General Solution of Maxwell Equations (in Lorenz gauge)

We want to learn to solve Maxwell equations in a general (dynamic) case:

$$\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu} \qquad \text{(Gauss units)}$$

$$F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu \quad \Rightarrow$$

$$\partial_\mu \partial^\mu A^\nu - \partial^\nu \partial_\mu A^\mu = \frac{4\pi}{c} J^\nu$$

Choose $\boxed{\partial_\mu A^\mu = 0}$ Lorenz gauge $\Rightarrow$

Maxwell equations become

$$\boxed{\Box A^\nu = \frac{4\pi}{c} J^\nu}$$

where $\Box \equiv \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$. We get

$$\boxed{\begin{aligned} \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right] \Phi &= 4\pi \rho \\ \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right] \vec{A} &= \frac{4\pi}{c} \vec{J} \end{aligned}}$$

Maxwell equations in Lorenz gauge (Gauss units).

In SI units get

$$\boxed{\begin{aligned} \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right]\Phi &= \frac{\rho}{\varepsilon_0} \\ \left[\frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2\right]\vec{A} &= \mu_0\vec{J} \end{aligned}}$$

~ inhomogeneous wave equations

To solve Maxwell equations for arbitrary ρ & $\vec{J}$ we need the Green function for the $\Box$ operator.