Last time | B. z-dependent geometry (cont'd)

$$\Phi(\rho, \varphi, z) = R(\rho)\, Q(\varphi)\, Z(z)$$

$$Q(\varphi) = e^{\pm i\nu\varphi}, \quad Z(z) = e^{\pm kz}$$

For $R(\rho)$ we have: $$R'' + \frac{1}{\rho} R' + \left(k^2 - \frac{\nu^2}{\rho^2}\right) R = 0$$

Bessel equation

The solution is $R(\rho) = J_\nu(k\rho)$, where

$$J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{n=0}^{\infty} \frac{(-1)^n}{n!\,\Gamma(n+\nu+1)} \left(\frac{x}{2}\right)^{2n}$$

Bessel function of 1st kind

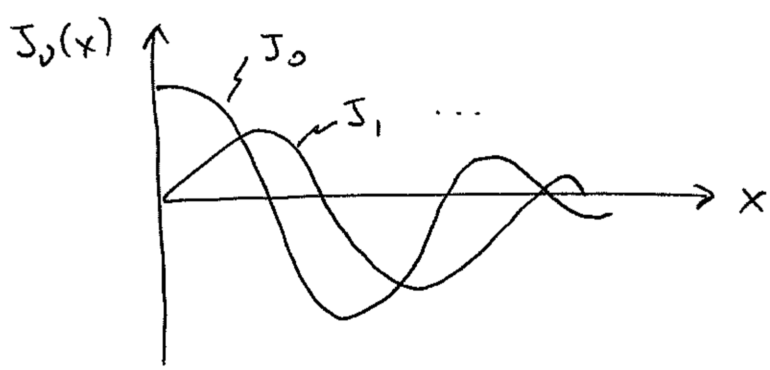
$$N_\nu(x) = \frac{J_\nu(x)\cos(\pi\nu) - J_{-\nu}(x)}{\sin \pi\nu}$$

Bessel function of the 2nd kind (aka Neumann ftn)

sometimes denoted $Y_\nu(x)$.

$$R(\rho) = A\, J_\nu(k\rho) + B\, N_\nu(k\rho)$$

~ general solution of Bessel eq'n



Bessel function roots are defined by

$$J_\nu(x_{\nu n}) = 0 \quad , n = 1, 2, 3, \ldots$$

where $x_{\nu n}$ is the nth root of $J_\nu(x)$.

At large $x \gg 1$, $J_\nu(x) \sim \cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right) \Rightarrow$

$\Rightarrow$ roots are given by $x_{\nu n} - \frac{\nu\pi}{2} - \frac{\pi}{4} = -\frac{\pi}{2} + \pi n$

$\Rightarrow \quad x_{\nu n} \approx \pi n + \frac{\pi}{2}\left(\nu - \frac{1}{2}\right)$; roots are known <u>precisely</u>.

$x_{01} \approx 2.405$, $x_{02} \approx 5.520$, ...

<u>Orthonormal set</u>: functions $\sqrt{\rho}\, J_\nu\left(x_{\nu n} \frac{\rho}{a}\right)$

form an orthogonal & complete set on $0 \le \rho \le a$.

<u>Proof</u>:

Start with $\frac{1}{\rho}\frac{d}{d\rho}\left(\rho \frac{d}{d\rho} J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)\right) + \left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2}\right)\cdot$

$\cdot\, J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) = 0 \quad \Rightarrow$ multiply by $\rho\, J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)$ & int.

$$\int_0^a d\rho\; J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)\frac{d}{d\rho}\left(\rho\frac{d}{d\rho} J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)\right) +$$

$$+ \int_0^a d\rho \left(\frac{x_{\nu n}^2}{a^2} - \frac{\nu^2}{\rho^2}\right)\rho\, J_\nu\left(x_{\nu n}\frac{\rho}{a}\right) J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right) = 0$$

parts =>

$$-\int_0^a d\rho\cdot\rho\cdot\left[\frac{d}{d\rho}J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)\right]\left[\frac{d}{d\rho}J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)\right]+$$

$$+\int_0^a d\rho\cdot\rho\cdot\left(\frac{x_{\nu n}^2}{a^2}-\frac{\nu^2}{\rho^2}\right)J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)=0$$

subtract the same thing with $n \leftrightarrow n'$ to get

$$\left(x_{\nu n}^2-x_{\nu n'}^2\right)\underbrace{\int_0^\infty d\rho\cdot\rho\cdot J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)}_{=0 \text{ if } n\neq n'}=0$$

In general one gets

$$\boxed{\int_0^a d\rho\cdot\rho\cdot J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)J_\nu\left(x_{\nu n'}\frac{\rho}{a}\right)=\frac{a^2}{2}\delta_{nn'}\left[J_{\nu+1}(x_{\nu n})\right]^2.}$$

assuming completeness

$\forall$ function $f(\rho)$ such that $f(0)=f(a)=0$

can be approximated as

$$\boxed{f(\rho)=\sum_{n=1}^{\infty}A_{\nu n}J_\nu\left(x_{\nu n}\frac{\rho}{a}\right)}$$

Fourier-Bessel series

with $A_{\nu n}=\dfrac{2}{a^2J_{\nu+1}^2(x_{\nu n})}\displaystyle\int_0^a d\rho\cdot\rho\cdot f(\rho)J_\nu\left(\frac{x_{\nu n}}{a}\rho\right).$

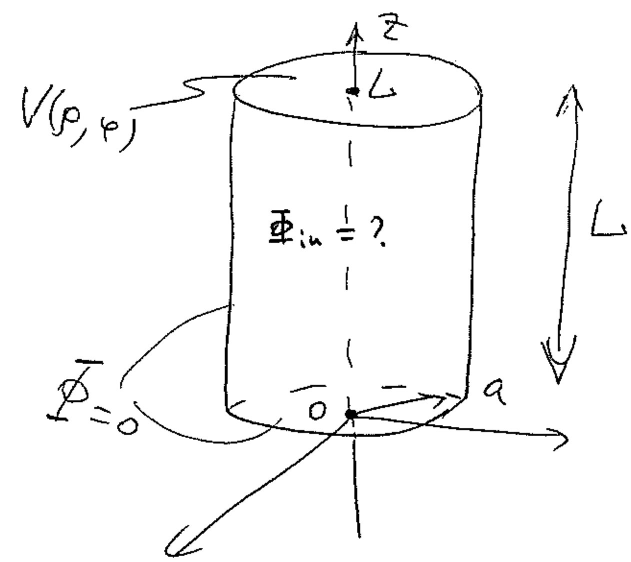
Another useful special functions are <u>modified Bessel functions</u> $I_\nu(z)$ & $K_\nu(z)$

Definition
$$I_\nu(x) = i^{-\nu} J_\nu(ix)$$
$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1} \left[J_\nu(ix) + i N_\nu(ix) \right]$$

obey $\frac{d^2R}{dx^2} + \frac{1}{x}\frac{dR}{dx} - \left(1 + \frac{\nu^2}{x^2}\right)R = 0$ diff. equation.

Very useful formula: $\boxed{\frac{1}{k}\delta(k-k') = \int_0^\infty dx \cdot x \cdot J_\nu(kx) J_\nu(k'x)}$

Example of a Boundary-Value Problem:



$$Q(\varphi) = A\sin(m\varphi) + B\cos(m\varphi)$$
$$Z(z) = C\sinh(Kz) + D\cosh(kz)$$
$$\Rightarrow D = 0 \text{ as } Z(0) = 0.$$
$$R(\rho) = E\, J_m(k\rho) + F N_m(k\rho)$$
$\Rightarrow$ finite at $\rho = 0 \Rightarrow F = 0$

$R(a) = 0 \Rightarrow K \to K_{mn} = \frac{x_{mn}}{a}$ ~ roots

$$\Rightarrow \Phi(\rho,\varphi,z) = \sum_{m=0}^\infty \sum_{n=1}^\infty J_m(K_{mn}\rho)\sinh(K_{mn}z)\cdot\left[A_{mn}\sin(m\varphi) + B_{mn}\cos(m\varphi)\right].$$

Finally, $\phi(\rho, \varphi, z = L) = V(\rho, \varphi)$

$$\Rightarrow V(\rho, \varphi) = \sum_{m,n} J_m(k_{mn}\rho)\sinh(k_{mn}L)\left[A_{mn}\sin(m\varphi) + B_{mn}\cos(m\varphi)\right]$$

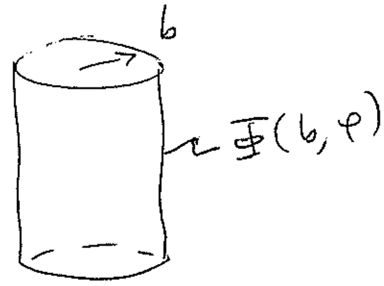
$\Rightarrow$ invert the Fourier (φ) and Fourier - Bessel (ρ) series to get

$$\begin{pmatrix} A_{mn} \\ B_{mn} \end{pmatrix} = \frac{2}{\pi a^2 \sinh(k_{mn}L)\, J_{m+1}^2(k_{mn}a)} \int_0^{2\pi} d\varphi \cdot \int_0^{a} d\rho \cdot \rho \cdot V(\rho, \varphi)\, J_m(k_{mn}\rho) \begin{pmatrix} \sin(m\varphi) \\ \cos(m\varphi) \end{pmatrix}.$$

for $m = 0$ use $\frac{1}{2} B_{0n}$ on the right-hand side

Jackson problem 2.12:

$$\Phi(\rho, \varphi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} a_n \rho^n \sin(n\varphi + \alpha_n) + \sum_{n=1}^{\infty} b_n \rho^{-n} \sin(n\varphi + \beta_n)$$



$\Rightarrow b_n = 0$ as ϕ is finite at $\rho = 0$ ($b_0 = 0$ too)

$$\Rightarrow \Phi(\rho, \varphi) = \frac{A_0}{2} + \sum_{m=1}^{\infty} \left(A_m \cos(m\varphi) + B_m \sin(m\varphi)\right)\rho^m$$

For $\rho = b$ we have