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SYMMETRY AND STABILITY IN TAYLOR-COUETTE FLOW*

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Abstract. We study the flow of a fluid between concentric rotating cylinders (the Taylor problem) by exploiting the symmetries of the system. The Navier–Stokes equations, linearized about Couette flow, possess two zero and four purely imaginary eigenvalues at a suitable value of the speed of rotation of the outer cylinder. There is thus a reduced bifurcation equation on a six-dimensional space which can be shown to commute with an action of the symmetry group $O(2)\times SO(2)$. We use the group structure to analyze this bifurcation equation in the simplest (nondegenerate) case and to compute the stabilities of solutions. In particular, when the outer cylinder is counterrotated we can obtain transitions which seem to agree with recent experiments of Andereck, Liu, and Swinney [1984]. It is also possible to obtain the "main sequence" in this model. This sequence is normally observed in experiments when the outer cylinder is held fixed.

Introduction. The flow of a fluid between concentric rotating cylinders, or *Taylor–Couette flow*, is known to exhibit a variety of types of behavior, the most celebrated being *Taylor vortices* (Taylor [1923]). The problem has been studied by a large number of authors: a recent survey is that of DiPrima and Swinney [1981]. The experimental apparatus has circular symmetry, and the standard mathematical idealization (periodic boundary conditions at the ends of the cylinder) introduces a further symmetry. As a result the Navier–Stokes equations for this problem are covariant with respect to the action of a symmetry group $O(2)\times SO(2)$. It has become clear that the symmetries inherent in bifurcating systems have a strong influence on their behavior. In this paper we study a series of bifurcations that occur in Taylor–Couette flow placing emphasis on the role of symmetry. (Schecter [1976] and Chossat and Iooss [1984] have also studied the problem from this viewpoint, and we discuss the relations between our work and theirs below.)

DiPrima and Grannick [1971] have found that when the outer cylinder is rotated in a direction opposite to that of the inner cylinder, the Navier–Stokes equations, linearized about Couette flow, possess six eigenvalues on the imaginary axis. It follows that aspects of the dynamics can be reduced (either by Lyapunov–Schmidt or center manifold reduction) to a vector field on $\mathbf{R}^6$; furthermore, this vector field commutes with an action of $O(2)\times SO(2)$. Moreover, as we explain in §7, recent experimental results due to Andereck, Liu, and Swinney [1984] seem to confirm the existence of the six-dimensional kernel.

We point out in particular that the six-dimensional kernel is a codimension one phenomenon, and hence it is not surprising that it should be possible to find it by varying only one parameter. Indeed, this degeneracy should occur relatively often in various circumstances, and so deserves detailed analysis.

We study the general class of bifurcation problems on $\mathbf{R}^6$ having this $O(2)\times SO(2)$ symmetry. We derive the general form possible for the vector field, and by classifying

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the possible ways to break symmetry, obtain equations for the bifurcating branches (subject to certain nondegeneracy conditions). We also obtain the (linearized orbital) stabilities of these branches.

By introducing an additional parameter α we split the kernel $\mathbf{R}^6$ into two subspaces $\mathbf{R}^2$ and $\mathbf{R}^4$ corresponding to a steady-state and a periodic bifurcation respectively. Depending on the sign of α, one or other of these bifurcations occurs first.

By inspecting the symmetries of the physically observed solutions we may tentatively identify them with various branches: in particular the flows known as Taylor vortices, wavy vortices, twisted vortices, helices (or spirals) seem to correspond naturally to solution branches; and there is also a branch described by DiPrima and Grannick [1971] as the "nonaxisymmetric simple mode".

The experimental results of Andereck, Liu, and Swinney may be summarized as follows. In the weakly counterrotating case (that is, when the speed of the outer cylinder Ω_0 is slightly less than the critical speed Ω_0^* where the six-dimensional kernel appears) the following transition sequence is observed as Ω_i, the speed of the inner cylinder, is increased.

$$\text{Couette flow} \to \text{Taylor vortices} \to \text{wavy vortices} \to \cdots$$

where the final state obtained when the wavy vortices lose stability seems not to be one representable in the six-dimensional kernel. In the strongly counterrotating case (that is, when Ω_0 is slightly greater than Ω_0^*) the observed transition sequence is:

$$\text{Couette flow} \to \text{spiral cells} \to \text{wavy spiral cells}.$$

We shall show in §7 that it is possible to make a nondegenerate choice of vector fields on $\mathbf{R}^6$ having $O(2)\times SO(2)$ symmetry which produces the same transition sequences in the following sense. It is possible to determine constraints on the Taylor expansion of this vector field, given only by inequalities on coefficients in this Taylor expansion, so that the solutions corresponding to these states are (orbitally) asymptotically stable and lose stability in a way that should produce the desired transitions. Moreover, when these inequalities are satisfied, no other solutions are asymptotically stable.

We also show in §7 that it is possible to choose these constraints differently, so that the "main sequence" of transitions occurs, namely,

$$\text{Couette flow} \to \text{Taylor vortices} \to \text{wavy vortices}$$
$$\to \cdot \text{modulated wavy vortices} \to \cdots.$$

This transition sequence is usually observed when the outer cylinder is held stationary ($\Omega_0 = 0$). What we show is that it is possible for the "wavy vortex solutions" to lose stability to a torus bifurcation, where two Floquet exponents cross the imaginary axis. This tertiary bifurcation has never been demonstrated theoretically hitherto. At this point, however, we cannot prove that the branch of "modulated wavy vortices" is asymptotically stable, though we hope that the results of Scheurle and Marsden [1984] will provide the techniques required to carry out this computation. We do show, moreover, that no other solutions are asymptotically stable when these constraints hold. In particular, stable spiral cells should not occur in this experimental situation.

The paper is organized as follows. In §1 we describe some of the flows observed in the Taylor experiment and review the evidence for the existence of a six-dimensional kernel. In §2 we discuss the symmetries that act on the six-dimensional kernel, and in §3 we discuss the symmetries of the observed flows. In §4 (and the Appendix) we discuss the reduction procedure and derive the exact form of the reduced mapping (or vector field) on the six-dimensional kernel prescribed by those symmetries. We classify the (conjugacy classes) of isotropy subgroups (which describe the type of symmetry-breaking that occurs at bifurcations). The heart of the paper is §5, where we analyze the branching equations and the stability of branches. In §6 we use these to obtain a list of the sixteen inequalities that must be imposed to ensure (what we mean by) nondegeneracy. Finally, in §7, we compare the six-dimensional model with experimental observations in both the counterrotating case and the case when the outer cylinder is held fixed.

1. The Taylor problem. By the term "Taylor problem" we mean the study of both the possible states of fluid flow between two rotating concentric cylinders, and the transitions between these states. The Taylor problem provides a beautiful example of a bifurcation problem with symmetry. In this paper we discuss how these symmetries affect the structure of the bifurcating solutions.

We denote the angular velocities of the inner and outer cylinders by Ω_i and Ω_0 respectively. To specify a direction, we assume that $\Omega_i \geqq 0$. In the standard experiments the outer cylinder is held fixed ($\Omega_0 = 0$) and the inner cylinder is speeded up in stages from $\Omega_i = 0$, at each stage allowing the flow to settle into a stable pattern; see Taylor [1923], Gollub and Swinney [1975]. Experiments have been performed in both the *corotating* case ($\Omega_0 > 0$), see Andereck, Dickman and Swinney [1983], and the *counter-rotating* case ($\Omega_0 < 0$), see Andereck, Liu, and Swinney [1984]. (These papers cite the earlier experimental work.) The experiments begin by rotating the outer cylinder at constant speed, and allowing the flow to stabilize; then the inner cylinder is speeded up as before. The experiments reveal a large number of fluid states, only some of which are understood on theoretical grounds. There can exist multiple steady states whose exploration requires different experimental procedures; see for example Coles [1965], Benjamin [1978a, b], Benjamin and Mullin [1982]. In our discussion we shall assume a fixed (but unspecified) value of Ω_0, and treat Ω_i (or the corresponding Reynolds number) as a bifurcation parameter. Our main concern will be with the series of bifurcations that occurs as Ω_i is increased steadily. We mention this because many numerical computations fix the ratio Ω_0/Ω_i, and hence do not correspond directly to the usual experimental procedure—a fact that, in the presence of multiple states, raises some problems of interpretation.

In the standard experiments, with $\Omega_0 = 0$, the first transition is from Couette (laminar) flow to (Taylor) *vortices*. Both flows are time-independent. This transition was first described, in terms of a steady state bifurcation, by Davey [1962]. He showed that as Ω_i is increased, the Navier–Stokes equations linearized about Couette flow have a double zero eigenvalue at the first bifurcation. At this eigenvalue Couette flow loses stability, and a branch of vortex solutions bifurcates. Davey's observations have been reproduced by several authors in different contexts, cf. the survey by DiPrima and Swinney [1981, §6.3]. Note that the appearance of a double zero eigenvalue might be surprising were it not for the existence of symmetries (which can couple eigenvalues together and force a degeneracy).

Again, in the standard experiments with $\Omega_0 = 0$, a second transition is observed, in which vortices lose stability to a time-periodic state known as *wavy vortices*. Presumably this transition takes place by way of a Hopf type bifurcation in which several eigenvalues (governing the stability of vortices) cross the imaginary axis as Ω_i is increased. However, this presumption has never been established directly. What has been shown (in Davey, DiPrima, and Stuart [1968]) is that along the Couette branch of solutions several eigenvalues of the linearized Navier–Stokes equations cross the imaginary axis as Ω_i is increased. In particular the next set of eigenvalues to cross the imaginary axis is a complex conjugate pair of purely imaginary eigenvalues, each of multiplicity two. Again, it would be surprising to see four eigenvalues crossing the imaginary axis simultaneously were it not for the symmetry. We note in passing (and amplify these remarks below) that the $O(2)$ symmetry which couples these four eigenvalues together forces the occurrence of two branches of time-periodic solutions bifurcating from the (unstable) main Couette branch: see Schecter [1976], and Golubitsky and Stewart [1985]. However, neither of these solutions can correspond to wavy vortex states, since their symmetries do not match those of wavy vortices. In fact, one of them has the symmetries of spiral cells (helices).

There are three additional facts which suggest that there might be a *relatively* simple local explanation for many of the observed states in the Taylor problem, at least in the counterrotating case $\Omega_0 < 0$. First, as observed in DiPrima and Grannick [1971], and Krueger, Gross, and DiPrima [1966], there is a critical speed of counterrotation $\Omega_0^* < 0$ such that, as Ω_i is increased, Couette flow loses linearized stability by having six eigenvalues cross the imaginary axis. These six eigenvalues are obtained by amalgamating the double zero eigenvalues and the complex conjugate pair of purely imaginary eigenvalues of multiplicity two, described above. Further, when Ω_0 is slightly less than Ω_0^*, the first bifurcation from Couette flow occurs when four eigenvalues (a complex conjugate pair each of multiplicity two) cross the imaginary axis; and there is a double zero eigenvalue at a higher value of Ω_i.

Second, in experiments in which Ω_0 is sufficiently negative, the primary bifurcation is not to the time-independent Taylor vortices, but to time-dependent spiral cells, see Andereck, Liu, and Swinney [1984].

Third, it is possible to produce a solution from the interaction of the four-dimensional center manifold (associated with the purely imaginary eigenvalues) and the two-dimensional center manifold (associated with the double zero eigenvalues) that has the same symmetry as wavy vortices. This suggests that it might be possible to prove the existence of a Hopf-type bifurcation from vortices to wavy vortices as a *secondary* bifurcation. This was observed by DiPrima and Sijbrand [1982] and again by Chossat and Iooss [1984].

Given these three facts, it would appear reasonable to study the Taylor problem in terms of perturbations of the degenerate case $\Omega_0 = \Omega_0^*$, using either a center manifold or a Lyapunov–Schmidt reduction from the Navier–Stokes equations. We call this degeneracy the *six-dimensional kernel* since the linearized equation has a kernel of dimension six and the reduced problem may therefore be posed on $\mathbf{R}^6$. The hope raised by the above facts is that one might be able to find a six-dimensional model which explains the observed prechaotic states and transitions in the counterrotating Taylor problem.

Let us consider the reduction in more detail. Rigorously, one can use the center manifold theorem to reduce the (infinite-dimensional) dynamics of the Navier–Stokes equations, near $\Omega_0 = \Omega_0^*$ and near Couette flow, to the study of some vector field g on a six-dimensional center manifold. Alternatively, one can focus only on time-independent and time-periodic solutions and use a reduction of the Lyapunov–Schmidt type to show the existence of a smooth (i.e. $\mathbf{C}^\infty$) mapping h: $\mathbf{R}^6 \to \mathbf{R}^6$ whose zeros are in one-to-one correspondence with the small-amplitude time-periodic (and time-independent) solutions of the Navier–Stokes equations. In either case, to study the dynamics $\dot{x} = g(x)$ on $\mathbf{R}^6$ or to solve $h(x) = 0$ in $\mathbf{R}^6$ would be a highly nontrivial task—were it not for the symmetries in the Taylor problem. Both reduction procedures can be performed so as to respect these symmetries. Therefore, g and h will commute with an action of the symmetry group $O(2)\times S^1$ as we explain below. This places considerable restrictions on the form that g and h may take. When, as here, we are studying only steady and periodic states, it is sufficient to use the simpler Lyapunov–Schmidt reduction. This is our approach. For a complete study of the dynamics, the same restrictions on the form of g will be true *provided* a smooth center manifold exists. (It is plausible that the symmetry might imply this, but we have not attempted to address this issue here.)

In this paper we give an explicit representation for all smooth mappings that commute with this action of $O(2)\times S^1$. We use the symmetries to show how to solve the equation $h=0$ (in the Lyapunov–Schmidt interpretation), and to determine (in most instances) the signs of the eigenvalues of the 6×6 Jacobian matrix $dh\,|_{h=0}$. In particular we compute these eigenvalues for the solutions corresponding to wavy vortices.

In this respect our results resemble those of a recent paper of Chossat and Iooss [1984]. However, instead of working on the six-dimensional kernel, Chossat and Iooss track the bifurcations step by step using the symmetry in the primary bifurcation to analyze the possible types of symmetry-breaking at secondary bifurcations, in terms of the linearized eigenfunctions. The types of solution that they find can all be expressed as combinations of the six linearized eigenfunctions that make up the six-dimensional kernel; but no reduction to $\mathbf{R}^6$ is used explicitly. Thus, although the various pieces of the bifurcation diagram are studied, their overall arrangement (and consistency) is not.

In our approach group theory is used to provide a coherent framework that organizes the analysis and in particular the computation of stabilities, leading to more detailed results. In particular we confirm, in our setting, a conjecture made by Chossat and Iooss [1984] about *tertiary* bifurcation to modulated wavy vortices. We show that (with suitable parameter values) the branch of wavy vortices loses stability by a torus bifurcation. In experiments this transition is observed, the new state being called *modulated wavy vortices*. See Rand [1982], Gorman, Swinney, and Rand [1981], Shaw et al. [1982].

The analysis of the simplest (nondegenerate) $O(2)\times S^1$-symmetric bifurcation problems on the six-dimensional kernel leads to a picture that includes branches corresponding to a variety of the observed flows: Couette, vortices, wavy vortices, twisted vortices, spiral cells, modulated wavy vortices, wavy spirals and an unstable flow found numerically by DiPrima and Grannick [1971] which they call the "non-axisymmetric simple mode." By "correspond" we mean that the solutions we find on the six-dimensional kernel appear to have the same symmetries as the experimentally determined states. As we indicated in the introduction, it is further possible to choose parameters in the model to mimic the observed transition sequences when the outer cylinder is held fixed, and also in the counterrotating case.

DiPrima, Eagles, and Sijbrand [1984] are currently making numerical calculations of certain of the Taylor coefficients of the vector field g obtained by a center manifold reduction. These or similar numerical results should make it possible to determine to what extent the six-dimensional model reflects the expected transitions in the Taylor problem, at least in the counterrotating case.

2. Symmetries on the six-dimensional kernel. Symmetries are introduced in the Taylor problem in three distinct ways:

(1) by the experimental apparatus,

(2) by the mathematical idealization,

(3) by the mathematical analysis.

Since each of these ways introduces a circle group of symmetries the result may seem confusing at first. However, these symmetries do affect the mathematically determined solutions and, moreover, seem to be present in the experimentally determined states.

The symmetries arising through the apparatus would appear to be the most natural. All formulations of the Taylor problem are invariant under rotation in the azimuthal plane, a plane perpendicular to the cylindrical axis. Rotation through θ in this plane moves one fluid state to another. We denote these symmetries by $SO(2)$.

Next we discuss the symmetries introduced by the mathematical idealization. In the experiments, when vortex flow is observed these vortices tend to have square cross-sections; that is, the height of each vortex is approximately equal to the distance between the cylinders. As a result, in an apparatus whose cylinder length is long compared with the distance between the cylinders, many vortices form at the initial bifurcation. Moreover, the vortex flow appears to be invariant under translation along the cylindrical axis by two band-widths, at least away from the ends of the cylinder. (In the cross-sectional regions vortex flow alternates between clockwise and counterclockwise.)

Thus in the mathematical idealization we assume that the cylinders have infinite length and look only for periodic solutions of period equal to two band-widths. As a result, the Navier–Stokes equations are invariant under both translations along the cylindrical axis and reflection of the cylinder through the azimuthal plane. Periodicity implies that translation by two band-widths acts as the identity. Thus the effective action of this group is by the (compact) group $O(2)$.

Finally, we consider a circle group of symmetries which is introduced into this problem by the technique we use to analyze the bifurcation structure. We use a Lyapunov–Schmidt reduction to determine time-periodic solutions of the Navier–Stokes equations which lie near Couette flow and the parameter values yielding the six-dimensional kernel. The circle group S^1, acting by change of phase on periodic functions, introduces symmetries into this problem. The addition of these S^1 symmetries by the Lyapunov–Schmidt procedure, to the symmetries mentioned above, is described in Sattinger [1983] and Golubitsky and Stewart [1985].

We summarize our discussion here as follows. The full group of symmetries of the Taylor problem on the six-dimensional kernel is:

$$O(2)\times SO(2)\times S^1 \tag{2.1}$$

where $O(2)$ acts by translation and flipping along the cylindrical axis, $SO(2)$ acts by rotation of the azimuthal plane and S^1 acts by change of phase of periodic solutions. For simplicity of notation we assume that the period of the cylindrical translations is 2π and that the period of patterns around the cylinder (in the azimuthal plane) is also 2π. In particular, rotation of the cylinder by half a period is $\pi \in SO(2)$. Moreover, we assume that solutions are 2π-periodic in time.

These assumptions do not affect the group-theoretic formulation of the problem, or its analysis; but they must be correctly interpreted in connection with the observed flows. Since the situation is potentially confusing, a few clarifying remarks may be in order. There is no problem in arranging period 2π for translations: we merely scale the distance along the axis. For periodic solutions in the azimuthal direction a little more caution is required. For example, it is commonly observed in experiments that wavy vortex solutions may appear with wave numbers 3 or 4 (say); that is, with 3 or 4 complete periods relative to a single turn of the cylinder. Provided only *one* such mode is present, we may scale the azimuthal angle to "factor out" this additional periodicity. The angle 2π then represents one period ($2\pi/3$ or $2\pi/4$ on the physical cylinder). In group-theoretic terms, an action of $SO(2)$ for which $\theta \in SO(2)$ produces a rotation by

$k\theta$, k an integer, can be viewed as the standard ($k=1$) action of $SO(2)/Z_k$ and this group may be *identified* with $SO(2)$.

On the six-dimensional kernel, only one such periodic mode occurs, and this procedure may be followed. If two modes with different wave numbers occur, it would be necessary to make $SO(2)$ act by $k\theta$ and $l\theta$ (where k, l are the respective wavenumbers) on the corresponding spaces of eigenfunctions and to carry out the analysis for the appropriate action of $O(2)\times SO(2)\times S^1$. See Chossat [1985].

3. Observed solutions and their symmetries. In the experiments a number of prechaotic states are observed. In this section we discuss the symmetries of each of the following states: Couette flow, Taylor vortices, wavy vortices, spiral cells, and twisted vortices.

Both the *Couette* and *vortex* flows are time-independent. As noted above, vortex flow produces bands along which the flow is in the azimuthal plane. See Fig. 3.1(a).

When $\Omega_0 \leqq 0$, vortex flow loses stability, and a time periodic state called *wavy vortices* appears. See Fig. 3.1(b). This periodic flow has the special form of rotating waves. More precisely, the solution $u(t)$ is a rotating wave if $u(t+\theta)=R_\theta u(t)$ where R_θ denotes rotation by angle θ in the azimuthal plane. We shall see in §4 that *all* periodic solutions obtained from the six-dimensional kernel must be rotating waves.

When $\Omega_0=\Omega$, wavy vortex solutions lose stability, and a new quasi-periodic solution with two independent frequencies appears. This new state is called *modulated wavy vortices*. It is interesting to observe, at this point, how the modulated wavy vortex solution might be detected by our proposed method using a Lyapunov–Schmidt reduction. The idea is to compute the Floquet exponents along the wavy vortex branch of solutions and show that certain of these exponents cross the imaginary axis. Then apply the Sacker–Neimark torus bifurcation theorem to conclude the existence of quasiperiodic solutions. We show in §7 that this scenario is possible. A similar remark holds for identifying wavy spiral states when $\Omega_0<0$.

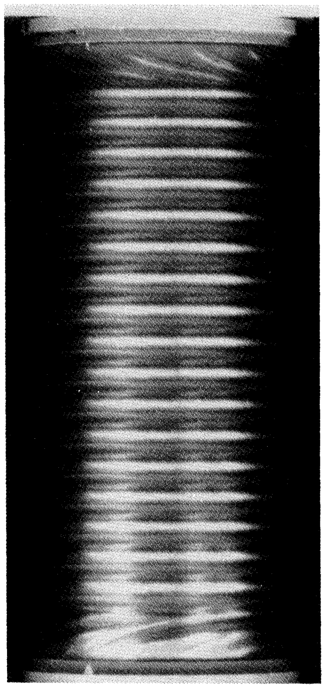
We note that the actual transition to chaos cannot be explained by our analysis. Nevertheless, chaotic behavior may be present in our model and this point deserves further investigation.

We also note here that in the corotating and counterrotating Taylor problems solutions with different planforms are observed. For example, in the corotating case Andereck, Dickman and Swinney [1983] have observed *twisted vortices*. See Fig. 3.1 (c). In the strongly counterrotating case, Couette flow loses stability to a helicoidal pattern called *spiral cells*, which are time-periodic rotating waves. See Fig. 3.1(d).

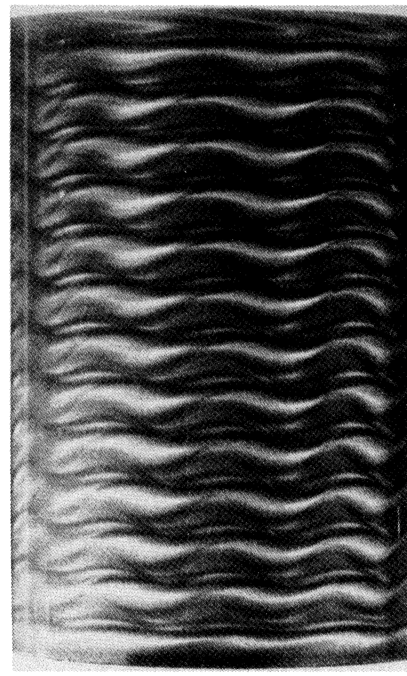
We can distinguish each of the states described above by their *isotropy subgroups*; that is, by the subgroup of (2.1) which leaves the given state invariant. In Table 3.1 we list the isotropy subgroups for each fluid state described above.

We now discuss the entries in Table 3.1. The steady-state solutions are invariant under change of phase (S^1); the periodic solutions are all rotating waves and are invariant under Δ since a change of phase may be compensated for by rotating the cylinder. Couette flow is invariant under all symmetries. Taylor vortices are invariant under all rotations ($SO(2)$) and the flip along the cylindrical axis κ. We denote by $Z_2(\kappa)$ the two-element group generated by κ.

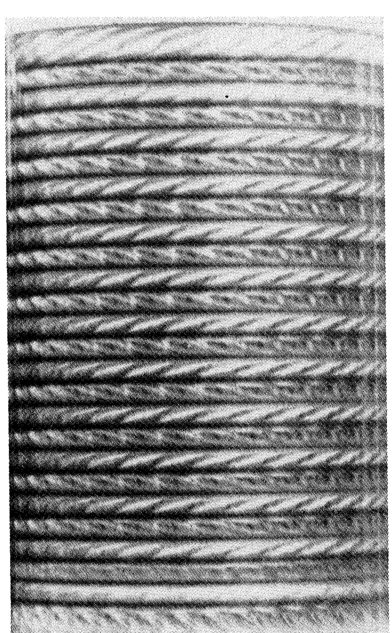
Isotropy subgroups for the periodic solutions are obtained as follows. The helical state, spiral cells, is invariant under $\widetilde{SO}(2)$ since a translation along the cylinder axis may be compensated for by a rotation of the cylinder. Next observe that wavy vortices are invariant under the group element obtained by composing the flip (κ) with rotation



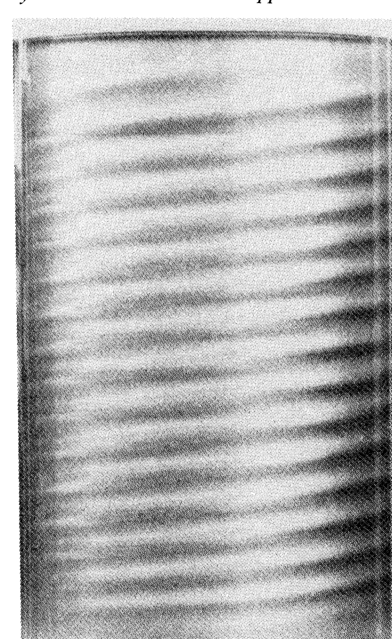
(a) *Taylor vortices* ($R_o = 1164$, $R_i = 1161$).



(b) *Modulated wavy vortices* ($R_o = -100, R_i = 350$). *In a still photograph wavy vortices and modulated wavy vortices have a similar appearance.*



(c) *Twisted vortices* ($R_o = 721$, $R_i = 1{,}040$).



(d) *Spiral cells* ($R_o = -295, R_i = 237$).

FIG. 3.1. *Observed flows in the Taylor experiment. Reynolds numbers for the inner* (R_i) *and outer* (R_o) *cylinders. Photographs kindly supplied by Harry Swinney and Randy Tagg. Similar photographs will appear in Andereck, Liu and Swinney* [1985].

TABLE 3.1
Isotropy subgroups of observed fluid states.

State	Isotropy subgroup
Couette flow	$O(2)\times SO(2)\times S^1$
Taylor vortices	$Z_2(\kappa)\times SO(2)\times S^1$
Wavy vortices	$Z_2(\kappa,\pi)\times\Delta$
Twisted vortices	$Z_2(\kappa)\times\Delta$
Spiral cells	$\widetilde{SO}(2)\times\Delta$

$\Delta=\{(\theta,-\theta)\in SO(2)\times S^1\}$
$\kappa\in O(2)$ is the flip $z\to -z$ along the cylindrical axis
$\pi\in SO(2)$ is rotation of the azimuthal plane by one-half period
$\widetilde{SO}(2)=\{(\psi,-\psi)\in O(2)\times SO(2)\}$

of the cylinder by half a period $\pi\in SO(2)$. Finally, twisted vortices are invariant under the flip κ.

It is worth noting that the first three bifurcations in the standard Taylor problem ($\Omega_0=0$) break symmetry in a simple way. Couette flow to Taylor vortices breaks the translational symmetries; Taylor vortices to wavy vortices breaks the rotational symmetries ($SO(2)$); and wavy vortices to modulated wavy vortices breaks the rotating wave symmetries (Δ).

4. Group theory and the six-dimensional kernel. In this section, we answer four questions:

(1) What is the exact form of the six-dimensional kernel?

(2) What is the action of the symmetries of the Taylor problem on this kernel?

(3) What is the form of the reduced mapping h, obtained by the Lyapunov–Schmidt procedure?

(4) What are the possible isotropy subgroups of points in the six-dimensional kernel?

We answer the first question by referring to DiPrima and Sijbrand [1982]. Let $\eta=R_i/R_o$ be the ratio of the radii of the inner and outer cylinders. We quote:

> Thus, for example, for $\eta=0.95$ and $\Omega_0/\Omega_i=-0.73976$, Couette flow is simultaneously unstable to an axisymmetric disturbance with wave numbers $(\lambda,m)=(3.482,0)$ and a nonaxisymmetric disturbance with wavenumbers $(\lambda,m)=(3.482,1)$. We also note that... there are 6 critical modes with axial (Z) and azimuthal (Θ) dependence as follows:

$$\cos\lambda Z,\ \sin\lambda Z,\ e^{\pm i\Theta}\cos(\lambda Z),\ e^{\pm i\Theta}\sin(\lambda Z). \tag{4.1}$$

The action of the translations in $O(2)$ on the eigenfunctions in (4.1) is generated by translations of the axial (angle) Z and the flip (κ) which acts by $Z\to -Z$. Rotations in the azimuthal plane act by translations in Θ. (We have omitted the radial dependence of the eigenfunctions here as the group $O(2)\times SO(2)$ acts trivially in the radial direction.) Observe that the resulting action of $O(2)\times SO(2)$ on the six-dimensional space generated by the eigenfunctions in (4.1) leads to the following equivalent action. We identify the six-dimensional kernel with

$$V=\mathbf{R}^2\oplus(\mathbf{R}^2\otimes\mathbf{C}) \tag{4.2}$$

and let elements of $O(2)$ act on $\mathbf{R}^2$ in the standard way and elements of $SO(2)$ act by multiplication on **C**. That is

$$(\theta,\psi)(v,w\otimes z)=\left(R_\theta v,(R_\theta w)\otimes(e^{i\psi}z)\right)$$

where R_θ is the usual rotation of $\mathbf{R}^2$ through the angle θ. Similarly, the flip κ acts by $(Kv, Kw\otimes z)$, where K is the matrix $\left(\begin{smallmatrix}1&0\\0&1\end{smallmatrix}\right)$. The action of the phase shifts S^1 in V turns out to be identical with the action of $SO(2)$ on V. This fact can be verified by direct computation. First observe that the $\mathbf{R}^2$ summand in V is spanned by $\{\cos(\lambda Z),\sin(\lambda Z)\}$, a steady-state kernel. As such, S^1 acts trivially on $\mathbf{R}^2$. Next, observe that S^1 commutes with $O(2)\times SO(2)$ and hence the actions of S^1 and $O(2)\times SO(2)$ on $\mathbf{R}^2\otimes\mathbf{C}$ commute. Since the only matrices acting on $\mathbf{R}^2\otimes\mathbf{C}$ commuting with $O(2)\times SO(2)$ are scalar multiples of matrices in $SO(2)$ (see Golubitsky and Stewart [1985, Lemma 3.2]), it follows that the elements of S^1 act in a fashion identical to elements of $SO(2)$. Without loss of generality, we may identify the actions of $SO(2)$ and S^1.

One consequence of this identification is that the subgroup $\Delta=\{(\psi,-\psi)\in SO(2)\times S^1\}$ is in the isotropy subgroup of every element in V. Thus, we have proved:

LEMMA 4.1. *Every periodic solution found in the six-dimensional kernel is a rotating wave.*

A second consequence of the identification of the actions of $SO(2)$ and S^1 is the simple form that the reduction bifurcation equation h: $V\to V$, obtained via a Lyapunov–Schmidt reduction, must take. (The function h depends on a number of extra parameters, Ω_0 for example. We suppress this dependence here.) Let the purely imaginary eigenvalues of the linearized Navier–Stokes equations be $\pm\omega i$. For simplicity, use a scaling argument to assume $\omega=1$. Then the idea behind the Lyapunov–Schmidt reduction is to look for small amplitude periodic solutions of period near 2π. One does this by rescaling time in the original equation by a perturbed period parameter τ and looking for precisely 2π-periodic solutions to the scaled equations. What results, after appropriate applications of the implicit function theorem, is a reduction equation

$$h(v,\tau)=0$$

where h: $V\times\mathbf{R}\to V$ is smooth and commutes with $O(2)\times SO(2)\times S^1$. We claim that we may assume that the dependence of h on τ is particularly simple. In fact,

$$h(v,\tau)=g(v)-(1+\tau)J \tag{4.3}$$

where J is the matrix form of the action by $\pi/2\in S^1$ on V.

To verify this claim, suppose for the moment that there exists a smooth center manifold. Let $g(v)$ be the reduction vector field on that center manifold. It was proved in Golubitsky and Stewart [1985] that if the Lyapunov–Schmidt reduction is applied to g, introducing τ, then the resulting function h has exactly the form (4.3). This fact relies on having the spatial symmetries $SO(2)$ identified with the temporal symmetries S^1.

If we perform the Lyapunov–Schmidt reduction directly from the Navier–Stokes equations, then the reduced function has the same form as (4.3), at least to first order in τ. In any case, the form (4.3) is used later only to solve certain equations for τ explicitly. If higher order terms are present, then these equations may be solved implicitly, which is sufficient for our purposes. Therefore we lose nothing by working with h in the form (4.3). Moreover, we note that g commutes with the action of $O(2)\times SO(2)\times S^1$ on V and may be identified with the mapping on V obtained by a center manifold reduction, at least up to any finite order in its Taylor expansion.

For the remainder of this section we describe precisely the form that mappings g which commute with $O(2)\times SO(2)\times S^1$ must have. Note that a third consequence of identifying the actions of $SO(2)$ and S^1 is that at this stage we may ignore one of them. Henceforth, we assume that

$$\Gamma=O(2)\times S^1 \tag{4.4}$$

is our group of symmetries and turn attention to the action of Γ on V.

At this point we choose coordinates on V. First write $V=V_1\oplus V_2$ where $V_1=\mathbf{R}^2\cong\mathbf{C}$ and $V_2=\mathbf{R}^2\otimes\mathbf{C}\cong M(2,\mathbf{R})$, the space of 2×2 matrices with real entries. Thus elements of V have the form

$$(z,A) \tag{4.5}$$

where

$$z=x+iy\in\mathbf{C} \quad\text{and}\quad A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in M(2,\mathbf{R}).$$

In these coordinates the group action of $\Gamma=O(2)\times S^1$ on (z,A) is defined as follows:

$$(\theta,\psi)(z,A)=\left(e^{i\theta}z,R_\theta AR_\psi\right) \tag{4.6}$$

where

$$R_\theta=\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}$$

is the rotation matrix. See Golubitsky and Stewart [1985, §3] for more detail.

We now answer the third question by describing in detail the invariant functions and the equivariant mappings corresponding to this group action. (Recall that Φ: $V\to V$ is equivariant if $\Phi(\gamma v)=\gamma\Phi(v)$ for all $\gamma\in\Gamma$, $v\in V$.) Proofs are found in the Appendix.

PROPOSITION 4.2. *Let ϕ: $V\to R$ be a smooth function defined in a neighborhood of the origin which is invariant with respect to the action of* Γ *in* (4.6). *Then there exists a smooth function h: $\mathbf{R}^5\to\mathbf{R}$ defined near* 0 *such that*

$$\phi(v)=h\left(\beta,N,\delta^2,\gamma,\sigma\right)$$

where

$$\begin{aligned}
&\beta\equiv z\bar z=x^2+y^2,\\
&N=a^2+b^2+c^2+d^2,\\
&\delta=\det A,\\
&\gamma=\operatorname{Re}\left(z^2\bar\zeta\right),\\
&\sigma=i\delta\operatorname{Im}\left(z^2\bar\zeta\right)
\end{aligned} \tag{4.7}$$

and

$$\zeta=a^2+b^2-c^2-d^2+2i(ac+bd).$$

THEOREM 4.3. *Let Φ: $V\to V$ be a smooth Γ-equivariant mapping defined near* 0. *Then there exist Γ-invariant functions*

$$p,q,r,s,P^1,P^2,Q^1,Q^2,Q^3,Q^4,R^1,R^2,R^3,R^4,M^3,M^4$$

such that

$$\Phi(z,A)=\left(pz+qi\delta z+r\bar{z}\zeta+si\delta\bar{z}\zeta,\ \sum_{j=1}^{4}\left(S^jK_j+T^jL_j\right)\right), \tag{4.8}$$

where

$$K_1=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\quad K_2=\begin{pmatrix} -b & a\\ -d & c\end{pmatrix},\quad K_3=\begin{pmatrix} a & b\\ -c & -d\end{pmatrix},\quad K_4=\begin{pmatrix} -b & a\\ d & -c\end{pmatrix}, \tag{4.9}$$

$$L_1=\begin{pmatrix} -c & -d\\ a & b\end{pmatrix},\quad L_2=\begin{pmatrix} d & -c\\ -b & a\end{pmatrix},\quad L_3=\begin{pmatrix} c & d\\ a & b\end{pmatrix},\quad L_4=\begin{pmatrix} -d & c\\ -b & a\end{pmatrix},$$

and

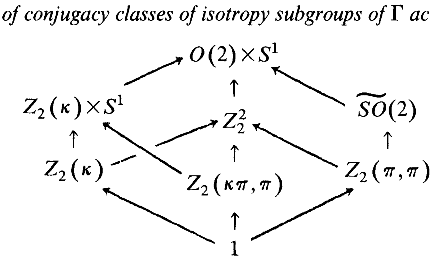
$$\begin{aligned} S^1&=P^1,\\ S^2&=P^2,\\ T^1&=R^1\delta+Q^1\operatorname{Im}(z^2\bar{\zeta}),\\ T^2&=R^2\delta+Q^2\operatorname{Im}(z^2\bar{\zeta}),\\ S^3&=Q^3\operatorname{Re}(\bar{z}^2)+R^3\delta\operatorname{Im}(\bar{\zeta})+M^3\delta\operatorname{Im}(\bar{z}^2),\\ S^4&=Q^4\operatorname{Re}(\bar{z}^2)+R^4\delta\operatorname{Im}(\bar{\zeta})+M^4\delta\operatorname{Im}(\bar{z}^2),\\ T^3&=-Q^3\operatorname{Im}(\bar{z}^2)+R^3\delta\operatorname{Re}(\bar{\zeta})+M^3\delta\operatorname{Re}(\bar{z}^2),\\ T^4&=-Q^4\operatorname{Im}(\bar{z}^2)+R^4\delta\operatorname{Re}(\bar{\zeta})+M^4\delta\operatorname{Re}(\bar{z}^2). \end{aligned} \tag{4.10}$$

We shall exploit the form of ϕ in (4.8) to solve explicitly the reduced bifurcation equation $g=0$. In order to understand what types of solutions one may find in $g=0$ we answer the fourth question of this section. By determining, up to conjugacy, the set of all isotropy subgroups of elements in V, we determine the symmetries that possible solutions to the Navier–Stokes equations found by reducing to V may have.

The lattice (of conjugacy classes) of isotropy subgroups for Γ acting on V is given in Table 4.1. Containment of one conjugacy class in another is indicated by arrows. In Table 4.2 we list these isotropy subgroups along with the states in the Taylor problem which have those symmetries. We use the notation Z_2 to indicate a two-element group and $Z_2(\alpha)$ to indicate the two-element group generated by $\alpha\in\Gamma$.

We emphasize that the containments in Table 4.1 are of conjugacy classes. For example, $Z(\kappa\pi,\pi)$ is not contained in $Z_2(\kappa)\times S^1$. However, $Z_2(\kappa\pi,\pi)$ is conjugate to $Z_2(\kappa,\pi)$ which is contained in $Z_2(\kappa)\times S^1$.

TABLE 4.1
Lattice of conjugacy classes of isotropy subgroups of Γ *acting on* V.



Note: Z_2^2 is generated by κ, $(\kappa\pi,\pi)$, (π,π).

TABLE 4.2
The symmetries associated with observed fluid states.

Isotropy subgroup	Solution type
$O(2)\times S^1$	Couette flow
$Z_2(\kappa)\times S^1$	Taylor vortices
$\widetilde{SO}(2)$	spiral cells
$Z_2(\kappa\pi,\pi)$	wavy vortices
$Z_2(\kappa)$	twisted vortices

We derive the lattice pictured in Table 4.1 by first considering orbit representatives. Begin by considering the action of $O(2)\times S^1$ on $\mathbf{R}^2\otimes\mathbf{C}\cong M(2,\mathbf{R})$. Let A be a 2×2 matrix. As shown in Golubitsky and Stewart [1985, §7], we can choose an element of $O(2)\times S^1$ so that A is conjugated to the diagonal matrix $\left(\begin{smallmatrix} a & 0\\ 0 & d\end{smallmatrix}\right)$ where $a\geqq d\geqq 0$. It is then easy to show that there are four types of orbits as shown in Table 4.3.

TABLE 4.3
Orbit representatives of $O(2)\times S^1$ acting on $M(2,\mathbf{R})$.

Orbit representative	Isotropy subgroup
0	$O(2)\times S^1$
$\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix},\ a>0$	Z_2^2
$\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix},\ a>0$	$\widetilde{SO}(2)$
$\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix},\ a>d>0$	$Z_2(\pi,\pi)$

Having put the matrices in $M(2,\mathbf{R})$ into normal form, we now use the isotropy subgroups of these matrices to conjugate the elements $z\in\mathbf{C}\cong\mathbf{R}^2$. In this way we obtain representatives for all the orbits of $O(2)\times S^1$ acting on $\mathbf{R}^2\oplus(\mathbf{R}^2\otimes\mathbf{C})$. These results are summarized in Table 4.4.

TABLE 4.4
Orbit representatives of $O(2)\times S^1$ acting on $\mathbf{R}^2\oplus(\mathbf{R}^2\otimes\mathbf{C})$.

Orbit representative (z,A)	Isotropy subgroup
$(0,0)$	$O(2)\times S^1$
$(x,0),\ x>0$	$Z_2(\kappa)\times S^1$
$\left(0,\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}\right),\ a>0$	Z_2^2
$\left(x,\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}\right),\ x>0,\ a>0$	$Z_2^2(\kappa)$
$\left(iy,\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}\right),\ y>0,\ a>0$	$Z_2^2(\kappa\pi,\pi)$
$\left(x+iy,\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}\right),\ x>0,\ y>0,\ a>0$	1
$\left(0,\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}\right),\ a>0$	$\widetilde{SO}(2)$
$\left(x,\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}\right),\ x>0,\ a>0$	1
$\left(0,\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}\right),\ a>d>0$	$Z_2(\pi,\pi)$
$\left(x+iy,\begin{pmatrix} a & 0\\ 0 & d\end{pmatrix}\right),\ \lvert x+iy\rvert\neq 0,\ x\geqq 0,\ a>d>0$	1

TABLE 4.5

Fixed-point subspaces of $O(2)\times S^1$ acting on V.

Isotropy subgroup	Fixed-point subspace	Dimension
$O(2)\times S^1$	0	0
$Z_2(\kappa)\times S^1$	$(x,0)$	1
Z_2^2	$\left(0,\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}\right)$	2
$\widetilde{SO}(2)$	$\left(0,\begin{pmatrix} a & b\\ -b & a\end{pmatrix}\right)$	2
$Z_2(\kappa)$	$\left(x,\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}\right)$	3
$Z_2(\kappa\pi,\pi)$	$\left(iy,\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}\right)$	3
$Z_2(\pi,\pi)$	$\left(0,\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)$	4
1	$\left(x+iy,\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)$	6

In the last table of this section we present the fixed-point subspaces of the various isotropy subgroups. More precisely, let $\Sigma\subset\Gamma$ be a subgroup. Define

$$V^{\Sigma}=\{v\in V\mid \sigma v=v \text{ for all } \sigma\in\Sigma\}. \tag{4.11}$$

Observe that if Φ: $V\to V$ commutes with Γ, then Φ maps V^{Σ} to itself (see Golubitsky and Stewart [1985, (1.6)]).

5. Branching and stability. Let $h(z,A,\lambda,\tau)$ be the mapping on the six-dimensional kernel obtained via the Lyapunov–Schmidt reduction. Note that h depends explicitly on the bifurcation parameter λ and the perturbed period τ. In addition, we know that h commutes with the symmetries in the Taylor problem. We shall use the consequences of this fact to explain how to compute both the solutions to $h=0$ and the eigenvalues of the 6×6 Jacobian matrix dh along branches of solutions to $h=0$. One consequence of the $O(2)\times SO(2)\times S^1$ symmetries is that the eigenvalues of dh determine the orbital asymptotic stability of solutions.

Let us be more precise. Recall the form of h in (4.3) with its simple τ-dependence, namely

$$h(z,A,\lambda,\tau)=g(z,A,\lambda)-(1+\tau)\left(0,\begin{pmatrix} -b & a\\ -d & c\end{pmatrix}\right) \tag{5.1}$$

where $A=\begin{pmatrix} a & b\\ c & d\end{pmatrix}$. In deriving this form we note that if the Navier–Stokes equations admit a smooth center manifold then h will have exactly the form (5.1), where g is the reduced vector field on that center manifold. We proved in Golubitsky and Stewart [1985, Thm. 8.2] that the eigenvalues of dh determine the orbital asymptotic stability of solutions to the vector field g on the center manifold. Moreover the center manifold reduction implies that the stabilities of solutions to the vector field g are the same as the stabilities of the corresponding solutions to the Navier–Stokes equations.

If there should not exist a smooth center manifold (which we doubt) then we are computing the correct stabilities for g, accurate to any finite order, by using the eigenvalues of $(dh)|_{h=0}$.

We now describe how to compute the eigenvalues of dh. Recall that (4.8) restricts the form of g in (5.1) to:

$$g(z,A,\lambda)=\left(pz+qi\delta z+r\bar{z}\zeta+si\delta\bar{z}\zeta,\ \sum_{j=1}^{4} S^jK_j+T^jL_j\right) \tag{5.2}$$

where p,q,r,s and the coefficients $P^1,\cdots,M^4$ appearing in (4.11) are invariant functions, hence functions of the five variables $\beta,N,\delta^2,\gamma,\sigma$ defined in (4.7) and λ. Moreover, since h is obtained via the Lyapunov–Schmidt reduction, the linear terms must vanish. Hence

$$p(0)=0,\quad P^1(0)=0,\quad P^2(0)=1. \tag{5.3}$$

Equivariance shows that to solve the equations $h=0$ we need only evaluate h on typical orbit representatives. The resulting equations are listed in Table 5.1. See Table 4.4 for the list of orbit representatives and their isotropy subgroups.

Remarks. (i) The equations involving τ serve only to determine the perturbed period of the associated periodic solution. Note that (5.1) allows us to eliminate τ by solving these equations explicitly (or implicitly if there does not exist a smooth center manifold, see §4 above).

(ii) Observe that τ is indeterminate on the $Z_2(\kappa)\times S^1$ branch, which is to be expected since the bifurcation is to a "steady state."

(iii) Observe that the theoretical basis of our explicit calculations is given by (4.10): fixed-point subspaces V^Σ are mapped to themselves by equivariant mappings. Therefore we may restrict h to V^Σ and seek solutions to $h\,|\,V^\Sigma=0$, considering each isotropy subgroup Σ in turn.

Table 5.1 also lists the coefficients that determine the signs of the (real parts of the) eigenvalues of dh. We consider the branching equations briefly first, and then describe in more detail the eigenvalue calculations.

By writing (4.9) in coordinates we obtain

$$h\left(x,y,\begin{pmatrix} a & b\\ c & d\end{pmatrix}\right)=\left(X,Y,\begin{pmatrix} A & B\\ C & D\end{pmatrix}\right), \tag{5.4}$$

where

$$\begin{aligned}
&\text{(a) } X=px-q\delta y+rx\,\mathrm{Re}(\zeta)+ry\,\mathrm{Im}(\zeta)+s\delta y\,\mathrm{Re}(\zeta)-s\delta x\,\mathrm{Im}(\zeta),\\
&\text{(b) } Y=py+q\delta x-ry\,\mathrm{Re}(\zeta)-rx\,\mathrm{Im}(\zeta)+s\delta x\,\mathrm{Re}(\zeta)+s\delta y\,\mathrm{Im}(\zeta),\\
&\text{(c) } A=(S^1+S^3)a+(-S^2-S^4)b+(-T^1+T^3)c+(T^2-T^4)d,\\
&\text{(d) } B=(S^2+S^4)a+(S^1+S^3)b+(-T^2+T^4)c+(-T^1+T^3)d,\\
&\text{(e) } C=(T^1+T^3)a+(-T^2-T^4)b+(S^1-S^3)c+(-S^2+S^4)d,\\
&\text{(f) } D=(T^2+T^4)a+(T^1+T^3)b+(S^2-S^4)c+(S^1-S^3)d.
\end{aligned} \tag{5.5}$$

The branching equations always take the form $X=Y=A=B=C=D=0$, evaluated on the appropriate orbit representative. The entries in the table follow readily. However, a few comments should be made regarding the last four entries of "unknown" type.

TABLE 5.1

Branching equations and eigenvalues for solutions with given symmetry.

Solution type; Isotropy; orbit	Branching equations (to be evaluated at orbit representative shown)	Signs of eigenvalues	Multiplicity
Couette flow $O(2)\times S^1$ $(0,0)$	None	p $P^1 \pm i(P^2-1-\tau)$	2 2, 2
Taylor vortices $Z_2(\kappa)\times S^1$ $(x,0)$	$p=0$ at: $(x^2,0,0,0,0,\lambda)$[1]	0 p_β $(P^1+x^2Q^3)\pm i(P^2-1-\tau+x^2Q^4)$ $(P^2-x^2Q^3)\pm i(P^2-1-\tau-x^2Q^4)$	1 1 1, 1 1, 1
unknown Z_2^2 $\left(0,\begin{pmatrix} a & 0 \\ 0 & 0\end{pmatrix}\right)$	$P^1=0$ $P^2=1+\tau$ at: $(0,a^2,0,0,0,\lambda)$	0 $p+ra^2$ $p-ra^2$ $P_N^1+a^2P_N^3$ $R^2+a^2R^4$	2 1 1 1 1
spiral cells $\widetilde{SO}(2)$ $\left(0,\begin{pmatrix} a & 0 \\ 0 & a\end{pmatrix}\right)$	$P^1+a^2R^2=0$ $P^2-a^2R^1=1+\tau$ at: $(0,2a^2,a^4,0,0,\lambda)$	0 $p\pm iqa^2$ $2P_N^1+R^2+a^2(P_{\delta^2}^1+2R_N^2)+a^4R_{\delta^2}^2$ $(R^2+2a^2R^4)\pm i(R^1+2a^2R^4)$	1 1, 1 1 1, 1
wavy vortices $Z_2(\kappa\pi,\pi)$ $\left(iy,\begin{pmatrix} a & 0 \\ 0 & 0\end{pmatrix}\right)$	$p-a^2r=0$ $P^1-y^2Q^3=0$ $P^2-y^2Q^4=1+\tau$ at: $(y^2,a^2,0,-a^2y^2,0,\lambda)$	0 r $R^2a^2+2Q^3y^2+ay^2Q_d^4$ $+a^4R^4-a^2y^2M^4$ $\begin{bmatrix} Y_y & Y_a \\ A_y & A_a\end{bmatrix}$[2] $\det = p_\beta P_N^1-(p_N-r)(P_\beta^1-Q^3)+\cdots$ $\text{trace} = p_\beta y^2+P_N^1a^2+\cdots$	2 1 1 1, 1
twisted vortices $Z_2(\kappa)$ $\left(x,\begin{pmatrix} a & 0 \\ 0 & 0\end{pmatrix}\right)$	$p+a^2r=0$ $P^1+x^2Q^3=0$ $P^2+x^2Q^4=1+\tau$ at: $(x^2,a^2,0,a^2x^2,0,\lambda)$	0 $-r$ $R^2a^2-2Q^3x^2$ $-ax^2Q_d^4+a^4R^4+a^2x^2M^4$ $\begin{bmatrix} X_x & X_a \\ A_x & A_a\end{bmatrix}$[2] $\det = p_\beta P_N^1-(p_N+r)(P_\beta^1-Q^3)+\cdots$ $\text{trace} = p_\beta x^2+P_N^1a^2+\cdots$	2 1 1 1, 1
unknown $Z_2(\pi,\pi)$ $\left(0,\begin{pmatrix} a & 0 \\ 0 & d\end{pmatrix}\right)$	$R^2+(a^2+d^2)R^4=0$ plus others at: $(0,a^2+d^2,a^2d^2,0,0,\lambda)$	not computed	

TABLE 5.1 (continued)

unknown 1 $\left(x,\begin{pmatrix} a & 0 \\ 0 & a \end{pmatrix}\right)$	$q=0$ plus others at: $(x^2,2a^2,a^4,0,0,\lambda)$	not computed	
unknown 1 $\left(x+iy,\begin{pmatrix} a & 0 \\ 0 & 0 \end{pmatrix}\right)$	$r=0$ plus others at: $(x^2+y^2,a^2,0,(x^2-y^2)a^2,0,\lambda)$	not computed	
unknown 1 $\left(x+iy,\begin{pmatrix} a & 0 \\ 0 & d \end{pmatrix}\right)$	$X=Y=A=B=C=D=0$ at: $(x^2+y^2,a^2+d^2,a^2d^2,$ $(x^2-y^2)(a^2-d^2),2xyad(a^2-d^2),\lambda)$	not computed	

Notes: [1] The equations must be evaluated at $(\beta,N,\delta^2,\gamma,\sigma,\lambda)$ where these in turn are evaluated on an orbit representative to yield the form stated.

[2] The remaining eigenvalues are those of the specified 2×2 matrix. Its determinant and trace are shown to lowest order (omitting positive factors) to determine their signs.

When the isotropy group is $Z_2(\pi,\pi)$ we take two of the equations, namely $A=0=D$, which reduce to:

$$(S^1+S^3)a+(T^2-T^4)d=0,$$

$$(T^2+T^4)a+(S^1-S^3)d=0.$$

Now observe that S^1+S^3 and T^2+T^4 have a factor d. Divide this out and subtract. The result has a factor (a^2-d^2), and the entry in the table follows. We do not require the remaining equations, because an appeal to nondegeneracy (§6) now rules out this case.

For the next two cases, the equations $X=Y=0$ lead, among other things, to the listed equation, which is also ruled out by nondegeneracy. The final case is extremely complicated and it remains possible that such a branch might occur: see §7 for further discussion.

The computation of the eigenvalues, particularly those along the $Z_2(\kappa)$ and $Z_2(\kappa\pi,\pi)$ branches, is the most difficult part of this section. This computation is facilitated by the use of several results in Golubitsky and Stewart [1985, §8b]. The first is that along a solution branch (v_0,λ_0,τ_0) with isotropy subgroup Σ, the derivative $(dh)_{v_0,\lambda_0,\tau_0}$ commutes with Σ. This implies (Lemma 8.4 of that paper) that $(dh)_{v_0,\lambda_0,\tau_0}$ leaves invariant the subspaces W_j of V formed by adding together all subspaces of V on which Σ acts by a fixed irreducible representation. We use the W_j to put dh into block diagonal form.

The second fact is that $(dh)_{v,\lambda,\tau}$ vanishes on all vectors tangent to the orbit of v under the action of $O(2)\times S^1$. These null-vectors are given by:

$$\text{(5.6)}\quad \begin{aligned} &\text{(a)} \quad (z,A)\to\frac{d}{d\theta}(z,AR_\theta)\Big|_{\theta=0}=\left(0,A\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}\right), \\ &\text{(b)} \quad (z,A)\to\frac{d}{d\psi}\left(e^{i\psi}z,R_\psi A\right)\Big|_{\psi=0}=\left(iz,\begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}A\right). \end{aligned}$$

We now outline the explicit computation of the eigenvalues listed in Table 5.1.

(a) $Z_2\times S^1$ (*Taylor vortices*). Decompose $V=\mathbf{R}^2\oplus(\mathbf{R}^2\otimes\mathbf{C})$ into irreducibles for $Z_2\times S^1$. We get $V=W_0\oplus W_1\oplus W_2\oplus W_3$ where

$$W_0=\{(x,0)\},\qquad W_2=\left\{\left(0,\begin{pmatrix} a & b\\ 0 & 0\end{pmatrix}\right)\right\},$$

$$W_1=\{(iy,0)\},\qquad W_3=\left\{\left(0,\begin{pmatrix} 0 & 0\\ c & d\end{pmatrix}\right)\right\}.$$

The actions are given by

	W_0	W_1	W_2	W_3
$\kappa\in O(2)$	1	-1	1	-1
$\psi\in SO(2)$	1	1	$\mathbf{R}_\psi$	$\mathbf{R}_\psi$

which are distinct irreducible actions. Therefore, dh leaves each W_j invariant. Let $\Phi_j=dh\,|\,W_j$, so that dh has the block form:

$$\begin{array}{c} \\ x\\ y\\ a\ b\\ c\ d\end{array}\;\begin{array}{c|c|c|c} x & y & a\ b & c\ d\\ \hline \Phi_0 & 0 & 0 & 0\\ \hline 0 & \Phi_1 & 0 & 0\\ \hline 0 & 0 & \Phi_2 & 0\\ \hline 0 & 0 & 0 & \Phi_3\end{array}$$

We evaluate the Φ_j at the orbit representative $(x,0)$, with the following results.

$\Phi_0=X_x=p+p_x x=p_x x$ since $p=0$ on this branch by Table 5.1. Now $x>0$ so we can divide it out.

$$\Phi_1=Y_y=p+p_y y=0,$$

$$\Phi_2=\begin{bmatrix} A_a & A_b\\ B_a & B_b\end{bmatrix}=\begin{bmatrix} S^1+S^3 & -S^2-S^4\\ S^2+S^4 & S^1+S^3\end{bmatrix}=\begin{bmatrix} P^1+x^2Q^3 & -(P^2-1-\tau+x^2Q^4)\\ P^2-1-\tau+x^2Q^4 & P^1+x^2Q^3\end{bmatrix}.$$

Since a matrix $\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$ has eigenvalues $\alpha\pm i\beta$, the entry in the table follows. Similarly

$$\Phi_3=\begin{bmatrix} C_c & C_d\\ D_c & D_d\end{bmatrix}=\begin{bmatrix} S^1-S^3 & -S^2+S^4\\ S^2-S^4 & S^1-S^3\end{bmatrix}=\begin{bmatrix} P^1-x^2Q^3 & -(P^2-1-\tau-x^2Q^4)\\ P^2-1-\tau-x^2Q^4 & P^1-x^2Q^3\end{bmatrix}.$$

The entries for vortices in Table 5.1 follow. Note that Φ_2 and Φ_3 have to be scalar multiples $\begin{pmatrix}\alpha & -\beta\\ \beta & \alpha\end{pmatrix}$ of rotations since dh commutes with S^1; direct calculation confirms this.

(b) Z_2^2 (*unknown*). We use the same decomposition $V=W_0\oplus W_1\oplus W_2\oplus W_3$ as in (a). The actions are:

	W_0	W_1	W_2	W_3
κ	1	-1	1	-1
(π,π)	-1	-1	1	1

So the irreducible components on these spaces are distinct. Therefore dh leaves each W_j invariant: set $\Phi_j = dh \mid W_j$. We have

$$\Phi_0 = p + ra^2, \qquad \Phi_1 = p - ra^2.$$

Now dh has one eigenvalue 0 on each of W_2 and W_3, so the remaining eigenvalues are $\operatorname{Tr}\Phi_2$, $\operatorname{Tr}\Phi_3$. Use (5.4) here. These are computed as follows.

$$\operatorname{Tr}\Phi_2 = A_a + B_b = (S^1 + S^3) + (S_a^1 + S_a^3)a + (S^1 + S^3) + (S_b^2 + S_b^4)a$$

since $b = c = d = 0$ on the orbit. The branching equations show that $S^1 + S^3 = 0$, so the sign is given by $S_a^1 + S_a^3 + S_b^2 + S_b^4$. Now on the orbit we have $z = 0$, $\delta = 0$, $\zeta = a^2$, $\delta_a = 0$, $\delta_b = 0$. So by (4.11) we compute this as $P_a^1 + P_b^2$. But $N_b = 0$ on the orbit, so this becomes $P_N^1 \cdot 2a$. Dividing by $2a > 0$ gives the table entry. (From now on, we omit such details from the calculation.)

Similarly,

$$\operatorname{Tr}\Phi_3 = C_c + D_d = S^1 - S^2 + (T_c^1 + T_c^3)a + (S^1 - S^3) + (T_d^2 + T_d^4)a.$$

But $S^1 + S^3 = 0$ on this branch, so this is

$$4S^1 + a(T_c^1 + T_c^3 + T_d^2 + T_d^4) = a^2R^2 + a^4R^4.$$

(c) $\widetilde{SO}(2)$ (*spiral cells*). Let

$$W_0 = \mathbf{C}, \quad W_1 = \left\{\begin{pmatrix} a & b \\ -b & a \end{pmatrix}\right\}, \quad W_2 = \left\{\begin{pmatrix} a & b \\ b & -a \end{pmatrix}\right\}.$$

These are irreducible under $\widetilde{SO}(2)$; and $(\theta, -\theta) \in \widetilde{SO}(2)$ acts on W_0 as $e^{i\theta}$, on W_1 as the identity, and on W_2 as $e^{2i\theta}$ (see Golubitsky and Stewart [1985, (10.5)]). Hence the W_j are invariant under dh. Let $\Phi_j = dh \mid W_j$ as usual. We compute

$$\Phi_0 = \begin{bmatrix} p & -qa^2 \\ qa^2 & p \end{bmatrix},$$

which has the form $\left(\begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha \end{smallmatrix}\right)$ required to commute with $\widetilde{SO}(2)$. Now Φ_1 has one zero eigenvalue on W_1, so the other is

$$\operatorname{Tr}\Phi_1 = A_a + A_d + B_b - B_c$$

by Golubitsky and Stewart [1985, (10.11)]. We compute this on the orbit $(0, \left(\begin{smallmatrix} a & 0 \\ 0 & a \end{smallmatrix}\right))$. The result is

$$\operatorname{Tr}\Phi_1 = 2a(S_a^1 + S_a^3 + T_a^2 - T_a^4 + S_d^1 + S_d^4 + T_d^2 - T_d^4).$$

In deriving this, note that $S^1 + S^3 + T^3 - T^4 = 0$ by the branching equations. Also, the b- and c-derivatives of the invariants are equal, so the b- and c-derivative terms cancel. The a- and d-derivatives of β, N, δ, γ, σ are equal, whereas $\zeta_a = 2a = -\zeta_d$. Hence the only terms that remain, on dividing out positive factors, are

$$2P_N^1 + R^2 + a^2(P_{\delta^2}^1 + 2R_N^2) + a^4R_{\delta^2}^2.$$

The matrix of Φ_2 can be computed as

$$\begin{bmatrix} A_a - A_d & A_b + A_c \\ B_a - B_d & B_b + B_c \end{bmatrix}$$

and must be of the form $\left(\begin{smallmatrix} \alpha & -\beta \\ \beta & \alpha \end{smallmatrix}\right)$ by Golubitsky and Stewart [1985, §10]. Much as above, we find that

$$A_a - A_d = -2a^2(R^2 + 2a^2R^4),$$
$$A_b + A_c = -2a^2(R^1 + 2a^2R^4),$$

as required for the entry in the table.

(d) $Z_2(\kappa\pi, \pi)$ (*wavy vortices*). We decompose $V = W_0 \oplus W_1$ where

$$W_0 = \langle y, a, b \rangle = +1 \text{ eigenspace},$$
$$W_1 = \langle x, c, d \rangle = -1 \text{ eigenspace},$$

and take a basis in the order

$$y, a, b; x, c, d.$$

Let $\Phi_j = dh \mid W_j$. The null-vectors for the two zero eigenvalues of dh may be found from (5.4) and are

$$\begin{bmatrix} 0 \\ 0 \\ -a \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ -y \\ a \\ 0 \end{bmatrix}$$

with respect to this basis, when evaluated on the orbit. So column b of dh is zero and columns x, c are linearly dependent. Direct calculation yields $C_x = D_x = 0$, whence by linear dependence $C_c = D_c = 0$. So dh is of the form

	y	a	b	x	c	d
y	*	*	0	0	0	0
a	*	*	0	0	0	0
b	*	*	0	0	0	0
x	0	0	0	*	*	*
c	0	0	0	0	0	*
d	0	0	0	0	0	*

The eigenvalues of Φ_2 are therefore

$$0,$$
$$X_x = 2ra^2,$$
$$D_d = a^2R^2 + 2y^2Q^3 + ay^2Q_d^4 + a^4R^4 - a^2y^2M^4,$$

and those of Φ_1 are given by

$$\begin{bmatrix} Y_y & Y_a \\ A_y & A_a \end{bmatrix}.$$

Now

$$\begin{aligned} Y_y &= p + p_y y - ra^2 - r_y y a^2 = p_y y - r_y a^2 y, \\ Y_a &= p_a y - (a^2 r_a + 2ar) y, \\ A_y &= (S_y^1 + S_y^3) a, \\ A_a &= (S_a^1 + S_a^3) a \quad \text{since } S^1 + S^3 = 0. \end{aligned}$$

To evaluate the y- and a-derivatives, note that on the orbit,

$$\begin{aligned} &N_y = 0, \quad \beta_y = 2y, \quad (\delta^2)_y = 0, \quad \gamma_y = -2ya^2, \quad \sigma_y = 0, \\ &N_a = 2a, \quad \beta_a = 0, \quad (\delta^2)_a = 0, \quad \gamma_a = -2ay^2, \quad \sigma_a = 0. \end{aligned}$$

The y-derivatives introduce a factor y, the a-derivatives a factor a. So the matrix is of the form

$$\begin{bmatrix} e_{11} y^2 & e_{12} ay \\ e_{21} ay & e_{22} a^2 \end{bmatrix}$$

for certain functions e_{ij}. We therefore evaluate the e_{ij} to lowest order. The result is

$$\begin{aligned} \begin{bmatrix} e_{11} y^2 & e_{12} ay \\ e_{21} ay & e_{22} a^2 \end{bmatrix} &= \begin{bmatrix} p_y y & p_a y - 2ary \\ (S_y^1 + S_y^3) a & (S_a^1 + S_a^3) a \end{bmatrix} \\ &= \begin{bmatrix} 2p_\beta y^2 & 2(p_N - r) ay \\ 2(P_\beta^1 - Q^3) ay & 2P_N^1 a^2 \end{bmatrix}. \end{aligned}$$

The determinant and trace of Φ_1 therefore have the signs indicated in the table.

(e) $Z_2(\kappa)$ (*twisted vortices*). This is similar to (d). The decomposition into invariant subspaces is now $V = W_0 \oplus W_1$ where

$$\begin{aligned} W_0 &= \langle x, a, b \rangle = +1 \text{ eigenspace for } \kappa, \\ W_1 &= \langle y, c, d \rangle = -1 \text{ eigenspace for } \kappa. \end{aligned}$$

We take a basis for V in the order

$$x, a, b; y, c, d.$$

Then dh has block form, and we let $dh \mid W_j = \Phi_j$.

There are two zero eigenvalues of dh given by (5.4). In the basis above the associated eigenvectors are

$$\begin{bmatrix} x \\ a \\ b \\ y \\ c \\ d \end{bmatrix} \to \begin{bmatrix} 0 \\ b \\ -a \\ 0 \\ d \\ -c \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} -y \\ -c \\ -d \\ x \\ a \\ b \end{bmatrix}.$$

Evaluation on the orbit representative $(x, (\begin{smallmatrix} a & 0 \\ 0 & 0 \end{smallmatrix}))$ yields the eigenvectors

$$\begin{bmatrix} 0 \\ 0 \\ -a \\ 0 \\ 0 \\ 0 \end{bmatrix} \quad \text{and} \quad \begin{bmatrix} 0 \\ 0 \\ 0 \\ x \\ a \\ 0 \end{bmatrix}.$$

Therefore column b of dh is zero and columns y and c are linearly dependent. Putting the zeros in column b, we get

$$d\phi = \begin{array}{c} \\ x \\ a \\ b \\ y \\ c \\ d \end{array} \begin{array}{c} \begin{array}{cccccc} x & a & b & y & c & d \end{array} \\ \left[\begin{array}{ccc|ccc} * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ * & * & 0 & 0 & 0 & 0 \\ \hline 0 & 0 & 0 & & & \\ 0 & 0 & 0 & & \Phi_1 & \\ 0 & 0 & 0 & & & \end{array}\right] \end{array}.$$

Direct calculation, evaluating at $y=b=c=d=0$, yields $C_y=0$, $D_y=0$, whence by linear dependence of columns y and c we also have $C_c=D_c=0$. So

$$\Phi_1 = \left[\begin{array}{cc|c} Y_y & Y_c & Y_d \\ \hline 0 & 0 & C_d \\ 0 & 0 & D_d \end{array}\right].$$

This is triangular, so its eigenvalues are

$$\begin{aligned} &0, \\ &Y_y = p - a^2 r = -2ra^2, \\ &D_d = a^2R^2 - 2x^2Q^3 - ax^2Q_d^4 + a^4R^4 + a^2x^2M^4, \end{aligned}$$

using the branching equations.

Since column b of Φ_0 is zero, the eigenvalues of Φ_0 are 0, together with those of

$$\begin{bmatrix} X_x\, X_a \\ A_x\, A_a \end{bmatrix}.$$

Now

$$\begin{aligned} X_x &= p_x x + r_x x a^2, \\ X_a &= p_a x + \left(a^2 r_a + 2ar\right) x, \\ A_x &= \left(S_x^1 + S_x^3\right) a, \\ A_a &= \left(S_a^1 + S_a^3\right) a. \end{aligned}$$

Again the matrix is of the form

$$\begin{bmatrix} f_{11} y^2 & f_{12} ay \\ f_{21} ay & f_{22} a^2 \end{bmatrix}$$

and we may retain only the lowest order terms in the f_{ij}. The result is

$$\begin{bmatrix} 2p_\beta x^2 & 2(p_N + r)ax \\ 2\left(P^1_\beta - Q^3\right)ax & 2P^1_N a^2 \end{bmatrix}$$

so the determinant and trace have the indicated signs.

Notice the "duality" between wavy and twisted vortices. This completes the verification of Table 5.1.

6. Nondegeneracy conditions. We now proceed to a detailed analysis of the solutions to the branching equations, and the signs of the real parts of the eigenvalues along branches, which determine (orbital asymptotic) stability. The main qualitative features of the bifurcation diagrams, and their associated stabilities, depend upon the signs of a number of coefficients. We therefore impose appropriate *nondegeneracy conditions*: these coefficients should be nonzero.

Recall from §1 that in order to obtain the six-dimensional kernel we had to fix the speed of counterrotation of the outer cylinder at some critical value Ω_0^*. At this speed we found a two-dimensional eigenspace associated with zero eigenvalues, $\mathbf{R}^2$, coalescing with a four-dimensional space associated with a pair of complex conjugate purely imaginary eigenvalues, $\mathbf{R}^2 \otimes \mathbf{C}$. Thus, in order to model the effects of counterrotation in the Taylor experiment, we must introduce a perturbation parameter α which will split apart the bifurcations corresponding to $\mathbf{R}^2$ (vortices) and $\mathbf{R}^2 \otimes \mathbf{C}$ (spiral cells and Z_2^2).

We choose to do this by replacing P^1 in (5.2) by $\alpha + P^1$. If $\alpha < 0$, then the bifurcation to vortices occurs second (in the bifurcation parameter $\lambda = \Omega_i$); if $\alpha > 0$, it occurs first. Thus, we may think of α as $\Omega_0 - \Omega_0^*$. See Fig. 6.1.

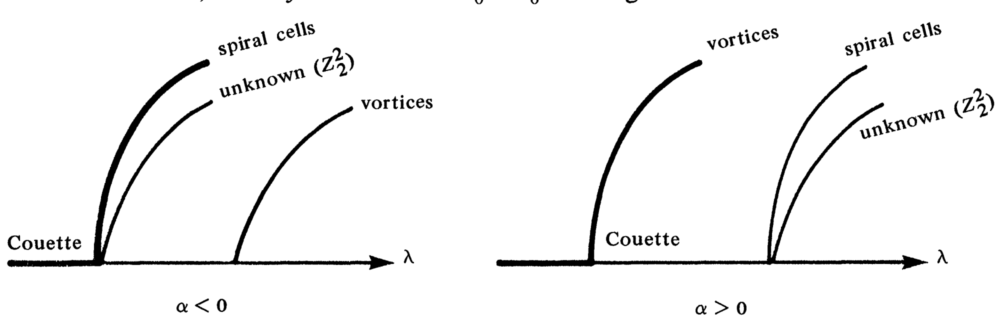


FIG. 6.1. *Schematic rendition of the effect of the perturbation α. The directions of the branches are chosen arbitrarily and secondary branches have been suppressed.*

The nondegeneracy conditions we impose on h are stated when $\alpha = 0$ and when $z = 0, A = 0$.

In Table 6.1 we list sixteen nondegeneracy conditions; a Γ-equivariant bifurcation

TABLE 6.1
Nondegeneracy conditions for h.

(a)	p_β	(h)	R^2
(b)	p_λ	(i)	$2P^1_N + R^2$
(c_1)	$(P^1_\beta + Q^3)p_\lambda - P^1_\lambda p_\beta$	(j)	$2(p_N P^1_\lambda - P^1_{Np\lambda}) - p_\lambda R^2$
(c_2)	$(P^1_\beta - Q^3)p_\lambda - P^1_\lambda p_\beta$	(k)	$P^1_N p_\beta - (P^1_\beta + Q^3)(p_N + r)$
(d)	P^1_N	(l)	$P^1_N p_\beta - (P^1_\beta - Q^3)(p_N - r)$
(e)	P^1_λ	(m)	r
(f)	$P^1_\lambda(p_N + r) - p_\lambda P^1_N$	(n)	Q^3
(g)	$P^1_\lambda(p_N - r) - p_\lambda P^1_N$	(o)	q

problem h is called *nondegenerate* if these sixteen expressions are nonzero for h. In Table 6.2 we list the lower order terms for each solution branch of $h=0$ and each eigenvalue of dh along these branches. For nondegenerate h these lower order terms determine the direction of branching (super or subcritical) and the (orbital) asymptotic stability of each solution. In Table 6.1 we use the convention that all quantities are to be evaluated at the origin. So, for example, P_N^1 means $P_N^1(0,0,0,0,0,0)$. The terms (a)–(n) in Table 6.2 refer to the corresponding expressions in Table 6.1.

TABLE 6.2
Branching equations and eigenvalues to lowest order for nondegenerate problems.

	Branching equations	Sign of real part of eigenvalues	Multiplicity
Couette flow $(O(2)\times S^1)$ $(0,0)$	None	$(b)\lambda$ $\alpha+(e)\lambda$	2 4
Taylor vortices $Z_2(\kappa)\times S^1$ $(x,0)$	$\lambda=-\frac{(a)}{(b)}x^2$	0 (a) $\alpha+\frac{(c_1)}{(b)}x^2$ $\alpha+\frac{(c_2)}{(b)}x^2$	1 1 2 2
unknown Z_2^2 $\left(0,\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}\right)$	$\lambda=\frac{-1}{(e)}[\alpha+(d)a^2]$	0 $-\frac{(b)}{(e)}\alpha+\frac{(f)}{(e)}a^2$ $-\frac{(b)}{(e)}\alpha+\frac{(g)}{(e)}a^2$ (d) (h)	2 1 1 1 1
spiral cells $\widetilde{SO}(2)$ $\left(0,\begin{pmatrix} a & 0\\ 0 & a\end{pmatrix}\right)$	$\lambda=-\frac{1}{(e)}[\alpha+(i)a^2]$	0 $-\frac{(b)}{(e)}\alpha+\frac{(j)}{(e)}a^2$ (i) $-(h)$	1 2 1 2
wavy vortices $Z_2(\kappa\pi,\pi)$ $\left(iy,\begin{pmatrix} a & 0\\ 0 & 0\end{pmatrix}\right)$	$-\frac{(c_2)}{(b)}y^2+\frac{(g)}{(b)}a^2=\alpha$ $\lambda=\frac{(a)}{(c_2)}\alpha+\frac{(l)}{(c_2)}a^2$	0 (m) $2(n)y^2+(h)a^2$ $\det\begin{bmatrix} Y_y & Y_a\\ A_y & A_a\end{bmatrix}=(l)$ $\operatorname{tr}\begin{bmatrix} Y_y & Y_a\\ A_y & A_a\end{bmatrix}=(a)y^2+(d)a^2$	2 1 1 2

TABLE 6.2 (continued)

twisted vortices $Z_2(\kappa)$ $\left(x,\begin{pmatrix}a & 0\\ 0 & 0\end{pmatrix}\right)$	$-\frac{(c_1)}{(b)}x^2+\frac{(f)}{(b)}a^2=\alpha$ $\lambda=\frac{(a)}{(c_1)}\alpha+\frac{(k)}{(c_1)}a^2$	0 $-(m)$ $2(n)x^2+(h)a^2$ $\det\begin{bmatrix}X_x & X_a\\ A_x & A_a\end{bmatrix}=(k)$ $\operatorname{tr}\begin{bmatrix}X_x & X_a\\ A_x & A_a\end{bmatrix}=(a)x^2+(d)a^2$	2 1 1 2
$Z_2(\pi,\pi)$	No solutions by (h)		
1	No solutions by (m), (o) except perhaps on the orbit	$\left(x+iy,\begin{pmatrix}a & 0\\ 0 & d\end{pmatrix}\right)$	

*The terms (a)–(o) are defined in Table 6.1 and required, by assumption of nondegeneracy, to be nonzero.

In our analysis of the bifurcation diagrams and the asymptotic stability of the associated solutions, we use only the equivariant form of the bifurcation equations and the implicit function theorem. Moreover, in each appeal to the implicit function theorem we find a neighborhood of the origin in (z,A,λ,α)-space on which its consequences are valid. Since we use the implicit function theorem only finitely many times, all of our conclusions hold simultaneously in some fixed neighborhood of $(0,0,0,0)$ in (z,A,λ,α)-space. This neighborhood does depend, however, on the particular values that enter into the nondegeneracy conditions.

The computation of the entries in Table 6.2 may be completed in a routine fashion using the entries in Table 5.1. We give the flavor of these computations by presenting the results for wavy vortices.

The branching equations for $Z_2(\kappa\pi,\pi)$ are

$$p(y^2,a,0^2,-a^2y^2,0,\lambda)-a^2r(y^2,a^2,0,-a^2y^2,0,\lambda)=0,$$
$$\alpha+P(y^2,a^2,0,a^2y^2,0,\lambda)+y^2Q^3(y^2,a^2,0,-a^2y^2,0,\lambda)=0.$$

See Table 5.1. (The third branching equation in that table is used only to eliminate τ.) Expanding to lowest order, we have

$$0=p_\beta(0)y^2+(p_N(0)-r(0))a^2+p_\lambda(0)\lambda+\cdots,$$

$$0=\alpha+P_\beta(0)y^2+P_N(0)a^2+P_\lambda(0)\lambda-Q^3(0)y^2+\cdots.$$

Using the implicit function theorem, we can solve for λ and y^2 as a function of a^2 if

$$p_\beta(0)P^1_\lambda(0)-p_\lambda(0)\left(P^1_\beta(0)-Q^3(0)\right)\neq 0.$$

This is condition (c_2) of Table 6.1. We then obtain

$$\lambda=\frac{p_\beta(0)}{(c_2)}\alpha+\frac{P_N^1(0)\,p_\beta(0)-\big(P_\beta^1(0)-Q^3(0)\big)\big(p_N(0)-r(0)\big)}{(c_2)}a^2+\cdots,$$

$$y^2=-\frac{p_\lambda(0)}{(c_2)}\alpha-\frac{p_\lambda(0)\,p_N^1(0)-P_\lambda^1(0)\big(p_N^1(0)-r(0)\big)}{(c_2)}a^2+\cdots.$$

Using the entries in Table 6.1 along with some rearrangement of terms, we obtain the entry in Table 6.2.

7. Comparison with experiment. In this section we discuss how the above model bifurcation problem(s) on the six-dimensional kernel compare with experimental observations in the two main settings.

(1) Experiments by Andereck, Liu, and Swinney [1984] in the counterrotating case, including parameter values near a point at which the six-dimensional kernel appears to occur.
(2) The standard "main sequence" of bifurcations in the case where the outer cylinder is held fixed:

Couette flow $\to$ Taylor vortices $\to$ wavy vortices $\to$ modulated wavy vortices $\to \cdots$

We will show below that it is possible to make choices for the signs of the coefficients that appear in Table 6.1 as nondegeneracy conditions, so that the resulting bifurcation sequences are in qualitative agreement with the experimentally observed bifurcation sequences. In the counterrotating case we have direct (numerical) evidence for the existence of the six-dimensional kernel through the work of DiPrima and Grannick [1971]; no evidence for this six-dimensional kernel currently exists when the outer cylinder is held fixed. We hasten to add, however, that the existence of the six-dimensional kernel is, because of symmetry, only a codimension one phenomenon; it should occur frequently in various forms of Taylor–Couette flow. We also note that, unfortunately, there are many different choices for the signs of the nondegeneracy conditions in Table 6.1 (over 10,000), so many different bifurcation sequences are possible besides the ones we consider here. However, the possibilities are not totally arbitrary, as we see below.

7.1. The counterrotating case. In a private communication, D. Andereck gave us the (qualitative) form of the experimental results for counterrotating Taylor–Couette flow, which have since appeared in Andereck, Liu, and Swinney [1984]. We present these results in Fig. 7.1. There are three features that deserve mention here.

(i) There is a critical speed of counterrotation Ω_0^* at which the primary bifurcation from Couette flow changes from Taylor vortices to spiral cells. This corresponds to the critical speed of counterrotation found numerically by DiPrima and Grannick [1971]. However, they presumably performed the calculations for values of the dimensions of the apparatus which differ from those used by Andereck, Liu and Swinney [1984].

(ii) In the weakly counterrotating case $\Omega_0<\Omega_2^*$—this corresponds to the case

($\alpha > 0$) where our perturbation parameter α is positive—the observed bifurcation sequence is:

Couette → vortices → wavy vortices

See Fig. 7.2.

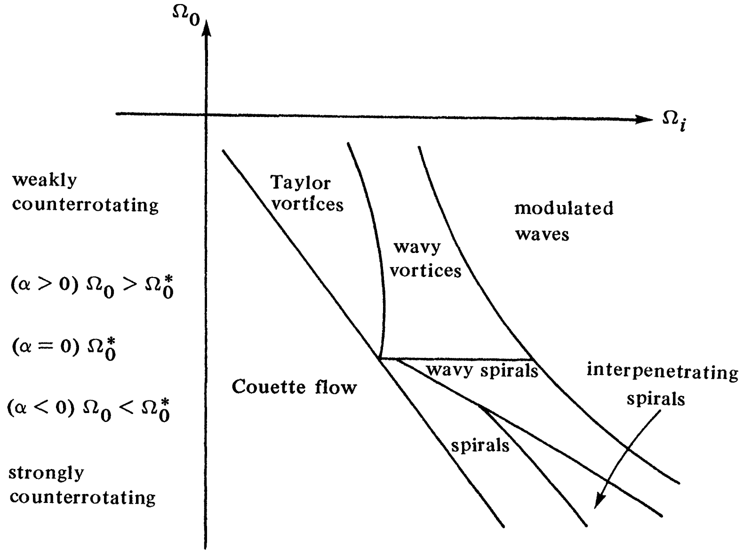


FIG. 7.1. *Qualitative version of the experimental results of Andereck, Liu, and Swinney* [1984] *showing observed transitions between stable states in the counterrotating Taylor–Couette system.*

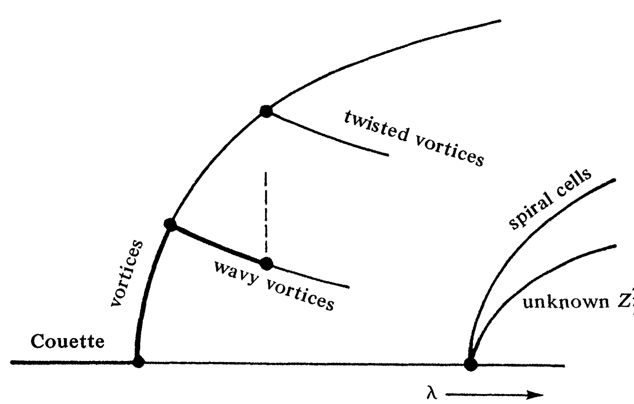


FIG. 7.2. *Schematic bifurcation diagram when $\alpha > 0$ corresponding to the observations of Andereck, Liu, and Swinney* [1984]. (*Secondary branches may or may not join other branches, depending on the values of the coefficients.*)

If the speed of the inner cylinder is increased further, then the wavy vortices lose stability to another state which does not appear to correspond to any state in our model.

(iii) In the strongly counterrotating case where Ω_0 is slightly greater than Ω_0^*—this corresponds to our $\alpha < 0$—the observed transition sequence is:

Couette → spiral cells → wavy spiral cells

See Fig. 7.3.

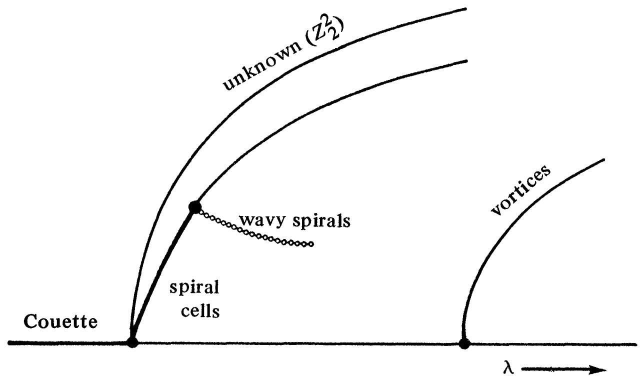


FIG. 7.3. *Schematic bifurcation diagram when* $\alpha<0$ *corresponding to the observations of Andereck* [1984].

In order to reproduce qualitatively the experimental results, we demand the following:

(7.1)
(a) Couette flow is stable for $\lambda \ll 0$.
(b) Vortices bifurcate supercritically and stably when $\alpha>0$.
(c) There is a secondary bifurcation from vortices to wavy vortices when $\alpha>0$.
(d) Wavy vortices are supercritical and stable at the initial bifurcation from vortices.
(e) Spiral cells bifurcate supercritically and stably when $\alpha<0$.
(f) Spiral cells lose stability to a Hopf bifurcation.

We claim that it is possible to choose signs for the nondegeneracy conditions in Table 6.1 so that each of the conditions in (7.1) is satisfied. Moreover, we claim that when these nondegeneracy conditions are satisfied, it follows that:

(7.2) Any bifurcation from vortices to Taylor vortices occurs after the bifurcation from vortices to wavy vortices.

The assumptions in (7.1) correspond, in order, to the following nondegeneracy conditions (cf. Table 6.1):

(7.3)
(a) $(e)<0, (b)<0$.
(b) $(a)>0$.
(c) $(c_2)>0$.
(d) $(l)>0; (m)>0, (n)>0$.
(e) $(i)>0, (h)<0$.
(f) $(j)>0$.

It is a simple matter to check that the conditions (7.3) are precisely the conditions needed to satisfy (7.1). The only point which requires comment is asymptotic stability of the wavy vortex branch. Observe that at the bifurcation from vortices to wavy vortices, $a=0$ and $y\neq 0$. It follows from Table 6.2 that the wavy vortices are stable if

$$(m)>0, \quad (n)>0, \quad (l)>0, \quad \text{and } (a)>0.$$

However, $(l)>0$ when the branch of wavy vortices is supercritical, and $(a)>0$ has already been assumed in (7.3b). Observe that the branch of wavy vortices can lose stability if either

$$(d)<0 \quad \text{or} \quad (h)<0. \tag{7.4}$$

If $(d)<0$ then this branch will lose stability to a torus bifurcation. However,

$$(d)=((i)-(h))/2, \tag{7.5}$$

and the assumption that spiral cells are stable implies that $(i)>0$, $(h)<0$ (cf. (7.3e)) so this possibility cannot occur. Nevertheless, $(h)<0$ is satisfied, and wavy vortices may lose stability by a single real eigenvalue passing through zero. This observation verifies (7.2a). It is possible that a new solution branch with isotropy subgroup 1 will appear at this bifurcation, but we have neither confirmed nor eliminated this possibility. See Table 5.1.

Finally, we verify (7.2b). The wavy vortex branch begins at $\lambda_w(a)/(c_2)$, while a branch of twisted vortices would begin at $\lambda_t=(a)/(c_2)$. We compute $\operatorname{sgn}(\lambda_t-\lambda_w)$. Now

$$\operatorname{sgn}(\lambda_t-\lambda_w)=\operatorname{sgn}\left(\frac{1}{(c_1)}-\frac{1}{(c_2)}\right), \tag{7.6}$$

since $(a)>0$ by (7.3b). However,

$$(c_2)-(c_1)=-2Q^3p_\lambda=-2(n)(b)>0$$

using (7.3a, d). Hence (7.6) implies that $\lambda_t>\lambda_w$ as claimed in (7.2b). Observe that it is possible, under different circumstances, for twisted vortices to bifurcate supercritically and stably from vortices. Such a transition has been observed in the corotating case. See Andereck, Dickman and Swinney [1983].

7.2. The main sequence. Here we verify that the main sequence of bifurcations can also occur in the six-dimensional model. This sequence of bifurcations is observed in experiments when the outer cylinder is held fixed. For this sequence to hold, we need Couette flow to lose stability first to vortices. This happens in our model when $\alpha>0$, and we concentrate on this case.

To obtain the main sequence, we need (7.1a, b, c, d) to hold. Of course, this is possible precisely when the nondegeneracy conditions (7.3a, b, c, d) hold. If we wish to show in this model that, in addition, the wavy vortex solutions lose stability to a torus bifurcation, then two complex conjugate eigenvalues must cross the imaginary axis along the branch of wavy vortices. This can happen only if $(d)<0$. Note that if $(d)<0$ then (7.5) implies that (7.3e) is not valid, and that spiral cells cannot be asymptotically stable.

As we saw above, wavy vortices can lose stability by a real eigenvalue crossing through 0. However, this eventuality cannot occur if $(h)>0$, which is possible, since we have assumed nothing about (i). Thus, assuming

$$(d)<0, \qquad (h)>0$$

leads to the main sequence. (See Fig. 7.4.) Other choices for the main sequence are possible.

We conclude that both the main sequence and certain regimes in the experiments of Andereck, Liu, and Swinney [1984] appear to be qualitatively consistent with our six-dimensional model, for suitable values of the coefficients. (Note that aside from the states discussed above, no other stable states occur except perhaps with isotropy group

1, as mentioned.) Since the coefficients can in principle be computed by Lyapunov–Schmidt reduction from the Navier–Stokes equations, further numerical work should be able to provide a more stringent test. It would also be of interest to determine, in terms of the physical parameters in the problem, the location of the codimension one set of values at which the six-dimensional kernel occurs.

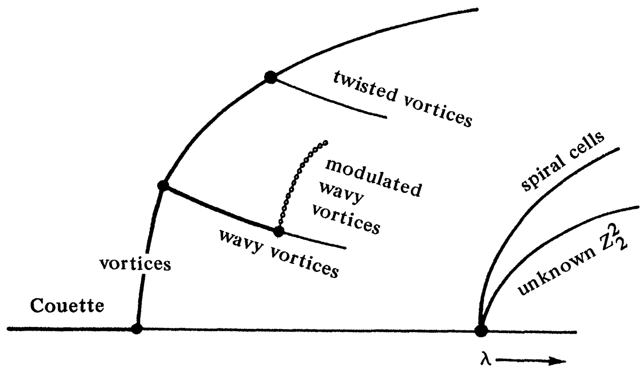


FIG. 7.4. *Schematic bifurcation diagram corresponding to (one occurrence of) the main sequence.*

Appendix. Equivariant mappings on the six-dimensional kernel. Let $V=\mathbf{R}^2\oplus(\mathbf{R}^2\otimes\mathbf{C})$ be the six-dimensional kernel described in §4. Recall that $\Gamma=O(2)\times S^1$ acts on V by

$$(\theta,\psi)(v,w\otimes z)=\left(R_\theta v,(R_\theta w)\otimes(e^{i\psi}Z)\right).$$

For computational purposes we choose coordinates by identifying the first $\mathbf{R}^2$ with $\mathbf{C}$, and $\mathbf{R}^2\otimes\mathbf{C}$ with 2×2 matrices as described in §4, so that an element of V is written (z,a) where

$$z\in\mathbf{C},\quad A=\begin{pmatrix} a & b\\ c & d\end{pmatrix},\quad a,b,c,d\in\mathbf{R}.$$

Recall that a (smooth) function ϕ: $V\to\mathbf{R}$ is *invariant* under Γ if

$$\phi(\gamma v)=\phi(v),\qquad \gamma\in\Gamma,\quad v\in V,$$

and a (smooth) mapping Φ: $V\to V$ is *equivariant* if it commutes with Γ, that is

$$\Phi(\gamma v)=\gamma\Phi(v),\qquad \gamma\in\Gamma,\quad v\in V.$$

The aim of this appendix is to describe completely these invariant functions and equivariant mappings, as promised in §4 above. The main result, which will yield Theorem 4.3 when appropriate terms are collected together, is:

PROPOSITION A.1. (a) *Every invariant function on V is of the form*

$$\phi(v)=h\left(\beta,N,\delta^2,\gamma,\sigma\right)\tag{A.1}$$

for a smooth function h: $\mathbf{R}^5\to\mathbf{R}$, *where*

$$\begin{aligned}\beta&=z\bar z=x^2+y^2,\\ N&=a^2+b^2+c^2+d^2,\\ \delta^2&=(ad-bc)^2,\\ \gamma&=\mathrm{Re}(z^2\bar\zeta),\\ \sigma&=\delta\,\mathrm{Im}(z^2\bar\zeta)\end{aligned}\tag{A.2}$$

and

$$\zeta=(a^2+b^2-c^2-d^2)+2i(ac+bd). \tag{A.3}$$

(b) *Every equivariant mapping* $V\to V$ *is of the form*

$$\begin{aligned}\Phi(z,A)=\big(&pz+qi\delta z+r\bar z\zeta+si\delta\bar z\zeta,\\ &P^1\operatorname{Re}(H_1)+P^2\operatorname{Re}(H_2)\\ &\quad+Q^1\operatorname{Im}(z^2\bar\zeta)\operatorname{Im}(H_1)+Q^2\operatorname{Im}(z^2\bar\zeta)\operatorname{Im}(H_2)\\ &\quad+Q^3\operatorname{Re}(\bar z^2H_3)+Q^4\operatorname{Re}(\bar z^2H_4)\\ &\quad+R^1\delta\operatorname{Im}(H_1)+R^2\delta\operatorname{Im}(H_2)+R^3\delta\operatorname{Im}(\bar\zeta H_3)+R^4\delta\operatorname{Im}(\bar\zeta H_4)\\ &\quad+M^3\delta\operatorname{Im}(\bar z^2H_3)+M^4\delta\operatorname{Im}(\bar z^2H_4)\big),\end{aligned} \tag{A.4}$$

where

$$H_1=\begin{pmatrix}a-ic & b-id\\ c+ia & d+ib\end{pmatrix},\qquad H_3=\begin{pmatrix}a+ic & b+id\\ -c+ia & -d+ib\end{pmatrix},$$
$$H_2=\begin{pmatrix}-b+id & a-ic\\ -d-ib & c+ia\end{pmatrix},\qquad H_4=\begin{pmatrix}-b-id & a+ic\\ d-ib & -c+ia\end{pmatrix},$$

and

$$p,q,r,s,P^1,P^2,Q^1,Q^2,Q^3,Q^4,R^1,R^2,R^3,R^4,M^3,M^4$$

are invariant functions.

In more abstract language, Proposition A.1 says that the ring of invariant functions is *generated* by $\beta,N,\delta^2,\gamma,\sigma$; and that the module of equivariant mappings is generated over the invariants by the twelve mappings in (A.4), whose coefficients are $p,q,r,s,P^1,\cdots,M^4$. By standard results of Schwarz [1975] and Poénaru [1976] we may assume ϕ and Φ are polynomials when proving Proposition A.1.

The computation comes in two stages. First we compute the (polynomial) S^1-invariants and -equivariants; then we use this information and the $O(2)$-action to obtain the $O(2)\times S^1$-invariants and -equivariants. Since S^1 acts trivially on $z\in\mathbf{R}^2$, we need consider only the action on $A\in\mathbf{R}^2\otimes\mathbf{C}$. We take complex coordinates

$$z_1=a+ib,\qquad z_2=c+id.$$

Then we may identify $\mathbf{R}^2\otimes\mathbf{C}$ with $\mathbf{C}\oplus\mathbf{C}$, where S^1 acts diagonally:

$$\theta(z_1,z_2)=(e^{i\theta}z_1,e^{i\theta}z_2).$$

LEMMA A.2. *The real S^1-invariants on $\mathbf{C}\oplus\mathbf{C}$ are generated by* $z_1\bar z_1$, $z_2\bar z_2$, $\operatorname{Re} z_1\bar z_2$, $\operatorname{Im} z_1\bar z_2$. *The S^1-equivariants are generated over the invariants by* $(z_1,0)$, $(0,z_1)$, $(z_2,0)$, $(0,z_2)$, $(i\bar z_1,0)$, $(0,i\bar z_1)$, $(i\bar z_2,0)$, $(0,i\bar z_2)$.

Proof. These results (which generalize easily to S^1 acting on $\mathbf{C}^n$) are no doubt well-known, but for completeness we sketch a proof. The idea is first to find the complex invariants and equivariants and then to read off the real ones.

Consider a $\mathbf{C}$-valued polynomial function

$$p(z_1,\bar z_1,z_2,\bar z_2)=\sum A_{\alpha\beta\gamma\delta}z_1^\alpha\bar z_1^\beta z_2^\gamma\bar z_2^\delta.$$

Since $\overline{e^{i\theta}z}=e^{-i\theta}\bar{z}$, we can use S^1-invariance to exclude all terms other than those for which

$$\alpha-\beta+\gamma-\delta=0.$$

So p is a polynomial in $z_1\bar{z}_1$, $z_2\bar{z}_2$, $\bar{z}_1z_2$, and $z_1\bar{z}_2$. If p is to be real in a,b,c,d, then we have $p=\bar{p}$, so $A_{\alpha\beta\gamma\delta}=\bar{A}_{\beta\alpha\delta\gamma}$. This leads to the real invariant generators stated.

For the equivariants, we consider a pair of functions p_1,p_2 of the above form. Equivariance excludes all terms other than those for which

$$\alpha-\beta+\gamma-\delta=1.$$

This yields equivariant generators which are *complex* scalar multiples of $(z_1,0)$, $(z_2,0)$, $(0,z_1)$, $(0,z_2)$. Taking real and imaginary parts, we obtain the stated real equivariant generators.

In $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$ coordinates, we have the invariant generators

$$\begin{aligned} z_1\bar{z}_1&=a^2+b^2,\\ z_2\bar{z}_2&=c^2+d^2,\\ \operatorname{Re}(z_1\bar{z}_2)&=ac+bd,\\ \operatorname{Im}(z_1\bar{z}_2)&=bc-ad=-\delta, \end{aligned} \tag{A.5}$$

and the equivariant generators $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)\to$

$$\text{(A.6)}\quad E_1=\begin{pmatrix} a & b \\ 0 & 0 \end{pmatrix},\quad E_2=\begin{pmatrix} 0 & 0 \\ a & b \end{pmatrix},\quad E_3=\begin{pmatrix} 0 & 0 \\ c & d \end{pmatrix},\quad E_4=\begin{pmatrix} c & d \\ 0 & 0 \end{pmatrix},$$

$$E_5=\begin{pmatrix} -b & a \\ 0 & 0 \end{pmatrix},\quad E_6=\begin{pmatrix} 0 & 0 \\ -b & a \end{pmatrix}\quad E_7=\begin{pmatrix} 0 & 0 \\ -d & c \end{pmatrix},\quad E_8=\begin{pmatrix} -d & c \\ 0 & 0 \end{pmatrix}.$$

Note that there is a relation

$$(a^2+b^2)(c^2+d^2)=(ac+bd)^2+(ad-bc)^2.$$

We are now ready for the:

Proof of Proposition A.1 (a). The calculations are easier if we use complex notation. Let

$$\zeta=(a^2+b^2-c^2-d^2)+2i(ac+bd).$$

Then every real-valued function of the four invariant generators can be written in terms of $N=a^2+b^2+c^2+d^2$, ζ, $\bar{\zeta}$, and δ. Note that N and δ^2 are $O(2)$-invariant. See Golubitsky and Stewart [1985, §9].

Since S^1 acts trivially on $z\in\mathbf{R}^2$, we can write the S^1-invariants on $\mathbf{R}^2\oplus(\mathbf{R}^2\otimes\mathbf{C})$ in the form

$$\phi(z,\bar{z},N,\zeta,\bar{\zeta},\delta).$$

Under the $O(2)$-action these transform as follows:

	z	$\bar{z}$	ζ	$\bar{\zeta}$	δ	N
κ	$\bar{z}$	z	$\bar{\zeta}$	ζ	$-\delta$	N
θ	$e^{i\theta}z$	$e^{-i\theta}\bar{z}$	$e^{2i\theta}\zeta$	$e^{-2i\theta}\bar{\zeta}$	δ	N

(The expressions ζ and $\bar{\zeta}$ are introduced because of this pleasant transformation behavior.)

Since $z\bar{z}$, $\zeta\bar{\zeta}$, N, and δ^2 are $O(2)$-invariant, we can write the general $O(2)\times S^1$-invariant in the form

$$\text{(A.7)}\qquad \begin{aligned}\phi &= az^\alpha\zeta^\beta + bz^\alpha\bar{\zeta}^\beta + c\bar{z}^\alpha\zeta^\beta + d\bar{z}^\alpha\bar{\zeta}^\beta + \delta\left(ez^\alpha\zeta^\beta + fz^\alpha\bar{\zeta}^\beta + g\bar{z}^\alpha\zeta^\beta + h\bar{z}^\alpha\bar{\zeta}^\beta\right)\\ &\equiv \phi_0 + \delta\phi_1\end{aligned}$$

where $a, b, \cdots, h \in \mathbf{C}[z\bar{z}, N, \delta^2]$. (Note: $\zeta\bar{\zeta} = N - 4\delta^2$ so no $\zeta\bar{\zeta}$ terms are required.) Reality of ϕ implies that

$$\bar{a} = d, \quad \bar{b} = c, \quad \bar{e} = h, \quad \bar{f} = g,$$

while κ-invariance leads to

$$a = b, \quad c = d, \quad e = -h, \quad f = -g.$$

Hence a, b, c, d are real and e, f, g, h are purely imaginary.

Finally we apply $SO(2)$-invariance. Since δ is $SO(2)$-invariant and is independent of $z, \bar{z}, \zeta, \bar{\zeta}$, we must have ϕ_0 and ϕ_1 separately $SO(2)$-invariant. This excludes all terms other than

$$\text{(A.8)}\qquad a\left(z^\alpha\zeta^\beta + \bar{z}^\alpha\bar{\zeta}^\beta\right) \quad \text{when } \alpha + 2\beta = 0,$$

$$\text{(A.9)}\qquad b\left(z^\alpha\bar{\zeta}^\beta + \bar{z}^\alpha\zeta^\beta\right) \quad \text{when } \alpha - 2\beta = 0,$$

$$\text{(A.10)}\qquad \delta e\left(z^\alpha\zeta^\beta - \bar{z}^\alpha\bar{\zeta}^\beta\right) \quad \text{when } \alpha + 2\beta = 0,$$

$$\text{(A.11)}\qquad \delta f\left(z^\alpha\bar{\zeta}^\beta - \bar{z}^\alpha\zeta^\beta\right) \quad \text{when } \alpha - 2\beta = 0,$$

Now (A.8) and (A.10) imply $\alpha = \beta = 0$, giving nothing new. The others yield $\alpha = 2\beta$.

We claim that only $\alpha = 2$, $\beta = 1$ yield new generators. For example

$$\left(z^{\alpha+2}\bar{\zeta}^{\beta+1} + \bar{z}^{\alpha+2}\zeta^{\beta+1}\right) = \left(z^\alpha\bar{\zeta}^\beta + \bar{z}^\alpha\zeta^\beta\right)\left(z^2\bar{\zeta} + \bar{z}^2\zeta\right) - (z\bar{z})^2(\zeta\bar{\zeta})\left(z^{\alpha-2}\bar{\zeta}^{\beta-1} + \bar{z}^{\alpha-2}\zeta^{\beta-1}\right).$$

Since b is real and f purely imaginary, we obtain generators

$$\operatorname{Re}\left(\bar{z}^2\zeta\right),\ i\delta\operatorname{Im}\left(z^2\bar{\zeta}\right)$$

in addition to $z\bar{z}$, N, δ^2. This proves part (*a*) of Proposition A.1.

Proof of Proposition A.1 (b). Write the general equivariant in the form

$$\Phi(z, A) = \left(\Phi_0(z, A), \Phi_1(z, A)\right)$$

where

$$\begin{aligned}\Phi_0 &: \mathbf{R}^2 \oplus (\mathbf{R}^2 \otimes \mathbf{C}) \to \mathbf{R}^2,\\ \Phi_1 &: \mathbf{R}^2 \oplus (\mathbf{R}^2 \otimes \mathbf{C}) \to \mathbf{R}^2 \otimes \mathbf{C}.\end{aligned}$$

We begin with Φ_0. Since the S^1-action on $\mathbf{R}^2$ is *trivial*, the S^1-equivariance condition implies that Φ_0 is S^1-invariant and hence can be written in the form (A.7) above.

However, this time there is no reality condition since we seek mappings into $\mathbf{R}^2$, not $\mathbf{R}$. The κ-equivariance again implies a, b, c, d are real, and e, f, g, h are purely imaginary. Replacing the latter by ie, if, ig, ih, we may assume all coefficients $a - h$ are real, and replace δ by $i\delta$. Write $\Phi_0 = \phi_0 + i\delta\phi_1$: again we can treat ϕ_0 and ϕ_1 separately.

Now $SO(2)$-equivariance excludes all terms other than

$$\text{(A.12)}\qquad z^\alpha\zeta^\beta,\ i\delta z^\alpha\zeta^\beta; \quad \alpha + 2\beta = 1,$$

$$\text{(A.13)}\qquad z^\alpha\bar{\zeta}^\beta,\ i\delta z^\alpha\bar{\zeta}^\beta; \quad \alpha - 2\beta = 1,$$

(A.14) $$\bar{z}^\alpha\zeta^\beta,\ i\delta\bar{\zeta}^\alpha\zeta^\beta; \quad -\alpha+2\beta=1,$$

(A.15) $$\bar{z}^\alpha\bar{\zeta}^\beta,\ i\delta\bar{z}^\alpha\bar{\zeta}^\beta; \quad -\alpha-2\beta=1.$$

In (A.12) we have $\alpha=1$, $\beta=0$, yielding z and $i\delta z$. In (A.13) we have $\alpha=2\beta+1$. As before, we may use the invariance of $z\bar{z}$ and $\zeta\bar{\zeta}$ to reduce the size of α and β:

$$\begin{aligned} z^{2\beta+1}\bar{\zeta}^\beta &= z^{2\beta-1}(z^2\bar{\zeta})\bar{\zeta}^{\beta-1} \\ &= (z^2\bar{\zeta}+\bar{z}^2\zeta)z^{2\beta-1}\bar{\zeta}^{\beta-1}-(z\bar{z})^2(\zeta\bar{\zeta})z^{2\beta-3}\bar{\zeta}^{\beta-2}. \end{aligned}$$

Thus we can reduce β by 1 and α by 2. The process stops when $\beta=1$, $\alpha=2$. But now

$$z^3\bar{\zeta}=zz^2\bar{\zeta}=(z^2\bar{\zeta}+\bar{z}^2\zeta)z-(z\bar{z})\bar{z}\zeta.$$

Thus we get new generators $\bar{z}\zeta$, $i\delta\bar{z}\zeta$. In (A.14) we can similarly assume $\beta\leqq 2$. But $\beta=2$ gives

$$\bar{z}^3\zeta^2=(\bar{z}^2\zeta+z^2\bar{\zeta})\bar{z}\zeta-(z\bar{z})(\zeta\bar{\zeta})z$$

so no new generator arises; and $\beta=1$ gives $\bar{z}\zeta$ which is already included. Finally (A.15) is not possible.

Thus we have found four generators $(z,0)(i\delta z,0)$, $(\bar{z}\zeta,0)$, $(i\delta\bar{z}\zeta,0)$ corresponding to mappings of $\mathbf{R}^2\oplus(\mathbf{R}^2\otimes\mathbf{C})$ into $\mathbf{R}^2$.

Now we look at

$$\Phi_1\colon \mathbf{R}^2\oplus(\mathbf{R}^2\otimes\mathbf{C})\to\mathbf{R}^2\otimes\mathbf{C}.$$

Again complex notation is more convenient. Define (in a notation consistent with the statement of Proposition A.1) the complex matrices

(A.16) $$\begin{aligned} H_1&=(E_1+E_3)+i(E_2-E_4),\\ H_2&=(E_5+E_7)+i(E_6-E_8),\\ H_3&=(E_1-E_3)+i(E_2+E_4),\\ H_4&=(E_5-E_7)+i(E_6+E_8). \end{aligned}$$

Then the S^1-equivariants on $\mathbf{R}^2\otimes\mathbf{C}$ are generated over $\mathbf{C}$ by H_k, $\bar{H}_k$ $(k=1,\cdots,4)$ and over $\mathbf{R}$ by the real and imaginary parts of H_k $(k=1,\cdots,4)$. Since S^1 acts trivially on $z\in\mathbf{R}^2$ we can think of the S^1-equivariants mapping $\mathbf{R}^2\oplus(\mathbf{R}^2\otimes\mathbf{C})\to\mathbf{R}^2$ as S^1-equivariants mapping $\mathbf{R}^2\otimes\mathbf{C}\to\mathbf{R}^2$ parametrized by z and $\bar{z}$. Thus they are linear combinations of H_k, $\bar{H}_k$, $(k=1,\cdots,4)$ with coefficients in $\mathbf{C}[N,\delta,\zeta,\bar{\zeta};z,\bar{z}]$.

We write the equivariance condition $\phi(\gamma v)=\gamma\Phi(v)$ in the form

(A.17) $$\Phi(v)=\gamma^{-1}\Phi(\gamma v).$$

Suppose

$$\Phi(v)=\rho(v)H(v)$$

where $\rho\colon \mathbf{R}^2\oplus(\mathbf{R}^2\otimes\mathbf{C})\to\mathbf{R}$, $H\colon \mathbf{R}^2\oplus(\mathbf{R}^2\otimes\mathbf{C})\to\mathbf{R}^2\otimes\mathbf{C}$. Then (A.17) is equivalent to

(A.18) $$\rho(v)H(v)=\gamma^{-1}\rho(\gamma v)H(\gamma v)=\rho(\gamma v)\gamma^{-1}H(\gamma v).$$

We compute this action on H_k $(k=1,\cdots,4)$ when $\gamma \in O(2)$. Using (A.16) and noting that

$$\kappa(z,A)=\left(\bar{z},\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix}A\right),$$
$$\psi(z,A)=\left(e^{i\psi}z,R_\psi A\right)$$

where

$$R_\psi=\begin{pmatrix}\cos\psi & -\sin\psi\\ \sin\psi & \cos\psi\end{pmatrix},$$

we find

(A.19)

$H(v)$	$\kappa^{-1}H(\kappa v)$	$\psi^{-1}H(\psi v)$
H_1	$\bar{H}_1$	H_1
H_2	$\bar{H}_2$	H_2
H_3	$\bar{H}_3$	$e^{2i\psi}H_3$
H_4	$\bar{H}_4$	$e^{2i\psi}H_4$

We can write the general S^1-equivariant Φ_1: $\mathbf{R}^2\oplus(\mathbf{R}\otimes\mathbf{C})\to\mathbf{R}^2\otimes\mathbf{C}$ in the form

$$\Phi_1=\operatorname{Re}\left\{\sum_{k=1}^{4}(\rho_k+\delta\sigma_k)H_k\right\} \tag{A.20}$$

where the ρ_k, σ_k are polynomials over $\mathbf{C}$ of the form

$$\rho_k=\rho_k\left(N,\delta^2,\zeta,\bar{\zeta};z,\bar{z}\right),\qquad \sigma_k=\sigma_k\left(N,\delta^2,\zeta,\bar{\zeta};z,\bar{z}\right).$$

Since $z\bar{z}$ and $\zeta\bar{\zeta}$ are $O(2)$-invariant, we can write the ρ_k and σ_k as

$$az^\alpha\zeta^\beta+bz^\alpha\bar{\zeta}^\beta+c\bar{z}^\alpha\zeta^\beta+d\bar{z}^\alpha\bar{\zeta}^\beta \tag{A.21}$$

with $a,b,c,d,\in\mathbf{C}[N,\delta^2]$.

We now apply κ- and ψ-equivariance in the form (A.17), writing

$$\tilde{\rho}=\rho(\kappa z,\kappa A),\qquad \hat{\rho}=\rho(\psi z,\psi A).$$

Now κ-equivariance (using (A.18) and (A.19)) implies that

$$\tilde{\rho}_k=\bar{\rho}_k, \tag{A.22}$$

$$\tilde{\sigma}_k=-\bar{\sigma}_k \tag{A.23}$$

and ψ-equivariance implies

(A.24), (A.25)
$$\hat{\rho}_k=\begin{cases}\rho_k & (k=1,2),\\ e^{-2i\psi}\rho_k & (k=3,4),\end{cases}$$

(A.26), (A.27)
$$\hat{\sigma}_k=\begin{cases}\sigma_k & (k=1,2),\\ e^{-2i\psi}\sigma_k & (k=3,4).\end{cases}$$

Now write the ρ_k and σ_k in the form (A.20) (we suppress unnecessary fine points of notation in the interests of clarity). From (A.22) we get

$$\text{(A.28)} \qquad (\text{for } \rho_k)\ a=\bar{a}, \quad b=\bar{b}, \quad c=\bar{c}, \quad d=\bar{d},$$

and from (A.23)

$$\text{(A.29)} \qquad (\text{for } \sigma_k)\ a=-\bar{a}, \quad b=-\bar{b}, \quad c=-\bar{c}, \quad d=-\bar{d}.$$

That is, the coefficients are real for ρ_k and purely imaginary for σ_k. We therefore replace σ_k by $i\sigma_k$, so that ρ_k and σ_k are real: now (A.20) takes the form

$$\text{(A.30)} \qquad \Phi=\operatorname{Re}\left\{\sum_{k=1}^{4}(\rho_k+i\delta\sigma_k)H_k\right\}.$$

The ψ-action multiplies z, ζ, and H_k by complex constants $e^{i\psi}$, $e^{2i\psi}$, $e^{2i\psi}$ respectively. Hence we may consider each of the eight terms in (A.30) separately. From (A.26) and (A.27) we obtain the following conditions on the exponents α, β, required for ψ-equivariance:

	Real part of:	$k=1,2$	$k=3,4$
(A.31)	$z^\alpha\zeta^\beta H_k$	$\alpha+2\beta=0$	$\alpha+2\beta=-2$
(A.32)	$z^\alpha\bar{\zeta}^\beta H_k$	$\alpha-2\beta=0$	$\alpha-2\beta=-2$
(A.33)	$\bar{z}^\alpha\zeta^\beta H_k$	$-\alpha+2\beta=0$	$-\alpha+2\beta=-2$
(A.34)	$\bar{z}^\alpha\bar{\zeta}^\beta H_k$	$-\alpha-2\beta=0$	$-\alpha-2\beta=-2$
(A.35)	$i\delta z^\alpha\zeta^\beta H_k$	$\alpha+2\beta=0$	$\alpha+2\beta=-2$
(A.36)	$i\delta z^\alpha\bar{\zeta}^\beta H_k$	$\alpha-2\beta=0$	$\alpha-2\beta=-2$
(A.37)	$i\delta\bar{z}^\alpha\zeta^\beta H_k$	$-\alpha+2\beta=0$	$-\alpha+2\beta=-2$
(A.38)	$i\delta\bar{z}^\alpha\bar{\zeta}^\beta H_k$	$-\alpha-2\beta=0$	$-\alpha-2\beta=-2$

We deal with these terms case by case, first for $k=1,2$; then for $k=3,4$. So let $k=1,2$.

(A.31) implies $\alpha=\beta=0$, leading to the generators

$$\text{(A.39)} \qquad \operatorname{Re}(H_k), \qquad k=1,2.$$

(A.32) requires $\alpha=2\beta$, so we get

$$\text{(A.40)} \qquad z^{2\beta}\bar{\zeta}^\beta H_k+\bar{z}^{2\beta}\zeta^\beta\bar{H}_k.$$

Similarly (A.33) requires $\alpha=2\beta$, and the result is

$$\text{(A.41)} \qquad \bar{z}^{2\beta}\zeta^\beta H_k+z^{2\beta}\bar{\zeta}^\beta\bar{H}_k.$$

Forming the sum and difference of (A.40) and (A.41) we may replace them by

$$x_\beta=(z^{2\beta}\bar{\zeta}^\beta+\bar{z}^{2\beta}\zeta^\beta)(H_k+\bar{H}_k),$$

$$y_\beta=(z^{2\beta}\bar{\zeta}^\beta-\bar{z}^{2\beta}\zeta^\beta)(H_k-\bar{H}_k).$$

We observe the following identities:

$$x_{\beta+1}=2\operatorname{Re}(z^2\bar{\zeta})x_\beta-(z^2\bar{z}^2\zeta\bar{\zeta})x_{\beta-1},$$

$$y_{\beta+1}=2\operatorname{Re}(z^2\bar{\zeta})y_\beta-(z^2\bar{z}^2\zeta\bar{\zeta})y_{\beta-1},$$

which have invariant functions as coefficients. Since

$$x_0=2\operatorname{Re}(H_k),\quad y_0=0,\quad x_1=2\operatorname{Re}(z^2\bar{\zeta})\operatorname{Re}(H_k),$$

and these may be obtained from the generators (A.39), an inductive argument shows that only y_1 need be retained in a list of generators. So we obtain the new generators

$$\text{(A.42)}\qquad \operatorname{Im}(z^2\bar{\zeta})\operatorname{Im}(H_k),\qquad k=1,2.$$

For (A.34) we have $\alpha=\beta=0$ and nothing new results.

For (A.35) we have $\alpha=\beta=0$, leading to new generators $\operatorname{Re}(i\delta H_k)$, or equivalently

$$\text{(A.43)}\qquad \delta\operatorname{Im}(H_k),\quad k=1,2.$$

From (A.36) and (A.37) we get $\alpha=2\beta$, yielding

$$i\delta z^{2\beta}\bar{\zeta}^{\beta}H_k-i\delta\bar{z}^{2\beta}\zeta^{\beta}\overline{H}_k,$$
$$i\delta\bar{z}^{2\beta}\zeta^{\beta}H_k-i\delta z^{2\beta}\bar{\zeta}^{\beta}\overline{H}_k.$$

Forming the sum and difference, we replace these by

$$v_\beta=i\delta(z^{2\beta}\bar{\zeta}^{\beta}+\bar{z}^{2\beta}\zeta^{\beta})(H_k-\overline{H}_k),$$
$$w_\beta=i\delta(z^{2\beta}\bar{\zeta}^{\beta}-\bar{z}^{2\beta}\zeta^{\beta})(H_k+\overline{H}_k).$$

We note the identities

$$v_{\beta+1}=2\operatorname{Re}(z^2\bar{\zeta})v_\beta-(z^2\bar{z}^2\zeta\bar{\zeta})v_{\beta-1},$$
$$w_{\beta+1}=2\operatorname{Re}(z^2\bar{\zeta})w_\beta-(z^2\bar{z}^2\zeta\bar{\zeta})w_{\beta-1},$$
$$v_1=\operatorname{Re}(z^2\bar{\zeta})\delta\operatorname{Im}(H_k),$$
$$w_1=\delta\operatorname{Im}(z^2\bar{\zeta})\operatorname{Re}(H_k).$$

It follows by induction that no new generators arise here.

Finally (A.38) leads to $\alpha=\beta=0$, and no new generators. This completes the analysis for $k=1,2$.

Next, we let $k=3,4$. The calculations follow a similar pattern.

(A.31) is impossible.

(A.32) and (A.33) lead to

$$t_\beta=z^{2\beta-2}\bar{\zeta}^{\beta}H_k+\bar{z}^{2\beta-2}\zeta^{\beta}\overline{H}_k,\qquad \beta\geqq 1,$$
$$u_\beta=\bar{z}^{2\beta+2}\zeta^{\beta}H_k+z^{2\beta+2}\bar{\zeta}^{\beta}H_k,\qquad \beta\geqq 0.$$

Now

$$t_{\beta+1}=2\operatorname{Re}(z^2\bar{\zeta})t_\beta-(z^2\bar{z}^2\zeta\bar{\zeta})t_{\beta-1},\qquad(\beta\geqq 2),$$
$$t_2=2\operatorname{Re}(z^2\bar{\zeta})t_1-(\zeta\bar{\zeta})u_0,$$
$$u_{\beta+1}=2\operatorname{Re}(z^2\bar{\zeta})u_\beta-(z^2\bar{z}^2\zeta\bar{\zeta})u_{\beta-1},\qquad(\beta\geqq 1),$$
$$u_1=2\operatorname{Re}(z^2\bar{\zeta})u_0-(z^2\bar{z}^2)x_1.$$

Hence inductively the only new generators are t_1 and u_0; that is,

$$(\mathrm{A}.44)\qquad \begin{aligned} &\bar{\zeta}H_k+\zeta\overline{H}_k, && k=3,4,\\ &\bar{z}^2H_k+z^2\overline{H}_k, && k=3,4. \end{aligned}$$

However, we observe that the identities

$$\begin{aligned} N\,\mathrm{Re}(H_1)-2\delta\,\mathrm{Im}(H_2)&=\mathrm{Re}(\bar{z}H_3),\\ 2\delta\,\mathrm{Im}(H_1)+N\,\mathrm{Re}(H_2)&=\mathrm{Re}(\bar{z}H_4) \end{aligned}$$

are valid. Thus the generators $\bar{z}^2H_k+z^2\overline{H}_k$ $(k=3,4)$ are redundant and can be omitted.

From (A.34) we have either $\alpha=0$, $\beta=1$ or $\alpha=2$, $\beta=0$. These lead to t_1 and u_0 again.

For convenience we now consider (A.38), for which $\alpha=2$, $\beta=0$ or $\alpha=0$, $\beta=1$. These lead to new generators

$$(\mathrm{A}.45)\qquad \begin{aligned} &\delta\,\mathrm{Im}(\bar{\zeta}H_k), && k=3,4,\\ &\delta\,\mathrm{Im}(\bar{z}^2H_k), && k=3,4. \end{aligned}$$

Finally we take (A.36) and (A.37), yielding $\alpha=2\beta-2$ $(\beta\geqq 1)$ and $\alpha=2\beta+2$ respectively. So we have terms

$$\begin{aligned} r_\beta&=i\delta z^{2\beta-2}\bar{\zeta}^\beta H_k-i\delta\bar{z}^{2\beta-2}\zeta^\beta\overline{H}_k \qquad (\beta\geqq 1),\\ s_\beta&=i\delta\bar{z}^{2\beta-2}\zeta^\beta H_k-i\delta z^{2\beta-2}\bar{\zeta}^\beta\overline{H}_k \qquad (\beta\geqq 0). \end{aligned}$$

As usual, we find that

$$\begin{aligned} r_{\beta+1}&=2\,\mathrm{Re}(z^2\bar{\zeta})r_\beta-(z^2\bar{z}^2\zeta\bar{\zeta})r_{\beta-1} \qquad (\beta\geqq 2),\\ r_2&=2\,\mathrm{Re}(z^2\bar{\zeta})\delta\,\mathrm{Im}(\bar{\zeta}H_k)-(\zeta\bar{\zeta})\delta\,\mathrm{Im}(\bar{z}^2H_k),\\ s_{\beta+1}&=2\,\mathrm{Re}(z^2\bar{\zeta})s_\beta-(z^2\bar{z}^2\zeta\bar{\zeta})s_{\beta-1},\\ s_1&=2\,\mathrm{Re}(z^2\bar{\zeta})s_0-(z^2\bar{z}^2)\delta\,\mathrm{Im}(\bar{\zeta}H_k),\\ s_0&=\delta\,\mathrm{Im}(\bar{z}^2H_k). \end{aligned}$$

Taking (A.45) into account, we find no new generators.

This completes the analysis. We have found twelve generators (A.39), (A.42), (A.43), (A.44), (A.45). Proposition A.1(b) now follows.

Note that the invariants (A.2) for $O(2)\times S^1$ do not form a polynomial ring: there is a relation

$$\delta^2\gamma^2-\sigma^2=(z\bar{z})^2(\zeta\bar{\zeta})=\beta^2(N-4\delta^2).$$

Further, the equivariants do not form a free module, although the relations have degree 9 or more. For example

$$[\sigma][\delta\,\mathrm{Im}(H_1)]=[\delta^2][\mathrm{Im}(z^2\bar{\zeta})\mathrm{Im}(H_1)].$$

(There are other relations too). In consequence, the singularity theory of $O(2)\times S^1$ on the six-dimensional kernel would be extremely complicated to compute.

Finally, we turn to the statement of Theorem 4.3. We obtain the form stated there for the equivariants from that used in Proposition A.1, by defining

$$K_j = \mathrm{Re}(H_j), \quad L_j = \mathrm{Im}(H_j), \quad j = 1,2,3,4,$$

and collecting terms according to the matrices K_j, L_j that occur.

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