Therefore,

$$\boxed{W = q\,\Phi(0) - \vec{p}\cdot\vec{E} - \frac{1}{6}\sum_{i,j} Q_{ij}\,\frac{\partial E_j}{\partial x_i}(0) + \dots}$$

Dielectrics.

Suppose we have two types of charges:

"free charges" and "bound charges".

The potential is then the sum of potentials of free and bound charges: $\Phi = \Phi_{free} + \Phi_{bound}$

$$\Phi_{free}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \,\frac{\rho_f(\vec{x}\,')}{|\vec{x}-\vec{x}\,'|}$$

Bound charges give charge-neutral media (e^- & p in the atoms & molecules). The dominant multipole is dipole. (It's easy to polarize a molecule.) Potential of a point dipole $\vec{p}$ (at $\vec{x}\,'$) is

$$\Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\,\frac{\vec{p}\cdot(\vec{x}-\vec{x}\,')}{|\vec{x}-\vec{x}\,'|^3}$$

Def. Defining <u>polarization</u> $\vec{P}(\vec{x})$ as dipole moment per unit volume, we write

$$\Phi_{bound}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int_V d^3x' \; \frac{\vec{P}(\vec{x}')\cdot(\vec{x}-\vec{x}')}{|\vec{x}-\vec{x}'|^3}$$

where V is the region containing polarization $\vec{P}$.

As $\dfrac{\vec{x}-\vec{x}'}{|\vec{x}-\vec{x}'|^3} = \vec{\nabla}' \dfrac{1}{|\vec{x}-\vec{x}'|} \Rightarrow$

$$\Phi_{bound}(\vec{x}) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \; \vec{P}(\vec{x}')\cdot\vec{\nabla}'\frac{1}{|\vec{x}-\vec{x}'|} = (\text{parts})$$

$$= -\frac{1}{4\pi\varepsilon_0}\int d^3x' \; \frac{1}{|\vec{x}-\vec{x}'|}\cdot\vec{\nabla}'\cdot\vec{P}(\vec{x}')$$

Finally, $\Phi(\vec{x}) = \Phi_{free}(\vec{x}) + \Phi_{bound}(\vec{x}) =$

$$= \boxed{\frac{1}{4\pi\varepsilon_0}\int d^3x' \; \frac{1}{|\vec{x}-\vec{x}'|}\left[\rho_f(\vec{x}') - \vec{\nabla}'\cdot\vec{P}(\vec{x}')\right] = \Phi(\vec{x})}$$

given ρ_f & $\vec{P}(\vec{x}) \Rightarrow$ get Φ.

$\Rightarrow$ There seems to be two components to total charge density: $\rho_{tot} = \rho_f - \vec{\nabla}\cdot\vec{P}$

Now, $\vec{E} = -\vec{\nabla}\Phi \Rightarrow \boxed{\vec{\nabla}\times\vec{E} = 0}$ true in dielectrics

$$\vec{\nabla}\cdot\vec{E} = -\nabla^2\Phi = \frac{1}{\varepsilon_0}\left[\rho_f(\vec{x}) - \vec{\nabla}\cdot\vec{P}(\vec{x})\right]$$

as $\nabla^2 \dfrac{1}{|\vec{x}-\vec{x}'|} = -4\pi\,\delta^3(\vec{x}-\vec{x}')$

Def. Define <u>electric displacement</u> $\vec{D}$

as $$\boxed{\vec{D} \equiv \varepsilon_0 \vec{E} + \vec{P}}$$

Since $\vec{\nabla} \cdot \vec{E} = \frac{1}{\varepsilon_0} \left[\rho_f - \vec{\nabla} \cdot \vec{P} \right]$

$$\Rightarrow \vec{\nabla} \cdot \left[\varepsilon_0 \vec{E} + \vec{P} \right] = \rho_f \Rightarrow \boxed{\vec{\nabla} \cdot \vec{D} = \rho_f}$$

$\vec{D}$ appears to have the meaning of electric field due to free charges.

<u>Linear Isotropic Homogeneous medium</u> (LIH):

$$\vec{P} = \varepsilon_0 \chi \vec{E}$$

linear ~ as $\vec{P} \propto \vec{E}$

homogeneous : χ is a constant, and not $\chi(\vec{x})$.

isotropic : χ is independent of direction

(i.e. could be $\vec{P}_i = \varepsilon_0 \sum_j \chi_{ij} E_j$ ...)

Def χ ~ electric susceptibility.

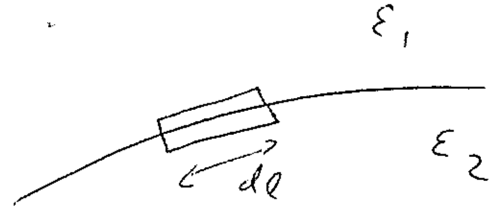
$$\vec{D} = \varepsilon_0 \vec{E} + \varepsilon_0 \chi \vec{E} = \varepsilon_0 (1+\chi) \vec{E} \equiv \varepsilon \vec{E}$$

Def $\varepsilon = \varepsilon_0 (1+\chi)$ ~ dielectric constant. $\Rightarrow$ as $\vec{\nabla} \cdot \vec{D} = \rho_f$ $\Rightarrow \boxed{\vec{\nabla} \cdot \vec{E} = \rho_f / \varepsilon}$ just like vacu $\varepsilon_0 \to \varepsilon$

(in vacuum, $\chi = 0$ and $\varepsilon = \varepsilon_0$); fix $\vec{D}$, take $\varepsilon \to \infty \Rightarrow \vec{E} = 0$ conductors

Boundary matching

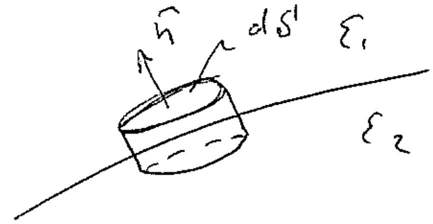
$\vec{\nabla} \times \vec{E} = 0. \quad \Rightarrow$



$$\oint_C \vec{E} \cdot d\vec{l} = 0 \Rightarrow \left(\vec{E}_1 - \vec{E}_2\right) \cdot \hat{t}\, dl = 0$$

$$\Rightarrow \boxed{E_{1t} = E_{2t}.}$$

$\vec{\nabla} \cdot \vec{D} = \rho_f \quad \Rightarrow$



$$\int_V \vec{\nabla} \cdot \vec{D}\, d^3x = \int \rho_f\, d^3x = \sigma_f \; dS'$$

$$\Rightarrow \boxed{D_{1n} - D_{2n} = \sigma_f}$$

σ_f ~ surface free charge density.

As we showed last term, $E_{1n} - E_{2n} = \dfrac{\sigma}{\varepsilon_0}$

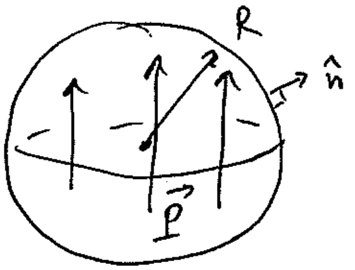
where $\sigma = \sigma_{free} + \sigma_{bound}$ ~ total surface charge density.

as $\vec{E} = \frac{1}{\varepsilon_0}\left(\vec{D} - \vec{P}\right) \Rightarrow$

$$E_{1n} - E_{2n} = \frac{1}{\varepsilon_0}\left(D_{1n} - D_{2n} - P_{1n} + P_{2n}\right) = \frac{\sigma_f}{\varepsilon_0} - \frac{P_{1n} - P_{2n}}{\varepsilon_0} = \frac{\sigma}{\varepsilon_0}$$

$$\Rightarrow \left\{ P_{1n} - P_{2n} = \sigma_f - \sigma = -\sigma_b \right\}$$

Example: uniformly polarized sphere:
find $\vec{E}$, $\vec{D}$



$$\Phi(\vec{x}) = -\frac{1}{4\pi\varepsilon_0}\int d^3x' \, \frac{\vec{\nabla}'\cdot\vec{P}(\vec{x}')}{|\vec{x}-\vec{x}'|}$$

$$= \frac{1}{4\pi\varepsilon_0}\int da' \, \frac{\hat{n}'\cdot\vec{P}}{|\vec{x}-\vec{x}'|}$$

as $-\vec{\nabla}\cdot\vec{P}$ is like ρ (charge density), $\hat{n}\cdot\vec{P}$ is like σ (surf. dens.)

Spherical coordinates $\hat{n}' = (\sin\theta'\cos\varphi', \sin\theta'\sin\varphi', \cos\theta')$

$$\vec{P} = \underset{x}{(0}, \underset{y}{0}, \underset{z}{P)}$$

$$\Rightarrow \hat{n}\cdot\vec{P} = P\cos\theta' = P\cdot P_1(\cos\theta') = P\sqrt{\frac{4\pi}{3}}\, Y_{10}(\theta',\varphi')$$

$$\frac{1}{|\vec{x}-\vec{x}'|} = \sum_{l=0}^{\infty}\sum_m \frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}} Y_{lm}^*(\theta',\varphi') Y_{lm}(\theta\varphi)$$

$$\Rightarrow \Phi(\vec{x}) = \frac{1}{4\pi\varepsilon_0} P\sqrt{\frac{4\pi}{3}}\sum_{l,m}\frac{4\pi}{2l+1}\frac{r_<^l}{r_>^{l+1}}\int d\cos\theta'\, d\varphi' \cdot R^2$$

$$\cdot Y_{lm}^*(\theta',\varphi')\, Y_{lm}(\theta,\varphi)\, Y_{10}(\theta',\varphi') = \frac{1}{4\pi\varepsilon_0} P\sqrt{\frac{4\pi}{3}}\cdot\frac{4\pi}{3}\cdot R^2$$

$$\cdot\frac{r_<}{r_>^2} Y_{10}(\theta,\varphi) = \frac{P R^2}{\cancel{4\pi}\varepsilon_0}\frac{\cancel{4\pi}}{3}\cdot\frac{r_<}{r_>^2}\cos\theta = \frac{PR^2}{3\varepsilon_0}\frac{r_<}{r_>^2}\cos\theta$$

$$\Rightarrow \underset{r>R}{\Phi_{out} = \frac{P}{3\varepsilon_0}\frac{R^3}{r^2}\cos\theta}, \quad \underset{r<R}{\Phi_{in} = \frac{P}{3\varepsilon_0}\, r\cos\theta}$$

$$\vec{E}_{out} = -\vec{\nabla}\Phi_{out} = \frac{R^3}{3\varepsilon_0}\left[\frac{3(\hat{n}\cdot\vec{P})\hat{n}-\vec{P}}{r^3}\right]$$

$$\vec{E}_{in} = -\vec{\nabla}\Phi_{in} = -\frac{\vec{P}}{3\varepsilon_0}$$

$$\vec{D}_{out} = \varepsilon_0\vec{E}_{out}, \qquad \vec{D}_{in} = \varepsilon_0\vec{E}_{in} + \vec{P} = \frac{2}{3}\vec{P}.$$

as $\Phi_{out} = \dfrac{R^3}{3\varepsilon_0}\dfrac{\vec{P}\cdot\vec{r}}{r^3}$ ~ just a dipole potential

$\Phi_{in} = \dfrac{P}{3\varepsilon_0} z = \dfrac{\vec{P}\cdot\vec{r}}{3\varepsilon_0}$ ~ uniform $\vec{E}$ field potential

Net dipole moment of the sphere

$$\vec{p} = \int d^3x\, \vec{P} = \frac{4}{3}\pi R^3 \vec{P}$$

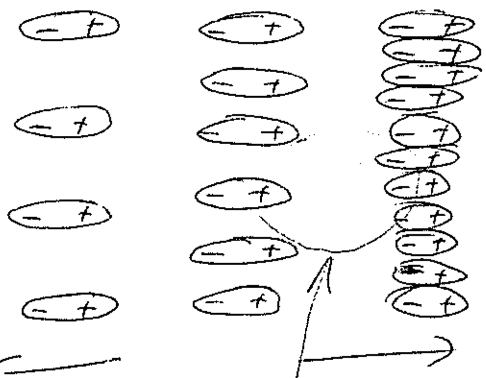
$$\Rightarrow \Phi_{out} = \frac{1}{4\pi\varepsilon_0}\frac{\vec{p}\cdot\vec{r}}{r^3}$$ ~ really a dipole potential

As the bound charge density is $\rho_b = -\vec{\nabla}\cdot\vec{P}$

$\Rightarrow P_{1n} - P_{2n} = -\sigma_b$.

Why is $\rho_b = -\vec{\nabla}\cdot\vec{P}$?

Pictorially:



lower density of $\vec{P}$

higher density of $\vec{P}$

more likely to find more negative charges in a volume element

$\Downarrow$

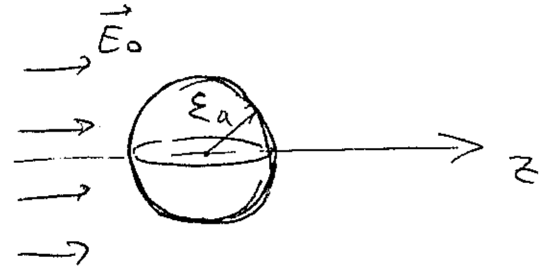
generates ρ_b!

Finally, if $\vec{D} = \varepsilon \vec{E}$

$$\Rightarrow \vec{\nabla}\cdot\vec{D} = \varepsilon\vec{\nabla}\cdot\vec{E} = \rho_f \Rightarrow$$

$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\varepsilon} \Rightarrow \boxed{\nabla^2\Phi = -\frac{\rho_f}{\varepsilon}}$$

<u>Example 2:</u> LIH dielectric sphere in external $\vec{E}$-field.



no free charges $\Rightarrow$

$\vec{\nabla} \cdot \vec{D} = 0$ inside & outside

$\vec{\nabla} \times \vec{E} = 0$ inside & outside

$\vec{D}_{out} = \varepsilon_0 \vec{E}_{out}$, $\vec{D}_{in} = \varepsilon \vec{E}_{in}$

$\Rightarrow$ as $\vec{\nabla} \times \vec{E} = 0 \Rightarrow \vec{E}_{out} = -\vec{\nabla} \Phi_{out}, \quad \vec{E}_{in} = -\vec{\nabla} \Phi_{in}$

$$0 = \vec{\nabla} \cdot \vec{D}_{out} = \varepsilon_0 \vec{\nabla} \cdot \vec{E}_{out} = -\varepsilon_0 \nabla^2 \Phi_{out} \Rightarrow \nabla^2 \Phi_{out} = 0$$