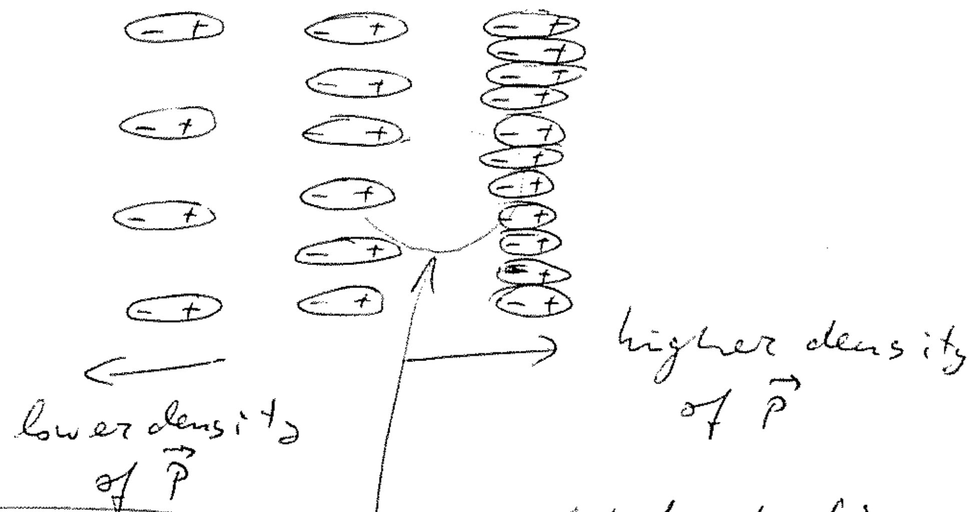
As the bound charge density is $\rho_b = -\vec{\nabla}\cdot\vec{P}$

$\Rightarrow \quad P_{1n} - P_{2n} = -\sigma_b$.

Why is $\rho_b = -\vec{\nabla}\cdot\vec{P}$?

Pictorially:



more likely to find more negative charges in a volume element

$\Downarrow$

generates ρ_b!

Finally, if $\vec{D} = \varepsilon\vec{E}$

$$\Rightarrow \vec{\nabla}\cdot\vec{D} = \varepsilon\vec{\nabla}\cdot\vec{E} = \rho_f \Rightarrow$$

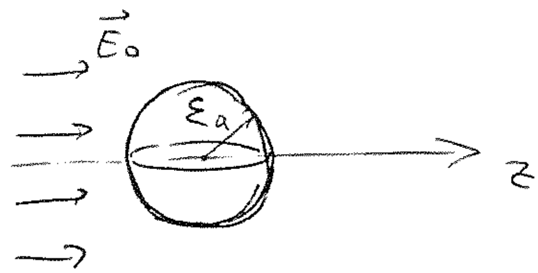
$$\Rightarrow \vec{\nabla}\cdot\vec{E} = \frac{\rho_f}{\varepsilon} \Rightarrow \boxed{\nabla^2\Phi = -\frac{\rho_f}{\varepsilon}}$$

Example 2: LIH dielectric sphere in external $\vec{E}$-field.

no free charges =>

$\vec{\nabla} \cdot \vec{D} = 0$ inside & outside

$\vec{\nabla} \times \vec{E} = 0$ inside & outside



$\vec{D}_{out} = \varepsilon_0 \vec{E}_{out}$, $\vec{D}_{in} = \varepsilon \vec{E}_{in}$

=> as $\vec{\nabla} \times \vec{E} = 0$ => $\vec{E}_{out} = -\vec{\nabla} \Phi_{out}$, $\vec{E}_{in} = -\vec{\nabla} \Phi_{in}$

$$0 = \vec{\nabla} \cdot \vec{D}_{out} = \varepsilon_0 \vec{\nabla} \cdot \vec{E}_{out} = -\varepsilon_0 \nabla^2 \Phi_{out} \Rightarrow \nabla^2 \Phi_{out} = 0$$

$$0 = \vec{\nabla}\cdot\vec{D}_{in} = \varepsilon\,\vec{\nabla}\cdot\vec{E}_{in} = -\varepsilon\nabla^2\Phi_{in} \Rightarrow \nabla^2\Phi_{in} = 0$$

$\Rightarrow$ We have $\nabla^2\Phi = 0$ everywhere (no free charges)

$\Rightarrow$ using the general solution of Laplace equation for problems with azimuthal symmetry in sphericall coordinates $\sum_\ell \left(A_\ell r^\ell + B_\ell r^{-\ell-1}\right) P_\ell(\cos\theta)$

we write:

$$\Phi_{in} = \sum_{\ell=0}^{\infty} A_\ell r^\ell P_\ell(\cos\theta) \quad , \; r < a$$

$$\Phi_{out} = \sum_{\ell=0}^{\infty} \left(B_\ell r^\ell + C_\ell r^{-\ell-1}\right) P_\ell(\cos\theta) \;, \; r > a.$$

We know that at $r \to \infty$ the potential should map onto that for the external field:

$$\Phi_{out}(r\to\infty) = -E_0 z = -E_0 r\cos\theta \Rightarrow$$

$\Rightarrow$ can fix B_ℓ's to write

$$\Phi_{out} = -E_0 r\cos\theta + \sum_{\ell=0}^{\infty} C_\ell r^{-\ell-1} P_\ell(\cos\theta).$$

Boundary conditions at the surface of the spere:

(1) $E_{in,t} = E_{out,t} \Rightarrow -\frac{1}{a}\left.\frac{\partial \Phi_{in}}{\partial \theta}\right|_{r=a} = -\frac{1}{a}\left.\frac{\partial \Phi_{out}}{\partial \theta}\right|_{r=a}$.

(2) $D_{in,n} = D_{out,n} \Rightarrow -\varepsilon \left.\frac{\partial \Phi_{in}}{\partial r}\right|_{r=a} = -\varepsilon_0 \left.\frac{\partial \Phi_{out}}{\partial r}\right|_{r=a}$

(1) $$\sum_{\ell=0}^{\infty} A_\ell a^\ell \frac{\partial}{\partial \theta} P_\ell(\cos\theta) = -E_0 a \frac{\partial}{\partial \theta} P_1(\cos\theta) +$$

$$+ \sum_{\ell=0}^{\infty} C_\ell a^{-\ell-1} \frac{\partial}{\partial \theta} P_\ell(\cos\theta)$$

associated Legendre function $P_\ell^m(x)$ with $m=1$.

as $P_\ell^1(\cos\theta) = \frac{\partial}{\partial \theta} P_\ell(\cos\theta)$ and P_ℓ^1's are all orthogonal $\Rightarrow$

$$\begin{cases} A_\ell a^\ell = C_\ell a^{-\ell-1}, & \ell \neq 1 \\ A_1 a = -E_0 a + C_1 a^{-2} \end{cases}$$

(may also impose continuity: $\Phi_{in}(r=a) = \Phi_{out}(r=a)$ $\Rightarrow$ same conditions) on A_ℓ & C_ℓ's

(2) $$\varepsilon \sum_{\ell=0}^{\infty} A_\ell \cdot \ell \cdot a^{\ell-1} P_\ell(\cos\theta) = -\varepsilon_0 E_0 P_1(\cos\theta) +$$

$$+ \varepsilon_0 \sum_{\ell=0}^{\infty} C_\ell (-\ell-1) a^{-\ell-2} P_\ell(\cos\theta)$$

$\Rightarrow P_\ell$'s are orthogonal $\Rightarrow$

$$\begin{cases} \varepsilon A_\ell \cdot \ell\, a^{\ell-1} = -\varepsilon_0 C_\ell (\ell+1) a^{-\ell-2}, & \ell \neq 1 \\ \varepsilon A_1 = -\varepsilon_0 E_0 \cancel{+} - \varepsilon_0\, 2 C_1 a^{-3} \end{cases}$$

$$\Rightarrow A_\ell = C_\ell = 0, \quad \text{for } \ell \neq 1.$$

$$\begin{cases} A_1 = -E_0 + C_1 a^{-3} \\ A_1 = -\frac{1}{\varepsilon}\left(\varepsilon_0 E_0 + \varepsilon_0 \cdot 2 C_1 a^{-3}\right) \end{cases}$$

$$C_1 a^{-3}\left(1 + 2\frac{\varepsilon_0}{\varepsilon}\right) = E_0\left(1 - \frac{\varepsilon_0}{\varepsilon}\right)$$

$$C_1 = E_0 a^3 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0}$$

$$A_1 = -E_0 \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0}$$

$$\Rightarrow \Phi_{in} = -E_0 \frac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} r\cos\theta$$

$$\Phi_{out} = \underbrace{-E_0 r\cos\theta}_{\text{external field}} + \underbrace{E_0 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \frac{a^3}{r^2}\cos\theta}_{\text{"image" dipole}}$$

"image" dipole $\vec{p} = 4\pi\varepsilon_0 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} \vec{E}_0 a^3$

Electric fields are $\vec{E}_{in} = \dfrac{3\varepsilon_0}{\varepsilon + 2\varepsilon_0} \vec{E}_0$

$$\vec{D}_{in} = \varepsilon_0 \vec{E}_{in} + \vec{P} = \varepsilon \cdot \vec{E}_{in}$$

$$\Rightarrow \vec{P} = (\varepsilon - \varepsilon_0)\, \vec{E}_{in} \Rightarrow \vec{P} = \frac{3\varepsilon_0(\varepsilon - \varepsilon_0)}{\varepsilon + 2\varepsilon_0} \vec{E}_0$$

$\Rightarrow$ total dipole moment of the sphere is

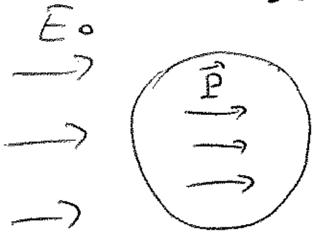
$$\vec{P} \cdot \frac{4}{3}\pi a^3 = 4\pi\varepsilon_0 \frac{\varepsilon - \varepsilon_0}{\varepsilon + 2\varepsilon_0} a^3 \vec{E}_0$$ ~ same as above.

Polarization surface-charge density:

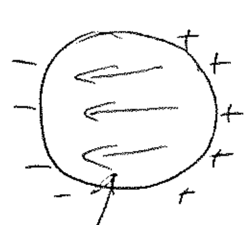
$$D_{in,n} = D_{out,n} \Rightarrow \varepsilon_0 E_{in,n} + P_{in,n} = \varepsilon_0 E_{out,n}$$

$$\Rightarrow \sigma_{pol} = \varepsilon_0 (E_{out,n} - E_{in,n}) = P_{in,n} = \frac{3\varepsilon_0(\varepsilon - \varepsilon_0)}{\varepsilon + 2\varepsilon_0} E_0 \cdot \cos\theta$$

or simply $\overset{=0}{P_{out,n}} - P_{in,n} = -\sigma_b \Rightarrow \sigma_b = P_{in,n}$



, but



~~the~~ electric field due to σ_{pol}. $\Rightarrow$

$\Rightarrow E_{in} < E_0$ (!)

Finally, $\varepsilon \to \infty$ $\vec{E}_{in} = 0$,

$$\Phi_{out} \to -E_0 r \cos\theta \left(1 - \frac{a^3}{r^3}\right)$$ ~ vacuum result with conducting sphere.