial conditions. The following definition is due to GUCKENHEIMER [13] (except that we speak of sensitive dependence of A rather than of f).

Definition 2.12. An invariant set A has *sensitive dependence* if there is a set $Y \supset A$ of positive (Lebesgue) measure and an $\varepsilon > 0$ such that for every $x \in Y$ and every $\delta > 0$ there is a point y that is δ-close to x and an integer $m > 0$ such that

$$|f^m(x) - f^m(y)| > \varepsilon .$$

We also introduce a weaker notion than sensitive dependence that is equivalent to Definition 2.12 for invariant sets of positive measure.

Definition 2.13. An invariant set A has *weak dependence* if there is an $\varepsilon > 0$ such that for every $x \in A$ and every $\delta > 0$ there is a point y that is δ-close to x and an integer $m > 0$ such that

$$|f^m(x) - f^m(y)| > \varepsilon .$$

Proposition 2.14. *Let $f: X \to X$ be continuous, and $x \in X$. Suppose that $S \subset X$ satisfies the following conditions:*

(a) $f(S) \subset S$,

(b) $\omega(x) \subset \overline{\mathscr{P}_S}$,

(c) $\omega(x) \not\subset \bar{S}$.

Then $\omega(x)$ has weak dependence. If, in addition, $\omega(x)$ has positive measure, then $\omega(x)$ has sensitive dependence.

Proof. Using (c) choose a point p in $\omega(x) - \bar{S}$. Let d be the distance from p to $\bar{S}$ and choose ε to be less than d. Let $y \in \omega(x)$ and let $\delta > 0$. In the δ-neighborhood of y there exists a point $x' = f^k(x)$, some k, and a point z that iterates to S (by (b)) – say in l iterates. Since $p \in \omega(x)$ there exists an $m \geqq l$ such that $f^m(x')$ is $(d - \varepsilon)$-close to p. By (a), $f^m(z) \in S$ and hence has distance at least ε from $f^m(x')$. □

3. One-dimensional maps

In this section we prove results about maps on the line which illustrate the methods of the previous section, in particular, Lemma 2.1 and Proposition 2.14. Our main results deal with the two issues of density of periodic orbits and sensitive dependence for ω-limit sets of these one-dimensional maps. Many authors have considered these issues but usually when assuming stronger hypotheses on the mappings. We focus on results that can be obtained for a general class of mappings using the topological methods of Section 2. We note that the results in this section are *not* required in subsequent sections.

Let $f : \mathbb{R} \to \mathbb{R}$ be continuous and let $\mathrm{Per}(f)$ denote the periodic points of f. Then $\mathscr{P}_{\mathrm{Per}(f)}$ is the set of eventually periodic points of f. An ω-limit set is *minimal* if it contains no proper closed invariant subsets.

We now state our two main theorems.

Theorem 3.1. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map, and $x \in \mathbb{R}$. Then*

(a) $\omega(x) \subset \overline{\mathscr{P}_{Per(f)}}$.

(b) *The limit points of $\omega(x)$ lie in $\overline{\mathrm{Per}(f)}$. In particular, if $\omega(x)$ is topologically transitive, then $\omega(x) \subset \overline{\mathrm{Per}(f)}$.*

Theorem 3.2. *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous map and $x \in \mathbb{R}$. If $\omega(x)$ is compact and is not minimal, then $\omega(x)$ has weak dependence. If, in addition, $\omega(x)$ has positive measure, then $\omega(x)$ has sensitive dependence.*

Corollary 3.3. *Suppose that $\omega(x) \subset \mathbb{R}$ consists of a finite union of closed intervals. Then $\omega(x)$ is topologically transitive, periodic points are dense in $\omega(x)$ and there is sensitive dependence.*

Proof. Topological transitivity is clear since $\omega(x)$ has interior in $\mathbb{R}$. Hence by Theorem 3.1(b), $\omega(x) \subset \overline{\mathrm{Per}(f)}$. Again since $\omega(x)$ has interior, the dense set of periodic orbits can be taken to lie in $\omega(x)$.

Suppose that $\omega(x)$ consists of k intervals. Then f^k maps a single interval into itself and has a fixed point. This implies that $\omega(x)$ contains a period k point and is not minimal. In addition, $\omega(x)$ has positive measure and sensitive dependence follows from Theorem 3.2. □

Remark 3.4. (a) Theorem 3.2 holds also for continuous circle maps. In addition, Theorem 3.1 is valid for mappings of the circle provided the set of periodic points is nonempty. The proofs are completely analogous to those for mappings on the line.

(b) Theorem 3.1 is reminiscent of Pugh's closing lemma [19, 20]. Note however that no genericity or differentiability assumptions on f are required, in contrast with mappings of the circle or higher-dimensional manifolds. However, even in $\mathbb{R}$, it is true only generically that the *nonwandering set* $\Omega(f)$ is equal to $\overline{\mathrm{Per}(f)}$; see Young [22]. Part (a) of Theorem 3.1 was proved previously by Block [5] using similar methods. We present a proof here to focus on the way Lemma 2.1 is used in the proof.

(c) Suppose that A is an attractor and that A has an open basin of attraction. Then Theorem 3.1 implies that A must contain a periodic point. But then by Theorem 2.10, A has finitely many connected components. That is, A must be a periodic orbit or a finite union of closed intervals and cannot be a Cantor set. Of course, our definition of attractor does not exclude Cantor sets. Indeed, the 'Feigenbaum' limit set is an attractor and a Cantor set but does not have an open basin of attraction.
(d) The 'Feigenbaum' limit set is minimal, has zero measure and does not display weak dependence (cf. [14]). This shows that the hypothesis of non-minimality in Theorem 3.2 is crucial. (In particular, assuming that $\omega(x)$ is not a periodic orbit is not sufficient to guarantee weak dependence.) In addition, there are examples of minimal Cantor sets with positive measure that are attractors and do not display weak dependence; see [15].

We end with a result that is useful in the computation of symmetry-increasing bifurcations; see [10]. Let p be a periodic point for f and let S denote the corresponding periodic orbit. We call p (or S) *unstable* if there exists a neighborhood U of S such that $\operatorname{dist}(f(x), S) \geqq \operatorname{dist}(x, S)$ for all $x \in U$. Note that if p has period k, f is differentiable and $|(f^k)'(p)| > 1$, then p is unstable.

Theorem 3.5. *Suppose that $x \in \mathbb{R}$ and that the orbit of x under f is bounded. If $p \in \omega(x)$ is an unstable periodic point, then*

$$\omega(x) \subset \overline{\mathscr{P}_p}.$$

Corollary 3.6. *Suppose that an odd continuous mapping $f : \mathbb{R} \to \mathbb{R}$ has a compact attractor A containing the unstable fixed point 0. Then*

(a) $A \subset \overline{\mathscr{P}_0}$.
(b) *A is connected.*
(c) *A has sensitive dependence.*
(d) *Periodic points are dense in A.*

Proof. Statement (a) follows from Theorem 3.5, (b) follows from Theorem 2.10, (c) follows from Theorem 3.2 and (d) follows from Theorem 3.1. □

We now turn to the proofs of Theorems 3.1, 3.2 and 3.5.

Proof of Theorem 3.1. (a) Setting $S = \operatorname{Per}(f)$ in Lemma 2.1 implies that either $\omega(x) \subset \overline{\mathscr{P}_{\operatorname{Per}(f)}}$ or $\omega(x) \subset \overline{C_0} \cup \cdots \cup \overline{C_{r-1}}$ where the C_j are connected components of $\mathbb{R} - \mathscr{P}_{\operatorname{Per}(f)}$ and are cyclically permuted by f. We show that the second alternative implies that $\omega(x) \subset \mathscr{P}_{\operatorname{Per}(f)}$, which proves part (a).

Let $\omega_j = \omega(x) \cap \overline{C_j}$. Then ω_j is an ω-limit set for f^r. By construction, f has no periodic points in C_j. Therefore f^r has no fixed points in C_j and either $f^r(x) > x$ for each $x \in C_j$ or $f^r(x) < x$ for each $x \in C_j$.

Since ω_j is an ω-limit set for f^r, ω_j lies in the boundary of C_j. It follows that $\omega(x)$ is finite (possibly empty). If $y \in \omega(x)$, then the orbit of y under f consists of finitely many points and so $y \in \mathscr{P}_{\mathrm{Per}(f)}$ as required.

(b) Suppose that y is a limit point in $\omega(x)$ and let (a, b) be an open interval containing y. We show that $(a, b) \cap \mathrm{Per}(f) \neq \emptyset$, thus proving the first statement in part (b). By part (a), there is an eventually periodic point $z \in (a, b)$. Suppose that z iterates to a periodic orbit of period r. Let $g = f^r$. Then z iterates under g to a fixed point p, $g^k(z) = p$ for some k. If $p = z$, then we are finished, so we may assume that $p > z$.

Let $\omega_j(x)$ denote the ω-limit set of $f^j(x)$ under g. Then y is a limit point of $\omega_j(x)$ for at least one j, $1 \leqq j \leqq r$. Hence, there are points $y_1 < y_2$ contained in $\omega_j(x) \cap (a, b)$. Let $\varepsilon = (y_2 - y_1)/2$. Since $y_1, y_2 \in \omega_j(x)$, $f^j(x)$ iterates under g to a point $x' \in (a, b)$ within distance ε of y_2 and then to a point $g^l(x')$ within distance ε of y_1 for some $l \geqq k$. Therefore, we have

$$g^l(x') < x', \quad g^l(z) = p > z.$$

It follows from the intermediate value theorem that g^l has a fixed point between x' and z and hence in (a, b). This is the required periodic point for f.

Suppose now that $\omega(x)$ is topologically transitive. To complete the proof of part (b) it is sufficient to show that $\omega(x)$ is either a periodic orbit or a perfect set. That is, either every point is periodic or every point is a limit point. Since $\omega(x)$ is topologically transitive, we can assume without loss of generality that $x \in \omega(x)$. If $f^{n_1}(x) = f^{n_2}(x)$ for positive integers $n_1 \neq n_2$, then $\omega(x)$ is a periodic orbit. Otherwise the orbit $\{f^n(x); n = 1, 2, \ldots\}$ consists of distinct points. Let $y \in \omega(x)$. Then there is an increasing sequence n_k such that $f^{n_k}(x) \to y$. The points $f^{n_k}(x)$ lie in $\omega(x)$ and are distinct, so that y is a limit point of $\omega(x)$ as required. □

Next we turn to the proof of Theorem 3.2. We require two preliminary results. The first of these, Lemma 3.7, is used also in the proof of Theorem 3.5. The second result, Lemma 3.8, contains the technical part of the proof of Theorem 3.2.

Lemma 3.7. *Suppose that $f: \mathbb{R} \to \mathbb{R}$ is continuous, $x \in \mathbb{R}$ and $\omega(x)$ is compact. If $z \in \omega(x) - \mathrm{Per}(f)$, then $\omega(x) \subset \overline{\mathscr{P}_z}$.*

Proof. Since $f(\omega(x)) = \omega(x)$, there exists a sequence $z_n \in \omega(x)$, such that $z_0 = z$ and $f(z_n) = z_{n-1}$. We have assumed that z is not periodic, so the points in the sequence are distinct. Hence $\omega(x) \cap \mathscr{P}_z$ is an infinite set. Suppose that $\omega(x) \not\subset \overline{\mathscr{P}_z}$. By Lemma 2.1, $\omega(x) \subset \overline{C_0} \cup \cdots \cup \overline{C_{r-1}}$ where each C_j is a connected component of $\mathbb{R} - \mathscr{P}_z$. But then $\omega(x) \cap \mathscr{P}_z$ consists at most of the union of the end points of the intervals $\overline{C_j}$ and hence has at most $2r$ points. This is a contradiction. □

Lemma 3.8. *Suppose that $f: \mathbb{R} \to \mathbb{R}$ is continuous and $x \in \mathbb{R}$. Suppose further that $\omega(x) \subset \overline{\mathrm{Per}(f)}$ and $\omega(x) \cap \mathrm{Per}(f) \neq \emptyset$. Then either $\omega(x)$ has weak dependence or $\omega(x)$ is a periodic orbit.*

Proof. The proof divides into two cases depending on whether or not the following property (†) is satisfied.

(†) For any positive integer l, there is a periodic orbit $S \subset \mathbb{R}$ such that at least l connected components of $\mathbb{R} - S$ intersect $\omega(x)$.

Suppose first that property (†) is valid and let y be a periodic point in $\omega(x)$ with period k. We may choose a periodic orbit $S \subset \mathbb{R}$ so that at least $k+1$ components of $\mathbb{R} - S$ intersect $\omega(x)$. Suppose that $\omega(x) \not\subset \overline{\mathscr{P}_S}$. Then there are at least $k+1$, but by Lemma 2.1 finitely many, components of $\mathbb{R} - \mathscr{P}_S$ that cover $\omega(x)$ and that are cyclically permuted by f. One of these components contains y and there is a contradiction. Hence $\omega(x) \subset \overline{\mathscr{P}_S}$. Either $\omega(x) = S$, in which case $\omega(x)$ is a periodic orbit, or S satisfies the hypotheses of Proposition 2.14, so that $\omega(x)$ has weak dependence.

It remains to consider the case when property (†) is not valid. Since $\omega(x) \subset \overline{\mathrm{Per}(f)}$, it follows that $\omega(x) \subset \overline{\mathrm{Fix}(f^k)}$ for some k. By continuity, $\omega(x) \subset \mathrm{Fix}(f^k)$ and consists entirely of periodic orbits. We show that if $\omega(x)$ contains more than one periodic orbit, then there is weak dependence.

It is sufficient to show that $\omega(x)$ has weak dependence under $g = f^k$. Suppose that P_1 and P_2 are distinct periodic orbits in $\omega(x)$ and define

$$\varepsilon = \tfrac{1}{4} \min_{p_1 \in P_1, p_2 \in P_2} |p_1 - p_2| .$$

Suppose that $y \in \omega(x)$ and $\delta > 0$. Choose an iterate $z = f^l(x)$ that is δ-close to y. There are integers $m_1, m_2 \geqq 0$ such that $g^{m_j}(z)$ is within distance ε of points $p_j \in P_j$, $j = 1, 2$. On the other hand, y is fixed by g so that $g^{m_j}(y) = y$. Hence $|g^{m_j}(y) - g^{m_j}(z)| = |y - g^{m_j}(z)| > \varepsilon$ for $j = 1$ or $j = 2$, thus proving weak dependence. □

Proof of Theorem 3.2. We prove weak dependence. Strong dependence then follows from the additional assumption that $\omega(x)$ has positive measure. The strategy of our proof is to reduce, under the assumption that $\omega(x)$ does not have weak dependence, to the situation where the hypotheses of Lemma 3.8 are valid, in which case the theorem follows.

The proof varies depending on whether or not $\omega(x)$ is topologically transitive. Suppose first that $\omega(x)$ is topologically transitive. Since $\omega(x)$ is not minimal, there is an invariant closed subset S contained properly in $\omega(x)$. If $z \in S$ is not periodic, then by Lemma 3.7, $\omega(x) \subset \overline{\mathscr{P}_z} \subset \overline{\mathscr{P}_S}$ and weak dependence follows from Proposition 2.14. So we may assume that S, and hence $\omega(x)$, contains periodic points. In addition, $\omega(x) \subset \overline{\mathrm{Per}(f)}$ by Theorem 3.1(b) so that the hypotheses of Lemma 3.8 are satisfied as required.

Next suppose that $\omega(x)$ is not topologically transitive. We assert that either $\omega(x) \subset \mathrm{Per}(f)$ or $\omega(x)$ has weak dependence. If the first possibility holds, the hypotheses of Lemma 3.8 are satisfied.

We prove the assertion by assuming that $\omega(x) \not\subset \mathrm{Per}(f)$ and showing that $\omega(x)$ has weak dependence. Choose $z \in \omega(x) - \mathrm{Per}(f)$ and let $S =$

$\overline{\{f^n(z); n \geqq 0\}}$. By Lemma 3.7, $\omega(x) \subset \overline{\mathscr{P}_z} \subset \overline{\mathscr{P}_S}$. Note that if S is a proper subset of $\omega(x)$, then we are finished by Proposition 2.14.

Choose $y \in \omega(x)$ such that $f(y) = z$. By the above discussion we may assume that $y \in S$. So either z iterates onto y under f or $z \in \omega(z)$. Since $z \notin \operatorname{Per}(f)$, it is the second possibility that is valid: $z \in \omega(z)$. Hence we may again apply Lemma 3.7 to deduce that $\omega(x) \subset \overline{\mathscr{P}_{\omega(z)}}$. Since $\omega(x)$ is not topologically transitive, $\omega(z)$ is a proper subset of $\omega(x)$, and weak dependence follows from Proposition 2.14. □

Proof of Theorem 3.5. Let S be the periodic orbit corresponding to p. We show that there are points in $\omega(x)$ that iterate to S and do not lie in S. Such a point q is not periodic and hence $\omega(x) \subset \overline{\mathscr{P}_q} \subset \overline{\mathscr{P}_p}$ by Lemma 3.7.

Since the orbit of x under f is bounded, $\omega(x)$ is compact. Hence, $\omega(x)$ lies in the interior of a closed interval I. Let U be the neighborhood of S in the definition of unstable periodic point. We may assume that $U \subset I$. Let $V \subset U$ be a smaller neighborhood of S and let $W \subset I$ be a neighborhood of $\omega(x)$.

We assert that there must be a point $q \in W - U$ such that $f(q) \in V$. Choose a sequence of neighborhoods V_j converging to S and a sequence of neighborhoods W_j converging to $\omega(x)$. By the assertion we obtain a sequence of points $q_j \in W_j - U$, such that $f(q_j) \in V_j$. The sequence q_j lies in $I - U$, which is compact, so passing to a convergent subsequence, we have that $q_j \to q$. Moreover, it follows from the construction of the sequence that $q \in \omega(x) - S$ and $f(q) \in S$ as required.

It remains to verify the assertion. First observe that there is an integer $K \geqq 0$ such that $f^k(x) \in W$ for all $k \geqq K$. For otherwise, $f^j(x) \in \mathbb{R} - W$ for infinitely many integers j, and since this set of points is bounded, there would be an ω-limit point in $\mathbb{R} - W$. This contradicts the fact that $\omega(x) \subset W$.

So without loss of generality, we may assume that $f^j(x) \in W$ for all $j \geqq 0$. Since $S \subset \omega(x)$, x eventually iterates into V. Let k be the least integer satisfying $f^k(x) \in V$. Then $q = f^{k-1}(x) \in W - U$. For if $q \in U$, then $q \in U - V$ and must iterate out of U before entering V. □

4. Symmetry of an attractor

Suppose that Γ is a finite group acting linearly on X, and that f is Γ-equivariant, that is,

$$f(\gamma x) = \gamma f(x).$$

If $x \in X$, we define the *isotropy subgroup* of x to be the subgroup

$$\Sigma_x = \{\gamma \in \Gamma : \gamma x = x\}.$$

If Σ is a subgroup, then it has a *fixed-point subspace*

$$\operatorname{Fix}(\Sigma) = \{x \in X : \sigma x = x \text{ for all } \sigma \in \Sigma\}.$$

We can now define the symmetry of a nonempty set $A \subset X$. The subgroup of Γ that fixes each point in A is denoted by

$$T_A = \bigcap_{x \in A} \Sigma_x,$$

and the *isotropy subgroup* of A consists of the group elements that preserve the set A, and is denoted by

$$\Sigma_A = \{\gamma \in \Gamma : \gamma A = A\}.$$

Proposition 4.1. *Let $A \subset X$ be nonempty. Then*
(a) $A \subset \mathrm{Fix}(T_A)$.
(b) T_A *is a normal subgroup of* Σ_A.

Proof. These statements follow easily from the definitions of T_A and Σ_A. □

By Proposition 4.1, Σ_A is contained in the normalizer $N(T_A)$ of T_A. The *symmetry group* S_A of A is defined to be the quotient group

$$S_A = \Sigma_A / T_A \subset N(T_A)/T_A.$$

Note that if A consists of a single point, then S_A is the trivial group.

Remark 4.2. It is the group S_A that plays the most significant role in the theoretical issues discussed in this paper. However it is interesting to compare the meaning of the three groups T_A, Σ_A and S_A in applications, particularly to nonequilibrium solutions of partial differential equations. The group T_A refers to the symmetries of a solution at each instant in time while Σ_A refers to symmetries of the time-average of that solution. The important observation for applications is that Σ_A can be larger than T_A [10]. In this sense, $S_A = \Sigma_A/T_A$ are the new symmetries that appear in solutions by taking time-averages.

If $T_A = 1$, then Σ_A can be identified with S_A and we say that A is *Σ_A-symmetric*. Note that $T_A = 1$ in the important case when A contains a point with trivial isotropy.

Having defined the kinds of symmetry that an attractor can have we now consider several different issues concerning these symmetries – each issue is considered in a separate subsection. In the first subsection we show that there are general group-theoretic restrictions on the symmetries of connected components of an attractor A given that we know the symmetry group S_A. In the second subsection we show that there are definite restrictions on the symmetry group S_A given by the representation of the full group of symmetries Γ on $\mathbb{R}^n$. These restrictions are controlled by the group elements in Γ that act as reflections across hyperplanes in $\mathbb{R}^n$. The final subsection is devoted to proving a technical but general result concerning the way connected components of complements of preimage sets are permuted by both the mapping (as in Lemma 2.1) and the group action. This result is used in the final section to prove more specific results about the symmetries of attractors and their components for planar mappings with dihedral symmetry.

(a) *Symmetries of connected components*

Proposition 4.3. *Assume*:
(a) *Y is a finite set with $|Y| = r$.*
(b) *The finite group $\mathbb{Z}_r$ acts transitively on Y.*
(c) *Σ is a group acting fixed-point freely on Y.*
(d) *The actions of Σ and $\mathbb{Z}_r$ commute.*
Then Σ is isomorphic to a subgroup of $\mathbb{Z}_r$, that is, $\Sigma \cong \mathbb{Z}_k$ where k divides r.

Proof. Let a be a generator of $\mathbb{Z}_r$, fix $y \in Y$, and let $\sigma \in \Sigma$. Since $\sigma y \in Y$, there is a unique $a^p \in \mathbb{Z}_r$ such that $\sigma y = a^p y$, by (a) and (b). Define $\chi : \Sigma \to \mathbb{Z}_r$ by $\chi(\sigma) = a^p$. We show that χ is a monomorphism. The proposition follows with $k = r/p$.

To see that χ is a homomorphism, suppose that $\chi(\sigma_j) = a^{p_j}$ for $j = 1, 2$. Then $\sigma_j y = a^{p_j} y$ and

$$\sigma_2 \sigma_1 y = \sigma_2 a^{p_1} y = a^{p_1} \sigma_2 y$$

since by (d) the actions of Σ and $\mathbb{Z}_r$ commute. Hence

$$\sigma_2 \sigma_1 y = a^{p_1} a^{p_2} y = a^{p_2} a^{p_1} y$$

since $\mathbb{Z}_r$ is abelian. It follows that

$$\chi(\sigma_2 \sigma_1) = \chi(\sigma_2)\, \chi(\sigma_1)\,.$$

We now show that χ is injective. Suppose that $\chi(\sigma) = 1$. Then $\sigma y = y$ *and* $\sigma = 1$ since by (c), Σ acts fixed-point freely. □

Corollary 4.4. *Suppose that $f : X \to X$ is Γ-equivariant and A is a periodic orbit of period r. Then $S_A \cong \mathbb{Z}_k$ where k divides r.*

Proof. Let $Y = A = \{x, f(x), \ldots, f^{r-1}(x)\}$. Note that the action of f on Y is a transitive $\mathbb{Z}_r$ action. Moreover, $\Sigma_{f^j(x)} = T_A$ for each j from which it follows that S_A acts fixed-point freely on Y. Since Γ-equivariance means that the actions of S_A and $\mathbb{Z}_r$ commute, the result follows from Proposition 4.3. □

Remark 4.5. Often we shall discuss properties of Γ-symmetric attractors A for Γ-equivariant mappings f so that $S_A = \Gamma$. This assumption can be verified in two distinct ways. First, when A contains a point with trivial isotropy, then $T_A = 1$ and we can identify S_A with $\Sigma_A \subset \Gamma$. If $\Sigma_A \neq \Gamma$, discard the elements of Γ that are not in Σ_A and redefine $\Gamma = \Sigma_A$. Then f is still Γ-equivariant and A is Γ-symmetric.

Second, even when A does not contain points with trivial isotropy, this hypothesis can be satisfied – if in addition we restrict f. Suppose that $T_A \neq 1$. Note that $A \subset \mathrm{Fix}(T_A)$ by Proposition 4.1(a) and $\mathrm{Fix}(T_A)$ is an f-invariant subspace. Let $g = f|_{\mathrm{Fix}(T_A)}$. Then g is a Δ-equivariant mapping where $\Delta = N(T_A)/T_A$. Inside Δ, $S_A = \Sigma_A$ and we are back in the first case.

Theorem 4.6. *Let $f: X \to X$ be a Γ-equivariant map with an attractor A, $S_A = \Gamma$. Suppose that A is the disjoint union of two compact sets A_1 and A_2. Then S_{A_1} is a normal subgroup of Γ and the quotient group Γ/S_{A_1} is cyclic.*

Proof. Choose S_0 to be a closed set separating A_1 and A_2 such that $A \cap S_0 = \emptyset$. Let $S = \bigcup_{\gamma \in \Gamma} \gamma S_0$. Since A is Γ-symmetric, we have that $A \cap S = \emptyset$. By Corollary 2.5, we have

$$A \subset C_0 \cup \cdots \cup C_{r-1}$$

where the C_j are connected components of $X - \mathscr{P}_S$. Moreover, the connected components are permuted cyclically by f and permuted by elements of Γ.

Define $\Delta_j = \{\gamma \in \Gamma : \gamma C_j = C_j\}$. Since f permutes the components C_j and is Γ-equivariant, it follows that $\Delta_0 = \cdots = \Delta_{r-1}$. Let $\Delta = \Delta_0$. Then Δ is the kernel of the action of Γ on $\{C_0, \ldots, C_{r-1}\}$ and is normal in Γ. Moreover, Γ/Δ acts fixed-point freely on $\{C_0, \ldots, C_{r-1}\}$ and hence is cyclic by Proposition 4.3.

We assert that S_{A_1} contains Δ. It is easy to check that S_{A_1} is a normal subgroup of Γ. It then follows that $\Gamma/S_{A_1} \cong (\Gamma/\Delta)/(S_{A_1}/\Delta)$ is cyclic (since Γ/Δ is cyclic). It remains to verify the assertion. If $\gamma \in \Delta$, then $\gamma(A \cap C_j) = A \cap C_j$ for each j. The choice of S_0 guarantees that if a C_j intersects A_1, then $C_j \cap A_2 = \emptyset$ so that $\gamma(A_1 \cap C_j) = A_1 \cap C_j$. Thus A_1 is made up of γ-symmetric pieces and is itself γ-symmetric. □

Remark 4.7. (a) Suppose in Theorem 4.6 that Γ is a simple noncyclic group and $S_A = \Gamma$. Then $S_{A_1} = \Gamma$. (Recall that a group G is *simple* if the only normal subgroups are G and 1.) An example of such a group Γ is given by the symmetry group of the icosahedron which is isomorphic to the alternating group A_5.

(b) Suppose that $\Gamma = \boldsymbol{D}_m$ in Theorem 4.6. Then S_{A_1} is either $\boldsymbol{D}_m$ or $\mathbb{Z}_m$ when m is odd, and S_{A_1} is either $\boldsymbol{D}_m$, $\mathbb{Z}_m$ or $\boldsymbol{D}_{m/2}$ when m is even. It is easily verified that these subgroups obey the hypotheses of the theorem. The remaining normal subgroups $\mathbb{Z}_k$, where k divides m, may be ruled out thanks to the isomorphism $\boldsymbol{D}_m/\mathbb{Z}_k \cong \boldsymbol{D}_{m/k}$.

(c) For certain representations of a group Γ, there may be further restrictions on the symmetry of disjoint parts of the attractor. For example, we show in Theorem 5.8 that if $\boldsymbol{D}_m$ acts faithfully on $\mathbb{R}^2$, $m \geqq 3$, then $S_{A_1} = \boldsymbol{D}_m$.

(b) *Symmetries of attractors*

We have seen that there are group-theoretic restrictions on the symmetry groups of periodic orbits (Corollary 4.4) and on the symmetry of connected components of symmetric attractors (Theorem 4.6). We now show that there are restrictions on the symmetry groups of attractors. In contrast to the previous results, these restrictions are not purely group-theoretic, but depend on the representation of the group. Suppose that $\Gamma \subset \boldsymbol{O}(n)$ is a compact Lie group and let Σ be a subgroup of Γ. Recall that $\tau \in \Gamma$ is a reflection if $\mathrm{Fix}(\tau)$ is a hyperplane in $\mathbb{R}^n$. Let K_Σ be the set of reflections in $\Gamma - \Sigma$ and define

$$L_\Sigma = \bigcup_{\tau \in K_\Sigma} \mathrm{Fix}(\tau).$$

The connected components of $\mathbb{R}^n - L_\Sigma$ are permuted by elements of Σ. We use the fact that a Σ-symmetric attractor for a Γ-equivariant map cannot intersect L_Σ. This is a consequence of the following result proved in [6, Proposition 1.1] under slightly different hypotheses.

Proposition 4.8. *Let $f : \mathbb{R}^n \to \mathbb{R}^n$ be continuous and commute with a matrix ρ. Let $A \subset \mathbb{R}^n$ be an attractor for f. If*

$$A \cap \rho(A) \neq \emptyset,$$

then

$$\rho(A) = A.$$

Proof. We show that $\rho(A) \subset A$. A similar argument gives the reverse inclusion. We assert that if $U \supset A$ is any open neighborhood, then $U \supset \rho(A)$. Since $\rho(A)$ is closed, it follows that $\rho(A) \subset A$, as desired.

We verify this assertion as follows. Since A is stable, there is an open neighborhood V of A such that $f^m(V) \subset U$ for all $m \geqq 0$. Since, by equivariance, $\rho(A)$ is also an attractor for f, there exists an $x \in \mathbb{R}^n$ such that $\rho(A) = \omega(x)$.

Let z be a point in $A \cap \rho(A) \subset V$. Since V is open, there is a $k \geqq 0$ such that $f^k(x)$ is near to z; hence $f^k(x) \in V$. It follows that $\rho(A) = \omega(x) = \omega(f^k(x)) \subset U$, as asserted. □

Proposition 4.9. *Let $\Gamma \subset \boldsymbol{O}(n)$ be a compact Lie group with subgroup Σ. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous Γ-equivariant mapping with a Σ-symmetric topologically mixing attractor. Then there is a connected component of $\mathbb{R}^n - L_\Sigma$ that is preserved by Σ.*

Proof. By Proposition 4.8, any Σ-symmetric attractor A must satisfy $A \cap L_\Sigma = \emptyset$. In addition, A is connected by Theorem 2.7. Hence A lies inside a single connected component C of $\mathbb{R}^n - L_\Sigma$. But Σ fixes A and hence C. □

If we drop the topological mixing assumption in Proposition 4.9, then the situation is more complicated, but there is still a representation-theoretic restriction on Σ.

Theorem 4.10. *Let $\Gamma \subset \boldsymbol{O}(n)$ be a compact Lie group with subgroup Σ. Suppose that $f : \mathbb{R}^n \to \mathbb{R}^n$ is a continuous Γ-equivariant mapping with a Σ-symmetric attractor. Then there is a subgroup Δ such that*

(a) *Δ is a normal subgroup of Σ,*
(b) *Σ/Δ is cyclic,*
(c) *Δ fixes a connected component of $\mathbb{R}^n - L_\Delta$.*

Proof. Let A be a Σ-symmetric attractor for a Γ-equivariant continuous mapping f. Let L be the union of the reflection hyperplanes that do not intersect A, that is,

$$L = \bigcup_{\mathrm{Fix}(\tau) \cap A = \emptyset} \mathrm{Fix}(\tau)$$

where τ is a reflection in Γ. We use L to define Δ.

Observe that Σ acts on L since elements of Σ fix A. Hence, Σ leaves $\mathscr{P}_L$ invariant and Σ permutes the connected components of $\mathbb{R}^n - \mathscr{P}_L$.

By Corollary 2.5, A is covered by finitely many connected components $C_0, \ldots, C_{r-1}$ of $\mathbb{R}^n - \mathscr{P}_L$ and these connected components are permuted cyclically by f. Let $\Delta_i = \{\delta \in \Sigma : \delta C_i = C_i\}$.

Next, observe that

$$\delta C_i = C_j \;\Rightarrow\; \delta C_{i+1} = C_{j+1}. \tag{4.1}$$

To see this, observe that

$$\delta C_i = C_j \Rightarrow \delta f(C_i) = f(C_j) \Rightarrow \delta C_{i+1} \cap C_{j+1} \neq \emptyset.$$

Implication (4.1) follows, since the C_is are connected components.

Thus $\Delta_i \subset \Delta_{i+1}$ and the Δ_is are all equal. We define

$$\Delta \equiv \Delta_0 = \cdots = \Delta_{r-1}.$$

Since Δ is the kernel of the action of Σ on the set $\mathscr{C} = \{C_0, \ldots, C_{r-1}\}$, Δ is a normal subgroup of Σ (thus verifying (a)) and Σ/Δ acts fixed-point freely on $\mathscr{C}$. It follows from (4.1) that this action commutes with cyclic permutations on $\mathscr{C}$ and from Proposition 4.3 that Σ/Δ is cyclic (thus verifying (b)).

Next, we assert that $L_\Delta \subset L$. Before proving the assertion we show that verifying this assertion will indeed prove (c) and complete the proof of this theorem. Since $L \subset \mathscr{P}_L$ and since Δ fixed a connected component of $\mathbb{R}^n - \mathscr{P}_L$, it follows that Δ must fix a connected component of $\mathbb{R}^n - L$. Hence, by the assertion, Δ must fix a connected component of $\mathbb{R}^n - L_\Delta$.

To prove the assertion, assume that $\mathrm{Fix}(\tau) \subset L_\Delta$, that is, assume that $\tau \notin \Delta$. We prove that $\mathrm{Fix}(\tau) \subset L$ by contradiction, that is, we assume that

$$\mathrm{Fix}(\tau) \cap A \neq \emptyset \tag{4.2}$$

and, to establish the contradiction, show that $\tau \in \Delta$.

Observe that (4.2) implies that $\tau(A) \cap A \neq \emptyset$. Hence, Proposition 4.8 implies that $\tau \in \Sigma$; thus, τ permutes the C_js.

Finally, observe that since A is covered by the C_js, (4.2) implies that $\mathrm{Fix}(\tau) \cap C_j \neq \emptyset$ for some j; hence, $\tau(C_j) \cap C_j \neq \emptyset$. Since $\tau \in \Sigma$, τ permutes the C_js, and $\tau(C_j) = C_j$. So, by definition, $\tau \in \Delta_j = \Delta$. □

Remark 4.11. (a) In Section 5 we apply Theorem 4.10 when $\Gamma = \boldsymbol{D}_m$ is acting on $\mathbb{R}^2$ and show that not all subgroups of $\boldsymbol{D}_m$ can be the symmetry group of an attractor for a $\boldsymbol{D}_m$-equivariant mapping.

(b) Suppose that Γ is finite. Then the representation-theoretic restriction obtained in Theorem 4.10 is necessary and sufficient; see ASHWIN & MELBOURNE [1]. In particular, there are no restrictions on cyclic subgroups of Γ

nor on subgroups of Γ that contain all the reflections in Γ. (For the case of cyclic subgroups, see also KING & STEWART [16].) Proposition 4.9 is also optimal except when Γ is a cyclic subgroup of $\boldsymbol{O}(2)$ (see [1]).

(c) *Permuting connected components of complements of preimage sets*

Definition 4.12. Let $\mathscr{D}$ be a finite collection of closed subsets of X. The collection $\mathscr{D}$ is a *fundamental decomposition* for the action of Γ if

(a) $X = \bigcup_{B \in \mathscr{D}} B$.

(b) $\overline{\operatorname{int}(B)} = B$, for each $B \in \mathscr{D}$.

(c) The sets $\operatorname{int}(B)$ are pairwise disjoint.

(d) The group Γ acts on $\mathscr{D}$, that is, $\gamma(B) \in \mathscr{D}$ for all $B \in \mathscr{D}$ and $\gamma \in \Gamma$.

(e) If $\gamma(B) = B$ for some $B \in \mathscr{D}$ and nontrivial $\gamma \in \Gamma$, then there is an element $\delta \in \Gamma$ such that $\gamma\delta B \neq \delta B$.

Remark 4.13. (a) Definition 4.12(e) states that Γ acts fixed-point freely on group orbits in $\mathscr{D}$.

(b) This definition is similar to that of a fundamental domain. However, we allow the possibility that $\gamma B = B$ for some $B \in \mathscr{D}$ and some nontrivial $\gamma \in \Gamma$.

(c) A natural way to produce fundamental decompositions is to choose a hyperplane in X passing through the origin. Let S_0 denote a half-plane inside this hyperplane, and let $S = \bigcup_{\gamma \in \Gamma} \gamma S_0$. Let $\mathscr{D}$ be the collection of closures of connected components of $X - S$. It is clear that $\mathscr{D}$ satisfies Definition 4.12(a)–(d). Condition (e) must be verified in each case.

Recall that $\mathscr{P}_S$ is defined to be the set of preimages of a set S under f.

Proposition 4.14. *Suppose that Γ is not cyclic and A is an ω-limit set with $S_A = \Gamma$. Let $\mathscr{D}$ be a fundamental decomposition for Γ and let S be the closed set $\bigcup_{B \in \mathscr{D}} \partial B$. Then*

(a) $A \subset \overline{\mathscr{P}_S}$.

(b) *If A is an attractor, then $A \cap S \neq \emptyset$.*

(c) *If A is an attractor and $\mathscr{D}$ is constructed as in Remark 4.13(c), then A intersects γS_0 for each $\gamma \in \Gamma$.*

Proof. (a) Suppose that $A \not\subset \overline{\mathscr{P}_S}$. Then by Lemma 2.1, $A - \mathscr{P}_S$ is covered by connected components $C_0, \ldots, C_{r-1}$ of $X - \mathscr{P}_S$, and these connected components are permuted cyclically by f. We assert that Γ acts fixed-point freely on the connected components. Then it follows from Proposition 4.3 that Γ is cyclic, which we had assumed not to be the case. When applying that proposition, set $\Sigma = \Gamma$ and $Y = \{C_0, \ldots, C_{r-1}\}$.

It remains to verify the assertion. Suppose that $\gamma C_0 \subset C_0$. Then the Γ-equivariance of f implies that $\gamma C_j \subset C_j$ for each j. Let B_j denote the unique subset of $\mathscr{D}$ that contains C_j – note that uniqueness follows from Definition 4.12(c). Then $\gamma B_j = B_j$ for every j. Since A is Γ-symmetric, the collection of subsets $\{B_j\}$ consists of a collection of group orbits of Γ by Definition 4.12(d), each of which is fixed by γ. But Definition 4.12(e) states that Γ acts fixed-point freely on group orbits, so $\gamma = 1$ as required.

(b) This follows from Proposition 2.4.

(c) If $S = \bigcup_{\gamma \in \Gamma} \gamma S_0$, then A intersects γS_0 for some $\gamma \in \Gamma$, and hence for all $\gamma \in \Gamma$ since $S_A = \Gamma$. □

5. Planar maps with dihedral symmetry

The dihedral group $\boldsymbol{D}_m$ consists of the symmetries of the regular m-sided polygon and is generated by a rotation θ through $2\pi/m$ and a reflection κ. The irreducible representations of $\boldsymbol{D}_m$ are one- or two-dimensional and the faithful representations are given on $\mathbb{C} \cong \mathbb{R}^2$ by

$$\theta \cdot z = e^{2l\pi i/m} z, \qquad \kappa \cdot z = \bar{z}$$

where l and m are coprime. We consider here only the standard two-dimensional representation $l = 1$; the results for the other two-dimensional irreducible representations are identical.

The subgroups of $\boldsymbol{D}_m$ are $\boldsymbol{D}_k$ and $\mathbb{Z}_k$, $k \geqq 1$, where k divides m. There are m axes of symmetry for $\boldsymbol{D}_m$ which we label $L_1, \ldots, L_m$.

We prove below that there are certain subgroups of $\boldsymbol{D}_m$ that cannot be the symmetries of attractors for $\boldsymbol{D}_m$-equivariant mappings. The proofs rely on the following simple remark.

Lemma 5.1. *Let $\Delta \subset \boldsymbol{D}_m$ be a subgroup satisfying Theorem 4.10(c), that is, Δ fixes a connected component of $\mathbb{R}^2 - L_\Delta$. Then Δ is either $\boldsymbol{D}_m$, $\boldsymbol{D}_1$ or* $\mathbf{1}$.

Proof. Suppose that $\Delta \neq \boldsymbol{D}_m$. Then there are reflections in $\boldsymbol{D}_m$ that do not lie in Δ so that L_Δ is nonempty and consists of lines through the origin. Observe that any nontrivial rotation in $\boldsymbol{D}_m$ cannot preserve a connected component of $\mathbb{R}^2 - L_\Delta$; so Δ cannot contain a proper rotation. The only subgroups of $\boldsymbol{D}_m$ that do not contain rotations are $\boldsymbol{D}_1$ and $\mathbf{1}$. □

Corollary 5.2. *Suppose that f is $\boldsymbol{D}_m$-equivariant with a Σ-symmetric topologically-mixing attractor. Then Σ is either $\boldsymbol{D}_m$, $\boldsymbol{D}_1$ or* $\mathbf{1}$.

Proof. Apply Proposition 4.9 and Lemma 5.1. □

Theorem 5.3. *Suppose that f is $\boldsymbol{D}_m$-equivariant, $m \geqq 2$. Suppose further that A is an attractor for f and $\Sigma_A = \boldsymbol{D}_k$.*

(a) *If m is odd, then $k = 1$ or $k = m$.*

(b) *If m is even, then $k = 1$, $k = 2$ or $k = m$.*

We note that the first nontrivial consequence of Theorem 5.3 occurs when $m = 6$. For $m < 6$ the only subgroups $\boldsymbol{D}_k$ of $\boldsymbol{D}_m$ have $k = 1$, $k = 2$ or $k = m$ and the theorem is trivially valid.

Proof. Let $\Sigma = \boldsymbol{D}_k \subset \boldsymbol{D}_m$ where $3 \leqq k < m$. We show that there is no subgroup $\Delta \subset \Sigma$ satisfying the conditions of Δ 4.10(a)–(c). Condition (c) together with Lemma 5.1 implies that Δ is either $\boldsymbol{D}_1$ or $\mathbf{1}$ (since $k < m$). Now when $k \geqq 3$, the group $\boldsymbol{D}_1$ is not a normal subgroup of $\boldsymbol{D}_k$ (so (a) fails) and the group $\boldsymbol{D}_k/\mathbf{1} = \boldsymbol{D}_k$ is not cyclic (so (b) fails). □

Remark 5.4. King & Stewart [16] prove that there exist attractors with cyclic symmetry for any cyclic subgroup. It is shown in Ashwin & Melbourne [1] that there exist attractors with $\boldsymbol{D}_m$ and $\boldsymbol{D}_2$ symmetry.

Lemma 5.5. *Suppose that A is a $\boldsymbol{D}_m$-symmetric attractor for a $\boldsymbol{D}_m$-equivariant mapping f. If $m \geqq 3$, then A intersects each half-line emanating from the origin. If $m = 2$, then A intersects at least one line of symmetry.*

Proof. Let S_0 be any half-line emanating from the origin and define $S = \bigcup_{\gamma \in \boldsymbol{D}_m} \gamma S_0$. The set S generally consists of $2m$ half-lines and is illustrated for the case $m = 4$ in Figure 1. Note that when S_0 lies on an axis of symmetry, then S consists of m half-lines. When $m \geqq 3$, it is easy to check that for any choice of S_0, the collection $\mathscr{D}$ of connected components of $\mathbb{R}^2 - S$ is indeed a fundamental decomposition. It follows immediately from Proposition 4.14, that if A is an attractor with $\Sigma_A = \boldsymbol{D}_m$ and $m \geqq 3$, then for any choice of S_0, $A \subset \overline{\mathscr{P}_S}$ and $A \cap S_0 \neq \emptyset$.

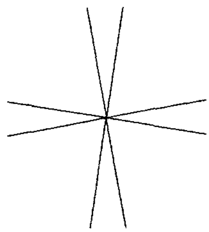

Figure 1. half-lines for $m = 4$.

In the case $m = 2$, let S be the union of the two axes of symmetry. Then the connected components of $\mathbb{R}^2 - S$ form a fundamental decomposition. It follows that $A \subset \overline{\mathscr{P}_S}$, and $A \cap S \neq \emptyset$. □

Corollary 5.6. *Let A be a $\boldsymbol{D}_m$-symmetric ω-limit set for a $\boldsymbol{D}_m$-equivariant mapping f where $m \geqq 2$. Then A has weak dependence. Moreover, if A has positive measure, then A has sensitive dependence.*

Proof. Let S be the union of the axes of symmetry for $\boldsymbol{D}_m$. Choosing S_0 to be a half-axis of symmetry in the proof of Lemma 5.5 shows that $A \subset \overline{\mathscr{P}_S}$. It follows that A and S satisfy the hypotheses of Proposition 2.14. □

Proposition 5.7. *Let f be $\boldsymbol{D}_m$-equivariant where m is even and $m \geqq 4$. Let A be an attractor with $\Sigma_A = \boldsymbol{D}_2$. Then A intersects precisely one axis of symmetry.*

Proof. Let L and M denote the two axes of symmetry for the subgroup $\boldsymbol{D}_2$. By Lemma 5.5, A intersects at least one of the axes, say L. Moreover, by Proposition 4.8, A does not intersect any other axis of symmetry with the possible exception of M.

Let S denote the union of the two axes of symmetry for $\boldsymbol{D}_m$ that are adjacent to L. Then $A \cap S = \emptyset$. Since A intersects L, A intersects a connected component of $\mathbb{R}^2 - S$ that intersects L. But such a connected component cannot intersect M. Therefore $A \cap M = \emptyset$ by Lemma 2.6. □

As promised, we can improve Theorem 4.6 for the faithful representations of $\boldsymbol{D}_m$ on $\mathbb{C} \cong \mathbb{R}^2$.

Theorem 5.8. *If A is a $\boldsymbol{D}_m$-symmetric attractor, $m \geqq 3$, and A is the disjoint union of two compact subsets A_1 and A_2, then these subsets are $\boldsymbol{D}_m$-symmetric.*

Proof. Let S_0 be a closed set with the property that A_1 and A_2 lie in distinct connected components of $\mathbb{R}^2 - S_0$. Define $S = \bigcup_{\gamma \in \boldsymbol{D}_m} \gamma S_0$. By Corollary 2.5, A is covered by finitely many connected components $C_0, \ldots, C_{r-1}$ of $\mathbb{R}^2 - \mathscr{P}_S$.

We assert that these components are $\boldsymbol{D}_m$-invariant. It is sufficient to show that reflections leave the components invariant since $\boldsymbol{D}_m$ is generated by reflections. Let L be an axis of symmetry corresponding to a reflection κ and observe that κ permutes connected components by the $\boldsymbol{D}_m$-invariance of S. But A intersects L by Lemma 5.5 and hence L intersects one of the connected components, say C_0. Since κ fixes L pointwise we have $\kappa C_0 = C_0$. In addition, the equivariant map f permutes the connected components so that $\kappa C_j = C_j$ for each j thus verifying the assertion.

Now let $\gamma \in \boldsymbol{D}_m$. Then $\gamma A = A$ and by the assertion, $\gamma C_j = C_j$ for each j. Hence $\gamma(A \cap C_j) = A \cap C_j$. But S is constructed so that only one of A_1 or A_2 may intersect a given C_j. If A_1, say, intersects C_j, then we have $\gamma(A_1 \cap C_j) = A_1 \cap C_j$. Thus A_1 and A_2 are unions of $\boldsymbol{D}_m$-symmetric subsets and are themselves $\boldsymbol{D}_m$-symmetric. □

The following result is useful for computing symmetry-increasing bifurcations. See [10].

Theorem 5.9. *Let f be a $\boldsymbol{D}_m$-equivariant mapping, $m \geqq 3$, with an attractor A.*

(a) *If $S_A = \boldsymbol{D}_m$, then $A \subset \overline{\mathscr{P}_S}$ where S is the union of any two lines through the origin.*

(b) *If $\Sigma_A = \boldsymbol{D}_2$, then $A \subset \overline{\mathscr{P}_L}$ for some line of symmetry L.*

Proof. (a) Suppose that $A \not\subset \overline{\mathscr{P}_S}$. Then by Lemma 2.1,

$$A \subset \overline{C_0} \cup \cdots \cup \overline{C_{r-1}},$$

where $C_0, \ldots, C_{r-1}$ are connected components of $\mathbb{R}^2 - \mathscr{P}_S$, and these connected components are permuted cyclically by f.

Since $m \geqq 3$, there is an axis of symmetry M, $M \not\subset S$. By Lemma 5.5, $A \cap M \neq \emptyset$. Hence one of the connected components, C_0, say, intersects M. It follows that $\overline{C_j} \cap M \neq \emptyset$ for $j = 0, \ldots, r-1$. In particular, A intersects only the two connected components of $\mathbb{R}^2 - S$ that intersect M. But A is $\boldsymbol{D}_m$-symmetric and hence intersects all four connected components of $\mathbb{R}^2 - S$ giving a contradiction.

(b) Let $S = L_1 \cup L_2$ where L_1 and L_2 are the axes of symmetry for $\boldsymbol{D}_2$. By the proof of Lemma 5.5, $A \subset \overline{\mathscr{P}_S}$. It follows from Proposition 2.4 that $A \subset \overline{\mathscr{P}_{A \cap S}}$. By Proposition 5.7, A intersects precisely one of these axes, L_1, say. In particular $A \cap S \subset L_1$. Therefore $A \subset \overline{\mathscr{P}_{L_1}}$. □

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Department of Mathematics
University of Houston
Houston, TX 77204-3476

and

Institut für Angewandte Mathematik
der Universität Hamburg
D-2000 Hamburg 13

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