Last time Worked out a couple of examples on cylindrical geometry (solving Laplace equation): the main principle was separation of variables

Cylindrical Geometry	Solution of $\nabla^2 \Phi = 0$
z-independent $\varphi \in [0, 2\pi]$	$\Phi(\rho, \varphi) = a_0 + b_0 \ln\rho + \sum_{n=1}^{\infty} \left[a_n \rho^n \sin(n\varphi + \alpha_n) + b_n \rho^{-n} \sin(n\varphi + \beta_n) \right]$
z-dependent	$\Phi(\rho, \varphi, z) = R(\rho)\, Q(\varphi)\, Z(z)$ $Q(\varphi) = e^{\pm i\nu\varphi}$ $\nearrow$ $Z(z) = e^{\pm kz} \Leftrightarrow R(\rho) = A\, J_\nu(k\rho) + B\, N_\nu(k\rho)$ $\searrow$ $Z(z) = e^{\pm ikz} \Leftrightarrow R(\rho) = C\, I_\nu(k\rho) + D\, K_\nu(k\rho)$

$$\Phi(b,\varphi) = \frac{A_0}{2} + \sum_{m=1}^{\infty}\left(A_m\cos(m\phi) + B_m\sin(m\varphi)\right)b^m$$

$$\Rightarrow \begin{pmatrix}A_m\\ B_m\end{pmatrix} b^m = \frac{1}{\pi}\int_0^{2\pi} d\varphi' \begin{pmatrix}\cos(m\varphi')\\ \sin(m\varphi')\end{pmatrix}\Phi(b,\varphi'), \quad A_0 = \frac{1}{\pi}\int_0^{2\pi} d\varphi'\,\Phi(b,\varphi')$$

$$\Rightarrow \Phi(\rho,\varphi) = \frac{1}{\pi}\int_0^{2\pi} d\varphi'\,\Phi(b,\varphi')\sum_{m=1}^{\infty}\left[\cos m\varphi \cos m\varphi' + \right.$$

$$\left. + \sin m\varphi \sin m\varphi'\right]\cdot b^{-m}\rho^m + \frac{1}{2\pi}\int_0^{2\pi} d\varphi'\,\Phi(b,\varphi').$$

Now, $[\ \ ] = \cos m(\varphi-\varphi') = \frac{1}{2}\left[e^{im(\varphi-\varphi')} + e^{-im(\varphi-\varphi')}\right]$

$$\Rightarrow \sum_{m=1}^{\infty} e^{im(\varphi-\varphi')}\left(\frac{\rho}{b}\right)^m = \frac{\rho}{b}\cdot e^{i(\varphi-\varphi')}\frac{1}{1-\frac{\rho}{b}e^{i(\varphi-\varphi')}} =$$

$$= \frac{1}{\frac{b}{\rho}e^{-i(\varphi-\varphi')}-1} \Rightarrow \sum_{m=1}^{\infty}\left(\frac{\rho}{b}\right)^m \cos m(\varphi-\varphi') =$$

$$= \frac{1}{2}\left[\frac{1}{\frac{b}{\rho}e^{-i(\varphi-\varphi')}-1} + \frac{1}{\frac{b}{\rho}e^{i(\varphi-\varphi')}-1}\right] = \frac{\frac{b}{\rho}\cos(\varphi-\varphi')-1}{1+\frac{b^2}{\rho^2}-2\frac{b}{\rho}\cos(\varphi-\varphi')}$$

$$= \frac{b\rho\cos(\varphi-\varphi')-\rho^2}{\rho^2+b^2-2\rho b\cos(\varphi-\varphi')}$$

So, $\Phi(\rho,\varphi) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi'\,\Phi(b,\varphi')\left[1+2\frac{b\rho\cos(\varphi-\varphi')-\rho^2}{b^2+\rho^2-2\rho b\cos(\varphi-\varphi')}\right]$

$$= \frac{1}{2\pi}\int_0^{2\pi} d\varphi' \,\Phi(b,\varphi') \,\frac{b^2-\rho^2}{b^2+\rho^2-2b\rho\cos(\varphi-\varphi')}$$ as desired!

Green function in cylindrical coordinates:

need to solve $\nabla^2 G(\vec{x},\vec{x}') = -4\pi\delta^3(\vec{x}-\vec{x}') =$

$$= -\frac{4\pi}{\rho}\,\delta(\rho-\rho')\,\delta(\varphi-\varphi')\,\delta(z-z').$$

write $\delta(z-z') = \frac{1}{2\pi}\int_{-\infty}^{\infty} dk\, e^{ik(z-z')} = \int_0^{\infty}\frac{dk}{\pi}\cos[k(z-z')]$

$$\delta(\varphi-\varphi') = \frac{1}{2\pi}\sum_{m=-\infty}^{\infty} e^{im(\varphi-\varphi')}$$

$$\Rightarrow G(\vec{x},\vec{x}') = \frac{1}{2\pi^2}\sum_{m=-\infty}^{\infty}\int_0^{\infty} dk\, e^{im(\varphi-\varphi')}\cos[k(z-z')]\cdot$$

$$\cdot g_m(k,\rho,\rho')$$

Plug this back into eqn for G to get

$$\frac{1}{\rho}\frac{d}{d\rho}\left(\rho\frac{d}{d\rho}g_m(k,\rho,\rho')\right) - \left(k^2+\frac{m^2}{\rho^2}\right)g_m = -\frac{4\pi}{\rho}\delta(\rho-\rho')$$

Same story as in rectangular coord's:

if $\rho<\rho'$ $\Rightarrow$ get $g_m \sim A I_m(k\rho) + B K_m(k\rho)$

$\Rightarrow$ we want regular behavior as $\rho\to 0$ $\Rightarrow$

$\Rightarrow g_m \sim I_m(k\rho)$ for $\rho<\rho'$

Similarly, for $\rho > \rho'$ we don't want ∞ at $\rho \to \infty$ $\Rightarrow$ $g_m \sim K_m(k\rho)$ for $\rho > \rho'$

$\Rightarrow$ as $g_m(\rho, \rho', k)$ is symmetric under $\rho \leftrightarrow \rho'$

$$\Rightarrow \quad g_m(\rho, \rho', k) = C \cdot I_m(k\rho_<)\, K_m(k\rho_>)$$

Discontinuity at $\rho = \rho'$ $\Rightarrow$ $\dfrac{\partial g_m}{\partial \rho}(\rho \to \rho'_+) - \dfrac{\partial g_m}{\partial \rho}(\rho \to \rho'_-) = -\dfrac{4\pi}{\rho'}$

small $k\rho$:

$$I_m(x)\Big|_{x \ll 1} \simeq \frac{1}{m!}\left(\frac{x}{2}\right)^m, \quad K_m(x)\Big|_{x \ll 1} \simeq \frac{\Gamma(m)}{2}\left(\frac{2}{x}\right)^m$$

$\Rightarrow$ $-\dfrac{m}{\cancel{\rho}} \dfrac{1}{\cancel{2m}} \cdot \cancel{\rho} \cdot C = -\dfrac{4\pi}{\cancel{\rho}}$ $\Rightarrow$ $\boxed{C = 4\pi}$

Using $C = 4\pi$ $\Rightarrow$

$$G(\vec{x}, \vec{x}') = \frac{1}{|\vec{x} - \vec{x}'|} = \frac{2}{\pi} \sum_{m=-\infty}^{\infty} \int_0^\infty dk\, e^{im(\varphi - \varphi')} \cos[k(z - z')] \cdot I_m(k\rho_<)\, K_m(k\rho_>).$$

[Take a differential equation of the Sturm-Liouville type:

$$\frac{d}{dx}\left[p(x)\frac{dy}{dx}\right] + g(x)\, y = 0$$

Suppose u and v are two linearly-independent solutions:

$$\left.\begin{array}{ll} \dfrac{d}{dx}[p u'] + g u = 0 & \Big| \cdot v \\ \dfrac{d}{dx}[p v'] + g v = 0 & \Big| \cdot u \end{array}\right\} \text{\& subtract}$$

$$v \frac{d}{dx}[p u'] - u \frac{d}{dx}[p v'] = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left[p\,(v u' - u v')\right] = 0$$

$$\Rightarrow p(u v' - v u') = \text{const} \Rightarrow u v' - v u' = \frac{\text{const}}{p(x)}$$

Def. The Wronskian: $\boxed{W[u,v] \equiv u v' - v u'}$

$$\Rightarrow \quad \boxed{W[u,v] = \frac{\text{const}}{p(x)}}$$

Modified Bessel eq'n: $R'' + \frac{1}{x} R' - \left(1 + \frac{\nu^2}{x^2}\right) R = 0$

$$\Rightarrow \quad x R'' + R' - x\left(1 + \frac{\nu^2}{x^2}\right) R = 0$$

$$\Rightarrow \quad \frac{d}{dx}\left[x R'\right] - x\left(1 + \frac{\nu^2}{x^2}\right) R = 0 \quad \Rightarrow \quad p(x) = x$$

$$\Rightarrow \quad W[I_\nu, K_\nu] = \frac{\text{const}}{x}$$

$I_\nu(x)$ & $K_\nu(x)$ are normalized such that const $= -1$

$$\Rightarrow \quad W[I_\nu(x), K_\nu(x)] = -\frac{1}{x}$$