Electrostatic Energy in Dielectrics.

$$W = \frac{1}{2}\int d^3x\, \rho_{\cancel{tot}\ free}\, \Phi$$ ~ doesn't apply anymore (it actually applies, but we need to subtract out the energy of the dielectric w/o free charges, etc.)

As we bring in charges from ∞, we also need to construct correct polarization of the medium. (effect of ρ_b)

Change $\rho_{free}(\vec{x})$ by small quantity $\delta\rho_f(\vec{x})$. $\Rightarrow$

$$\delta W = \int_V d^3x\, \delta\rho_f(\vec{x})\, \Phi(\vec{x})$$

As $\rho_{free} = \vec{\nabla}\cdot\vec{D} \Rightarrow \delta\rho_f = \vec{\nabla}(\delta\vec{D})$

$$\Rightarrow \delta W = \int_V d^3x\, \vec{\nabla}(\delta\vec{D})\Phi = (\text{parts}) = -\int d^3x\, \delta\vec{D}\cdot\vec{\nabla}\Phi = \int d^3x\, \vec{E}\cdot\delta\vec{D} \Rightarrow \boxed{W = \int d^3x \int_0^D \vec{E}\cdot\delta\vec{D}}$$

If medium is linear and isotropic, then $\vec{D}_{(x)} = \varepsilon_{(x)}\vec{E}_{(x)}$

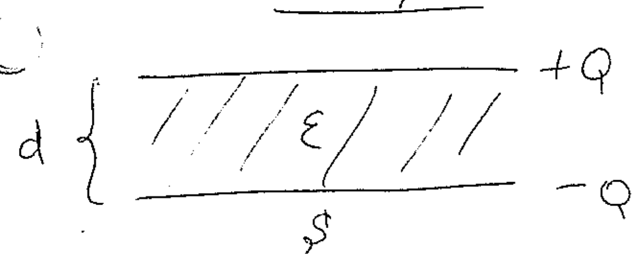
$$\Rightarrow \delta W = \int d^3x\, \vec{E}\cdot\varepsilon\cdot\delta\vec{E} = \delta\left(\int d^3x\, \varepsilon\cdot\frac{1}{2}\vec{E}^2\right) =$$

$$= \delta\left(\int d^3x\, \frac{1}{2}\vec{E}\cdot\vec{D}\right) \Rightarrow \boxed{W = \frac{1}{2}\int d^3x\, \vec{E}\cdot\vec{D}}$$

also, $\vec{E} = -\vec{\nabla}\Phi$, $\vec{\nabla}\cdot\vec{D} = \rho_{free} \Rightarrow W = \frac{1}{2}\int d^3x\, \rho_{free}\cdot\Phi$.

$\Rightarrow$ pt. charge q: $W = q\,\Phi(\vec{x}_0) \Rightarrow \vec{F} = -\vec{\nabla}W = +q\cdot\vec{E}$

Example capacitor with dielectric in it.



$\vec{\nabla}\cdot\vec{D} = \rho_{free} \Rightarrow D = Q/S$

$\vec{\nabla}\cdot\vec{E} = \frac{\rho_{free}}{\varepsilon} \Rightarrow E = \frac{1}{\varepsilon}\frac{Q}{S}$.

$$W = \frac{1}{2} \underbrace{S \cdot d}_{\text{volume}} \frac{1}{\varepsilon} \frac{Q^2}{S^2} \Rightarrow \boxed{W = \frac{1}{2} \frac{d\,Q^2}{\varepsilon S}}$$

Capacitance $C = \frac{Q}{\Delta V} = \frac{Q}{E \cdot d} = \frac{Q}{\frac{Q}{S}\frac{1}{\varepsilon} \cdot d} = \frac{\varepsilon S}{d}$

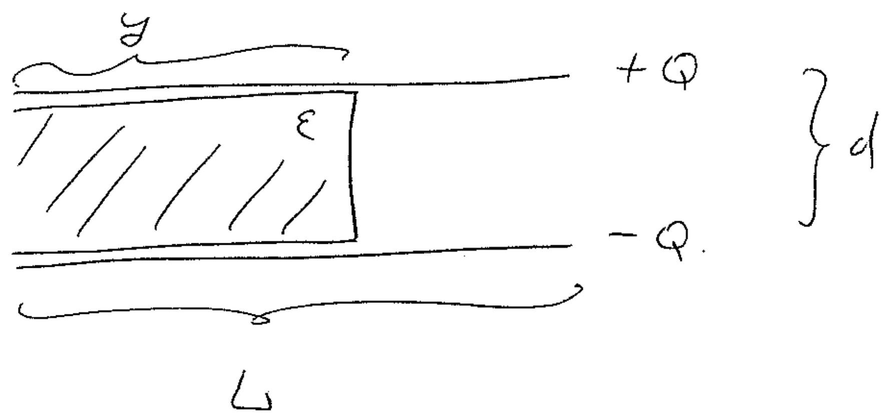
$\Rightarrow \boxed{\frac{C}{S} = \frac{\varepsilon}{d}}$ in vacuum $\varepsilon = \varepsilon_0 \Rightarrow \frac{C}{S} = \frac{\varepsilon_0}{d}$

works!

Forces: $F_\xi = -\left(\frac{\partial W}{\partial \xi}\right)_Q$ [$\delta E_{tot} = 0 \Rightarrow \delta W = +\delta E_{pot.}$ for body]

Force due to displacement in ξ-direction with sources Q fixed. (insulated from external world)

Example:



$L \times L$ square plates.

In general, surface charge density is different in vacuum & dielectric parts:

$\sigma_d = \varepsilon E_d$, $\sigma_v = \varepsilon_0 E_v$.

at the interface $E_{d,t} = E_{v,t} \Rightarrow E_d = E_v \equiv E$

$$\Rightarrow Q = L y \sigma_d + L\cdot(L-y)\sigma_v = L\left(y\cdot\varepsilon + (L-y)\varepsilon_0\right)E$$

$$\Rightarrow E = \frac{Q}{L\left[\varepsilon y + \varepsilon_0(L-y)\right]}$$

in dielectric $D = \varepsilon E$, in vacuum $D = \varepsilon_0 E$

$$\Rightarrow \text{total energy } W = \frac{1}{2}\cdot d y L\cdot D_d\cdot E_d +$$

$$+ \frac{1}{2} d(L-y)L\, D_v E_v = \frac{1}{2} d y L\cdot\varepsilon\cdot\left(\frac{Q}{L[\varepsilon y + \varepsilon_0(L-y)]}\right)^2 +$$

$$+ \frac{1}{2} d(L-y)L\cdot\varepsilon_0\left(\frac{Q}{L[\varepsilon y + \varepsilon_0(L-y)]}\right)^2 =$$

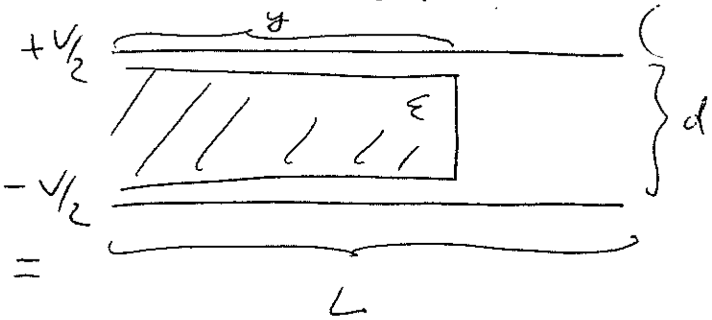
$$= \frac{1}{2} d\frac{Q^2}{L[\varepsilon y + \varepsilon_0(L-y)]} \Rightarrow F = -\left(\frac{\partial W}{\partial y}\right)_Q \Rightarrow$$

$$\Rightarrow \boxed{\frac{1}{2}\frac{dQ^2(\varepsilon-\varepsilon_0)}{L[\varepsilon y + \varepsilon_0(L-y)]^2} = F} > 0$$

$F > 0 \Rightarrow$ the force pulls the slab inside the capacitor

if $\varepsilon = \varepsilon_0 \Rightarrow F = 0$ no force in vacuum.

The problem is different if capacitor plates are held at constant potential difference V:



$$V = E \cdot d \Rightarrow E = \frac{V}{d}$$

$$W = \frac{1}{2} L d \, E^2 \left[\varepsilon y + \varepsilon_0 (L-y) \right] =$$

$$= \frac{1}{2} \frac{L V^2}{d} \left[\varepsilon y + \varepsilon_0 (L-y) \right]$$

$$\Rightarrow F = + \left(\frac{\partial W}{\partial y} \right)_V = \frac{1}{2} \frac{L V^2}{d} (\varepsilon - \varepsilon_0) > 0$$

force still pulls dielectric in

note the sign! The system is not isolated anymore.

When we move the dielectric, we first fix the charges $\Rightarrow \delta W_1 = \frac{1}{2} \int \rho \, \delta \Phi_1 \, d^3x$.

Then we let the charges exit/enter the system to keep potential constant

$$\delta W_2 = \frac{1}{2} \int d^3x \left[\rho \, \delta \Phi_2 + \Phi \, \delta \rho_2 \right]$$

Now, to keep Φ constant, we need $\delta \Phi_1 = - \delta \Phi_2$, $\Rightarrow$

$\delta W_2 = - \delta W_1 + \frac{1}{2} \int d^3x \, \Phi \, \delta \rho_2$. Now, as $\nabla^2 \Phi = - \frac{\rho}{\varepsilon}$ $\Rightarrow$

$\Rightarrow$ both terms in δW_2 are equal $\Rightarrow \delta W_2 = -2 \delta W_1$

$$\Rightarrow \delta W_V = \delta W_1 + \delta W_2 = - \delta W_1 = - \delta W_Q \Rightarrow F = \left(\frac{\partial W}{\partial \xi} \right)_V.$$