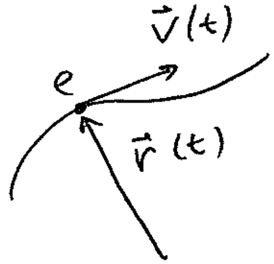
Last time

Radiation by Moving Charges (cont'd)



~ consider a charge moving along a fixed trajectory $\vec{r}(t)$

~ we need to find A^μ, radiated power

Solved Maxwell equations $\Box A^\mu = \frac{4\pi}{c} J^\mu$ to get

$$A^\mu(x) = \frac{1}{c^2}\int d^4x' \; \frac{1}{|\vec{x}-\vec{x}'|} \; \delta\left(t-t'-\frac{|\vec{x}-\vec{x}'|}{c}\right) J^\mu(x')$$

Here we have
$$\begin{cases} \rho(\vec{x},t) = e\,\delta^3(\vec{x}-\vec{r}(t)) \\ \vec{J}(\vec{x},t) = e\vec{v}\;\delta^3(\vec{x}-\vec{r}(t)) \end{cases}$$

$\Rightarrow$ plugged $J^0 = c\rho$ in to obtain

$$\Phi(\vec{x},t) = \int_{-\infty}^{\infty} dt' \; \frac{e}{|\vec{x}-\vec{r}(t')|} \; \delta\left(t-t'-\frac{|\vec{x}-\vec{r}(t')|}{c}\right).$$

Hence, the field of a point charge is

$$\Phi(\vec{x},t) = \frac{1}{c^2}\int d^4x' \, \frac{\theta(x_0 - x_0')}{|\vec{x}-\vec{x}'|} \, .$$

$$\cdot \, c \cdot \delta(x_0 - x_0' - |\vec{x}-\vec{x}'|) \; c \; e \; \delta(\vec{x}' - \vec{r}(t')) =$$

$$= \int_{-\infty}^{\infty} dx_0' \, \frac{\theta(x_0 - x_0')}{|\vec{x} - \vec{r}(t')|} \, e \, \delta(x_0 - x_0' - |\vec{x} - \vec{r}(t')|) =$$

$$= \int_{-\infty}^{\infty} dt' \, \frac{e}{|\vec{x}-\vec{r}(t')|} \, \delta\left(t - t' - \frac{1}{c}|\vec{x} - \vec{r}(t')|\right)$$

$\Rightarrow$ t' has to be determined from the implicit equation: (label $t' = t_{ret}$)

$$t_{ret} = t - \frac{1}{c}|\vec{x} - \vec{r}(t_{ret})|$$

To integrate denote $F(t,t') \equiv t - t' - \frac{1}{c}|\vec{x} - \vec{r}(t')|$

$$\Rightarrow \Phi(\vec{x},t) = \int_{-\infty}^{\infty} dt' \, \frac{e}{|\vec{x} - \vec{r}(t')|} \, \delta(F(t,t')) =$$

$$= \frac{e}{|\vec{x} - \vec{r}(t_{ret})|} \, \frac{1}{|\partial F/\partial t'|}\Bigg|_{t'=t_{ret}}.$$

$$\left.\frac{\partial F}{\partial t'}\right|_{t'=t_{ret}} = -1 + \frac{1}{c}\,\frac{(\vec{x}-\vec{r}(t_{ret}))\cdot V(t_{ret})}{|\vec{x}-\vec{r}(t_{ret})|}\quad, \text{where } \vec{V}(t_{ret}) = \frac{d\vec{r}(t_{ret})}{dt_{ret}}$$

Defining $\hat{n} \equiv \dfrac{\vec{x}-\vec{r}(t_{ret})}{|\vec{x}-\vec{r}(t_{ret})|}$ and $\vec{\beta}(t_{ret}) \equiv \dfrac{\vec{V}(t_{ret})}{c}$

get $\left.\dfrac{\partial F}{\partial t'}\right|_{t'=t_{ret}} = -1 + \hat{n}\cdot\vec{\beta} \Rightarrow \left|\left.\dfrac{\partial F}{\partial t'}\right|_{t'=t_{ret}}\right| = 1-\hat{n}\cdot\vec{\beta}$

as $|\vec{\beta}| \leq 1$. $\Rightarrow$ $\Phi(\vec{x},t) = \left[\dfrac{e}{(1-\hat{n}\cdot\vec{\beta})R}\right]_{ret}$

where $R(t) \equiv |\vec{x}-\vec{r}(t)|$ and the subscript of "ret" means that we need to evaluate everything at $t = t_{ret}$.

Similarly $\vec{A}(\vec{x},t) = \left[\dfrac{e\vec{\beta}}{(1-\hat{n}\cdot\vec{\beta})\,R}\right]_{ret}$.

These are Lienard-Wiechert potentials!

To find $\vec{E}$ & $\vec{B}$ need to use

$$\vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\frac{\partial\vec{A}}{\partial t}\quad,\quad \vec{B} = \vec{\nabla}\times\vec{A}.$$

To differentiate one needs to remember that

$$\frac{\partial t_{ret}}{\partial t} = -\frac{\partial F/\partial t}{\partial F/\partial t_{ret}} = \frac{1}{1-\hat{n}\cdot\vec{\beta}(t_{ret})} \qquad \vec{R}_{(t)} = \vec{x} - \vec{r}(t)$$

$$\frac{\partial R(t_{ret})}{\partial t} = \frac{\partial t_{ret}}{\partial t}\frac{\partial R}{\partial t_{ret}} = \frac{1}{1-\hat{n}\cdot\vec{\beta}}\,\frac{(\vec{x}-\vec{r})\cdot(-\vec{v})}{R} =$$

$$= -\frac{\vec{R}\cdot\vec{v}}{R}\,\frac{1}{1-\hat{n}\cdot\vec{\beta}} = -\frac{\hat{n}\cdot\vec{v}}{1-\hat{n}\cdot\vec{\beta}} = -c\,\frac{\hat{n}\cdot\vec{\beta}}{1-\hat{n}\cdot\vec{\beta}}$$

As $t_{ret} = t - \frac{R(t_{ret})}{c} \Rightarrow \vec{\nabla} t_{ret} = -\frac{1}{c}\vec{\nabla} R(t_{ret}) =$

$$= -\frac{1}{c}\frac{\vec{R}}{R} - \frac{1}{c}\underbrace{\frac{\partial R}{\partial t_{ret}}}_{= -c\frac{\vec{R}\cdot\vec{\beta}}{R} = -c\hat{n}\cdot\vec{\beta}}\vec{\nabla} t_{ret} \Rightarrow \vec{\nabla} t_{ret} = -\frac{\vec{R}}{cR(1-\vec{\beta}\cdot\hat{n})}$$

$$\Rightarrow \vec{\nabla} t_{ret} = \frac{-\hat{n}}{c(1-\hat{n}\cdot\vec{\beta})} = -\frac{1}{c}\vec{\nabla} R.$$

$\Rightarrow$ after straightforward (but tedious) calculations get

Yefimenko equations

$$\boxed{\vec{E} = e\left[\frac{\hat{n}-\vec{\beta}}{\gamma^2(1-\vec{\beta}\cdot\hat{n})^3R^2}\right]_{ret} + \frac{e}{c}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3R}\right]_{ret} \qquad \vec{B} = \left[\hat{n}\times\vec{E}\right]_{ret}.}$$

first term ~ just due to velocity, "velocity field" $\sim \frac{1}{R^2}$ static

the 2nd term is due to acceleration $\Rightarrow$

$\Rightarrow$ "acceleration field" $\sim \frac{1}{R} \Rightarrow |\vec{S}| \sim \frac{1}{R^2}$ radiation field.

$\Rightarrow$ if a particle moves with constant velocity $\Rightarrow$ 2nd term is absent and the 1st term can be obtained by a simple boost from the charge's rest frame, where one only has Coulomb $\vec{E}$ - field.

Power Radiated by an Accelerated Charge.

Imagine a non-relativistic motion, $|\vec{\beta}| \ll 1$, but with non-negligible acceleration: $|\dot{\vec{\beta}}| \sim$ large.

Radiation is given by the $\dot{\vec{\beta}}$ - term:

$$\vec{E}_{rad} = \frac{e}{c}\left[\frac{\hat{n}\times[(\hat{n}-\vec{\beta})\times\dot{\vec{\beta}}]}{(1-\vec{\beta}\cdot\hat{n})^3 R}\right]_{ret} \simeq \Big|\text{ as } |\vec{\beta}| \ll 1$$

$$\simeq \frac{e}{c}\left[\frac{\hat{n}\times(\hat{n}\times\dot{\vec{\beta}})}{R}\right]_{ret}$$

The Poynting vector is given by $(\vec{B} = \hat{n}\times\vec{E})$

$$\vec{S} = \frac{c}{4\pi}\vec{E}\times\vec{B} = \frac{c}{4\pi}\vec{E}\times(\hat{n}\times\vec{E}) = \frac{c}{4\pi}\Big[\hat{n}\,|\vec{E}|^2 -$$

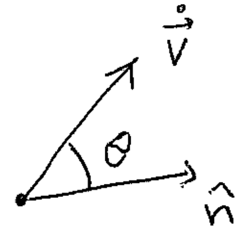
$$-\vec{E}\,\underbrace{(\hat{n}\cdot\vec{E})}_{=0}\Big] \Rightarrow \quad \vec{S} = \frac{c}{4\pi}\hat{n}\,|\vec{E}|^2 \quad \Rightarrow$$

for $\vec{E}_{rad}$.

$$\frac{dP}{d\Omega} = R^2\,\hat{n}\cdot\vec{S}$$

$$\frac{dP}{d\Omega} = \frac{c}{4\pi} |\vec{E}|^2 R^2 = \frac{e^2}{4\pi c} \left|\left[\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})\right]\right|^2$$

$= |(\hat{n} \times \dot{\vec{\beta}})^2|$



$$\Rightarrow |\hat{n} \times (\hat{n} \times \dot{\vec{\beta}})|^2 = |\dot{\vec{\beta}}|^2 \sin^2\theta$$

$$\Rightarrow \boxed{\frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} |\dot{\vec{V}}|^2 \sin^2\theta}$$

<u>Example</u>: imagine a charge oscillating along x-axis: $\vec{X}(t) = d \cdot \sin(\omega t)\, \hat{x} \Rightarrow$

$$\Rightarrow \vec{V} = +d\omega \cos(\omega t)\, \hat{x}, \quad \dot{\vec{V}} = -d\omega^2 \sin(\omega t)\, \hat{x}$$

$\Rightarrow$ time averaged $\left\langle \frac{dP}{d\Omega} \right\rangle = \frac{e^2}{4\pi c^3} \frac{1}{2} d^2 \omega^4 \sin^2\theta$

dipole moment $|\vec{p}| = ed$, $k = \frac{\omega}{c} \Rightarrow$

$$\Rightarrow \left\langle \frac{dP}{d\Omega} \right\rangle = \frac{c}{8\pi} k^4 |\vec{p}|^2 \sin^2\theta$$

(Gaussian units)

$\leftarrow$ just dipole radiation, as expected!

$$P = \int d\Omega \frac{dP}{d\Omega} = \frac{e^2}{4\pi c^3} |\dot{\vec{v}}|^2 \cdot 2\pi \cdot \frac{4}{3} = \frac{2}{3} \frac{e^2}{c^3} |\dot{\vec{v}}|^2$$

$$\boxed{P = \frac{2}{3} \frac{e^2}{c^3} |\dot{\vec{V}}|^2}$$

Larmor formula (non-relativistic limit)

⇒ To find the relativistic generalization of this formula, need to use the complete $\vec{E}_{rad}$ in the expression for $dP/d\Omega$ ~ tedious...

⇒ Insted use the following line of arguments:

$$P = \frac{2}{3}\frac{e^2}{m^2c^3}\frac{d\vec{p}}{dt}\cdot\frac{d\vec{p}}{dt}$$

P is a Lorentz-invariant quantity (why?)

$P = \frac{d\mathcal{E}}{dt}$, $\mathcal{E}$ is 0th component of p^μ
t is 0th component of x^μ.

Generalize Larmor formula by writing

$$P = -\frac{2}{3}\frac{e^2}{m^2c^3}\frac{dp_\mu}{d\tau}\frac{dp^\mu}{d\tau}$$

(this is also in charge's own time, as in $\frac{dE}{d\alpha}$)

(in NR case $\tau \simeq t$ ⇒ get back $|\dot{\vec{p}}|^2$)

$$\Rightarrow P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{d\vec{p}}{d\tau}\frac{d\vec{p}}{d\tau} - \frac{1}{c^2}\left(\frac{d\mathcal{E}}{d\tau}\right)^2\right) = \Bigg/ \begin{array}{l} \vec{p} = \gamma m\vec{v} \\ \mathcal{E} = mc^2\gamma \\ d\tau = dt/\gamma \end{array}$$

$$= \frac{2}{3}\frac{e^2}{m^2c^3}\left(\gamma^2m^2\left|\frac{d(\vec{v}\gamma)}{dt}\right|^2 - m^2c^2\gamma^2\left(\frac{d\gamma}{dt}\right)^2\right) =$$

$$= \frac{2}{3}\frac{e^2}{m^2c^3}\Big[\gamma^4m^2|\dot{\vec{v}}|^2 + \gamma^2m^2\dot{\gamma}^2\underbrace{(v^2-c^2)}_{-c^2\frac{1}{\gamma^2}} +$$

$$+ 2\gamma^2m^2\,\vec{v}\cdot\dot{\vec{v}}\,\gamma\cdot\dot{\gamma}\Big] \Rightarrow$$

as $\frac{d\gamma}{dt} = \dot{\gamma} = -\frac{1}{2}\gamma^3 \cdot (-2) \frac{\vec{V}\cdot\dot{\vec{V}}}{c^2} = \gamma^3 \vec{\beta}\cdot\dot{\vec{\beta}}$

$$\Rightarrow P = \frac{2}{3}\frac{e^2}{\cancel{m^2}c^3}\left[\gamma^4 \cancel{m^2}c^2 |\dot{\vec{\beta}}|^2 - \cancel{m^2}c^2\gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2 + \right.$$

$$\left. + 2\cancel{m^2}c^2\gamma^6(\vec{\beta}\cdot\dot{\vec{\beta}})^2\right] = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[\frac{|\dot{\vec{\beta}}|^2}{\gamma^2} + \right.$$

$$\left. + (\vec{\beta}\cdot\dot{\vec{\beta}})^2\right]$$

$|\vec{\beta}\times\dot{\vec{\beta}}|^2$ ''

Finally, as $(\vec{\beta}\times\dot{\vec{\beta}})\cdot(\vec{\beta}\times\dot{\vec{\beta}}) = \vec{\beta}^2\dot{\vec{\beta}}^2 - (\vec{\beta}\cdot\dot{\vec{\beta}})^2$

$$\Rightarrow \dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2 = \dot{\vec{\beta}}^2\underbrace{(1-\beta^2)}_{1/\gamma^2} + (\vec{\beta}\cdot\dot{\vec{\beta}})^2$$

$$\Rightarrow \boxed{P = \frac{2}{3}\frac{e^2}{c}\gamma^6\left[\dot{\vec{\beta}}^2 - (\vec{\beta}\times\dot{\vec{\beta}})^2\right]}$$

Lienard (1898)

$\Rightarrow$ acceleration leads to radiation

$\Rightarrow$ in particle accelerators energy loss due to radiation limits maximum attainable energy...

$$P \sim \frac{1}{m^2}\left(\frac{d\vec{p}}{dt}\right)^2 \quad \text{and} \quad \vec{F} = \frac{d\vec{p}}{dt} \sim \text{force} \Rightarrow$$

$\Rightarrow$ for the same applied force $P \sim \frac{1}{m^2} \Rightarrow$ the lighter the particle the more it radiates $\Rightarrow$ electrons radiate <u>most</u>.

Examples: linear accelerator

$$\vec{\beta} \parallel \dot{\vec{\beta}}\ ,\quad \dot{\vec{\beta}} = \text{const} \Rightarrow P = \frac{2}{3}\frac{e^2}{c}\gamma^6\,\dot{\vec{\beta}}^2$$

$$\frac{d\overset{\swarrow\ \text{momentum}}{p}}{dt} = m\frac{d}{dt}(\gamma v) = m\gamma\dot{v} + mv\gamma^3\frac{v\cdot\dot{v}}{c^2} =$$

$$= m\gamma\dot{v}\left(1 + \frac{v^2}{c^2}\gamma^2\right) = m\gamma^3\dot{v} = mc\gamma^3\dot{\beta}$$

$$\Rightarrow \dot{\beta}^2 = \frac{1}{m^2c^2\gamma^6}\left(\frac{dp}{dt}\right)^2 \Rightarrow P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dp}{dt}\right)^2$$

$$\frac{dp}{dt} = \text{force} = \frac{dE}{dx} \Rightarrow P = \frac{2}{3}\frac{e^2}{m^2c^3}\left(\frac{dE}{dx}\right)^2$$

force $\cdot\, dx$ = work   ↖ energy gain (or loss)

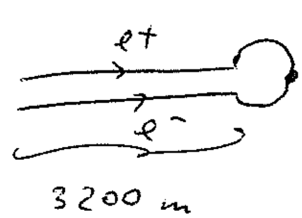
$$\frac{\text{power radiated}}{\text{power supplied}} = \frac{P}{v\cdot\frac{dE}{dx}} = \frac{2}{3}\frac{e^2}{m^2c^3v}\frac{dE}{dx} \xrightarrow[v\to c]{}$$

$$\to \frac{2}{3}\frac{e^2}{m^2c^4}\frac{dE}{dx} = \frac{2}{3}\frac{e^2}{mc^2}\frac{1}{mc^2}\frac{dE}{dx} =$$

$$= \frac{2}{3}\underbrace{2.8\cdot 10^{-15}\,m}_{e^2/mc^2}\,(0.511\ \text{MeV})^{-1}\cdot\frac{dE}{dx} \Rightarrow \text{to have this} \sim 1$$

$$\text{need } \frac{dE}{dx} \sim \frac{3}{2}\frac{.5}{2.8\cdot 10^{-15}}\frac{\text{MeV}}{m} \approx 2.5\cdot 10^{14}\frac{\text{MeV}}{m}$$

Usually $\frac{dE}{dx} < 50\ \text{MeV}/m \Rightarrow$ losses are tiny $\Rightarrow$ good to build linear colliders. (SLAC



$\sqrt{s} \simeq 90\ \text{GeV} \Rightarrow \frac{dE}{dx} \simeq \frac{45\ \text{GeV}}{3200\,m} \simeq 14\ \frac{\text{MeV}}{m}$.)