

Electromagnetic Field Theory (Physics 7401)

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grading: 30% HW, 30% Midterm, 40% Final

Midterm: week of Feb. 17

Final: April 24

Class Notes: posted online (to be updated)

Syllabus: online

Textbook: J.D. Jackson "Classical Electrodynamics"

H.C. Ohanian " — — "

A. Zangwill "Modern — — "

A. Gary "Classical Electromagnetism in a Nutshell"

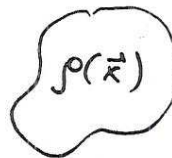
First Semester Review (E&M part)

I

Coulomb's Law: $F = \frac{1}{4\pi\epsilon_0} \frac{q_1 q_2}{|\vec{r}_1 - \vec{r}_2|^2}$



$$\Rightarrow \vec{E}(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|^3} (\vec{x} - \vec{x}')$$



$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int d^3x' \frac{\rho(\vec{x}')}{|\vec{x} - \vec{x}'|} \quad (\text{SI})$$

δ -functions: (i) $\delta(x) = \begin{cases} 0, & x \neq 0 \\ \infty, & x = 0 \end{cases}$ (ii) $\int_{-\infty}^{\infty} dx f(x) \delta(x) = f(0)$

$$\nabla^2 \frac{1}{|\vec{x} - \vec{x}'|} = -4\pi \delta^3(\vec{x} - \vec{x}')$$

$$\vec{\nabla} \cdot \vec{E} = \rho/\epsilon_0 \quad \text{Gauss's Law}$$

$$\oint_S \vec{E} \cdot \hat{n} da = \frac{1}{\epsilon_0} \int_V d^3x \rho(\vec{x}) = \frac{Q}{\epsilon_0}$$

integral form

$$\vec{E} = -\vec{\nabla} \Phi \Rightarrow$$

$$\nabla^2 \Phi = -\rho/\epsilon_0$$

Poisson eqn.

$$\nabla^2 \Phi = 0$$

Laplace eqn. ($\epsilon \neq \rho = 0$)

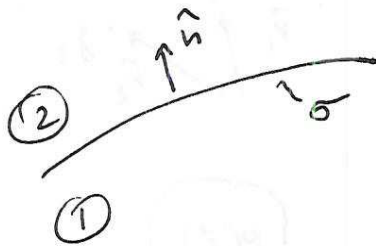
$$\vec{\nabla} \times \vec{E} = -\vec{\nabla} \times \vec{\nabla} \Phi = 0 \Rightarrow \vec{\nabla} \times \vec{E} = 0$$

boundaries:

$$\boxed{(\vec{E}_2 - \vec{E}_1) \cdot \hat{n} = \frac{1}{\epsilon_0} \sigma}$$

Gauss's Law

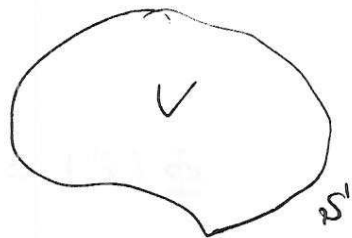
II



$$\boxed{E_{2t} = E_{1t}}$$

$(\vec{\nabla} \times \vec{E} = 0)$ & Stokes's theorem

Dirichlet b.c. problem:



Φ given on S'

solution

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V d^3x' G_D(\vec{x}, \vec{x}') \rho(\vec{x}') - \frac{1}{4\pi} \oint_{S'} \Phi(\vec{x}') \frac{\partial G_D(\vec{x}, \vec{x}')}{\partial n'} da'$$

where $G_D(\vec{x}, \vec{x}')$ is Dirichlet Green ftn:

$$\nabla'^2 G_D(\vec{x}, \vec{x}') = -4\pi \delta^3(\vec{x} - \vec{x}')$$

$$G_D(\vec{x}, \vec{x}') = 0 \text{ for } \vec{x}' \in S'$$

Neumann Problem: $\frac{\partial \Phi}{\partial n}$ is given on S' .

solution

$$\Phi(\vec{x}) = \frac{1}{4\pi\epsilon_0} \int_V d^3x' G_N(\vec{x}, \vec{x}') \rho(\vec{x}') + \frac{1}{4\pi} \oint_S da' G_N(\vec{x}, \vec{x}') \frac{\partial \Phi}{\partial n'} + \langle \Phi \rangle_{\text{surface}}$$

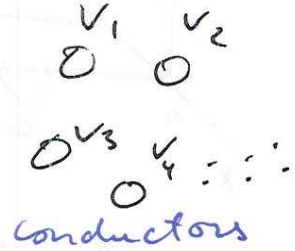
$$\nabla'^2 G_N(\vec{x}, \vec{x}') = -4\pi \delta^3(\vec{x} - \vec{x}'), \quad \frac{\partial G_N}{\partial n'} = \frac{-4\pi}{\epsilon_0} \text{ for } \vec{x}' \in S'$$

Electrostatic Energy

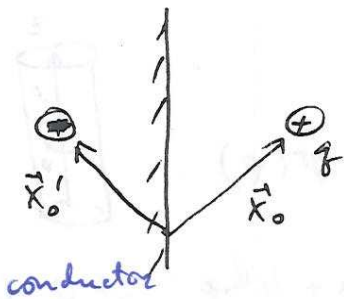
$$W = \frac{1}{2} \int d^3x \rho(\vec{x}) \Phi(\vec{x}) = \frac{\epsilon_0}{2} \int d^3x |\vec{E}|^2(\vec{x})$$

capacitance

$$Q_i = \sum_{j=1}^n C_{ij} V_j$$

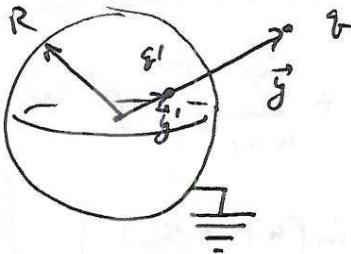


Method of Images



$$\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\vec{x} - \vec{x}'|} - \frac{1}{|\vec{x} - \vec{x}''|} \right]$$

used to satisfy b.c.



$$q' = -q \frac{R}{a}, \quad y' = \frac{R^2}{a}$$

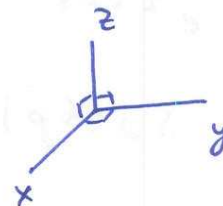
$$\Phi(\vec{x}) = \frac{q}{4\pi\epsilon_0} \left[\frac{1}{|\vec{x} - \vec{y}|} - \frac{R}{a} \frac{1}{|\vec{x} - \frac{R^2}{a^2} \vec{y}|} \right]$$

Separation of Variables

① Rectangular Coordinates: $\Phi(x, y, z) = X(x) Y(y) Z(z)$

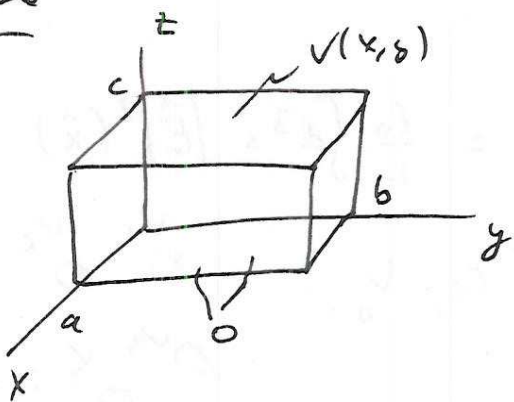
$$\nabla^2 \Phi = 0 \Rightarrow$$

$$\begin{cases} X(x) = e^{\pm i\alpha x} \\ Y(y) = e^{\pm i\beta y} \\ Z(z) = e^{\pm \gamma z} \end{cases}$$



$$\gamma^2 = \alpha^2 + \beta^2$$

Example



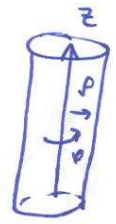
$$\Phi(x, y, z) = \sum_{n, m=1}^{\infty} A_{nm} \sin\left(\frac{\pi n}{a} x\right) \cdot \sin\left(\frac{\pi m}{b} y\right) \sinh\left(z \sqrt{\left(\frac{\pi n}{a}\right)^2 + \left(\frac{\pi m}{b}\right)^2}\right)$$

" γ_{nm} "

where $A_{nm} = \frac{4}{ab \sinh(\gamma_{nm} c)} \int_0^a dx \int_0^b dy V(x, y) \sin\left(\frac{\pi n}{a} x\right) \sin\left(\frac{\pi m}{b} y\right)$.

\Rightarrow can also construct Green functions by separation of variables

② Cylindrical Coordinates:



A. z - indep. case $\Phi(\rho, \varphi) = R(\rho) \Psi(\varphi)$

$$\begin{cases} \Psi(\varphi) = e^{\pm i\nu\varphi} \\ R(\rho) = a\rho^\nu + b\rho^{-\nu} \end{cases} \quad (\nu \neq 0) \quad R(\rho) \sim a + b \ln \rho \quad \text{if } \nu = 0$$

\Rightarrow solution is $\Phi(\rho, \varphi) = a_0 + b_0 \ln \rho + \sum_{n=1}^{\infty} [a_n \rho^n \sin(n\varphi + \alpha_n) + b_n \rho^{-n} \sin(n\varphi + \beta_n)]$

for $0 \leq \varphi < 2\pi$, single-valued function

B. z - dep. case

$\Phi(\rho, \varphi, z) = R(\rho) Q(\varphi) Z(z)$

$$\begin{cases} Z(z) = e^{\pm kz} \\ Q(\varphi) = e^{\pm i\nu\varphi} \\ R(\rho) = J_\nu(k\rho) \text{ or } N_\nu(k\rho) \end{cases} \quad \text{Bessel ftns of the 1st/2nd kind}$$

or

$$\begin{cases} Z(z) = e^{\pm ikz} \\ Q(\varphi) = e^{\pm i\nu\varphi} \\ R(\rho) = I_\nu(k\rho) \text{ or } K_\nu(k\rho) \end{cases} \quad \text{modified Bessel ftns}$$

normalization:

$$\int_0^a \rho \cdot \rho \cdot J_0(x_{0n} \frac{\rho}{a}) J_0(x_{0n'} \frac{\rho}{a}) = \frac{a^2}{2} \delta_{nn'} [J_{0+1}(x_{0n})]^2 \frac{1}{\sqrt{}}$$

where $J_0(x_{0n}) = 0$, $n = 1, 2, \dots$ roots of Bessel J_0 .

completeness: $\int_0^a dx \cdot x \cdot J_m(kx) J_m(k'x) = \frac{1}{k} \delta(k-k')$.

③ Spherical coordinates: $\Phi(r, \theta, \varphi) = \frac{u(r)}{r} P(\theta) Q(\varphi)$

$$\Rightarrow Q(\varphi) = e^{\pm im\varphi}, \quad \frac{u(r)}{r} = A_{em} r^l + B_{em} r^{-l-1}$$

A. Azimuthally symmetric case: $m = 0$, $\varphi = \text{const.}$

$P(\cos \theta) = P_l(\cos \theta)$, $l = 0, 1, 2, \dots$ Legendre polynomials

$P_l(x) = \frac{1}{2^l l!} \frac{d^l}{dx^l} (x^2 - 1)^l$ Rodrigues formula.

$$\int_{-1}^1 dx P_l(x) P_{l'}(x) = \frac{2}{2l+1} \delta_{ll'}$$

(normalization)

(polynomials \Rightarrow complete)

solution of Laplace eqn:

$$\Phi(r, \theta) = \sum_{l=0}^{\infty} [A_l r^l + B_l r^{-l-1}] P_l(\cos \theta)$$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \frac{r_l^l}{r_l^{l+1}} P_l(\cos \gamma)$$

, γ = angle between \vec{x} & \vec{x}'

$$r_l = \begin{cases} \max\{r, r'\} \\ \min\{r, r'\} \end{cases}$$

"generating functional" of Legendre polynomials

B. azimuthally - dependent case: $m \neq 0$

Spherical harmonics: associated Legendre function

$$Y_{lm}(\theta, \varphi) = \sqrt{\frac{2l+1}{4\pi} \frac{(l-m)!}{(l+m)!}} P_l^m(\cos \theta) e^{im\varphi}$$

where $P_l^m(x) = (-1)^m (1-x^2)^{m/2} \frac{d^m}{dx^m} P_l(x)$.

$$\int_{-1}^1 d\cos \theta \int_0^{2\pi} d\varphi Y_{l'm'}^*(\theta, \varphi) Y_{lm}(\theta, \varphi) = \delta_{ll'} \delta_{mm'} \quad (\text{orthogonality})$$

$$\sum_{l=0}^{\infty} \sum_{m=-l}^l Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi) = \delta(\varphi - \varphi') \delta(\cos \theta - \cos \theta')$$

(completeness)

Solution of Laplace eqn:

$$\Phi(r, \theta, \varphi) = \sum_{l=0}^{\infty} \sum_{m=-l}^l [A_{lm} r^l + B_{lm} r^{-l-1}] Y_{lm}(\theta, \varphi)$$

$$\frac{1}{|\vec{x} - \vec{x}'|} = \sum_{l=0}^{\infty} \sum_{m=-l}^l \frac{4\pi}{2l+1} \frac{r_l^l}{r_l^{l+1}} Y_{lm}^*(\theta', \varphi') Y_{lm}(\theta, \varphi)$$

addition theorem for spherical harmonics