General Solution of Maxwell Equations (in Lorenz gauge)

We want to learn to solve Maxwell equations in a general (dynamic) case:

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu} \quad \text{(Gauss units)}$$

$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \Rightarrow$

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J^\nu$$

Choose $\boxed{\partial_\mu A^\mu = 0}$ Lorenz gauge $\Rightarrow$

Maxwell equations become

$$\boxed{\Box A^\nu = \frac{4\pi}{c} J^\nu}$$

where $\Box \equiv \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$. We get

$$\boxed{\begin{aligned} \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right] \Phi &= 4\pi \rho \\ \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right] \vec{A} &= \frac{4\pi}{c} \vec{J} \end{aligned}}$$

Maxwell equations in Lorenz gauge (Gauss units).

In SI units get

$$\boxed{\begin{aligned}\left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right]\Phi &= \frac{\rho}{\varepsilon_0}\\ \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right]\vec{A} &= \mu_0 \vec{J}\end{aligned}}$$

~ inhomogeneous wave equations

To solve Maxwell equations for arbitrary ρ & $\vec{J}$ we need the Green function for the $\Box$ operator.

Complex Analysis 101 (see Arfken, Ch. 11)

$i = \sqrt{-1}$ or $i^2 = -1$ imaginary unit number

$z = x + iy$; $\bar{z} = x - iy$ ~ complex conjugate
(x, y ~ real)

Def. $f(z)$ is analytic at a point z_0 if it is differentiable and single-valued in a neighborhood of z_0. ($\frac{\partial f}{\partial z}$ exists).

$$\frac{\Delta f}{\Delta z} = \frac{f(z_0+\Delta z) - f(z_0)}{\Delta z}$$ => assume $f(z) = u(x,y) + iv(x,y)$
u, v ~ real

$$\frac{\Delta f}{\Delta z} = \frac{\left(\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}\right)\Delta x + \left(\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}\right)\Delta y}{\Delta x + i\Delta y} \Rightarrow$$

want this independent of direction of the derivative =>

$$\frac{\frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x}}{\frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y}} = \frac{1}{i} \Rightarrow$$

$$\Rightarrow i\frac{\partial u}{\partial x} - \frac{\partial v}{\partial x} = \frac{\partial u}{\partial y} + i\frac{\partial v}{\partial y} \Rightarrow$$

$$\Rightarrow \boxed{\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y}} \text{ and } \boxed{\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}}$$ Cauchy-Riemann conditions

note that if C-R conditions are satisfied =>

$$\Rightarrow \frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \frac{\partial^2 v}{\partial x \partial y} - \frac{\partial^2 v}{\partial x \partial y} = 0 \Rightarrow \nabla^2 u = 0 \text{ also } \nabla^2 v = 0$$

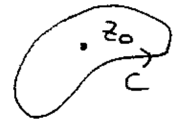
harmonic functions

$$\Rightarrow \frac{\partial^2}{\partial z \partial \bar{z}} f = 0$$

Cauchy Theorem: if $f(z)$ is analytic $\Rightarrow$ $\oint_C f(z)\,dz = 0$

Cauchy Formula: $f(z_0) = \frac{1}{2\pi i} \oint_C \frac{f(z)}{z - z_0}\,dz$

$f(z)$ ~ analytic function

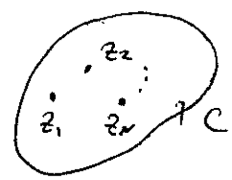


Define a residue: residue of $f(z)$ at z_0 is

$$\frac{1}{2\pi i} \oint_C f(z)\,dz = \mathrm{Res}\, f(z_0)$$

Residue Theorem | $f(z)$ is analytic except for a finite number of isolated singularities $z_1, \ldots, z_N$.

Then $$\oint_C f(z)\,dz = 2\pi i \sum_{n=1}^{N} \mathrm{Res}\, f(z_n)$$



How to find residues: simple pole ~ $\frac{1}{z-a}$

$$\Rightarrow \mathrm{Res}\, f(a) = \lim_{z \to a} (z-a) f(z)$$

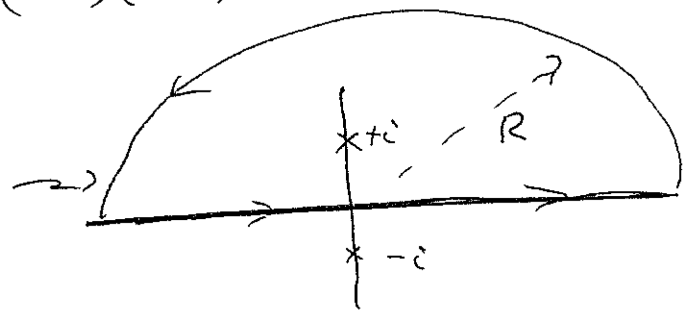
nth order pole: $\frac{1}{(z-a)^n}$

$$\mathrm{Res}\, f(a) = \lim_{z \to a} \frac{1}{(n-1)!} \frac{d^{n-1}}{dz^{n-1}} \left[(z-a)^n f(z) \right]$$

Example $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \arctan x \Big|_{-\infty}^{\infty} = \frac{\pi}{2} - \left(-\frac{\pi}{2}\right) = \pi$

Residues: $\displaystyle\int_{-\infty}^{\infty} \frac{dx}{x^2+1} = \int_{-\infty}^{\infty} \frac{dx}{(x-i)(x+i)}$

close the contour



if $f(z) \to 0$ as $z \to \infty$

faster than $\frac{1}{z}$ (i.e. $f(z) \sim \frac{1}{z^{1+\Delta}}$, $\Delta > 0$ as $z \to \infty$

or $z f(z) \to 0$ as $z \to \infty$)

$\Rightarrow$ integral over the semi-circle is zero when $R \to 0$ $\Rightarrow$ turned line integral into a contour integral

$\Rightarrow$ using residue th'm get $\displaystyle\int_{-\infty}^{\infty} dx \frac{1}{(x-i)(x+i)} =$

$$= 2\pi i \lim_{z \to i} \frac{z-i}{(z-i)(z+i)} = 2\pi i \frac{1}{2i} = \pi$$, as desired!

Proof of Cauchy theorem:

$$\oint_C f(z)\,dz = \oint_C [u+iv][dx+idy] = \oint_C [u\,dx - v\,dy]$$

$$+ i\oint_C [v\,dx + u\,dy]$$

Stokes' theorem: $\oint_C \vec{A}\cdot d\vec{l} = \int da\ \hat{n}\cdot(\vec{\nabla}\times\vec{A})$

$\Rightarrow$ if C is in the x-y plane $\Rightarrow$

$$\oint_C [A_x\,dx + A_y\,dy] = \int dx\,dy\,[\partial_x A_y - \partial_y A_x]$$

$$\Rightarrow \oint_C [u\,dx - v\,dy] = \int dx\,dy\,[-\partial_x v - \partial_y u] = 0$$

due to Cauchy-Riemann conditions

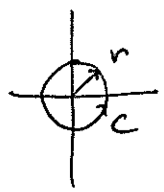
$$\oint_C [v\,dx + u\,dy] = \int dx\,dy\,[\partial_x u - \partial_y v] = 0$$

$\Rightarrow$ for an analytic function

$$\oint_C f(z)\,dz = 0,$$ as required.

Proof of Cauchy formula:

Warmup: $\oint_C dz \cdot z^n \underset{\substack{\uparrow \\ z = re^{i\varphi}}}{=} \int_0^{2\pi} re^{i\varphi} d\varphi \cdot i \; r^n e^{ni\varphi} = i r^{n+1} \int_0^{2\pi} d\varphi \, e^{i(n+1)\varphi}$



$$= i r^{n+1} \frac{1}{i(n+1)} \left[e^{2\pi i (n+1)} - 1 \right] = \begin{cases} 0, & n \neq -1, \; n \text{ integer} \\ 2\pi i, & n = -1 \end{cases}$$

$$\Rightarrow \oint_C dz \, (z - z_0)^n = 2\pi i \; \delta_{n,-1}$$

$n = -1 \Rightarrow$ simple pole

proof: $f(z)$ is analytic on and inside contour C.

$$\oint_C \frac{f(z)}{z - z_0} dz \underset{\substack{\uparrow \\ z = z_0 + re^{i\varphi}}}{=} \int_0^{2\pi} \frac{f(z_0 + re^{i\varphi})}{\cancel{re^{i\varphi}}} \cancel{re^{i\varphi}} \, i \, d\varphi = i \int_0^{2\pi} d\varphi \, f(z_0 + re^{i\varphi})$$

$$\xrightarrow[r \to 0]{} 2\pi i \, f(z_0) \Rightarrow \oint_C dz \frac{f(z)}{z - z_0} = 2\pi i \, f(z_0)$$

as desired.

Taylor series: $f(z) = \sum_{n=0}^{\infty} \frac{f^{(n)}(z_0)}{n!} (z - z_0)^n$

if $f(z)$ is expandable in the Taylor series $\Leftrightarrow$ analytic

Laurent series: $f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n$

(if $a_n \neq 0$ for $n \leq -1 \Rightarrow f(z)$ is not analytic at $z = z_0$)

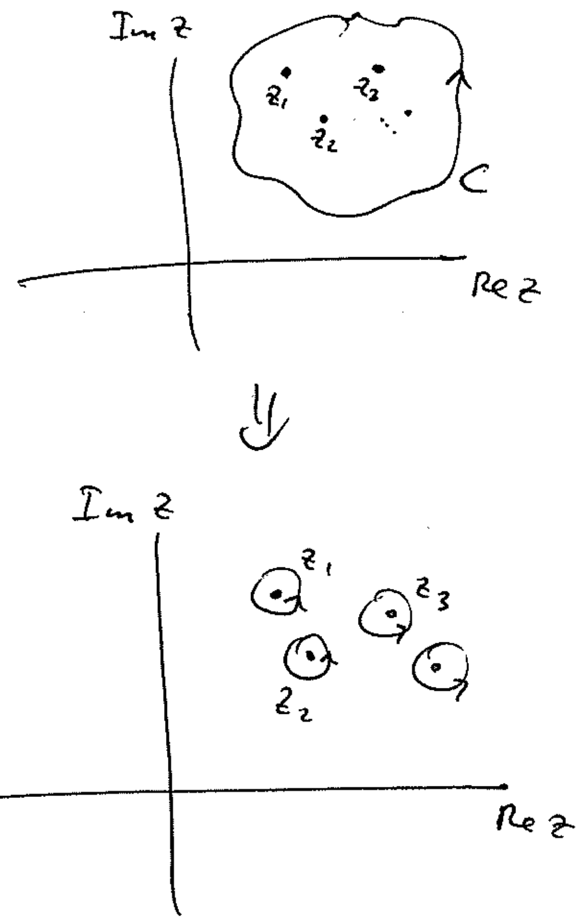
$$\oint_C dz\, f(z) = \sum_{n=-\infty}^{\infty} a_n \oint_C dz\,(z-z_0)^n$$

$$= \sum_{n=-\infty}^{\infty} a_n \cdot 2\pi i\, \delta_{n,-1} = 2\pi i\, a_{-1}$$

$\Rightarrow$ a_{-1} is called the residue of $f(z)$ at $z = z_0$.

Residue = coefficient of $\frac{1}{z-z_0}$ in Laurent series.

Proof of Residue Theorem:



Can deform the initial contour C to small circles around all the poles inside C:

Hence set

$$\oint_C dz\, f(z) = 2\pi i \left[a_{-1}^{(1)} + a_{-1}^{(2)} + \dots \right]$$

$$\Rightarrow \boxed{\oint_C dz\, f(z) = 2\pi i \sum_{i=1}^{N} \operatorname{Res} f(z_i)}$$