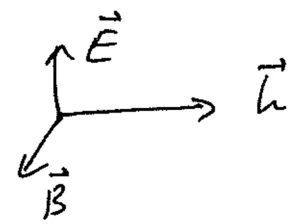
Final Review

Plane Electromagnetic Waves

$$\left[\nabla^2 - \mu\varepsilon \frac{\partial^2}{\partial t^2}\right]\begin{pmatrix}\vec{E}\\ \vec{B}\end{pmatrix} = 0$$

$\Rightarrow$ for monochromatic plane wave write

$$\begin{cases} \vec{E} = \vec{E}_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} \\ \vec{B} = \frac{1}{\omega}\, \vec{k}\times\vec{E} \end{cases}$$



In the end one has to take Re parts for $\vec{E}$ & $\vec{B}$.

Energy density $\langle u\rangle = \frac{1}{2}\varepsilon E_0^2$ (time-averaged)

Poynting vector $\langle\vec{S}\rangle = \frac{1}{2}\sqrt{\frac{\varepsilon}{\mu}}\, E_0^2\, \hat{k}$. (time-ave.)

$$u = \frac{1}{2}\left(\vec{E}\cdot\vec{D} + \vec{B}\cdot\vec{H}\right) \qquad \vec{S} = \vec{E}\times\vec{H}$$

phase velocity

$$V_{ph} = \frac{\omega}{k} = \frac{1}{\sqrt{\mu\varepsilon}} = \frac{c}{n}$$

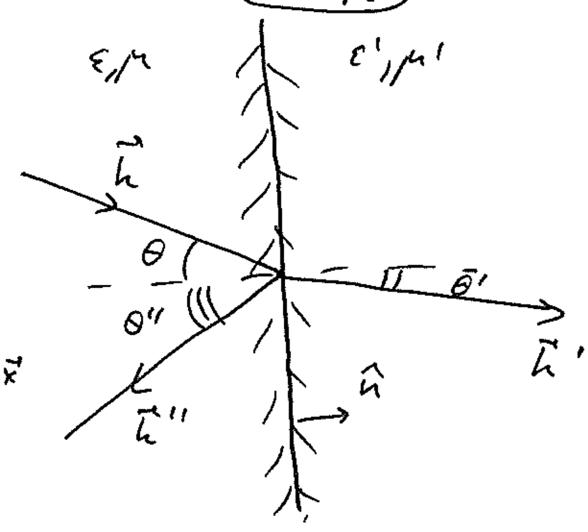
index of refraction

$$n = \sqrt{\frac{\varepsilon\mu}{\varepsilon_0\mu_0}}$$

Reflection and Refraction

incoming wave:

$$\begin{cases} \vec{E} = \vec{E}_0\, e^{-i\omega t + i\vec{k}\cdot\vec{x}} \\ \vec{B} = \frac{1}{\omega}\, \vec{k}\times\vec{E} \end{cases}$$



refracted wave

$$\begin{cases} \vec{E}' = \vec{E}_0'\, e^{-i\omega' t + i\vec{k}'\cdot\vec{x}} \\ \vec{B}' = \frac{1}{\omega'}\, \vec{k}'\times\vec{E}' \end{cases}$$

reflected wave

$$\begin{cases} \vec{E}'' = \vec{E}_0''\, e^{-i\omega'' t + i\vec{k}''\cdot\vec{x}} \\ \vec{B}'' = \frac{1}{\omega''}\, \vec{k}''\times\vec{E}'' \end{cases}$$

Need to impose 4 boundary conditions (B2)

$B_n, D_n \sim$ continuous , $E_t, H_t \sim$ continuous

$$\begin{cases} \hat{n}\cdot\left[\varepsilon(\vec{E}_0+\vec{E}_0'')-\varepsilon'\vec{E}_0'\right]=0 & D_n \\ \hat{n}\cdot\left[\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}_0''-\vec{k}'\times\vec{E}_0'\right]=0 & B_n \\ \hat{n}\times\left[\vec{E}_0+\vec{E}_0''-\vec{E}_0'\right]=0 & E_t \\ \hat{n}\times\left[\frac{1}{\mu}(\vec{k}\times\vec{E}_0+\vec{k}''\times\vec{E}_0'')-\frac{1}{\mu'}\vec{k}'\times\vec{E}_0'\right]=0 & H_t \end{cases}$$

Matching the phases: $\omega=\omega'=\omega''$

$\theta=\theta''$ $\quad$ $n\sin\theta=n'\sin\theta'$ $\sim$ geometric optics.

Snell's law

$\Rightarrow$ we solved for E_0' & E_0'' in terms of $E_0, n, n', \theta, \mu, \mu'$

for (I) $\vec{E}_0 \perp$ plane of incidence and (II) $\vec{E}_0 \parallel$ plane of incidence

$\Rightarrow$ got 2 sets of complicated formulas (see notes)

$\Rightarrow$ observed that total internal reflection happens at $\theta \geqslant \sin^{-1}\left(\frac{n'}{n}\right)$

$\Rightarrow$ defined & calculated transmission & reflection coefficients: $T=\dfrac{\langle|\vec{S}'|\rangle}{\langle|\vec{S}|\rangle}$ $\quad$ $R=\dfrac{\langle|\vec{S}''|\rangle}{\langle|\vec{S}|\rangle}$.

Frequency-dependent ε, μ, σ

worked out a simple model with the molecule being a harmonic oscillator to derive

$$\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{ne^2}{m\varepsilon_0(\omega_0^2 - i\omega\gamma - \omega^2)}$$

high frequency $\omega \gg \omega_0 \Rightarrow \dfrac{\varepsilon(\omega)}{\varepsilon_0} = 1 - \dfrac{\omega_p^2}{\omega^2}$

$\omega_p^2 = \dfrac{ne^2}{m\varepsilon_0}$ ~ plasma frequency

$k = \omega\sqrt{\mu_0\varepsilon(\omega)} = \dfrac{\omega}{c}\sqrt{1 - \dfrac{\omega_p^2}{\omega^2}} \Rightarrow$ for $\omega < \omega_p$ waves are damped (evanescent)

$$e^{ikz} \to e^{-|k|z}$$

conductors: $\varepsilon(\omega) = \varepsilon + \dfrac{i\sigma}{\omega}$

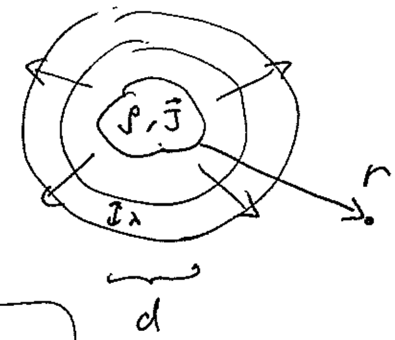
Kramers-Kronig relations:

$$\mathrm{Re}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = 1 + \frac{1}{\pi} P\int_{-\infty}^{\infty} d\omega' \frac{1}{\omega' - \omega}\,\mathrm{Im}\,\frac{\varepsilon(\omega')}{\varepsilon_0}$$

$$\mathrm{Im}\,\frac{\varepsilon(\omega)}{\varepsilon_0} = -\frac{1}{\pi}\int_{-\infty}^{\infty} d\omega'\, P\frac{1}{\omega' - \omega}\left[\mathrm{Re}\,\frac{\varepsilon(\omega')}{\varepsilon_0} - 1\right]$$

$\Rightarrow$ not free, Re & Im parts of $\varepsilon(\omega)$ are related! $\Rightarrow$ group velocity $\vec{V}_{gr} = \vec{\nabla}_k\,\omega\big|_{\vec{k} = \vec{k}_0}$

Radiation



radiation zone:

$d \ll \lambda \ll r$

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \frac{e^{ikr}}{r} \int d^3x' \, \vec{J}(\vec{x}') \, e^{-ik\hat{n}\cdot\vec{x}'}$$

$$\vec{H} = \frac{1}{\mu_0} \vec{\nabla} \times \vec{A} \qquad \vec{E} = \frac{i}{\omega\varepsilon_0} \vec{\nabla} \times \vec{H}$$

all $\otimes e^{-i\omega t}$ & Re

expand in $kd \sim d/\lambda$

Electric dipole radiation: $\vec{A} = -\frac{i\mu_0\omega}{4\pi} \vec{p} \frac{e^{ikr}}{r}$

$$\vec{H} = \frac{ik}{\mu_0} \hat{n} \times \vec{A} \qquad \vec{E} \approx -i\omega \, \hat{n} \times (\hat{n} \times \vec{A})$$

~ true for any multipole moment

$\vec{p} = \int d^3x \, \vec{x} \, \rho(\vec{x})$ ~ dipole moment

$$\frac{dP}{d\Omega} = \frac{c^2}{32\pi^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \, k^4 \, |\hat{n} \times \vec{p}|^2$$

angular distribution of electric dipole radiation

magnetic dipole: $\vec{A} = \frac{ik\mu_0}{4\pi} \hat{n} \times \vec{m} \frac{e^{ikr}}{r}$

$$\frac{dP}{d\Omega} = \frac{k^4}{32\pi^2} \sqrt{\frac{\mu_0}{\varepsilon_0}} \, |\hat{n} \times \vec{m}|^2$$

$\vec{m} = \int d^3x \, \frac{1}{2} \vec{x} \times \vec{J}$

magnetic dipole moment

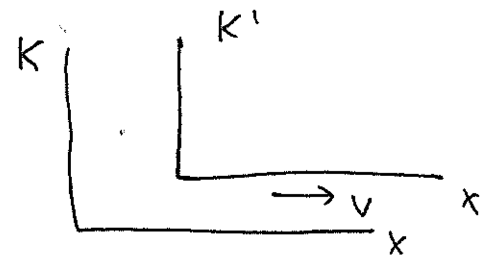
Quadrupole radiation: $Q_{ij} = \int d^3x\, \rho(\vec{x}) \left[3x_i x_j - \delta_{ij} \vec{x}^2 \right]$

$$\frac{dP}{d\Omega} = \frac{c k^6}{2(24\pi)^2 \varepsilon_0} \left[Q_{ij} n_j Q^*_{ik} n_k - Q_{ij} n_i n_j Q^*_{k\ell} n_k n_\ell \right]$$

$$P = \frac{c k^6}{1440 \pi \varepsilon_0} Q_{ij} Q^*_{ij}$$

Special Theory of Relativity

Lorentz transformation



$$x'^\mu = \Lambda^\mu{}_\nu x^\nu \quad , \mu, \nu = 0, 1, 2, 3$$

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\beta\gamma & 0 & 0 \\ -\beta\gamma & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$ for a boost along the x-axis

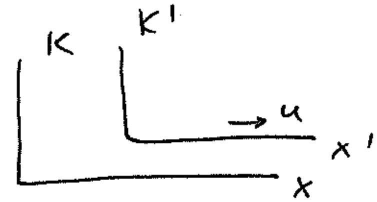
Interval: $ds^2 = c^2 dt^2 - d\vec{x}^2$ Lorentz-invariant

Proper time $d\tau = \frac{ds}{c}$ (time in rest frame)

$dt = \gamma\, d\tau$ ~ time dilation

Lorentz contraction: $\ell = \ell_0 \sqrt{1 - \frac{v^2}{c^2}} = \ell_0 / \gamma$

Velocity Transformation:



$$\boxed{V_x = \frac{V_x' + u}{1 + \frac{uV_x'}{c^2}}, \quad V_{y,z} = \gamma \frac{V'_{y,z}}{1 + \frac{uV_x'}{c^2}}}$$

$$\boxed{\tan\theta = \frac{v' \sin\theta'}{\gamma(v'\cos\theta' + u)}}$$ angle transformation

Doppler shift: $\boxed{\omega' = \gamma\,\omega(1 - \beta\cos\theta)}$

$$\boxed{\tan\theta' = \frac{\sin\theta}{\gamma(\cos\theta - \beta)}}$$ light abberation

4-vectors: $A'^{\mu} = \frac{\partial x'^{\mu}}{\partial x^{\nu}} A^{\nu}$ (contravariant), $A'_{\mu} = \frac{\partial x^{\nu}}{\partial x'^{\mu}} A_{\nu}$ (covariant)

$A_\mu B^\mu$ ~ scalar product (Lorentz-inv.)

Metric tensor: $g_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} = g^{\mu\nu}$

$\Rightarrow ds^2 = g_{\mu\nu}\, dx^\mu dx^\nu$, $A^\mu = g^{\mu\nu} A_\nu$, $A_\nu = g_{\nu\mu} A^\mu$ ~raises & lowers indices

$g_{\mu\alpha} g^{\alpha\nu} = \delta_\mu^{\ \nu}$.

$\partial_\mu = \frac{\partial}{\partial x^\mu}$, $\partial^\mu = \frac{\partial}{\partial x_\mu}$ $\Rightarrow$ $\Box \equiv \partial_\mu \partial^\mu = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \vec{\nabla}^2$

is Lorentz-invariant!

4-velocity: $\boxed{u^\mu \equiv \frac{dx^\mu}{d\tau} = \gamma(c, \vec{v})}$

Relativistic mechanics: $L = -mc^2\sqrt{1-\frac{v^2}{c^2}}$

Lagrangian of a free particle

4-momentum: $p^\mu = m u^\mu = \left(\frac{E}{c}, \vec{p}\right)$

with $\vec{p} = m\gamma\vec{v}$ (3-momentum), $E = m\gamma c^2$ (energy), $p_\mu p^\mu = m^2c^2$

$\Updownarrow$

$E^2 = p^2c^2 + m^2c^4$

Covariance of Electrodynamics:

current 4-vector: $J^\mu = (c\rho, \vec{J})$

$\partial_\mu J^\mu = 0$ continuity (current conservation)

4-vector potential: $A^\mu = (\Phi, \vec{A})$

Field strength tensor: $F^{\mu\nu} = \partial^\mu A^\nu - \partial^\nu A^\mu$

$$F^{\mu\nu} = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}$$

Dual tensor: $\tilde{F}^{\mu\nu} = \frac{1}{2}\varepsilon^{\mu\nu\rho\sigma} F_{\rho\sigma}$

Maxwell equations:

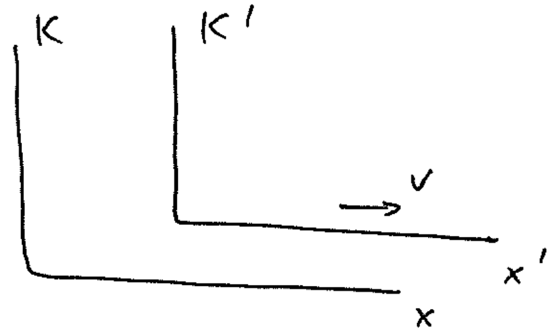
$$\boxed{\begin{aligned} \partial_\mu F^{\mu\nu} &= \frac{4\pi}{c} J^\nu \\ \partial_\mu \tilde{F}^{\mu\nu} &= 0 \end{aligned}}$$

in $\partial_\mu A^\mu = 0$ Lorenz gauge have $\boxed{\Box A^\mu = \frac{4\pi}{c} J^\mu}$.

Lorentz-invariants: $I_1 = F_{\mu\nu} F^{\mu\nu} = 2(B^2 - E^2)$

$I_2 = F_{\mu\nu} \tilde{F}^{\mu\nu} = -4\, \vec{E} \cdot \vec{B}$.

Under boosts:



$$\boxed{\begin{aligned} E_x' &= E_x & B_x' &= B_x \\ E_y' &= \gamma(E_y - \beta B_z) & B_y' &= \gamma(B_y + \beta E_z) \\ E_z' &= \gamma(E_z + \beta B_y) & B_z' &= \gamma(B_z - \beta E_y) \end{aligned}}$$

Relativistic Particles in Electromagnetic Fields

$$\boxed{\frac{d\vec{p}}{dt} = q\left[\vec{E} + \frac{1}{c}\vec{v} \times \vec{B}\right], \quad \frac{dE}{dt} = q\, \vec{v} \cdot \vec{E}}$$

$$\boxed{L = -mc^2 \sqrt{1 - \frac{v^2}{c^2}} - q\Phi + \frac{q}{c}\vec{v} \cdot \vec{A}}$$

→ need to solve these equations for given $\vec{E}, \vec{B}$.

Lagrangian for E&M fields: $\boxed{S = \frac{1}{c}\int d^4x\, \mathcal{L}}$ ← action

with the Lagrangian density

$$\boxed{\mathcal{L} = \mathcal{L}_{EM} + \mathcal{L}_{int} = -\frac{1}{16\pi} F_{\mu\nu} F^{\mu\nu} - \frac{1}{c} J_\mu A^\mu}$$

Euler-Lagrange eqn's give $\boxed{\partial_\mu F^{\mu\nu} = \frac{4\pi}{c} J^\nu}$ Maxwell equations.