Last time / Green Function for Wave Equation (cont'd)

We looked for Green function

$$\boxed{\Box G(x, x') = 4\pi\, \delta^3(\vec{x} - \vec{x}')\, \delta(t - t')}$$

of Maxwell equations $\Box A^\mu = \frac{4\pi}{c} J^\mu$.

Requiring that $G(\vec{r}, t) = 0$ for $t < 0$ we obtained <u>retarded</u> Green function

$$G_{ret}(\vec{r}, t) = -4\pi \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t + i\vec{k}\cdot\vec{r}}\, \frac{1}{\left(\frac{\omega}{c} - k + i\varepsilon\right)\left(\frac{\omega}{c} + k + i\varepsilon\right)}$$

$$= -4\pi \int \frac{d^3k}{(2\pi)^3} \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, e^{-i\omega t + i\vec{k}\cdot\vec{r}}\, \frac{1}{\frac{\omega^2}{c^2} - \vec{k}^2 + i\varepsilon\omega}$$

Integrating over $\vec{k}$ and ω we get

$$\boxed{G_{ret}(\vec{r}, t) = \frac{1}{r}\, \delta\left(t - \frac{r}{c}\right)}$$

$$\cdot \left[e^{-ikct} - e^{ikct} \right] = \frac{c}{2\pi r}\,\theta(t)\int_0 dk \left[e^{ik(r-ct)} + \right.$$

$$\left. + e^{-ik(r-ct)} - e^{-ik(r+ct)} - e^{ik(r+ct)} \right]\cdot e^{-\delta k}$$

(δ is some regulator at $k \to +\infty$, needed in Fourier-transform of the δ-function on the right of Green ftn eqn $\Rightarrow$ have it in G too)

$$= \frac{c}{2\pi r}\,\theta(t)\cdot\left\{ \frac{-1}{i(r-ct)-\delta} + \frac{-1}{-i(r-ct)-\delta} - \frac{-1}{-i(r+ct)-\delta} - \right.$$

$$\left. - \frac{-1}{+i(r+ct)-\delta} \right\} = \frac{ci}{2\pi r}\,\theta(t)\left\{ \frac{1}{r-ct+i\delta} - \frac{1}{r-ct-i\delta} + \right.$$

$$\left. + \frac{1}{r+ct-i\delta} - \frac{1}{r+ct+i\delta} \right\} = \left|\ \text{using}\quad \frac{1}{x-i\delta} - \frac{1}{x+i\delta} = 2\pi i\,\delta(x) \right.$$

regulator   Dirac delta-ftn. — do not confuse!

$$= \frac{ci}{2\pi r}\,\theta(t)\left\{ -2\pi i\,\delta(r-ct) + 2\pi i\,\underbrace{\delta(r+ct)}_{=\,0 \text{ as } r>0,\ t>0} \right\}$$

$$= \frac{c}{r}\,\theta(t)\,\delta(r-ct) = \frac{1}{r}\,\delta\!\left(t-\frac{r}{c}\right) \quad \text{as desired!}$$

$$\Rightarrow \boxed{G_{ret}(\vec r, t) = \frac{1}{r}\,\delta\!\left(t - \frac{r}{c}\right)}$$

B. Advanced Green function

(can be evaluated in a similar way)

$$G_{adv}(\vec r, t) = -4\pi \int \frac{d^3k}{(2\pi)^3}\frac{d\omega}{2\pi}\, e^{-i\omega t + i\vec k\cdot\vec r}\,\frac{1}{\frac{\omega^2}{c^2} - k^2 - i\omega\varepsilon}$$

$$\Rightarrow \boxed{G_{ret}(\vec{r}, t) = \frac{1}{r}\,\delta\left(t - \frac{r}{c}\right)} \quad \text{or}$$

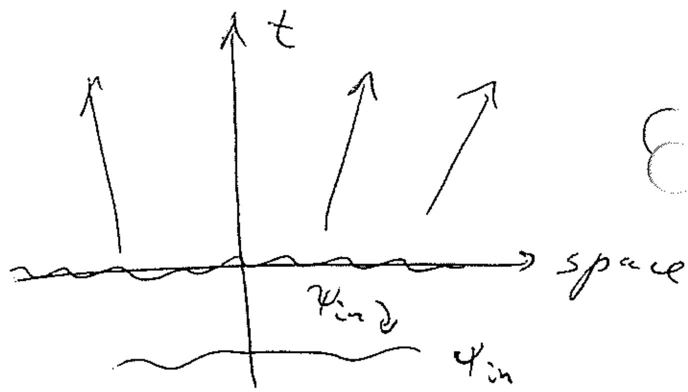
$$\boxed{G_{ret}(\vec{x}, t; \vec{x}\,', t') = \frac{1}{|\vec{x} - \vec{x}\,'|}\,\delta\left(t - t' - \frac{|\vec{x} - \vec{x}\,'|}{c}\right)}$$

retarded Green function

Given the source (localized in space-time) $f(\vec{x}, t)$ and initial condition $\Psi_{in}(\vec{x}, t)$ (satisfying homog. eq., $\Box\Psi_{in} = 0$. at $t = -\infty$) we can write the solution

$$\Psi(\vec{x}, t) = \Psi_{in}(\vec{x}, t) + \int d^3x'\,dt'\,G_{ret}(\vec{x}, t; \vec{x}\,', t')\,f(\vec{x}\,', t')$$

Retarded Green ftn is causal ~ gives the solution in the future due to sources in the past.



<u>Advanced Green function</u>:

$$G_{adv}(\vec{r}, t) = G^*_{ret}(-t, -\vec{r}) \quad \Rightarrow \quad \text{as } |\vec{r}| = |-\vec{r}|$$

$$\Rightarrow G_{adv}(\vec{r}, t) = \frac{1}{r}\,\delta\left(-t - \frac{r}{c}\right) = \frac{1}{r}\,\delta\left(t + \frac{r}{c}\right)$$

$$\Rightarrow \boxed{G_{adv}(\vec{r}, t) = \frac{1}{r}\,\delta\left(t + \frac{r}{c}\right)}$$

Therefore

$$\boxed{G_{adv}(\vec{x},t;\vec{x}',t') = \frac{1}{|\vec{x}-\vec{x}'|}\,\delta\left(t-t'+\frac{|\vec{x}-\vec{x}'|}{c}\right)}$$

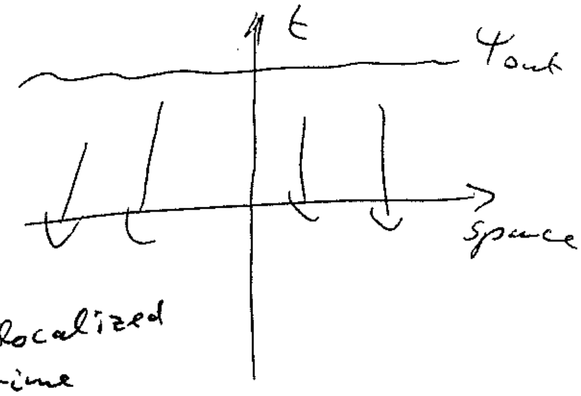
Work opposite to G_{ret} ~ acausal:

also satisfies $\Box \Psi_{out} = 0$.

$$\Psi(\vec{x},t) = \Psi_{out}(\vec{x},t) +$$

$$+ \int d^3x'\,dt'\; G_{adv}(\vec{x},t;\vec{x}',t')\, f(\vec{x}',t').$$

source is localized in space-time



Solution of Maxwell equations in Lorenz gauge:

$$\begin{cases} \nabla^2 \Phi - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\Phi = -\dfrac{\rho}{\varepsilon_0} \\[2ex] \nabla^2 \vec{A} - \dfrac{1}{c^2}\dfrac{\partial^2}{\partial t^2}\vec{A} = -\mu_0 \vec{J} \end{cases}$$

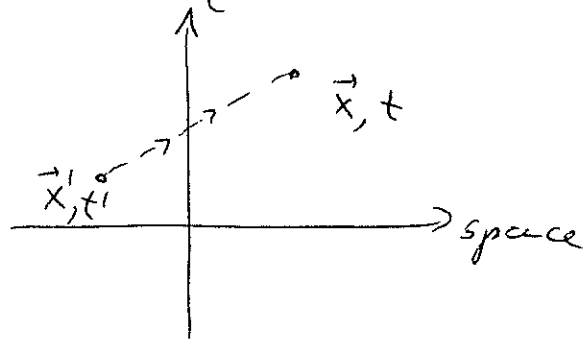
$\Rightarrow$ assume $\Phi_{in} = 0$, $\vec{A}_{in} = 0$ and use retarded Green ftn.

$$\Rightarrow \boxed{\begin{aligned} \Phi(\vec{x},t) &= \frac{1}{4\pi\varepsilon_0}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\rho\left(\vec{x}',\, t-\frac{|\vec{x}-\vec{x}'|}{c}\right) \\ \vec{A}(\vec{x},t) &= \frac{\mu_0}{4\pi}\int d^3x'\,\frac{1}{|\vec{x}-\vec{x}'|}\,\vec{J}\left(\vec{x}',\, t-\frac{|\vec{x}-\vec{x}'|}{c}\right) \end{aligned}}$$

where we integrated over t' the δ-ftn:

$\delta\left(t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right)$ to get $t' = t - \frac{|\vec{x}-\vec{x}'|}{c}$.

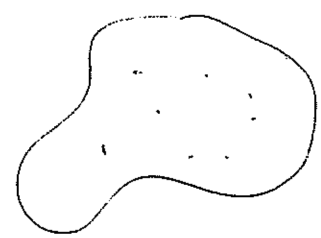
Physical meaning of $t' = t - \frac{|\vec{x} - \vec{x}'|}{c}$: for a source at time t' to affect the field at time t they need to be $c(t - t')$ away from each other ~ just far enough for light to travel!



~~Poynting's Theorem and Conservation of Energy and Momentum.~~

~~Energy:~~

~~Consider several point charges $q_1, \ldots, q_N$ located at $\vec{x}_1, \ldots, \vec{x}_N$ & moving with velocities $\vec{v}_1, \ldots, \vec{v}_N$ in external electromagnetic field:~~



~~The work on these charges due to EM field per unit time is~~

$$\cancel{\sum_{n=1}^{N} \vec{F}_n \cdot \vec{v}_n = \sum_{n=1}^{N} q_n \vec{E}(\vec{x}_n, t) \cdot \vec{v}_n =}$$

$$\cancel{= \int d^3x \left(\sum_{n=1}^{N} q_n \vec{v}_n \delta(\vec{x} - \vec{x}_n) \right) \cdot \vec{E}(\vec{x}, t) =}$$

$$\cancel{= \int d^3x \ \vec{J} \cdot \vec{E}}$$ ~~~ work due to $\vec{E}$-field on the charges $\vec{J}$~~