Final Review

General Solution of Maxwell Equations

$$\boxed{\Box A^\mu = \frac{4\pi}{c} J^\mu}$$ maxwell equations (Gauss units)

Looked for Green function: $\Box G(x, x') = 4\pi\, \delta^3(\vec{x}-\vec{x}')\,\delta(t-t')$

$\Rightarrow$ found $$G_{ret}(t, \vec{r}) = -4\pi \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{e^{-i\omega t + i\vec{k}\cdot\vec{r}}}{\left(\frac{\omega}{c}\right)^2 - \vec{k}^2 + i\varepsilon\omega}$$

($G_{ret}(t, \vec{r}) = 0$ for $t<0$, retarded Green function)

$\Rightarrow$ $$\boxed{G_{ret}(t, \vec{r}) = \frac{1}{r}\,\delta\!\left(t - \frac{r}{c}\right)}$$

Solution of Maxwell equations: (SI units)

$$\boxed{\begin{aligned} \Phi(\vec{x}, t) &= \frac{1}{4\pi\varepsilon_0} \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\, \rho\!\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right) \\ \vec{A}(\vec{x}, t) &= \frac{\mu_0}{4\pi} \int d^3x' \frac{1}{|\vec{x}-\vec{x}'|}\, \vec{J}\!\left(\vec{x}', t - \frac{|\vec{x}-\vec{x}'|}{c}\right) \end{aligned}}$$

know $\rho, \vec{J}$ $\Rightarrow$ get $\Phi, \vec{A}$

Complex Analysis

$f(z)$ is analytic at z_0 if it is differentiable (and single-valued) in a neighbourhood of z_0.

$$f(z) = u + i v \Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \quad \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$ Cauchy-Riemann conditions

Cauchy Formula: $$f(z_0) = \frac{1}{2\pi i} \oint_C dz \, \frac{f(z)}{z - z_0}$$

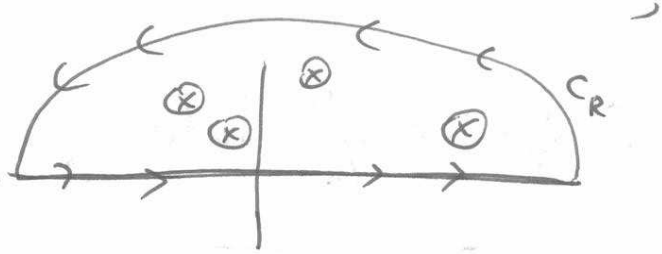
Residue Theorem: $$\boxed{\oint_C dz\, f(z) = 2\pi i \sum_{n=1}^{N} \mathrm{Res}\, f(z_n)}$$

Simple pole: $$\mathrm{Res}\, f(a) = \lim_{z \to a} \left[(z-a) f(z) \right]$$

n^{th} order pole: $$\mathrm{Res}\, f(a) = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]$$

$\int_{-\infty}^{\infty} dx\, f(x)\, e^{ikx}$, $k > 0 \Rightarrow$ Jordan's Lemma: if

$\lim_{R \to \infty} f(Re^{i\varphi}) = 0$ for $\forall \varphi \Rightarrow$ can close the contour, neglecting C_R:



Regulating singularities:

$$\boxed{PV \frac{1}{x} \equiv \frac{1}{2} \left[\frac{1}{x - i\varepsilon} + \frac{1}{x + i\varepsilon} \right]}$$ ~ principal value

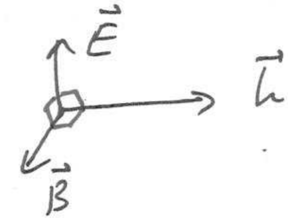
$$\boxed{\frac{1}{x - i\varepsilon} - \frac{1}{x + i\varepsilon} = 2\pi i\, \delta(x)}.$$

Plane Electromagnetic Waves

$$\left[\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}\right]\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0$$

$\Rightarrow$ for monochromatic plane wave write

$$\begin{cases}\vec{E} = \vec{E}_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} \\ \vec{B} = \frac{1}{\omega}\,\vec{k}\times\vec{E} \quad , \; k = \omega\sqrt{\mu\varepsilon}\end{cases}$$



In the end one has to take Re parts for $\vec{E}$ & $\vec{B}$.

Energy density $\langle u\rangle = \frac{1}{2}\,\varepsilon E_0^2$ (time-averaged)

Poynting vector $\langle\vec{S}\rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\,E_0^2\,\hat{k}$. (time-ave.)

$$u = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right) \qquad \vec{S} = \vec{E}\times\vec{H}$$

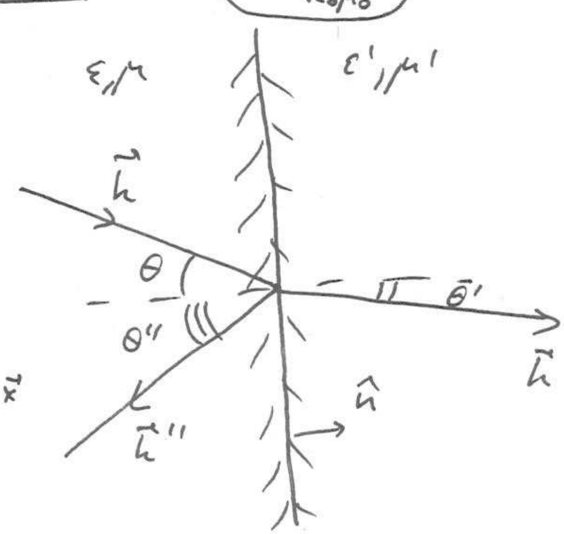
phase velocity

$$V_{ph} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$$

index of refraction

$$n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

Reflection and Refraction



incoming wave:

$$\begin{cases}\vec{E} = \vec{E}_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} \\ \vec{B} = \frac{1}{\omega}\,\vec{k}\times\vec{E}\end{cases}$$

refracted wave

$$\begin{cases}\vec{E}' = \vec{E}_0'\, e^{-i\omega' t + i\vec{k}'\cdot\vec{x}} \\ \vec{B}' = \frac{1}{\omega'}\,\vec{k}'\times\vec{E}'\end{cases}$$

reflected wave

$$\begin{cases}\vec{E}'' = \vec{E}_0''\, e^{-i\omega'' t + i\vec{k}''\cdot\vec{x}} \\ \vec{B}'' = \frac{1}{\omega''}\,\vec{k}''\times\vec{E}''\end{cases}$$

Need to impose 4 boundary conditions

B_n, D_n ~ continuous , E_t, H_t ~ continuous

$$\begin{cases} \hat{n}\cdot\left[\varepsilon(\vec{E}_0+\vec{E}_0'')-\varepsilon'\vec{E}_0'\right]=0 & D_n \\ \hat{n}\cdot\left[\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}_0''-\vec{k}'\times\vec{E}_0'\right]=0 & B_n \\ \hat{n}\times\left[\vec{E}_0+\vec{E}_0''-\vec{E}_0'\right]=0 & E_t \\ \hat{n}\times\left[\frac{1}{\mu}(\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}_0'')-\frac{1}{\mu'}\vec{k}'\times\vec{E}_0'\right]=0 & H_t \end{cases}$$

Matching the phases: $\omega=\omega'=\omega''$ $\quad k=k''=\sqrt{\mu\varepsilon}\,\omega,\ k'=\sqrt{\mu'\varepsilon'}\,\omega$

$\theta=\theta''$ $\quad n\sin\theta=n'\sin\theta'$ ~ geometric optics.

Snell's law

=> we solved for E_0' & E_0'' in terms of $E_0, n, n', \theta, \mu, \mu'$

for (I) $\vec{E}_0 \perp$ plane of incidence and (II) $\vec{E}_0 \parallel$ plane of incidence

=> got 2 sets of complicated formulas (see notes)

=> observed that total internal reflection happens at $\theta \geqslant \sin^{-1}\left(\frac{n'}{n}\right)$

=> defined & calculated transmission & reflection coefficients:

$$T=\frac{\langle|\vec{S}'|\rangle}{\langle|\vec{S}|\rangle} \qquad R=\frac{\langle|\vec{S}''|\rangle}{\langle|\vec{S}|\rangle}$$

Frequency-dependent ε, μ, σ

worked out a simple model with the molecule being a harmonic oscillator to derive

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{n e^2}{m \varepsilon_0 \left(\omega_0^2 - i\omega\gamma - \omega^2\right)}$$

high frequency $\omega \gg \gamma$ $\Rightarrow$ $\dfrac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \dfrac{\omega_p^2}{\omega^2}$
$\omega_0 = 0$

$\omega_p^2 = \dfrac{n e^2}{m \varepsilon_0}$ ~ plasma frequency

$k = \omega \sqrt{\mu_0 \varepsilon(\omega)} = \dfrac{\omega}{c}\sqrt{1 - \dfrac{\omega_p^2}{\omega^2}}$ $\Rightarrow$ for $\omega < \omega_p$ waves are damped (evanescent)

$$e^{ikz} \to e^{-|k|z}$$

conductors: $\varepsilon(\omega) = \varepsilon + \dfrac{i\sigma(\omega)}{\omega}$

Kramers-Kronig relations:

note that $\varepsilon(-\omega) = \varepsilon^*(\omega^*)$

$$\mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{1}{\pi} P \int_{-\infty}^{\infty} d\omega' \frac{1}{\omega' - \omega}\, \mathrm{Im}\,\frac{\varepsilon(\omega')}{\varepsilon_0}$$

$$\mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = -\frac{1}{\pi} \int_{-\infty}^{\infty} d\omega'\, P \frac{1}{\omega' - \omega} \left[\mathrm{Re}\,\frac{\varepsilon(\omega')}{\varepsilon_0} - 1\right]$$

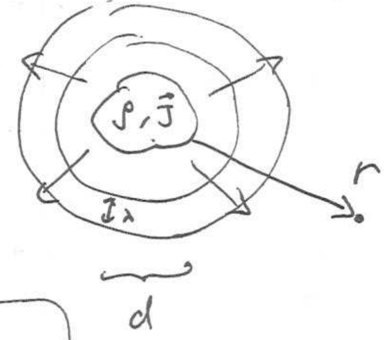
$\Rightarrow$ not free, Re & Im parts of $\varepsilon(\omega)$ are related!

$\Rightarrow$ We also defined group velocity $\vec{V}_{gr} = \vec{\nabla}_K \omega \big|_{\vec{k} = \vec{k}_0}$

Radiation

radiation zone:

$$d \ll \lambda \ll r$$



$$\boxed{\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \vec{J}(\vec{x}') \, e^{-ik\hat{n}\cdot\vec{x}'}}$$

$$\boxed{\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A}} \qquad \boxed{\vec{E} = \frac{i}{\omega\varepsilon_0} \vec{\nabla} \times \vec{H}}$$

all $\otimes\, e^{-i\omega t}$ & Re

expand in $kd \sim d/\lambda$

Electric dipole radiation: $\boxed{\vec{A} = -\frac{i\mu_0\omega}{4\pi} \vec{p} \, \frac{e^{ikr}}{r}}$

$$\boxed{\vec{H} \simeq \frac{ik}{\mu_0} \hat{n} \times \vec{A}} \qquad \boxed{\vec{E} \approx -i\omega \, \hat{n} \times (\hat{n} \times \vec{A})}$$

true for any multipole moment

$$\vec{p} = \int d^3x \, \vec{x} \, \rho(\vec{x})$$ electric dipole moment

$$\boxed{\frac{dP}{d\Omega} = \frac{c^2}{32\pi^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \, k^4 \, |\hat{n} \times \vec{p}|^2}$$

angular distribution of electric dipole radiation

magnetic dipole: $\boxed{\vec{A} = \frac{ik\mu_0}{4\pi} \hat{n} \times \vec{m} \, \frac{e^{ikr}}{r}}$

$$\boxed{\frac{dP}{d\Omega} = \frac{k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \, |\hat{n} \times \vec{m}|^2}$$

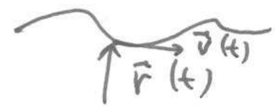
$\vec{m} = \int d^3x \, \frac{1}{2} \vec{x} \times \vec{J}$ magnetic dipole moment

Quadrupole radiation: $Q_{ij} = \int d^3x\, \rho(\vec{x}) \left[3x_i x_j - \delta_{ij} \vec{x}^{\,2} \right]$

$$\frac{dP}{d\Omega} = \frac{c k^6}{2(24\pi)^2 \varepsilon_0} \left[Q_{ij} n_j Q^*_{ik} n_k - Q_{ij} n_i n_j Q^*_{kl} n_k n_l \right]$$

$$P = \frac{c k^6}{1440 \pi \varepsilon_0} Q_{ij} Q^*_{ij}$$

Radiation by moving charges:



$$\Phi(\vec{x}, t) = \left[\frac{e}{(1 - \hat{n} \cdot \vec{\beta}) R} \right]_{ret} ,$$

$R(t) = |\vec{x} - \vec{r}(t)|$

$\hat{n} = (\vec{x} - \vec{r}(t))/R(t)$

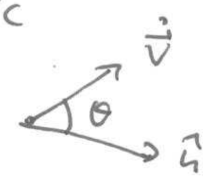
Lienard-Wiechert potentials

$$\vec{A}(\vec{x}, t) = \left[\frac{e \vec{\beta}}{(1 - \hat{n} \cdot \vec{\beta}) R} \right]_{ret}$$

$t_{ret} = t - \frac{1}{c} |\vec{x} - \vec{r}(t_{ret})|$

$\vec{\beta} = \vec{v}(t)/c$

Radiated power: $\dfrac{dP}{d\Omega} = \dfrac{e^2}{4\pi c^3} |\dot{\vec{v}}|^2 \sin^2\theta$



$$P = \frac{2}{3} \frac{e^2}{c^3} |\dot{\vec{v}}|^2$$

Larmor f-la (valid in the non-relativistic case only)

$$P = \frac{2}{3} \frac{e^2}{c} \gamma^6 \left[\dot{\vec{\beta}}^2 - (\vec{\beta} \times \dot{\vec{\beta}})^2 \right]$$

Lienard (relativistic case)