Last time | Transmission coefficient:

$$T = \sqrt{\frac{\mu \varepsilon'}{\mu' \varepsilon}} \left| \frac{E_0'}{E_0} \right|^2$$

Reflection coefficient
$$R = \left| \frac{E_0''}{E_0} \right|^2$$

Electromagnetic Waves in Conductors (cont'd)

Ohm's law $\vec{J} = \sigma \vec{E}$, $\rho = 0$ on average

get waves with the wave vector

$$k = \omega \sqrt{\mu \varepsilon} \sqrt{1 + \frac{\varepsilon \sigma}{\varepsilon \omega}} \equiv k_1 + i k_2$$

$\frac{\sigma}{\varepsilon \omega} >> 1$ good conductor

$\frac{\sigma}{\varepsilon \omega} << 1$ bad conductor

if $\vec{k} \parallel \hat{z}$ $\Rightarrow$
$$\begin{cases} \vec{E} = E_0 \hat{x}\, e^{-i\omega t + i k_1 z - k_2 z} \\ \vec{B} = B_0 \hat{y}\, e^{-i\omega t + i k_1 z - k_2 z} \end{cases}$$

$B_0 = \frac{k}{\omega} E_0$. $\vec{k} \cdot \vec{B}_0 = 0$, $\vec{k} \cdot \vec{E}_0 = 0$

$B_0 = \frac{k_1 + i k_2}{\omega} E_0$ ~ out of phase

$\Rightarrow$ we get $(\nabla^2 + k^2)\vec{E} = 0$ with $k^2 = \mu\varepsilon\omega^2 + i\omega\mu\sigma$

$$\Rightarrow k = \pm\sqrt{\mu\varepsilon}\,\omega\sqrt{1+\frac{i\sigma}{\varepsilon\omega}} \equiv \boxed{k_1 + ik_2}$$

As $\vec{\nabla}\cdot\vec{E} = 0 \Rightarrow \vec{k}\cdot\vec{E}_0 = 0$. ~ still transverse.

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \vec{k}\cdot\vec{B}_0 = 0$.

Good conductor: $\dfrac{\sigma}{\varepsilon\omega} \gg 1$

Bad conductor: $\dfrac{\sigma}{\varepsilon\omega} \ll 1$

Assume that $\vec{k} \parallel \hat{z}$ and $\vec{E} \parallel \hat{x}$ (linear polarization

$$\vec{E} = \hat{x}E_0 e^{i(kz-\omega t)} = \hat{x}E_0 e^{ik_1z - k_2z - i\omega t} = \vec{E}$$

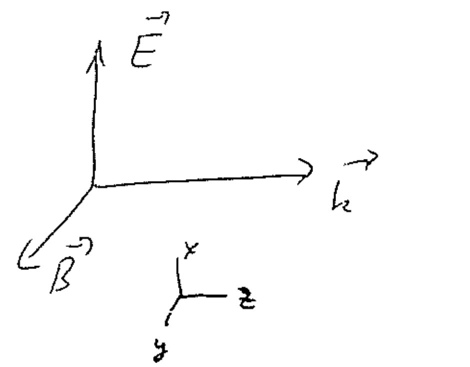
$\Rightarrow$ wavelength $\lambda = \dfrac{2\pi}{k_1}$

$E \sim e^{-k_2 z}$ ~ exponentially decaying

$\vec{B}$-field: $\vec{B} = -\dfrac{i}{\omega}\vec{\nabla}\times\vec{E} = \dfrac{1}{\omega}\vec{k}\times\vec{E}_0 e^{i(kz-\omega t)} \Rightarrow$

$$\vec{B} = \frac{k}{\omega}\hat{y}E_0 e^{i(kz-\omega t)}$$

$\Rightarrow \vec{B}\perp\vec{k}$, $\vec{B}\perp\vec{E}$



but: as k is complex, $\vec{B}$ & $\vec{E}$ are <u>out of phase</u>.

Time-averaged Poynting vector:

$$\langle S_z\rangle = \frac{1}{2}\,\mathrm{Re}\left(\vec{E}\times\vec{H}^*\right)_z = \frac{1}{2\mu}\mathrm{Re}\left(\frac{k^*}{\omega}|E_0|^2 e^{-2k_2z}\right) =$$

$$= \frac{1}{2\mu} \frac{k_1 |E_0|^2}{\omega} e^{-2k_2 z} \propto e^{-z/\delta} \Rightarrow \delta = \frac{1}{2k_2}$$ "skin depth"

$$k_2 = \mathrm{Im}\left[\sqrt{\mu\varepsilon}\,\omega\sqrt{1+\frac{i\sigma}{\varepsilon\omega}}\right]$$

Bad conductor. $k_2 \simeq \sqrt{\mu\varepsilon}\,\omega\,\frac{\sigma}{2\varepsilon\omega} = \sqrt{\frac{\mu}{\varepsilon}}\,\frac{\sigma}{2}$

$\Rightarrow \delta = \frac{1}{\sigma}\sqrt{\frac{\varepsilon}{\mu}}$ (if σ is small $\Rightarrow \delta$ is large)

Good conductor. $k_2 \simeq \sqrt{\mu\varepsilon}\,\omega\,\frac{1}{\sqrt{2}}\sqrt{\frac{\sigma}{\varepsilon\omega}} = \sqrt{\frac{\sigma\mu\omega}{2}}$

$\Rightarrow \delta = \sqrt{\frac{2}{\sigma\mu\omega}}$. (if σ is large $\Rightarrow \delta$ is small).

Frequency-dependent ε, μ, σ.

we just showed that $k = \sqrt{\mu\varepsilon}\,\omega\sqrt{1+\frac{i\sigma}{\varepsilon\omega}} =$

$= \omega\sqrt{\mu\left(\varepsilon+\frac{i\sigma}{\omega}\right)} \Rightarrow$ if we want $k = \sqrt{\mu\varepsilon}\,\omega$

as in non-conductors, we have to, in general, assume that $\varepsilon = \varepsilon(\omega)$, $\mu = \mu(\omega)$, $\sigma = \sigma(\omega)$

and, here redefine $\varepsilon \to \varepsilon(\omega) = \varepsilon_b + \frac{i\sigma}{\omega}$ due to bound charges

"complex dielectric function" (not a constant!)

$n(\omega) = \sqrt{\frac{\mu(\omega)\,\varepsilon(\omega)}{\mu_0\,\varepsilon_0}}$ ~ "complex index of refraction"

$$V_{ph} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n(\omega)} \text{ still.} \Rightarrow k = \frac{n(\omega)\cdot\omega}{c}$$

(real $n(\omega)$)

<u>a simple model for $\varepsilon(\omega)$</u>: consider an electron in external electric field:

$$\underbrace{m}_{\text{mass}}\ddot{\vec{x}} = \vec{F} = -\underbrace{K}_{\text{spring const}}\vec{x} - m\underbrace{\gamma}_{\text{damping}}\dot{\vec{x}} - e\vec{E}$$

$\vec{E}$: constant within the "atom"

$$\omega_0^2 = \frac{K}{m} \Rightarrow m\left(\ddot{\vec{x}} + \gamma\dot{\vec{x}} + \omega_0^2\vec{x}\right) = -e\vec{E}$$

if $\vec{E} = \vec{E}_0 e^{-i\omega t} \Rightarrow \vec{x} = \vec{x}_0 e^{-i\omega t} \Rightarrow$

$$\Rightarrow m\left(-\omega^2 - i\omega\gamma + \omega_0^2\right)\vec{x}_0 = -e\vec{E}_0$$

$$\Rightarrow \vec{x}_0 = \frac{e\vec{E}_0}{m\left(\omega^2 + i\omega\gamma - \omega_0^2\right)}$$

$\Rightarrow$ the amplitude of the ~~atomic~~ molecular dipole moment

$$\vec{p} = -e\vec{x}_0 = \frac{e^2\vec{E}_0}{m(\omega_0^2 - i\omega\gamma - \omega^2)}$$

$\Rightarrow$ if there are n electrons per unit volume

$$\Rightarrow \vec{P}(t) = n\vec{p}(t) = \frac{ne^2\overbrace{\vec{E}_0 e^{-i\omega t}}^{=\vec{E}}}{m(\omega_0^2 - i\omega\gamma - \omega^2)} \Rightarrow$$

$$\Rightarrow \vec{D} = \varepsilon_0\vec{E} + \vec{P} = \left[\varepsilon_0 + \frac{ne^2}{m(\omega_0^2 - i\omega\gamma - \omega^2)}\right]\vec{E} \equiv \varepsilon(\omega)\vec{E}$$

$$\Rightarrow \boxed{\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{ne^2}{m\varepsilon_0(\omega_0^2 - i\omega\gamma - \omega^2)}}$$

~ frequency dependent!

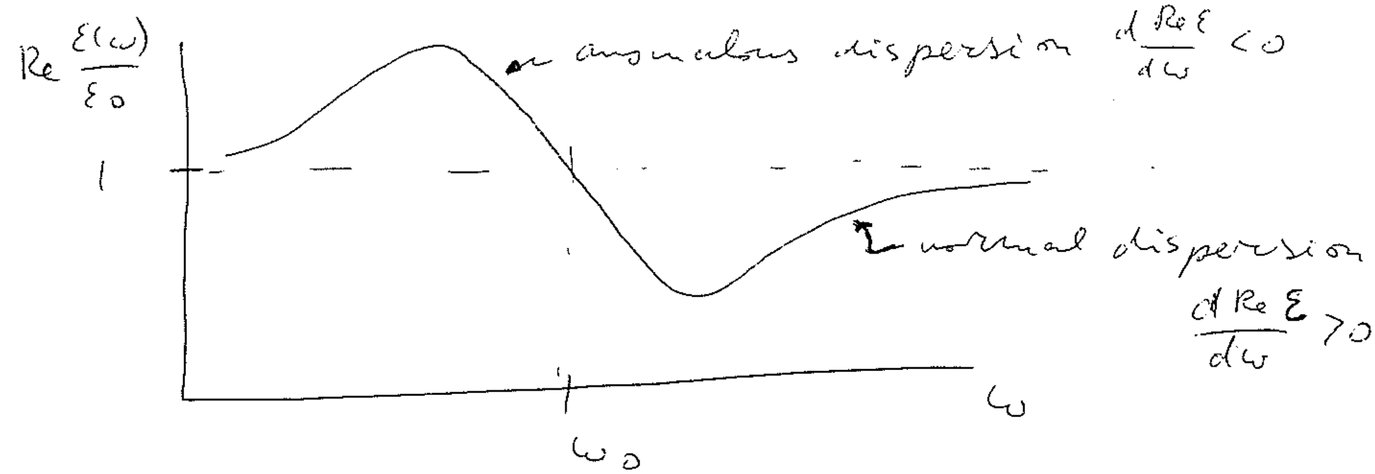
In general various electron states have different frequencies ω_i, γ_i & damping const's $\Rightarrow$ $\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{ne^2}{\varepsilon_0 m} \sum_i \frac{f_i}{\omega_i^2 - i\omega\gamma_i - \omega^2}$

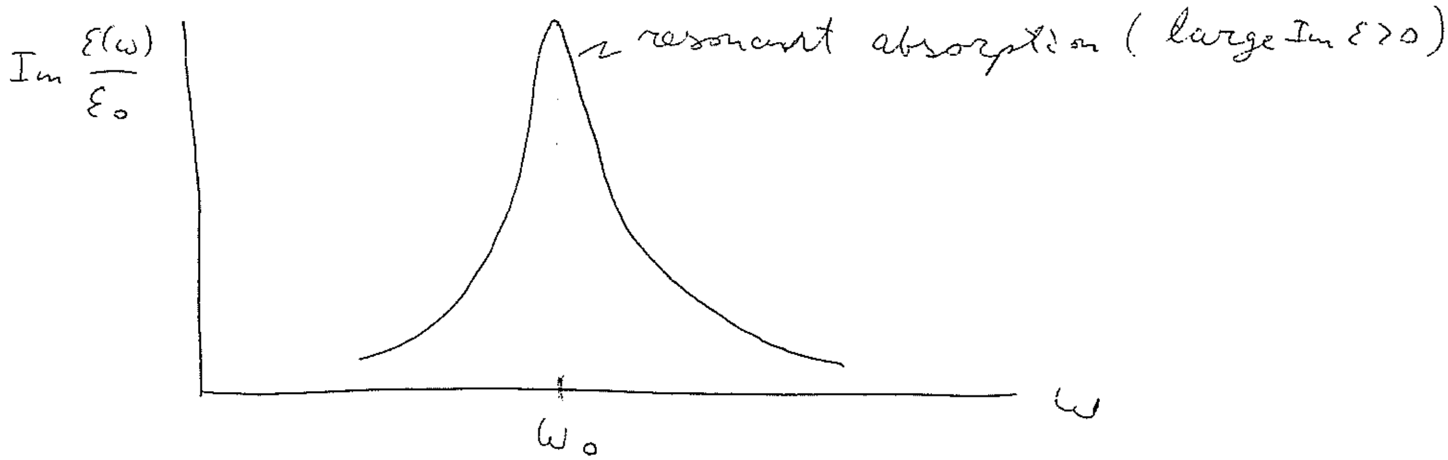
$$\mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{ne^2}{m\varepsilon_0} \frac{\omega_0^2 - \omega^2}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$

Oscillator strength f_i (# electrons/molecule with ω_{0i} & γ_i)

$\sum_i f_i = Z$

$$\mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = \frac{ne^2}{m\varepsilon_0} \frac{\omega\gamma}{(\omega_0^2 - \omega^2)^2 + \omega^2\gamma^2}$$





$$k = k_1 + ik_2 \Rightarrow |\vec{E}|^2 \sim e^{-2k_2 z} \equiv e^{-\alpha z}$$

$\alpha = 2k_2 = \frac{1}{\delta}$ = absorption coefficient (attenuation const.)

$k = \omega\sqrt{\mu(\omega)\varepsilon(\omega)}$ if $\mu(\omega) = \mu_0$ $\Rightarrow$

$$\Rightarrow k = \omega\sqrt{\mu_0\varepsilon_0}\,\sqrt{1 + \frac{ne^2}{m\varepsilon_0(\omega_0^2 - i\omega\gamma - \omega^2)}}$$

$\Rightarrow k_2 \neq 0$ is due to $\gamma \neq 0$ $\Rightarrow$ absorption is due to damping

due to $\mathrm{Im}\,\varepsilon \neq 0$, which is

<u>Low frequency</u>: if electrons are free

$$\Rightarrow \omega_0 = 0 \Rightarrow \frac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \frac{ne^2}{m\varepsilon_0\omega(\omega + i\gamma)} =$$

$$= 1 + \frac{ne^2 i}{m\varepsilon_0\omega(\gamma - i\omega)} = 1 + \frac{i\sigma}{\varepsilon_0\omega} \Rightarrow$$

on the other hand, by definition

$$\Rightarrow \sigma_{(\omega)} = \frac{ne^2}{m}\,\frac{1}{\gamma - i\omega}$$

Drude model (1900) of conductivity

if $\omega \to 0 \Rightarrow \varepsilon = \mathrm{Im}$, $\varepsilon \sim \frac{i}{\omega} \Rightarrow n' \sim \sqrt{\frac{i}{\omega}}$
$\Rightarrow R = \left|\frac{1-n'}{1+n'}\right|^2 \approx 1 \Rightarrow$ metals are shiny!

<u>High frequency</u>: $(\omega \gg \omega_0,\ \omega \gg \gamma$ too$)$

$$\frac{\varepsilon(\omega)}{\varepsilon_0} \simeq 1 - \frac{ne^2}{m\varepsilon_0\omega^2} = 1 - \frac{\omega_p^2}{\omega^2}$$

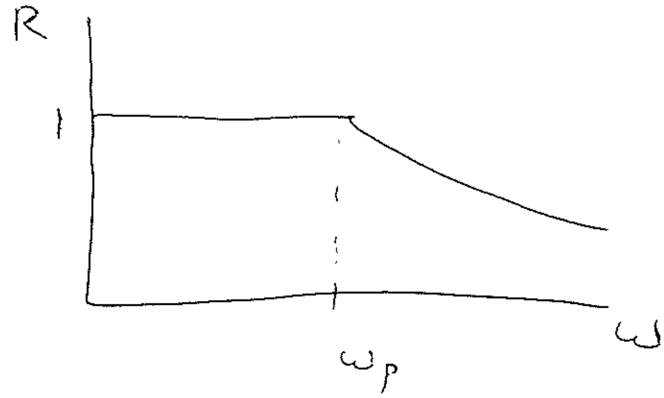
where $\omega_p^2 = \frac{ne^2}{m\varepsilon_0}$ is <u>the plasma frequency</u>

$$k = \omega\sqrt{\mu_0\varepsilon_0}\,\sqrt{1 - \frac{\omega_p^2}{\omega^2}} = \frac{1}{c}\sqrt{\omega^2 - \omega_p^2}$$

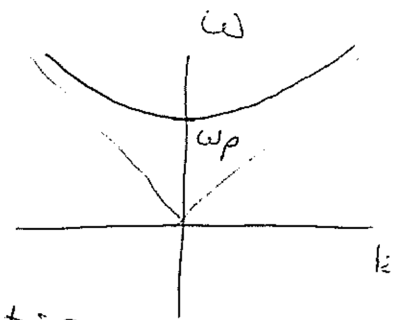
$\Rightarrow$ if $\omega < \omega_p \Rightarrow k = \frac{i}{c}\sqrt{\omega_p^2 - \omega^2}$ ~ imaginary $\Rightarrow$

$\Rightarrow$ waves do not propagate! ~ screening

Reflectivity
$$R = \left|\frac{1 - n(\omega)}{1 + n(\omega)}\right|^2 = \left|\frac{1 - \sqrt{1 - \frac{\omega_p^2}{\omega^2}}}{1 + \sqrt{1 - \frac{\omega_p^2}{\omega^2}}}\right|^2 = \begin{cases} 1, & \omega < \omega_p \\ <1, & \omega > \omega_p \end{cases}$$



most energy is <u>reflected</u>! (at $\omega < \omega_p$)



$$\omega^2 = c^2 k^2 + \omega_p^2 \Rightarrow \omega = \sqrt{c^2 k^2 + \omega_p^2}$$

dispersion relation

cf. $E^2 = c^2 k^2 + m^2 c^4$ for relativistic particle of mass m: ω_p is like a "mass" for photons in the medium!

<u>Kramers - Kronig Relations</u>

Is $\varepsilon(\omega)$ arbitrary? No. In fact, due to causality $\varepsilon(\omega)$ is an analytic function of ω!

Suppose $\vec{D}(\vec{x}, \omega) = \varepsilon(\omega)\, \vec{E}(\vec{x}, \omega)$

$$\Rightarrow \vec{D}(\vec{x}, t) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, D(\vec{x}, \omega)\, e^{-i\omega t} =$$

$$= \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \varepsilon(\omega) \vec{E}(\vec{x}, \omega)\, e^{-i\omega t} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} d\omega\, \varepsilon(\omega) e^{-i\omega t} \cdot$$

$$\frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} dt'\, e^{i\omega t'} \vec{E}(\vec{x}, t') = \int_{-\infty}^{\infty} dt' \quad \vec{E}(\vec{x}, t') \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \varepsilon(\omega) \cdot$$

$$\cdot\, e^{i\omega(t'-t)} =$$