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TIME-REVERSIBILITY AND PARTICLE SEDIMENTATION*

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Abstract. This paper studies an ordinary differential equation (ODE) model, called the Stokeslet model, and describes sedimentation of small clusters of particles in a highly viscous fluid. This model has a trivial solution in which the n particles arrange themselves at the vertices of a regular n-sided polygon. When $n=3$, Hocking [*J. Fluid Mech.*, 20 (1964), pp. 129-139] and Caflisch et al. [*Phys. Fluids*, 31 (1988), pp. 3175-3179] prove the existence of periodic motion (in the frame moving with the center of gravity in the cluster) in which the particles form an isosceles triangle. The study of periodic and quasiperiodic solutions of the Stokeslet model is continued, with emphasis on the spatial and time reversal symmetry of the model (time reversibility is due to infinite viscosity and spatial (dihedral) symmetry is due to the assumption of identical particles and the symmetry of the trivial solution). For three particles, the existence of a second family of periodic solutions and a family of quasiperiodic solutions is proved. It is also indicated how the methods generalize to the case of n particles.

Key words. bifurcation, symmetry, time-reversibility, sedimentation, Lyapunov Center Theorem

AMS(MOS) subject classifications. 58F22, 70K15, 76D07

Introduction. Jayaweera, Mason, and Slack [JMS] and Hocking [H] study the sedimentation of small clusters of small spheres in a highly viscous fluid experimentally and analytically. Experimentally, Jayaweera, Mason, and Slack find that when the cluster consists of fewer than seven falling particles, the spheres arrange themselves in a regular polygon. When the cluster consists of seven or more spheres, they find that the regular polygon is unstable. When the cluster size is three, they also find evidence for periodic motion where the spheres arrange themselves at the vertices of (a time-varying) isosceles triangle, at least on a short-to-moderate timescale. This periodic motion takes place in a frame moving with the center of gravity of the cluster.

To explain the transition in observed phenomena when the cluster size reaches seven, Hocking [H] studies an ordinary differential equation (ODE) model called the point particle or Stokeslet model. In the Stokeslet model, spheres are replaced by point particles with gravity acting as a delta function, Stokes flow for individual particles is assumed (an infinite viscosity limit), and an interaction between particles is assumed (caused by the flow fields of the individual particles). In this model, the regular polygon is an equilibrium. By direct calculation of the Jacobian matrix at the regular polygon, Hocking [H] shows that this equilibrium is elliptic (all eigenvalues on the imaginary axis) when the cluster size is less than seven and hyperbolic for cluster sizes between seven and 12, with the implication that the equilibria are hyperbolic for all cluster sizes greater than seven. Stability is equated with ellipticity. Hocking also constructs

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the periodic solution for the three particle Stokeslet model consisting of time-varying isosceles triangles.

Caflisch et al. [CLLS] continue the study of the sedimentation problem by analyzing the motion of three spheres in a Stokes fluid. They show that when the cluster size is three, the periodic solution where the particles arrange themselves as isosceles triangles also exists in this model. The [CLLS] proof of existence of the isosceles triangle periodic solution uses time-reversibility as does the proof in [H]. One interest of [CLLS] in finding this periodic solution is to use its existence to test for numerical codes developed to solve the more complicated finite sphere model.

Independently, Pickard and Tory [PT] have found a periodic solution in the four sphere model where the projection of the spheres in a horizontal plane forms a time-varying rhombus.

In this paper we continue the study of periodic and quasi-periodic solutions for the Stokeslet model for all cluster sizes. In our work we emphasize the existence of both time-reversal and spatial symmetries in the Stokeslet model. The time-reversal symmetry is due to the assumption of infinite viscosity and the spatial symmetries are due to the assumption of identical particles with symmetric coupling.

Using symmetry arguments, we will show that, when the cluster size is three, there are two families of periodic solutions to the Stokeslet model and one family of two-frequency motions. The periodic solutions form isosceles triangles; in one family the vertices of the side of unequal length move synchronously and in the other family these vertices move with a half-period phase shift. The family of two-frequency motions consists of equilateral triangles whose vertices move with a third of a period phase lag while rotating slowly on a circle (ponies on a merry-go-round). Our theory shows, moreover, that only two calculations using the Stokeslet model are needed to actually prove the existence of these solutions: one is the computation of eigenvalues done in [H], and the other is the verification of a transversality condition showing that ponies actually rotate on the merry-go-round.

Our methods enable us to show that solutions corresponding to the three solutions just described should exist for the n particle cluster. To actually prove this existence we would have to verify nonresonance on the linear terms (using the results in [H]) and verify the transversality condition for general n (to show that the ponies actually rotate). This we have not attempted.

Moreover, since our methods depend mainly on the abstract structure of the equations, identical results should hold for models with finite, but small, particle size. The calculations, however, will be more complicated.

The mathematical theory we develop is one of time-reversible systems with spatial symmetries. We begin in § 1 by describing Devaney's [D] Lyapunov center theorem for reversible systems following the proof given in Vanderbauwhede [V]. This theorem assumes both simple imaginary eigenvalues and nonresonance in the Jacobian matrix, neither of which is valid for the Stokeslet model. In §§ 2 and 3 we generalize this theorem to include zero eigenvalues in the Jacobian and 1:1 resonances caused by symmetry. Our main result is summarized in Theorem 3.1. Our method follows ideas in [GSt] for finding periodic solutions in symmetric systems undergoing Hopf bifurcation and ideas in [MRS] for finding periodic solutions in symmetric Hamiltonian systems.

In § 4 we describe in detail the symmetries of the Stokeslet model (whose derivation is described briefly in an Appendix) and in § 5 we describe how these symmetries affect the linear (Jacobian) analysis. The basic group theory of the Stokeslet model is presented in these sections. In § 6 we describe how to use Theorem 3.1 and symmetries to find periodic solutions for the Stokeslet model. We emphasize here how the symmetries of the model determine the nature of the periodic solutions we find. The periodic solutions for $n \leqq 6$ are described in § 7.

In § 8 we show how the two-frequency ponies on a merry-go-round are formed. Here we use results of Krupa [K] on bifurcation from group orbits of equilibria to organize the discussion. The remainder of this paper is devoted to the three particle Stokeslet model. The calculations mentioned previously verifying that the ponies actually rotate when $n = 3$ are also given in this section. Genericity arguments tell us that this result is expected; indeed, we expect the corresponding statement concerning ponies in the n particle case to be valid.

Finally, in § 9 we comment on the linearized stability of the three families of solutions for $n = 3$. We prove that generically one of the two families of isosceles triangles is elliptic and the other hyperbolic and that the family of ponies is always elliptic. These remarks are based on theoretical normal form calculations (which introduce a temporal phase shift symmetry), spatial symmetries, and time-reversal symmetry. Our linear stability results are similar to those of [MRS].

1. Lyapunov center theorem for time-reversible systems. Vanderbauwhede [V] proves a Lyapunov center theorem for time-reversible systems. We sketch his proof here so that we may generalize it to systems with spatial symmetries.

Consider the system of ODEs

$$\frac{dx}{dt} = f(x), \qquad x \in V, \tag{1.1}$$

where $f: V \to V$ is C^∞ and $V \cong \mathbb{R}^n$ for some n. A *time-reversal* symmetry of (1.1) is a reflection $R: V \to V$ satisfying

$$f(Rx) = -Rf(x). \tag{1.2}$$

Thus, when $x(t)$ is a solution to (1.1) so is

$$(T_R x)(t) \equiv Rx(-t). \tag{1.3}$$

We assume that x_0 is an equilibrium of (1.1) that is invariant under the time-reversal symmetry, i.e.,

$$f(x_0) = 0 \quad \text{and} \quad Rx_0 = x_0. \tag{1.4}$$

Observe that (1.4) coupled with differentiation of (1.2) leads to the antisymmetry

$$(df)_{x_0} R = -R(df)_{x_0}. \tag{1.5}$$

Hence, if λ is an eigenvalue of $(df)_{x_0}$ with eigenvector v, then $-\lambda$ is an eigenvalue with eigenvector Rv. Note that if $n = 2$, then λ must be either real or purely imaginary (depending on the sign of det $(df)_{x_0}$).

This eigenvalue dichotomy occurs frequently in higher dimensions, especially when spatial symmetries are present. In this section we assume:

$$\begin{aligned} &\pm\omega_0 i \text{ are simple eigenvalues of } (df)_{x_0}, \text{ and} \\ &k\omega_0 i \text{ is not an eigenvalue of } (df)_{x_0} \ (k = 0, 2, 3, \cdots). \end{aligned} \tag{1.6}$$

The Lyapunov center theorem is the following.

THEOREM 1.1. *Assuming* (1.2), (1.4), *and* (1.6) *there exists a smooth one-parameter family of periodic solutions* $x_a(t)$ *of* (1.1) *with periods near* $2\pi/\omega_0$ *where* $x_0(t) \equiv x_0$.

Proof. The proof of this theorem is an adaptation of the Lyapunov–Schmidt reduction proof of Hopf bifurcation where the role of a bifurcation parameter is replaced by time-reversibility. We follow the exposition in [V, Chap. 7, § 5.3] and [GS, Chap. VIII, § 2].

Begin by rescaling time in (1.1) so that $\pm i$ are eigenvalues of $(df)_{x_0}$. Define $F: \mathscr{C}^1_{2\pi}\times\mathbb{R}\to\mathscr{C}_{2\pi}$ by

$$F(x,\tau)=(1+\tau)\frac{dx}{dt}-f(x), \tag{1.7}$$

where $\mathscr{C}_{2\pi}$ is the Banach space of V-valued, 2π-periodic mappings and $\mathscr{C}^1_{2\pi}$ consists of those $x(t)\in\mathscr{C}_{2\pi}$ that are continuously differentiable. Observe that solutions to $F(x,\tau)=0$ correspond to $2\pi/(1+\tau)$-periodic solutions of (1.1).

We solve $F=0$ by Lyapunov–Schmidt reduction. Hypothesis (1.6) implies that $\ker(dF)_{x_0,0}$ is two-dimensional, being spanned by $\operatorname{Re}(e^{it}v)$ and $\operatorname{Im}(e^{it}v)$ where $(df)_{x_0}v=iv$. Indeed, we can identify $\ker(dF)_{x_0,0}$ with $\mathbb{C}$ by $z\to\operatorname{Re}(z\,e^{it}v)$. Lyapunov–Schmidt reduction proves the existence of a smooth mapping

$$\phi:\mathbb{C}\times\mathbb{R}\to\mathbb{C}$$

whose zeros are in 1:1 correspondence with the (near 2π)-periodic solutions of (1.1).

An essential ingredient in the proof of both Hopf bifurcation and of this theorem is the existence of S^1 phase shift symmetry in the reduced equations ϕ. Observe that $\theta\in S^1$ acts on $x(t)\in\mathscr{C}_{2\pi}$ by

$$\theta\cdot x(t)=x(t-\theta), \tag{1.8}$$

and that F is S^1-equivariant with respect to this action. Hence, when the Lyapunov–Schmidt reduction is done correctly, the reduced function ϕ is also S^1-equivariant. The action induced on $\ker(dF)_{x_0,0}$ by (1.8) is just the natural action

$$\theta\cdot z=e^{i\theta}z. \tag{1.9}$$

It follows from (1.9) and S^1-equivariance that

$$\phi(z,\tau)=p(z\bar z,\tau)z+q(z\bar z,\tau)iz, \tag{1.10}$$

where p and q are smooth, real-valued functions. See [GS, Chap. VIII, Prop. 2.3]. The details of Lyapunov–Schmidt reduction show that

$$p(0,0)=0,\quad q(0,0)=0,\quad q_\tau(0,0)\neq 0. \tag{1.11}$$

In Hopf bifurcation f and hence F, ϕ, p, and q depend smoothly on a bifurcation parameter μ. Solving $\phi=0$ for nontrivial solutions ($z\neq 0$) is equivalent by (1.10) to solving $p=q=0$. By (1.11) and the implicit function theorem the equation $q=0$ can be solved locally near the origin for $\tau=\tau(z\bar z,\mu)$, leading to the single equation $p(z\bar z,\tau(z\bar z,\mu),\mu)=0$. Finally, in Hopf bifurcation, we show that $p_\mu(0,0,0)\neq 0$ if complex conjugate eigenvalues of dF cross the imaginary axis with nonzero speed as μ is varied. Thus, the implicit function theorem can be applied once again to solve $p=0$ locally for $\mu=\mu(z\bar z)$, thus giving the desired branch of periodic solutions.

In this theorem we use the existence of the time-reversible symmetry R to solve $\phi=0$. Observe that F, as defined in (1.7), anticommutes with T_R, as defined in (1.3), that is,

$$F(T_Rx,\tau)=-T_RF(x,\tau). \tag{1.12}$$

Differentiation of (1.12) leads to the anticommutativity

$$(dF)_{x_0,0}T_R = -T_R(dF)_{x_0,0}, \tag{1.13}$$

and hence that $T_R : \ker(dF)_{x_0,0} \to \ker(dF)_{x_0,0}$. Let $R' = T_R | \ker(dF)_{x_0,0}$.

We assert that R' and S^1 do not commute on $\ker(dF)_{x_0,0}$, and hence that $R' \notin S^1$, since S^1 is commutative. To prove this assertion we let $x(t)$ be in $\mathscr{C}_{2\pi}$ and compute

$$(\theta \cdot T_R \cdot x)(t) = (T_R \cdot x)(t-\theta) = Rx(-t+\theta),$$

$$(T_R \cdot \theta \cdot x)(t) = R(\theta \cdot x)(-t) = Rx(-t-\theta).$$

Thus, as long as $x(t) \neq 0$ and $\theta \neq 0$ or π, $\theta \cdot T_R \cdot x \neq T_R \cdot \theta \cdot x$.

Since R' and S^1 act orthogonally on $\mathbb{C}$, they generate $O(2)$. In follows that there exists $\theta \in S^1$ such that $R'' = \theta \cdot R'$ acts as complex conjugation on $\mathbb{C}$. If we define $T' = \theta \cdot T_R$ acting on $\mathscr{C}_{2\pi}$, then T' is also an antisymmetry of F.

With these comments, it is now straightforward to check that Lyapunov–Schmidt reduction respects time-reversible symmetries, that is,

$$\phi(\bar{z}, \tau) = -\overline{\phi(z, \tau)}. \tag{1.14}$$

Identity (1.14) applied to (1.10) implies that $p \equiv 0$, that is,

$$\phi(z, \tau) = q(z\bar{z}, \tau)iz. \tag{1.15}$$

It follows from (1.11) and (1.15) that $\phi = 0$ can always be solved, using the implicit function theorem, for $\tau = \tau(z\bar{z})$, $\tau(0) = 0$. This solution translates to a one-parameter family of periodic solutions to (1.1) parametrized by the amplitude $z\bar{z}$.

2. Lyapunov center theorem with zero eigenvalues. As noted in § 1, eigenvalues of $(df)_{x_0}$ come in pairs $\pm\lambda$. When $Rv = v$, where v is an eigenvector with eigenvalue λ, λ must equal zero. For this reason, time-reversibility often leads to zero eigenvalues, but usually when $R|\ker(df)_{x_0}$ is the identity map. In this section, we generalize Theorem 1.1 to include this possibility.

Define $V_0 \equiv \ker(dF)_{x_0}$ and let $m = \dim V_0$.

THEOREM 2.1. *Assume that* (1.1) *satisfies* (1.2), (1.4),

(2.1) $\pm\omega_0 i$ *are simple eigenvalues of* $(df)_{x_0}$, *and*

$k\omega_0 i$ *is not an eigenvalue of* $(df)_{x_0}$ $(k = 2, 3, \cdots)$,

and

$$R \,|\, V_0 = I_{V_0}. \tag{2.2}$$

Then there exists a unique $(m+1)$*-parameter family of periodic solutions* $x_a(t)$ *of* (1.1) *with period near* $2\pi/\omega_0$.

Remark. Hypothesis (2.2) was used by Scheurle [Sc] when developing a KAM theory for reversible systems. See also Sevryuk [Se].

Proof. The proof of this theorem is similar to the proof of Theorem 1.1; we indicate the changes.

Assumption (2.1) implies that

$$\ker(dF)_{x_0,0} = V_i \oplus V_0, \tag{2.3}$$

where V_i is the two-dimensional real eigenspace of $(df)_{x_0}$ associated with the eigenvalues $\pm i$ and, as in § 1, can be identified with $\mathbb{C}$. Note that S^1 phase shift symmetry acts as in (1.9) on V_i and trivially on V_0 (since V_0 consists of constant functions).

Hence, the Lyapunov–Schmidt reduction leads to a smooth S^1-equivariant mapping

$$\phi: \mathbb{C} \times V_0 \times \mathbb{R} \to \mathbb{C} \times V_0 \tag{2.4}$$

whose zeros are in 1:1 correspondence with the desired periodic solutions to (1.1). Writing $\phi = (\phi_\iota, \phi_0)$ in coordinates and using S^1-equivariance leads to

$$\phi_i(e^{i\theta}z, w, \tau) = e^{i\theta}\phi_i(z, w, \tau), \tag{2.5a}$$

$$\phi_0(e^{i\theta}z, w, \tau) = \phi_0(z, w, \tau). \tag{2.5b}$$

From (2.5) we see that the w-coordinates may be treated as parameters. Hence ϕ has the form

$$\phi_i(z, w, \tau) = p(z\bar{z}, w, \tau)z + q(z\bar{z}, w, \tau)iz, \tag{2.6a}$$

$$\phi_0(z, w, \tau) = r(z\bar{z}, w, \tau), \tag{2.6b}$$

where p, q, and r are smooth, real-valued functions.

Next, we consider the effects of the time-reversal symmetry R. As in § 1, ϕ anticommutes with the action of T_R on $\ker (dF)_{x_0,0}$, which after a possible phase shift, we may write as

$$R''(z, w) = (\bar{z}, w). \tag{2.7}$$

Hence

$$\phi_i(\bar{z}, w, \tau) = -\overline{\phi_i(z, w, \tau)}, \tag{2.8a}$$

$$\phi_0(\bar{z}, w, \tau) = -\phi_0(z, w, \tau). \tag{2.8b}$$

It follows from (2.6) and (2.8) that

$$\phi_i(z, w, \tau) = q(z\bar{z}, w, \tau)iz, \tag{2.9a}$$

$$\phi_0(z, w, \tau) = 0. \tag{2.9b}$$

Thus $q(0) = 0$ and $q_\tau(0) \neq 0$ together with the implicit function theorem allow us to solve $\phi = 0$ for $\tau = \tau(z\bar{z}, w)$. This yields the desired $(m+1)$-parameter family of solutions.

3. Lyapunov center theorem with spatial symmetry. In this section we generalize Theorem 2.1 to the situation where f in (1.1) commutes with a compact group of spatial symmetries Γ. That is,

$$f(\gamma x) = \gamma f(x) \quad \forall \gamma \in \Gamma. \tag{3.1}$$

We also assume

$$\gamma x_0 = x_0 \quad \forall \gamma \in \Gamma. \tag{3.2}$$

The existence of spatial symmetries often causes the eigenvalues of $(df)_{x_0}$ to be multiple. This arises from the commutativity relation

$$(df)_{x_0}\gamma = \gamma (df)_{x_0} \quad \forall \gamma \in \Gamma, \tag{3.3}$$

which is obtained from (3.1) by differentiation.

The prototypical situation occurs when

$$V = W \oplus W$$

and Γ acts absolutely irreducibly on W. (*Absolute* irreducibility means that the only linear maps on W commuting with Γ are scalar multiples of the identity.) Since $(df)_{x_0}$ commutes with Γ, absolute irreducibility implies

$$(df)_{x_0}=\begin{pmatrix} aI & bI \\ cI & dI \end{pmatrix}.$$

Hence the eigenvalues of $(df)_{x_0}$ are the eigenvalues of $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right)$, each with multiplicity equal to dim W.

Suppose now that R commutes with Γ. Then the ± 1 eigenspaces of R are invariant under Γ. Assuming that *both* eigenspaces are nontrivial allows us, after a linear change of coordinates, to write $R(w_1, w_2)=(w_1, -w_2)$. Anticommutativity of $(df)_{x_0}$ with R implies $a=d=0$. Hence the eigenvalues of $(df)_{x_0}$ are either real or purely imaginary. This is the generalization of the eigenvalue dichotomy referred to in § 1.

In our generalization of Theorem 2.1, we assume

$$V_i = W \oplus W, \tag{3.4}$$

where Γ acts absolutely irreducibly on W. See [GSS, Chap. XVI, § 1]. The idea behind this assumption is the generalization of Hopf bifurcation to systems with symmetry obtained by detecting solutions by their symmetries [GSS, Chap. XVI, Thm. 4.1]. Their symmetries, however, contain both a spatial (Γ) component and a temporal (S^1) component. More precisely, $(\gamma, \theta) \in \Gamma \times S^1$ acts on $x(t)$ by

$$(\gamma, \theta) \cdot x(t) = \gamma x(t-\theta), \tag{3.5}$$

and the group of symmetries of a periodic mapping is a subgroup $\Sigma \subset \Gamma \times S^1$.

Observe that $V_i \oplus V_0$ is $\Gamma \times S^1$-invariant. The S^1-invariance follows from the discussion in § 2, while the Γ-invariance follows from the fact that (3.3) implies Γ-invariance of the eigenspaces of $(df)_{x_0}$. Let $\Sigma \subset \Gamma \times S^1$ be a subgroup. Define

$$\operatorname{Fix}(\Sigma) = \{v \in V_i \oplus V_0 \colon \sigma v = v \quad \forall \sigma \in \Sigma\}, \tag{3.6a}$$

$$\operatorname{Fix}_i(\Sigma) = \operatorname{Fix}(\Sigma) \cap V_i, \tag{3.6b}$$

$$\operatorname{Fix}_0(\Sigma) = \operatorname{Fix}(\Sigma) \cap V_0 \text{ and let } m = \dim \operatorname{Fix}_0(\Sigma). \tag{3.6c}$$

Then $\operatorname{Fix}(\Sigma) = \operatorname{Fix}_i(\Sigma) \oplus \operatorname{Fix}_0(\Sigma)$.

We now prove the generalization of Theorem 2.1 to Γ-equivariant systems that we will use in later sections. For completeness we restate all hypotheses here.

THEOREM 3.1. *Assume that the Γ-equivariant system of* ODE (1.1)

$$\frac{dx}{dt} = f(x)$$

has a Γ-invariant equilibrium x_0. Assume

(3.7) $\pm\omega_0 i$ *are nonzero eigenvalues of* $(df)_{x_0}$, *and*

$k\omega_0 i$ *is not an eigenvalue of* $(df)_{x_0}$ $(k=2, 3, \cdots)$.

Assume that the generalized eigenspace V_i corresponding to the eigenvalues $\pm\omega_0 i$ has the form (3.4)

$$V_i = W \oplus W,$$

where Γ acts absolutely irreducible on W.

Assume that (1.1) *has a time-reversal symmetry* R *that fixes* x_0 *and let* $\Sigma \subset \Gamma \times S^1$ *be a subgroup satisfying*:

(3.8a) $$\dim \operatorname{Fix}_i(\Sigma) = 2,$$

(3.8b) $$R(\operatorname{Fix}_i(\Sigma)) = \operatorname{Fix}_i(\Sigma),$$

(3.8c) $$R \mid \operatorname{Fix}_0(\Sigma) = I.$$

Then there exists an $(m+1)$*-parameter family of periodic solutions to* (1.1) *with period* $2\pi/\omega_0$ *and symmetry* Σ, *where* $m = \dim \operatorname{Fix}_0(\Sigma)$.

Proof. The proof of this theorem is similar to that of Theorem 2.1; we indicate the changes.

The Lyapunov–Schmidt reduction leads to a smooth $\Gamma \times S^1$-equivariant mapping

(3.9) $$\phi: V_i \times V_0 \times \mathbb{R} \to V_i \times V_0,$$

the Γ-equivariance following as in the proof of [GSS, Chap. XVI, Thm. 4.1]. This $\Gamma \times S^1$-equivariance implies that

(3.10) $$\phi: \operatorname{Fix}(\Sigma) \times \mathbb{R} \to \operatorname{Fix}(\Sigma).$$

Writing $\operatorname{Fix}(\Sigma) = \operatorname{Fix}_i(\Sigma) \oplus \operatorname{Fix}_0(\Sigma)$, we see that the hypotheses of Theorem 2.1 are satisfied if we identify $\operatorname{Fix}_i(\Sigma)$ with $\mathbb{C}$ in Theorem 2.1 and $\operatorname{Fix}_0(\Sigma)$ with V_0. □

Remarks 3.2. (a) Let H be the projection of $\Sigma \subset \Gamma \times S^1$ in Γ. Since S^1 acts trivially on V_0, we see that

$$\operatorname{Fix}_0(\Sigma) = \operatorname{Fix}_0(H) \equiv \{v \in V_0 : hv = v \quad \forall h \in H\}.$$

Thus, we have an alternative way of computing m in Theorem 3.1.

(b) The hypotheses of Theorem 3.1 involve both Σ and R. Therefore, it is possible to find an appropriate Σ but to have the conditions (3.8b), (3.8c) on R fail. In these cases, R can sometimes be replaced by a time-reversal symmetry $R' = \gamma R$ for some $\gamma \in \Gamma$ where R' satisfies (3.8b), (3.8c).

4. Abstract structure of the Stokeslet model. Let $x_1, \cdots, x_n \in \mathbb{R}^3$ denote the consecutive edges of an n-sided polygon in $\mathbb{R}^3$. Thus, the $3n$-vector $(x_1, \cdots, x_n)$ lies in the space

(4.1) $$V = \{(x_1, \cdots, x_n) \in \mathbb{R}^{3n} : x_1 + \cdots + x_n = 0\}.$$

Let e_1, e_2, e_3 be an orthonormal basis of $\mathbb{R}^3$ where e_1 indicates the vertical direction. The Stokeslet model, whose derivation is sketched in the Appendix, is

(4.2) $$\dot{x}_j = \sum_{k=1}^{n-2} [U(x_{j+1} + \cdots + x_{j+k}) - U(x_{j-1} + \cdots + x_{j-k})]$$

for $j = 1, \cdots, n$ where the indices are taken mod n and where

(4.3) $$U(x) = \frac{e_1}{|x|} + (e_1, x)\frac{x}{|x|^3}, \qquad x \in \mathbb{R}^3.$$

U is called the *Stokeslet*. Note that gravity is assumed to act in the x_1-direction.

We make three observations concerning this model.

(i) $D_n \times O(2)$ is a group of symmetries of (4.2).

(ii) Regular horizontal n-gons are equilibria that are fixed by a subgroup $\Gamma \subset D_n \times O(2)$ isomorphic to D_n.

(iii) There is a time-reversal symmetry fixing regular n-gons.

We discuss these points in order.

An action of $O(2)$ on $\mathbb{R}^3$ is generated by

$$R_\theta = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos(\theta) & -\sin(\theta) \\ 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} \quad \forall \theta \in SO(2), \tag{4.4a}$$

$$\kappa = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{4.4b}$$

This action extends to V by

$$\theta \cdot (x_1, \cdots, x_n) = (R_\theta x_1, \cdots, R_\theta x_n), \tag{4.5a}$$

$$\kappa \cdot (x_1, \cdots, x_n) = (\kappa x_1, \cdots, \kappa x_n). \tag{4.5b}$$

The dihedral group D_n, the symmetries of a regular n-gon, has $2n$ elements and is generated by an element of order n and a reflection of order 2. An action of D_n on V is defined by

$$\begin{aligned} &(x_1, \cdots, x_n) \to (x_2, \cdots, x_n, x_1), \\ &(x_1, \cdots, x_n) \to (x_n, x_{n-1}, \cdots, x_1). \end{aligned} \tag{4.6}$$

The Stokeslet model (4.2) is equivariant with respect to these actions of $O(2)$ and D_n. This is clear for the permutation symmetries (4.6). $O(2)$-equivariance is verified by first observing that

$$U(\gamma x) = \gamma U(x) \quad \forall \gamma \in O(2). \tag{4.7}$$

Using this group action it is simple to write down the regular n-gon. Let

$$\theta_n = 2\pi/n, \tag{4.8}$$

fix x_1 in the horizontal e_2, e_3 plane, and set

$$x_{j+1} = R_{\theta_n} x_j \qquad (j = 1, \cdots, n-1). \tag{4.9}$$

Using (4.7) it is now a simple matter to show that the regular n-gon (4.9) is an equilibrium for (4.2).

The precise subgroup of $D_n \times O(2)$ fixing (4.9) depends on the choice of x_1, though it is always isomorphic to D_n. Henceforth, we assume

$$x_1 = R_{\theta_n/2} e_3$$

and call the symmetries of this regular n-gon Γ. Γ is generated by

$$\mathscr{C}(x_1, \cdots, x_n) = (-\theta_n) \cdot (x_2, \cdots, x_n, x_1), \tag{4.10a}$$

$$\mathscr{T}(x_1, \cdots, x_n) = -\kappa \cdot (x_n, x_{n-1}, \cdots, x_1). \tag{4.10b}$$

Finally,

$$R = \begin{pmatrix} -1 & & \\ & 1 & \\ & & 1 \end{pmatrix}$$

extends to a time-reversal symmetry of (4.2) by

$$R \cdot (x_1, \cdots, x_n) = (Rx_1, \cdots, Rx_n). \tag{4.11}$$

Note that R fixes regular horizontal n-gons.

5. Group theoretic restrictions on $(df)_{x_0}$. Spatial and time-reversal symmetries greatly restrict the form of the matrix

$$L=(df)_{x_0}. \tag{5.1}$$

In this section we use the specific group structure described in § 4 to analyze L. This analysis is based on the following commutativity properties:

$$L\gamma=\gamma L \quad \forall \gamma \in \Gamma, \tag{5.2a}$$

$$LR=-RL. \tag{5.2b}$$

Recall from [GSS Chap. XII, § 2] that an *isotypic component* of V corresponding to an irreducible representation W of Γ is the sum of all irreducible subspaces of V that are Γ-isomorphic to W. Standard theorems show that L can be block diagonalized with respect to the isotypic decomposition of V into isotypic components. To describe these isotypic components we begin by listing (up to isomorphism) all of the irreducible representations of D_n. See Miller [M] for a proof that what follows is a complete list.

The irreducible representations of $\Gamma \cong D_n$ are either one- or two-dimensional. The actual number and type of these representations depends on the parity of n. When n is even there are four nonisomorphic one-dimensional irreducible representations of Γ, determined by whether $\mathscr{C}$ and $\mathscr{T}$ (see (4.10)) act by $+1$ or -1. We denote these representations by $W_{\varepsilon\delta}$ where $\mathscr{C}$ acts by εI and $\mathscr{T}$ acts by δI. (Of course, δ, $\varepsilon=\pm 1$.) W_{++} is the *trivial* representation.

When n is odd there are only two nonisomorphic one-dimensional irreducible representations of Γ, namely, W_{++} and W_{+-}. This follows since $\mathscr{C}$ is a group element of order n and hence $\mathscr{C}^n=I$. Thus, if $\mathscr{C}$ acts by $-I$, then $(-1)^n$ must be one, that is, n must be even.

There are $[(n-1)/2]$ distinct two-dimensional irreducible representations of Γ. We denote these by $W_k \cong \mathbb{C}$ for $k=1,\cdots,[(n-1)/2]$. On W_k

$$\mathscr{C}z=e^{ki\theta}z, \tag{5.3a}$$

$$\mathscr{T}z=\bar{z}. \tag{5.3b}$$

Note that Γ acts absolutely irreducibly on each W_k, and that W_{-k}, using the obvious extension to negative k in (5.3), is Γ-isomorphic to W_k.

The isotypic decomposition of V is given by Theorem 5.1.

THEOREM 5.1. (a) *When n is odd*

$$V\cong W_{++}\oplus W_{+-}\oplus W_1^2\oplus W_2^3\oplus\cdots\oplus W_{(n-1)/2}^3.$$

(b) *When n is even*

$$V\cong W_{++}\oplus W_{+-}\oplus W_{--}\oplus W_{-+}^2\oplus W_1^2\oplus W_2^3\oplus\cdots\oplus W_{(n-2)/2}^3.$$

(c) *On the repeated components W^s, the time-reversal symmetry R acts by $-I$ on the first copy of W and $+I$ on the remaining copies. R acts as $+I$ on the other (one-dimensional) representations.*

Before proving Theorem 5.1 we describe explicitly the consequences of Theorem 5.1 for $L=(df)_{x_0}$.

THEOREM 5.2. (a) *$L=0$ on W_{++}, W_{+-}, and W_{--}.*

(b) *$L|W_{-+}^2$ has a pair of simple eigenvalues $\pm\lambda$, where λ is either real or purely imaginary.*

(c) *$L|W_k^s$ has a pair of eigenvalues $\pm\lambda$, each of multiplicity two, where λ is either real or purely imaginary. When $s=3$, L has a pair of zero eigenvalues and $R|(\ker L\cap W_k^s)=+I$.*

Remark. It follows that L has $[n/2]+1$ zero eigenvalues forced by symmetry and $[n/2]$ independent pairs of eigenvalues with each pair either real or purely imaginary. In the generic case, when all of these independent eigenvalue pairs are nonzero, $R|\ker L=+I$.

Proof. Since L commutes with Γ, L leaves isotypic components invariant (see [GSS, Chap. XII, Thm. 3.5]).

(a) As observed earlier (5.2b) implies that L has eigenvalues $\pm\lambda$. On a one-dimensional space λ must equal zero.

(b), (c) Suppose that γ acts absolutely irreducible on W and diagonally on the isotypic component W^s. Then $L|W^s$ has the form

$$\begin{pmatrix} a_{11}I & \cdots & a_{1s}I \\ \vdots & & \vdots \\ a_{s1}I & \cdots & a_{ss}I \end{pmatrix},$$

where each $a_{ij}\in\mathbb{R}$. It follows that the eigenvalues of $L|W^s$ are just the eigenvalue of the $s\times s$ matrix

$$A=\begin{pmatrix} a_{11} & \cdots & a_{1s} \\ \vdots & & \vdots \\ a_{s1} & \cdots & a_{ss} \end{pmatrix}$$

each with multiplicity $\dim W$. Theorem 5.1(c) states that $R(w_1, w_2,\cdots, w_s)=(-w_1, w_2,\cdots, w_s)$. Thus (5.2b) implies that

$$A=\begin{pmatrix} 0 & a_{12} & \cdots & a_{1s} \\ a_{21} & & & \\ \vdots & & 0 & \\ a_{s1} & & & \end{pmatrix}.$$

Any vector in $(a_{12},\cdots, a_{1s})^{\perp}\in\mathbb{R}^{s-1}$ yields two null-vectors for $L|W^s$. Moreover, these null-vectors lie in the second through sth components of W^s and hence $R=+I$ on these null-vectors.

Thus there is only one set of nonzero independent eigenvalues in A and since $\operatorname{tr} A=0$, these eigenvalues must be $\pm\lambda$ where λ is either real or purely imaginary. □

The remainder of this section is devoted to the proof of Theorem 5.1.

Proof of Theorem 5.1. We begin by decomposing $V=V_1\oplus V_2$

$$V_1=\{(x_1,\cdots,x_n)\in V\colon x_j=a_je_1\}, \tag{5.4a}$$

$$V_2=\{(x_1,\cdots,x_n)\in V\colon x_j\cdot e_1=0\}. \tag{5.4b}$$

Observe that V_1 and V_2 are Γ-invariant subspaces, and that $R=-I$ on V_1 and $+I$ on V_2.

We divide the proof of Theorem 5.1 into two lemmas.

LEMMA 5.3. (a) *When n is odd*

$$V_2\cong W_{++}\oplus W_{+-}\oplus W_1\oplus W_2^2\oplus\cdots\oplus W^2_{(n-1)/2}.$$

(b) *When n is even*

$$V_2\cong W_{++}\oplus W_{+-}\oplus W_{-+}\oplus W_{--}\oplus W_1\oplus W_2^2\oplus\cdots\oplus W^2_{(n-2)/2}.$$

LEMMA 5.4. (a) *When n is odd*

$$V_1=W_1\oplus\cdots\oplus W_{(n-1)/2}.$$

(b) *When n is even*

$$V_1 = W_{-+} \oplus W_1 \oplus \cdots \oplus W_{(n-2)/2}.$$

We prove both of these lemmas by explicitly producing the irreducible subspaces making up V_1 and V_2 and counting the number of isomorphic representations.

For $1 \leqq \ell \leqq n-1$ define

$$V_{2,\ell} = \{(x_1, \cdots, x_n) \in V_2 : x_{j+1} = R_{\ell\theta_n} x_j\}. \tag{5.5}$$

Observe that

$$V_2 = V_{2,1} \oplus \cdots \oplus V_{2,\ell} \tag{5.6}$$

and that each $V_{2,\ell}$ is Γ-invariant. Indeed, if we identify a vector $x \in \mathbb{R}^2$ with $x_1 = x$ in $V_{2,\ell}$, we may explicitly compute the action of $\mathscr{C}$ and $\mathscr{T}$ on $V_{2,\ell}$. We find

$$\mathscr{C}x = R_{(\ell-1)\theta_n} x, \tag{5.7a}$$

$$\mathscr{T}x = -\kappa R_{-\ell\theta_n} x. \tag{5.7b}$$

It follows from (5.7) that $\mathscr{C}$ acts trivially on $V_{2,1}$ and that $V_{2,1}$ decomposes into two one-dimensional irreducible representations (the eigenvectors of $\mathscr{T}$) isomorphic to W_{++} and W_{+-}. When n is even, $\mathscr{C}$ acts as $-I$ on $V_{2,n/2}$ and this space also decomposes into two one-dimensional irreducible representations isomorphic to W_{-+} and W_{--}.

Γ acts irreducibly on the remaining two-dimensional invariant subspaces in (5.6). Now (5.7a) shows that $V_{2,\ell}$ is isomorphic to $W_{\ell-1}$ for $\ell = 2, \cdots, [(n-1)/2]+1$; this yields one copy each of the distinct W_j's in V_2. For $\ell > [(n-1)/2]+1$, the representation of Γ on $V_{2,\ell}$ is isomorphic to $W_{n-\ell-1}$. (Here we use $W_{-k} \cong W_k$.) Since $\ell \leqq n-1$ and $n-\ell+1 \geqq 2$ it follows that $W_2, \cdots, W_{[(n-1)/2]}$ each make a second appearance as irreducibles in V_2. This proves Lemma 5.3.

To prove Lemma 5.4 we define

$$V_{1,k} = \mathbb{R}\{v_k^r, v_k^i\}, \quad \text{where} \tag{5.8a}$$

$$v_k^r = (1, \cos(k\theta), \cdots, \cos((n-1)k\theta)), \tag{5.8b}$$

$$v_k^i = (0, \sin(k\theta), \cdots, \sin((n-1)k\theta)). \tag{5.8c}$$

Here we use $\theta = \theta_n = 2\pi/n$. We assert that

$$V_1 = V_{1,1} \oplus \cdots \oplus V_{1,[n/2]}. \tag{5.9}$$

To verify (5.9) we first show that each $V_{1,j} \subset V_1$. Let $\phi_k = e^{ik\theta}$ for $1 \leqq k \leqq n-1$. Then

$$v_k^r = \operatorname{Re}\Phi_k \quad \text{and} \quad v_k^i = \operatorname{Im}\Phi_k,$$

where

$$\Phi_k = (1, \phi_k, \cdots, \phi_k^{n-1}).$$

Since $\phi_k \neq 1$ is an nth root of unity

$$1 + \phi_k + \cdots + \phi_k^{n-1} = 0.$$

It follows that $v_k^r, v_k^i \in V_1$.

Note that each $V_{1,k}$ is two-dimensional with the single exception that when n is even $V_{1,n/2}$ is one-dimensional (since $v_{n/2}^i = 0$).

Next we show that each subspace $V_{1,k}$ is Γ-invariant and Γ-irreducible. Since all of the irreducibles are distinct representations of Γ, they must be linearly independent. Hence (5.9) will follow by a simple dimension count.

We compute:

$$\mathscr{C}v_k^r = \cos(k\theta)v_k^r - \sin(k\theta)v_k^i, \tag{5.10a}$$

$$\mathscr{C}v_k^i = \sin(k\theta)v_k^r + \cos(k\theta)v_k^i, \tag{5.10b}$$

$$\mathscr{T}v_k^r = -\cos(k\theta)v_k^r + \sin(k\theta)v_k^i, \tag{5.10c}$$

$$\mathscr{T}v_k^i = \sin(k\theta)v_k^r + \cos(k\theta)v_k^i. \tag{5.10d}$$

It follows that each $V_{1,k}$ is Γ-invariant. Moreover, in the basis $\{v_k^r, v_k^i\}$, $\mathscr{C}$ has the matrix form

$$\begin{pmatrix} \cos(k\theta) & \sin(k\theta) \\ -\sin(k\theta) & \cos(k\theta) \end{pmatrix}$$

and hence the action of Γ on $V_{1,k}$ is isomorphic to W_k. □

Remark 5.5. Let K be that part of $\ker(df)_{x_0}$ that is forced to exist due to spatial and time-reversal symmetry. The space K consists of $W_{++} \oplus W_{+-}$ plus W_{--} if n is even plus the subspaces of W_j^3 specified in the proof of Theorem 5.2(c). Observe that

$$\operatorname{Fix}_K(\mathscr{C}) = W_{++} \oplus W_{+-} \tag{5.11}$$

since $\mathscr{C}$ acts nontrivially on W_{--} and by a nontrivial rotation on the relevant W_j^3's.

6. Two-dimensional fixed-point subspaces. Theorem 5.2 states that in the Stokeslet model the eigenvalues of $(df)_{x_0}$ are either real or purely imaginary. We expect, since it is true generically, that the real eigenspace V_i associated with a purely imaginary eigenvalue of $(df)_{x_0}$ will have the form (3.4), that is, $V_i = W \oplus W$ where the group $\Gamma \cong D_n$ acts absolutely irreducibly on W. Moreover, in the Stokeslet model, the possible choices for W are any of the two-dimensional irreducibles W_k $(1 \leqq k \leqq [(n-1)/2])$ and, when n is even, the one-dimensional irreducible W_{-+}.

In this section we recall the classification—up to conjugacy—of all (isotropy) subgroups $\Sigma \subset D_n \times S^1$ that have two-dimensional fixed-point subspaces. This classification is derived in [GSt] (see also [GSS, Chap. XVIII, § 1]) and is implicit in [vGV]. We also describe briefly the implications of having symmetry Σ for each family of periodic solutions.

When n is even and $W = W_{-+}$ the eigenspace V_i is automatically two-dimensional. The symmetry of periodic solutions consists of those elements in $D_n \times S^1$ that act trivially on W_{-+}^2. By definition $\mathscr{C}$ acts by -1 and $\mathscr{T}$ by $+1$ on W_{-+}. Hence the group Σ generated by $(\mathscr{C}, \pi)$ and $\mathscr{T}$ acts trivially on W_{-+} and Σ is the group of symmetries of the associated periodic solutions. Note that Σ is isomorphic to D_n.

These symmetries force solutions $(x_1(t), \cdots, x_n(t))$ to have the form

$$(x, -R_{2\theta}\kappa x, R_{2\theta}x, -R_{4\theta}\kappa x, R_{4\theta}x, \cdots, -\kappa x) \tag{6.1}$$

for all time t, where $\theta = 2\pi/n$. To verify (6.1) observe that if $(\mathscr{C}, \pi)$ fixes a 2π-periodic solution, then

$$R_{-\theta}x_{k+1}(t+\pi) = x_k(t)$$

and solutions have the form

$$(x(t), R_\theta x(t+\pi), R_{2\theta}x(t), \cdots, R_{(n-1)\theta}x(t+\pi)).$$

Invariance under $\mathscr{T}$ implies $-\kappa R_{(n-1)\theta}x(t+\pi) = x(t)$. Thus

$$x(t+\pi) = -R_\theta \kappa x(t).$$

Substitution yields (6.1).

Note that when $n = 4$ and $x \cdot e_1 = 0$ (6.1) reduces to the rhombus

$$(x, \kappa x, -x, -\kappa x).$$

In general, for $n=4$, the horizontal projection of this periodic solution is a (time-dependent) rhombus whose opposite vertices move synchronously in the vertical direction e_1. For general even n, these types of solutions project onto n-gons in the horizontal plane with sides of equal lengths, and alternate vertices move synchronously in the vertical direction e_1. The actual horizontal n-gon, however, may be somewhat complicated.

Next we consider solution types that may be associated with two-dimensional irreducible representations of D_n. To simplify the discussion we first consider the case $n=3$ and then indicate some of the differences when $n>3$. The subgroups of $D_3 \times S^1$ that have two-dimensional fixed point subspaces are classified in [GSS, Chap. XVIII, Table 1.1] and are:

$$\mathbb{Z}_2(\mathcal{T}), \quad \mathbb{Z}_2(\mathcal{T}, \pi), \quad \mathbb{Z}_3(\mathcal{C}, 2\pi/3). \tag{6.2}$$

Solutions (x_1, x_2, x_3) having symmetry $\mathbb{Z}_2(\mathcal{T})$ satisfy

$$x_1 = -\kappa x_3 \quad \text{and} \quad x_2 = -\kappa x_2. \tag{6.3}$$

It follows that such solutions form isosceles triangles where the vertices of the side with unequal length move synchronously in the vertical direction. These are the type of solution found in the Stokeslet model by [H], [CLLS].

Solutions having symmetry $\mathbb{Z}_2(\mathcal{T}, \pi)$ satisfy

$$x_1(t) = -\kappa x_3(t+\pi) \quad \text{and} \quad x_2(t) = -\kappa x_2(t+\pi). \tag{6.4}$$

Since $|x_1| = |x_3|$ these solutions also form isosceles triangles. In this case, however, the endpoints of the side with unequal length move in the vertical direction with a precise half-period phase lag.

Solutions having symmetry $\mathbb{Z}_3(\mathcal{C}, 2\pi/3)$ satisfy

$$R_\theta x_{j+1}(t+2\pi/3) = x_j(t), \qquad j=1,2. \tag{6.5}$$

Such solutions form equilateral triangles whose vertices move with precise one-third period phase lags. These solutions resemble three ponies on a (stationary) merry-go-round.

Two complications arise when generalizing to the case $n>3$. First, for the standard irreducible representation the isotropy subgroups differ slightly depending on whether n is odd or congruent to zero or two mod four. Second, there are $[(n-1)/2]$ different two-dimensional irreducible representations, each leading to solutions with slightly different geometric form. We consider these complications in order.

Let $W = W_1$ be the standard two-dimensional irreducible representation of D_n. For each $n \geqq 3$ there are three (nonconjugate) isotropy subgroups of $D_n \times S^1$ acting on W^2 which have two-dimensional fixed-point subspaces and those are analogous to the $n=3$ case. First, for all n there is a subgroup $\mathbb{Z}_n(\mathcal{C}, 2\pi/n)$ whose solutions (as in the case $n=3$) correspond to n ponies on a stationary merry-go-round.

When n is odd there are solutions with $\mathbb{Z}_2(\mathcal{T})$ symmetry. They have $(n-1)/2$ pairs of sides with equal lengths which move synchronously, the pairing of sides being given by $\mathcal{T}$. We call these solutions *isosceles* n-gons. The unpaired side has vertices that move synchronously. When n is even, the corresponding solution has symmetry $\mathbb{Z}_2(\mathcal{T}) \oplus \mathbb{Z}_2(\mathcal{C}^{n/2}, \pi)$. So when n is even solutions are paired across a line connecting two opposite vertices and there is no unpaired side. Opposite sides (those with $(n/2)-1$ sides between them) move with a half period phase lag.

Finally, when n is odd there are solutions with $\mathbb{Z}_2(\mathcal{T}, \pi)$ symmetry. Again sides are paired, as in the $\mathbb{Z}_2(\mathcal{T})$ case, but for these solutions paired sides move with a half period phase lag. When $n \equiv 2 \bmod 4$ solutions of this type persist but with symmetry

$\mathbb{Z}_2(\mathcal{T}, \pi) \oplus \mathbb{Z}_2(\mathcal{C}^{n/2}, \pi)$. Again sides are paired across a line connecting opposite vertices and there is no unpaired side when n is even. Opposite sides move with a half-period phase lag. The two symmetries together force all but two (opposite) sides to come in quartets with two sides moving synchronously and two with the phase lag. When $n \equiv 0 \bmod 4$ isotropy of the third solution type is $\mathbb{Z}_2(\mathcal{T}R_{2\pi/n}) \oplus \mathbb{Z}_2(\mathcal{C}^{n/2}, \pi)$. Here solutions are paired across a line connecting the midpoints of two opposite sides. These pairs move synchronously while opposite sides move with a phase lag of one-half period.

Periodic solutions corresponding to $V_i = W_k^2$, where $k > 1$ differ from those with $k = 1$ in two ways. First when k divides n, $\mathcal{C}^{n/k}$ acts trivially on W_k. Thus these periodic solutions all divide the n-gon into n/k groups of evenly spaced sides. All of the sides in each group move synchronously. The D_n symmetry reduces to $D_{n/k}$ acting on these groups of sides—rather than individual sides.

Second, when n and k are relatively prime, solutions behave similarly to the $k = 1$ case with two exceptions. The ponies on a merry-go-round move to a different tune. You must move k ponies around to get to the one which is $2\pi/n$ out of phase. Similarly, when n is even, the sides which move with half-period time lag are not opposite but rather are separated by $\ell - 1$ sides where $k\ell \equiv n/2 \bmod n$.

7. Periodic solutions in the Stokeslet model. In this section we apply Theorem 3.1 to the Stokeslet model. It follows from Theorems 5.1 and 5.2 that there are $[n/2]$ pairs of eigenvalues, $\pm\lambda$, of the Jacobian at an equilibrium regular n-gon which can be either real or purely imaginary. These eigenvalues have been computed by [H] for $n \leqq 12$. He finds that when $n \geqq 7$ there is at least one pair which is real and nonzero. In such a case the equilibrium regular polygons are saddles and are asymptotically unstable; this observation is consistent with experiments where the regular n-gons, for $n \geqq 7$, seem to break apart. For this reason, we focus here on the case $3 \leqq n \leqq 6$.

To apply Theorem 3.1 with the isotropy subgroups described in § 6 we need two additional pieces of information. First, the various distinct, purely imaginary eigenvalues must be nonresonant. This point is trivially true for $n = 3$ and may be checked for the cases $n = 4, 5, 6$. It follows immediately that the simple eigenvalues corresponding to W_{+-} yield a family of periodic solutions when n equals four or six.

Second, in the case of the two-dimensional irreducibles W_k, we need to observe whether the time-reversal symmetry R preserves $\mathrm{Fix}_i(\Sigma)$. It turns out that R does preserve $\mathrm{Fix}_i(\Sigma)$ for either Σ that is generated by reflections, but does not for $\Sigma \cong \mathbb{Z}_n$. Consequently, using Theorem 3.1, we will prove the existence of two families of periodic solutions (generalizing the synchronous and asynchronous isosceles triangles) for $3 \leqq n \leqq 6$. We will show in the next section that there are two-frequency, quasi-periodic solutions corresponding to $\mathbb{Z}_n$, but additional remarks are necessary to verify this point. These quasi-periodic solutions may be thought of as ponies on a (rotating) merry-go-round.

The proof that the time-reversal symmetry R preserves $\mathrm{Fix}_i(\Sigma)$ when $\Sigma = \mathbb{Z}_2(\mathcal{T})$ or $\mathbb{Z}_2(\mathcal{T}, \pi)$ is straightforward since R commutes with all spatial symmetries, including $\mathcal{T}$, and with the phase shifts 0 and π. This commutativity ensures that $\mathrm{Fix}_i(\Sigma)$ is an invariant subspace for R. Note that R does not commute with $(\mathcal{C}, 2\pi/n)$ and it is for this reason that $\mathrm{Fix}_i(\mathbb{Z}_n)$ is not invariant under R.

In Table 7.1 we list all of those families of periodic solutions whose existences are proved by combining the calculations of [H] with Theorems 3.1 and 5.1.

8. Quasiperiodic solutions in the Stokeslet model. Krupa [K] considers bifurcation from group orbits of equilibria and, indeed, in the Stokeslet model the basic equilibrium,

TABLE 7.1
Periodic solutions determined for n particles in the Stokeslet model with $3 \leqq n \leqq 6$. *(See* § 8 *for quasi-periodic solutions.)*

n	W	Solution type	Isotropy
3	W_1	Synchronous isosceles triangles	$\mathbb{Z}_2(\mathcal{T})$
		Asynchronous isosceles triangle	$\mathbb{Z}_2(\mathcal{T}, \pi)$
4	W_{-+}	Synchronous rhombus	$D_4((\mathscr{C}, \pi), \mathcal{T})$
	W_1	Rectangles	$\mathbb{Z}_2(\mathcal{T}R_{\pi/2})$
		Isosceles quadrilateral	$\mathbb{Z}_2(\mathcal{T})$
5	W_1	Synchronous isosceles pentagon 1	$\mathbb{Z}_2(\mathcal{T})$
		Asynchronous isosceles pentagon 1	$\mathbb{Z}_2(\mathcal{T}, \pi)$
	W_2	Synchronous isosceles pentagon 2	$\mathbb{Z}_2(\mathcal{T})$
		Asynchronous isosceles pentagon 2	$\mathbb{Z}_2(\mathcal{T}, \pi)$
6	W_{-+}	Synchronous equilateral hexagon	$D_6((\mathscr{C}, \pi), \mathcal{T})$
	W_1	Synchronous isosceles hexagon 1	$\mathbb{Z}_2(\mathcal{T})$
		Asynchronous isosceles hexagon 1	$\mathbb{Z}_2(\mathcal{T}, \pi)$
	W_2	Synchronous isosceles hexagon 2	$\mathbb{Z}_2(\mathcal{T}) \oplus \mathbb{Z}_2(\mathscr{C}^3)$
		Asynchronous isosceles hexagon 2	$\mathbb{Z}_2(\mathcal{T}, \pi) \oplus \mathbb{Z}_2(\mathscr{C}^3)$

the regular n-gon, lies on an $O(2)$-orbit of such equilibria. Until now, we have neglected these symmetries having needed only to concentrate on the discrete D_n symmetry to obtain our results. We will use Krupa's results to reduce the dimension of the Stokeslet model by one—essentially we quotient by the space W_{+-} in the isotypic decomposition of Theorem 5.1. On this reduced space, we use Theorem 3.1 to prove the existence of the periodic ponies on stationary merry-go-round solutions. Finally, we show that the periodic ponies correspond to two-frequency trajectories in the full phase space. These solutions may be interpreted as ponies on a rotating merry-go-round.

We begin by describing Krupa's results along with their adaptation to the time-reversible situation. Let Δ be a compact Lie group acting linearly on $\mathbb{R}^n$. Let $f(x)$ be a smooth Δ-equivariant vector field with equilibrium x_0. Let $X = \Delta x_0$ be the group orbit of equilibria containing x_0 and let Γ be the isotropy subgroup of x_0 in Δ.

THEOREM 8.1. *There exists a Δ-invariant tubular neighborhood N of X and a decomposition on N*

$$f = f_T + f_N, \tag{8.1}$$

where

(i) *f_T and f_N are smooth and Δ-equivariant,*
(ii) *f_T is tangent to group orbits of Δ,*
(iii) *f_N is normal to X.*

Remark. By "normal to X" we mean the following. Let $\pi: N \to X$ be the projection associated with the tubular neighborhood and let N_x be the fiber $\pi^{-1}(x)$ for all $x \in X$. (We can certainly choose N so that N_x is normal to X in $\mathbb{R}^n$.) Then "f_N is normal to X" means that $f_n(y) \in N_{\pi(y)}$ for each $y \in N$.

The second result concerns the relationship between solutions of the differential equations $dx/dt = f(x)$ and $dy/dt = f_N(y)$.

THEOREM 8.2. *Let $x(t)$ be the solution to $dx/dt = f(x)$ with initial condition $x(0) = y_0 \in N_{x_0}$. Let $y(t)$ be the solution in N_{x_0} to $dy/dt = f_N(y)$ with the same initial condition $y(0) = y_0$. Then there exists a smooth curve $\delta(t) \in \Delta$ with $\delta(0) = 1$ satisfying*

$$x(t) = \delta(t)y(t).$$

Remarks 8.3. (a) We can think of $\delta(t)$ as representing a drift in the solution $x(t)$ along the group orbit through $y(t)$. In the Stokeslet model, group orbits of $\Delta = O(2) \times D_n$ are finite unions of circles. So if $y(t)$ is periodic and $\delta(t)$ is nonzero, we expect $x(t)$ to be a flow on a 2-torus.

(b) In general, $\delta(t)$ can be chosen to be a one-parameter group in Δ. Hence, the closure of the image of δ is Abelian and hence a torus.

(c) Whether drifting actually occurs ($\delta(t) \not\equiv 1$) depends on whether f_T is nonzero on $y(t)$. Generically, it can be determined from group theoretic calculations alone whether drifting is possible. For any given f, however, such as the Stokeslet model, we need to check explicitly that f_T is nonzero. Not surprisingly, for the Stokeslet model, these algebraic conditions show that drifting is not possible (that is, $f_T(y(t)) \equiv 0$) for the periodic solutions constructed in § 7. We show below that $f_T \neq 0$ for the ponies on a merry-go-round solution.

We now show how the decomposition (8.1) respects time-reversibility. We assume

$$f(Rx) = -Rf(x)$$

for some orthogonal matrix R that commutes with Δ. We also assume $Rx_0 = x_0$ for the equilibrium x_0 of f. We assert:

$$R(T_x\Delta x) = T_{Rx}\Delta(Rx) \quad \text{and} \quad RN_x = N_x, \tag{8.2a}$$

$$f_N(Rx) = -Rf_N(x), \tag{8.2b}$$

$$f_T(Rx) = -Rf_T(x). \tag{8.2c}$$

We actually prove (8.2) only when the dimension of group orbits in N is constant, though this restriction is not necessary. To verify (8.2a) let $z(t) = \delta(t)x$ be a curve in the group orbit of Δ through x. The typical element of $T_x \Delta x$ is $v = dz/dt(0)$. Observe that

$$Rv = \frac{d}{dt} R\delta(t)x = \frac{d}{dt} \delta(t)Rx \in T_{Rx} \Delta Rx,$$

since R commutes with Δ. Note that $Rx = x$ since $Rx_0 = x_0$ and R commutes with Δ. Hence $R(T_xX) = T_xX$ and since R is orthogonal $R(N_x) = N_x$.

To verify (8.2b) observe that

$$0 = Rf(x) + f(Rx) = [Rf_N(x) + f_N(Rx)] + [Rf_T(x) + f_T(Rx)],$$

where the first sum lies in $N_{\pi(x)}$ and the second sum lies in $T_{Rx}\Delta(Rx)$. By assumption these spaces are transverse; hence each sum is zero.

Remarks 8.4. (a) In the Stokeslet model f_T is tangent to $SO(2)$ group orbits. Tangent vectors to such orbits are horizontal, that is, perpendicular to e_1. R acts trivially on f_T. Thus (8.2c) is $f_T(Rx) = -f_T(x)$.

(b) Indeed, if we define

$$J \equiv \left.\frac{dR_\theta}{d\theta}\right|_{\theta=0}, \tag{8.3}$$

then tangent vector fields all have the form $a(x)Jx$ where

$$a(x) = \frac{f(x) \cdot Jx}{|Jx|^2}. \tag{8.4}$$

Note that by abuse of notation

$$Jx = (Jx_1, \cdots, Jx_n), \quad \text{where } J = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & -1 & 0 \end{pmatrix}.$$

In the Stokeslet model, we fixed an equilibrium regular n-gon x_0 by (4.9) and (4.10). For this equilibrium the isotropy subgroup $\Gamma \cong D_n$ is defined by (4.11). It is easy to check that the $\Delta = D_n \times O(2)$ group orbit through x_0 is one-dimensional and the tangent space to this group orbit is W_{+-}. (To verify this assertion we show that $\mathscr{C}$ acts trivially on the tangent space while $\mathscr{T}$ acts as $-I$.) Thus N_{x_0} is the sum of all isotypic components of V under Γ as listed in Theorem 5.1 except for W_{+-}. We now show, using Theorem 3.1, that there is a family of periodic solutions for $f_N \mid N_{x_0}$ corresponding to isotopy $\Sigma = \mathbb{Z}_n(\mathscr{C}, 2\pi/n)$. These are the ponies on a stationary merry-go-round.

As indicated in the last section, we cannot use the time-reversal symmetry R to prove directly the existence of the periodic solutions. The obstacle is that R does not leave the subspace $\mathrm{Fix}_i\,(\Sigma)$ invariant. To see this, observe

$$R \circ (\mathscr{C}, 2\pi/n) = (\mathscr{C}, -2\pi/n)R. \tag{8.5}$$

Hence, if $v \in \mathrm{Fix}_i\,(\Sigma)$, then Rv is fixed by $(\mathscr{C}, -2\pi/n)$. As shown in [GSS, Chap. XVIII] the subgroups Σ and $\Sigma' = \mathbb{Z}_n(\mathscr{C}, -2\pi/n)$ are conjugate (indeed $\mathscr{T}: \mathrm{Fix}\,(\Sigma) \to \mathrm{Fix}\,(\Sigma')$) and $\mathrm{Fix}\,(\Sigma) \perp \mathrm{Fix}\,(\Sigma')$.

We now apply Theorem 3.1 to the time-reversal symmetry $R' = \mathscr{T}R$. This time-reversal symmetry does leave $\mathrm{Fix}\,(\Sigma)$ invariant, but now we must check that (3.8c) is valid for R'. Namely, we must verify that

$$R' \mid \mathrm{Fix}_0\,(\Sigma) = I. \tag{8.6}$$

Remark 3.2 implies that $\mathrm{Fix}_0\,(\Sigma) = \mathrm{Fix}_0\,(\mathscr{C})$ and Remark 5.5 states that $\mathrm{Fix}_0\,(\mathscr{C}) = W_{++} \oplus W_{+-}$. Clearly, $R' \mid W_{++} = I$ since both R and $\mathscr{T}$ act trivially on W_{++}. The same is not true for W_{+-} since $\mathscr{T}$ acts nontrivially on W_{+-}; that is why we are obliged to use the normal vector field f_N and Theorem 8.1. Indeed, W_{+-} is not in the domain of f_N and (8.4) is satisfied for f_N. Thus, ponies on a stationary merry-go-round exist for f_N.

In the remainder of this section we verify that these ponies drift around the $O(2)$ group orbits, as described in Theorem 8.2. To do this we must first give a precise definition of drift. Write the Stokeslet model abstractly as

$$\frac{dx}{dt} = F(x), \tag{8.7}$$

where $F = F_N + F_T$, as in Theorem 8.1. Let $y(t)$ denote ponies on a merry-go-round solution for F_N which is T-periodic, and let $x(t)$ denote the corresponding trajectory for F. It follows from Theorem 8.2 that there exists a curve $\delta(t)$ in $SO(2)$ such that $x(t) = \delta(t)y(t)$.

DEFINITION 8.5. *Drift* occurs on $y(t)$ if $\delta(t) \not\equiv 1$. The *net drift* of $y(t)$ is $\delta(T)$.

If the net drift is unequal to one, then the solution $x(t)$ will drift around a 2-torus. If $\delta(T)$ and T are rationally independent, then $x(t)$ will be a true two-frequency motion and its trajectory will be dense on a 2-torus.

So to determine whether $x(t)$ has two-frequencies, we must compute $\delta(T)$. It follows from (8.7) that

$$\frac{d\delta}{dt} y = \delta F_T(y). \tag{8.8}$$

Using Remark 8.4(b), we write $F_T(y) = a(y)Jy$ for some smooth function $a(y)$. We can also rewrite $\delta(t) = R_{\phi(t)}$ with $\phi(0) = 0$. This gives δ explicitly as a curve in the action of $SO(2)$. Direct computation using the definition of R_θ leads to

$$\frac{dR_\phi}{dt} = \dot{\phi} R_\phi J. \tag{8.9}$$

Substitution of (8.9) into (8.8) yields

$$\dot{\phi} = -a(y(t)). \tag{8.10}$$

Integration of (8.10) yields a formula for the net drift:

$$\phi(T) = -\int_0^T a(y(t))\,dt. \tag{8.11}$$

Thus, to compute $\phi(T)$ we need formulas for $a(y)$ and $y(t)$. The formula for a in terms of F is given by (8.4); the more complicated part of this calculation involves finding an approximate formula for $y(t)$. To simplify the exposition we restrict our attention to the three particle Stokeslet model.

For $n = 3$, (4.2) states

$$F(y) = (U(y_2) - U(y_3),\ U(y_3) - U(y_1),\ U(y_1) - U(y_2)).$$

Using (4.3) and (8.4), we find that

$$\begin{aligned} a(y) = \frac{1}{|Jy|^2} \bigg[& \frac{e_1 \cdot y_1}{|y_1|^3}\, y_1 \cdot J(y_3 - y_2) \\ & + \frac{e_1 \cdot y_2}{|y_2|^3}\, y_2 \cdot J(y_1 - y_3) + \frac{e_1 \cdot y_3}{|y_3|^3}\, y_3 \cdot J(y_2 - y_1) \bigg]. \end{aligned} \tag{8.12}$$

Recall from (6.5) that ponies satisfy

$$y(t) = \left(p(t),\ R_\theta p\left(t + \frac{T}{3}\right),\ R_{2\theta} p\left(t + \frac{2T}{3}\right) \right),$$

where $\theta = 2\pi/3$. It follows that integrating each of the summands in (8.12) yields the same value. Hence

$$\phi(T) = -3 \int_0^T \frac{1}{|Jy|^2} \frac{1}{|y|^3} (e_1, y_1)(y_1, J(y_3 - y_2))\,dt. \tag{8.13}$$

Recall that Theorem 3.1 guarantees that F_N has a continuous family of T_ε-periodic ponies $y_\varepsilon(t)$ with $y_0(t) \equiv x_0$. Hence

$$y_\varepsilon(t) = x_0 + \varepsilon z(\omega_\varepsilon t) + O(\varepsilon^2), \tag{8.14}$$

where $z(t)$ is 2π-periodic, $\omega_\varepsilon = T_\varepsilon / 2\pi$, and from (4.9)

$$x_0 = (R_{\theta/2} e_3,\ R_{3\theta/2} e_3,\ R_{5\theta/2} e_3) \equiv (x_1^0, x_2^0, x_3^0). \tag{8.15}$$

Indeed, if we rescale time in the Stokeslet model so that the imaginary eigenvalues of $(dF)_{x_0}$ are $\pm i$, then the proof of Theorem 3.1 shows that $T_\varepsilon = 2\pi + O(\varepsilon^2)$.

THEOREM 8.6. *$\phi(T_\varepsilon) = K\varepsilon^2 + O(\varepsilon^3)$, where $K \neq 0$.*

Remark. Theorem 8.6 implies that $\phi(T_\varepsilon)$ is nonzero for small values of ε and hence net drift is nonzero. Since both ϕ and T vary continuously with ε, it follows that for most values of ε near zero $\phi(T_\varepsilon)/T_\varepsilon$ is irrational. Hence, the corresponding trajectories $x_\varepsilon(t)$ for the full Stokeslet model are dense in 2-tori.

The proof of Theorem 8.6 is based on the explicit calculation of $z(t) = (z_1(t), z_2(t), z_3(t))$, which we summarize by Lemma 8.7.

LEMMA 8.7. *There exist a horizontal vector h and a 2π-periodic function $c(t)$ such that*

(8.16a) $$z_1(t) = c(t)e_1 + R_t h,$$

(8.16b) $$z_2(t) = c(t+\theta)e_1 + R_{t+2\theta}h,$$

(8.16c) $$z_3(t) = c(t+2\theta)e_1 + R_{t+\theta}h.$$

Proof of Theorem 8.6. Since $e_1 \cdot x_1^0 = 0$, $\phi(T_\varepsilon)$ is at least order ε. Indeed, $\phi(T_\varepsilon)$ is of order at least ε^2 since from (8.16a) the order ε term is

$$C\int_0^T c(\omega_\varepsilon t)\,dt = 0,$$

where $C \equiv x_1^0 \cdot J(x_3^0 - x_2^0) \neq 0$.

Up to nonzero constant multiples, there are four terms contributing to the order ε^2 term in $\phi(T_\varepsilon)$:

(i) $\int_0^T e_1 \cdot (\text{order } \varepsilon^2 \text{ term in } y_1)\,dt$,

(ii) $\int_0^T c(\omega_\varepsilon t) \cdot (\text{order } \varepsilon \text{ term in } y_1 \cdot J(y_3 - y_2))\,dt$,

(iii) $\int_0^T c(\omega_\varepsilon t) \cdot (\text{order } \varepsilon \text{ term in } |Jy|^{-2})\,dt$,

(iv) $\int_0^T c(\omega_\varepsilon t) \cdot (\text{order } \varepsilon \text{ term in } |y_1|^{-3})\,dt$.

We show that the first three of these are zero and that the fourth is nonzero.

To verify that (i) is zero, recall that y is in V and hence that $y_1 + y_2 + y_3 = 0$. Thus, from (8.12), the sum of the contributions of the three integrands is zero.

To compute (ii), observe that the order ε term in $y_1 \cdot J(y_3 - y_2)$ is $z_1 \cdot J(x_3^0 - x_2^0) + x_1^0 \cdot J(z_3 - z_2)$ which, after substitution of (8.15) and (8.16), is identically zero.

Contribution (iii) is shown to be zero by verifying that

$$|Jy|^2 = 3 + O(\varepsilon^2).$$

Finally, we compute (iv). By direct calculation

$$|y_1|^{-3} = 1 - 3\varepsilon c(\omega_\varepsilon t) + O(\varepsilon^2).$$

Since $\int_0^T c(\omega_\varepsilon t)^2\,dt > 0$, we have proved the theorem. □

Proof of Lemma 8.7. The basic idea behind the Lyapunov-Schmidt reduction proof of the existence of periodic solutions in Theorem 3.1 is to restrict the reduced operator to the invariant subspace Fix (Σ) and to find solutions on that subspace. The difficulty in actually finding these periodic solutions in the phase space V occurs when Σ is a subgroup of $\Gamma \times S^1$ having elements with a nontrivial temporal (S^1) part. Since S^1 does not act directly on V (it does act by phase shift on $\mathscr{C}_{2\pi}$), it is somewhat difficult to find Fix (Σ) in V. Indeed, for the ponies solutions, $\Sigma = \mathbb{Z}_3(\mathscr{C}, \pm 2\pi/3)$, with either choice of sign possible.

In the Lyapunov-Schmidt reduction, however, we find periodic solutions by implicitly solving certain equations for higher order terms that are functions of periodic solutions of the linearized equations. More precisely, let $\mathscr{W}$ be the space of 2π-periodic solutions to $du/dt = (dF)_{x_0}u$. That is, $\mathscr{W}$ is spanned by $e^{it}v$ where v is an eigenvector of $(dF)_{x_0}$ with eigenvalue i. On the space $\mathscr{W}$, S^1 acts naturally by phase shifts and we can find Fix (Σ). Moreover, the periodic solutions in $\mathscr{W}$ are, after rescaling the period by ω_ε, the first order approximations to the actual periodic solutions of the Stokeslet model. Hence, we can prove the lemma if we can identify Fix (Σ) inside $\mathscr{W}$.

Recall from Theorem 5.1 that $(dF)_{x_0}$ has an invariant subspace isomorphic to W_1^2, which is $V_1 \oplus V_{2,2}$ in coordinates. See (5.8), (5.6). According to Theorem 5.2, the eigenvalues of $L \equiv (dF)_{x_0}|W_1^2$ are either real or purely imaginary, and the remaining eigenvalues are zero. Indeed, [H] shows that the nonzero eigenvalues are purely imaginary, and we have rescaled time so that they are $\pm i$. In addition, Theorem 5.2 implies that

$$L = \begin{pmatrix} 0 & aA \\ bA^{-1} & 0 \end{pmatrix},$$

where $A: V_{2,2} \to V_1$ is a fixed Γ-equivariant isomorphism and $ab = -1$ since the imaginary eigenvalues of L are $\pm i$. Replacing aA by $-A$, we see that

$$L = \begin{pmatrix} 0 & -A \\ A^{-1} & 0 \end{pmatrix}.$$

The periodic solutions in $\mathcal{W}$ are generated by

$$\exp(tL) = \begin{pmatrix} \cos(t)I & -\sin(t)A \\ \sin(t)A^{-1} & \cos(t)I \end{pmatrix}. \tag{8.17}$$

Note that (8.17) determines the action of S^1 explicitly.

We now use the symmetry of ponies solutions and (8.17) to determine the function $z(t)$ explictly. Let $(p, q) \in V_{2,2} \oplus V_1$ and let

$$z(t) = e^{tL}(p, q) \tag{8.18}$$

be a periodic function in $\mathcal{W}$. Then $z(t)$ is a ponies periodic function if $\mathcal{C}z(t+\theta) = z(t)$. It follows that

$$e^{-\theta L}(p, q) = (\mathcal{C}p, \mathcal{C}q). \tag{8.19}$$

Using (8.17), we can parametrize solutions to (8.19) by

$$p = -\frac{1}{\sqrt{3}} A(I + 2\mathcal{C})q \equiv Aq^{\perp}, \tag{8.20}$$

where $q \in V_{2,2}$ is arbitrary.

Since q is in $V_{2,2}$, it has the form $(h, R_\theta h, R_{2\theta} h)$ for some horizontal vector h. See (5.5). Using the action of $\mathcal{C}$ on $V_{2,2}$ (see (5.7)) it follows that $q^{\perp} = Jq$. We can now rewrite (8.20) as

$$p = AJq. \tag{8.21}$$

Next, we use (8.17) and (8.18) to show that

$$z(t) = (AJ(\cos(t)I + \sin(t)J)q, (\cos(t)I + \sin(t)J)q).$$

Finally, we observe that on $V_{2,2}$

$$R_t = \cos(t)I + \sin(t)J.$$

Hence,

$$z(t) = (AJR_t q, R_t q),$$

which proves the lemma since the first component of $AJR_t q$ is both a periodic function in t and a vertical vector (being in V_1), and the first component of $R_t q$ is $R_t h$ and h is a horizontal vector. □

9. Remarks on stability. In this section we discuss the stability of the families of periodic and quasi-periodic solutions that we found for the three-particle Stokeslet model. Although many of our comments will be valid for the general n particle model, we consider here only the technically less complicated three-particle case.

Our discussion will focus on linear stability, though we will remark briefly on the issue of nonlinear stability at the end of the section. Our results are similar to those of [MRS], who consider linear stability for periodic solutions of general Hamiltonian systems with spatial symmetry, and specifically consider D_3 symmetry in the Henon–Heiles system.

As in Hamiltonian systems, periodic solutions in time-reversible systems that are fixed by a time-reversal symmetry cannot be asymptotically stable, since if one trajectory approaches the cycle in forward time another must approach the cycle in backward time. Thus linear stability of such periodic solutions must correspond to Floquet exponents on the imaginary axis, the *elliptic* case, and instability to Floquet exponents with nonzero real part, the *hyperbolic* case.

When $n=3$ the Stokeslet model is posed on the six-dimensional space $V=W_{++}\oplus W_{+-}\oplus W^2$ where $W\cong\mathbb{C}$. In fact, continuous spatial symmetries force stability questions onto the four-dimensional subspace W^2. Besides the rotational symmetry $SO(2)$, whose tangent space at the equilibrium x_0 is W_{+-}, there is a scaling symmetry for the Stokeslet model (which until now has not been used explicitly):

$$F(rx)=\frac{1}{r}F(x). \tag{9.1}$$

See (4.3) to verify (9.1). The tangent space to the scaling symmetry is W_{++}, corresponding to equilibrium equilateral triangles of varying side length. Thus, when considering orbital stability, we need only view the stability of periodic solutions to the normal vector field F_N (defined on $W_{++}\oplus W^2$) restricted to W^2.

Our discussion in § 6 shows that there are three families of periodic solutions to F_N: ponies on a merry-go-round, $\mathbb{Z}_3(\mathscr{C}, 2\pi/3)$; synchronous isosceles triangles, $\mathbb{Z}_2(\mathscr{T})$; and asynchronous isosceles triangles, $\mathbb{Z}_2(\mathscr{T}, \pi)$. We prove the following theorem.

THEOREM 9.1. *Generically,* $\mathbb{Z}_3(\mathscr{C}, 2\pi/3)$ *periodic solutions are elliptic, and one of the isosceles triangles solutions is elliptic while the other is hyperbolic.*

Sketch of the proof. Our proof relies on many of the calculations presented in [GSS, Chap. XVIII], which we summarize here. The Poincaré–Birkhoff normal form of F_N on $W^2\cong\mathbb{C}^2$ has $D_3\times S^1$ equivariance—up to any finite order. Following [GSS, Chap. XVIII, Prop. 2.1] the normal form vector field has the form:

$$g(z_1, z_2)=A_1\begin{bmatrix} z_1 \\ z_2 \end{bmatrix}+A_2\begin{bmatrix} z_1^2\bar{z}_1 \\ z_2^2\bar{z}_2 \end{bmatrix}+A_3\begin{bmatrix} \bar{z}_1^2 z_2^3 \\ z_1^3\bar{z}_2^2 \end{bmatrix}+A_4\begin{bmatrix} z_1^4\bar{z}_2^3 \\ \bar{z}_1^3 z_2^4 \end{bmatrix}, \tag{9.2}$$

where the A_j are complex-valued functions of

$$\begin{aligned} &N=|z_1|^2+|z_2|^2, \quad P=|z_1|^2|z_2|^2, \quad S=(z_1\bar{z}_2)^3+(\bar{z}_1z_2)^3, \\ &T=i(|z_1|^2-|z_2|^2)((z_1\bar{z}_2)^3-(\bar{z}_1z_2))^3. \end{aligned} \tag{9.3}$$

Normal form can be achieved respecting the time-reversal symmetry $R(z_1, z_2)=(\bar{z}_1, \bar{z}_2)$. The anticommutativity of (9.2) with respect to R implies that

$$A_j=a_jT+i\alpha_j \qquad (j=1, \cdots, 4), \tag{9.4}$$

where the a_j and α_j are real-valued functions of N, P, S since

$$T^2=(4P-N^2)(S^2-4P^3).$$

See [GSS, Chap. XVIII, (2.2)].

For the normal form equations, the Floquet equations can be transformed to constant coefficient equations and solved explicitly [GSS, Chap. XVI, Prop. 6.4]. We denote the Floquet matrix for a periodic solution $u(t)$ by M_u.

By general theory, we know that both eigenvalues of M_u restricted to the two-dimensional subspace Fix (Σ), where Σ is the isotropy subgroup corresponding to u, are zero. One zero eigenvalue corresponds to the eigenvector u'; the other zero eigenvalue follows from the fact that Fix (Σ) is foliated by periodic trajectories (Theorem 3.1). Thus, linear stability is determined in the two-dimensional Σ-invariant subspace V_1 transverse to Fix (Σ).

For the $\mathbb{Z}_3$ solutions, M_u is a rotation matrix on V_1 [GSS, Chap. XVIII, Table 3.4] and time-reversal symmetry implies that this rotation must be by $\pi/2$. Hence the eigenvalues of $M_u | V_1$ lie on the imaginary axis. Assuming that these eigenvalues are nonzero, which is implied by a nondegeneracy condition on the third-order terms in the normal form equations g, the ponies are elliptic—at least in normal form.

When normal form is broken at high order, the ponies still remain elliptic. The two zero eigenvalues of the Floquet equations are still forced by u' and the family of periodic solutions: by continuity the other two eigenvalues must be nonzero and close to purely imaginary. Time-reversal symmetry implies that these eigenvalues are $\pm\lambda$ even when the equations are not in normal form, and hence these eigenvalues must remain purely imaginary.

For the $\mathbb{Z}_2(\mathcal{T})$, $\mathbb{Z}_2(\mathcal{T}, \pi)$ solutions, time-reversal symmetry implies that the transverse eigenvalues are $\pm\lambda$, and hence either elliptic or hyperbolic. Which occurs depends on the sign of det $(M_u | V_1)$. These determinants are computed for normal form equations in [GSS, Chap. XVIII, Table 3.2] and, at lowest order, have opposite signs. Indeed, the sign of this determinant is determined by the sign of $\alpha_2(0)\alpha_3(0)$, thus yielding a nondegeneracy condition involving terms of fifth order in the normal form equations. When normal form symmetry is broken at order greater than five, the argument given for the ponies solutions applies here also. Thus, generically, one of the isosceles triangles solutions will be elliptic and one hyperbolic. □

We end by discussing briefly nonlinear stability. As noted above, we know that these periodic solutions will never be asymptotically stable. There is, however, a KAM theory for time-reversible systems indicating the existence of invariant 2-tori for the normal vector field F_N surrounding the periodic solutions. See Sevryuk [Se] and Scheurle [Sc]. These results do not apply to cases of 1:1 resonance in the eigenvalues of the Jacobian $(dF_N)_{x_0}$. Because of D_3 symmetry, this 1:1 resonance is forced.

Let us speculate that the source of the multiple eigenvalues, the spatial D_3 symmetry, restricts the form of the equations sufficiently so that Scheurle's proof can be adapted to this symmetry case. Then, typical trajectories of F_N would lie on invariant 2-tori. We also speculate that due to the drift, computed in § 8, 2-tori near the ponies for F_N would lead to three-frequency motion in the full six-dimensional Stokeslet model.

Appendix. Derivation of the Stokeslet model. Under the assumption of small Reynolds number based on sphere radius, the Stokes equations:

(A1a) $$\mu\Delta\mathbf{u} - \nabla p = 0,$$

(A1b) $$\boldsymbol{\nabla} \cdot \mathbf{u} = 0$$

with appropriate boundary conditions, describe the sedimentation of solid spheres. In (A1) $\mathbf{u}$ is the fluid velocity, μ is the viscosity, and p is the pressure. Let U be the fundamental solution (Green's function) of (A1a) given in (4.3). The fluid velocity at position $\mathbf{y}$ due to a point load $mg\mathbf{e}_1$ at position x is given by the *Stokeslet* $\tau U(\mathbf{y}-\mathbf{x})$ where $\tau = mg/8\pi\mu$.

Consider a system of n identical spheres with radius a, sedimenting under the action of gravity. Let $\mathbf{z}_j$ be the center of the jth sphere. If $\mathbf{u}$ is the velocity of the fluid, then Faxen's first law (cf. [CR], [HB]) states that the velocity $\mathbf{v}$ of a sphere with radius a, under the action of a force $\mathbf{F}$, is given by

$$\mathbf{v} = \mathbf{u} + \mathbf{F}/6\pi\mu a.$$

Therefore, the relative motions of the system of n spheres are governed by

$$\frac{d\mathbf{x}}{dt} = \mathbf{u}_j - \mathbf{u}_{j-1},$$

where $x_j = \mathbf{z}_{j+1} - \mathbf{z}_j$; $\mathbf{u}_j$ is the fluid velocity experienced by the jth sphere which depends on the positions of the other $n-1$ spheres. In the point particle limit, i.e., $a \to 0$, and after rescaling time, the governing equations become (4.2).

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