III) ~~Other notable gauges:~~ $\cancel{x_\mu A^\mu = 0}$ ~~(Schwinger gauge)~~

$$\cancel{A_0 + A_z = 0 \Rightarrow \Phi + c A_z = 0} \quad \text{(light cone gauge)}$$

$$\cancel{A_0 = \Phi = 0} \quad \text{(axial gauge)}$$

$$\boxed{\Box A^\mu = \frac{4\pi}{c} J^\mu}$$

Green Function for Wave Equation.

In solving Maxwell equations in, say, Lorenz gauge one often encounters equations of the type:

$$\boxed{\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi f(\vec{x},t)}$$

Inhomogeneous wave eqn.

with $f(\vec{x},t)$ some known function (source).

The strategy for solving those is the same as in electrostatics: find the Green function.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{x},t;\vec{x}',t') = -4\pi\, \delta^3(\vec{x}-\vec{x}')\,\delta(t-t')$$

Then

$$\Psi(\vec{x},t) = \int d^3x'\,dt'\; G(\vec{x},t;\vec{x}',t')\, f(\vec{x}',t')$$

is a solution of the inhomogeneous wave equation. (in unlimited space-time).

In empty space $G(\vec{x},t;\vec{x}',t') \equiv G(\vec{x}-\vec{x}', t-t') \Rightarrow$ need to solve

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{r},t) = -4\pi\, \delta^3(\vec{r})\, \delta(t)$$

write $G(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}\, \tilde{G}(\vec{k},\omega)$

as $\delta^3(\vec{r})\,\delta(t) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}$

$$\Rightarrow \left(-\vec{k}^2 + \frac{\omega^2}{c^2}\right)\tilde{G} = -4\pi$$

$$\Rightarrow \tilde{G} = \frac{-4\pi}{\frac{\omega^2}{c^2} - \vec{k}^2}$$ photon propagator

$$\Rightarrow \boxed{G(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{i\vec{k}\cdot\vec{r} - i\omega t}\, \frac{-4\pi}{\frac{\omega^2}{c^2} - \vec{k}^2}}$$

Does this expression make sense? It has poles at $\frac{\omega}{c} = \pm|\vec{k}|$. There are several ways to regulate it:

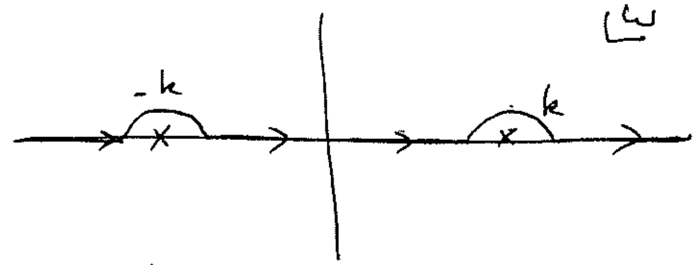
A. Retarded (causal) Green function

demand that $G(\vec{r},t) = 0$ for $t<0$

$$G(\vec{r},t) = \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, e^{-i\omega t + i\vec{k}\cdot\vec{r}}\, \frac{-4\pi}{\left(\frac{\omega}{c} - k\right)\left(\frac{\omega}{c} + k\right)}$$ with $k = |\vec{k}|$

Need $G = 0$ for $t<0$: if $t<0$ have to close the ω-contour into the upper half-plane.

$\Rightarrow$ need to have poles in the lower half-plane:



$$\Rightarrow \quad G_{ret}(\vec r, t) = -4\pi \int \frac{d^3k}{(2\pi)^3} \int \frac{d\omega}{2\pi}\, e^{-i\omega t + i\vec k\cdot\vec r}\, \frac{1}{\left(\frac{\omega}{c} - k + i\varepsilon\right)\left(\frac{\omega}{c} + k + i\varepsilon\right)}$$

$$\Rightarrow \quad \boxed{G_{ret}(\vec r, t) = -4\pi \int \frac{d^3k}{(2\pi)^3}\, \frac{d\omega}{2\pi}\, e^{-i\omega t + i\vec k\cdot\vec r}\, \frac{1}{\frac{\omega^2}{c^2} - k^2 + i\varepsilon\omega}}$$

advanced ~ change signs of $i\varepsilon$'s.

Do the Fourier transform

$$G_{ret}(\vec r, t) = -4\pi\, \theta(t)\, (-2\pi i)\, \frac{c^{\cancel{2}}}{\cancel{2\pi}} \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec k\cdot\vec r} \left[\frac{e^{-ikct}}{2k\cancel{c}} - \frac{e^{ikct}}{2k\cancel{c}} \right] = 2\pi i\, \theta(t)\, c \int \frac{d^3k}{(2\pi)^3}\, e^{i\vec k\cdot\vec r}\, \frac{1}{k} \left[e^{-ikct} - e^{ikct} \right]$$

$$= \left| \vec r \parallel \hat z \right| = \frac{2\pi i c\, \theta(t)}{(2\pi)^3} \int_0^\infty dk\, k^{\cancel{2}} \int_{-1}^{1} d\cos\theta \underbrace{\int_0^{2\pi} d\varphi}_{2\pi}\, e^{ikr\cos\theta}\, \cancel{\frac{1}{k}} \left[e^{-ikct} - e^{ikct} \right] = \frac{\cancel{i}c}{2\pi}\, \theta(t) \int_0^\infty dk\, \cancel{k}\, \frac{1}{\cancel{ik}r} \left[e^{ikr} - e^{-ikr} \right]$$

$$\cdot\left[e^{-ikct}-e^{ikct}\right]=\frac{c}{2\pi r}\theta(t)\int_0^\infty dk\left[e^{ik(r-ct)}+\right.$$

$$\left.+e^{-ik(r-ct)}-e^{-ik(r+ct)}-e^{ik(r+ct)}\right]\cdot e^{-\delta k}$$

(δ is some regulator at $k\to+\infty$, needed in Fourier-transform of the δ-function on the right of Green ftn eqn $\Rightarrow$ have it in G too)

$$=\frac{c}{2\pi r}\theta(t)\cdot\left\{\frac{-1}{i(r-ct)-\delta}+\frac{-1}{-i(r-ct)-\delta}-\frac{-1}{-i(r+ct)-\delta}-\right.$$

$$\left.-\frac{-1}{+i(r+ct)-\delta}\right\}=\frac{ci}{2\pi r}\theta(t)\left\{\frac{1}{r-ct+i\delta}-\frac{1}{r-ct-i\delta}+\right.$$

$$\left.+\frac{1}{r+ct-i\delta}-\frac{1}{r+ct+i\delta}\right\}=\left|\text{using } \frac{1}{x-i\delta}-\frac{1}{x+i\delta}=2\pi i\,\delta(x)\right.$$

regulator — Dirac delta-ftn. — do not confuse!

$$=\frac{ci}{2\pi r}\theta(t)\left\{-2\pi i\,\delta(r-ct)+2\pi i\underbrace{\delta(r+ct)}_{=0 \text{ as } r>0,\,t>0}\right\}$$

$$=\frac{c}{r}\theta(t)\,\delta(r-ct)=\frac{1}{r}\delta\left(t-\frac{r}{c}\right)\quad\text{as desired!}$$

$$\Rightarrow\boxed{G_{ret}(\vec{r},t)=\frac{1}{r}\delta\left(t-\frac{r}{c}\right)}$$

B. Advanced Green function (can be evaluated in a similar way)

$$G_{adv}(\vec{r},t)=-4\pi\int\frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}e^{-i\omega t+i\vec{k}\cdot\vec{r}}\frac{1}{\frac{\omega^2}{c^2}-k^2-i\omega\varepsilon}$$

$$\Rightarrow \boxed{G_{ret}(\vec{r}, t) = \frac{1}{r}\,\delta\left(t - \frac{r}{c}\right)} \quad \text{or}$$

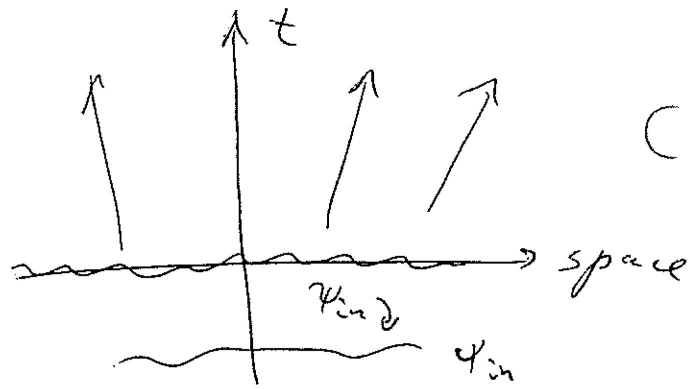
$$\boxed{G_{ret}(\vec{x}, t; \vec{x}', t') = \frac{1}{|\vec{x} - \vec{x}'|}\,\delta\left(t - t' - \frac{|\vec{x} - \vec{x}'|}{c}\right)}$$

retarded Green function

Given the source (localized in space-time) $f(\vec{x}, t)$ and initial condition $\Psi_{in}(\vec{x}, t)$ ← satisfying homog. eq., $\Box\Psi_{in} = 0$. at $t = -\infty$ we can write the solution

$$\Psi(\vec{x}, t) = \Psi_{in}(\vec{x}, t) + \int d^3x'\,dt'\,G_{ret}(\vec{x}, t; \vec{x}', t')\,f(\vec{x}', t')$$

Retarded Green ftn is <u>causal</u> ~ gives the solution in the future due to sources in the past.



<u>Advanced Green function</u>:

$$G_{adv}(\vec{r}, t) = G^{*}_{ret}(-t, -\vec{r}) \Rightarrow \text{as } |\vec{r}| = |-\vec{r}|$$

$$\Rightarrow G_{adv}(\vec{r}, t) = \frac{1}{r}\,\delta\left(-t - \frac{r}{c}\right) = \frac{1}{r}\,\delta\left(t + \frac{r}{c}\right)$$

$$\Rightarrow \boxed{G_{adv}(\vec{r}, t) = \frac{1}{r}\,\delta\left(t + \frac{r}{c}\right)}$$