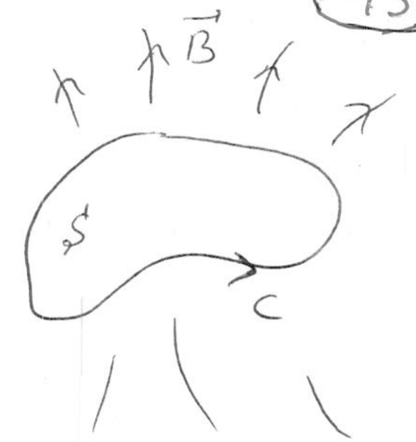
Faraday's Law of Induction.



Suppose we have a current loop C in external magnetic field.

Faraday observed that changes in $\vec{B}$ generate current in the loop.

Namely he observed that, if we define magnetic flux $\Phi = \int_S da \; \vec{B}\cdot\hat{n}$, then

$$\boxed{\mathcal{E} = \oint_C \vec{E}'\cdot d\vec{l} = -k\,\frac{d\Phi}{dt}}$$

where $\mathcal{E}$ is called electromotive force.

k is a proportionality constant, $k=1$ in SI units

"$-$" sign is due to Lenz's law ~ the system tries to oppose changes.

$\vec{E}'$ ~ electric field at the element $d\vec{l}$ in its rest frame (!)

k is fixed from Galilean invariance:

Φ changes either due to

(i) changes in $\vec{B}$

or

(ii) changes in circuit S ~ motion of C.

$$\frac{d\vec{B}}{dt} = \frac{\partial \vec{B}}{\partial t} + (\vec{v}\cdot\vec{\nabla})\vec{B} = \frac{\partial \vec{B}}{\partial t} + \vec{\nabla}\times(\vec{B}\times\vec{v}) + \vec{v}\underbrace{(\vec{\nabla}\cdot\vec{B})}_{=0}$$

$$\Rightarrow \frac{d}{dt}\int_S \vec{B}\cdot\hat{n}\, da = \int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, da + \int_S da\ \hat{n}\cdot\vec{\nabla}\times(\vec{B}\times\vec{v}) =$$

↓ Stokes's th'm

$$= \int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, da + \oint_C d\vec{l}\cdot(\vec{B}\times\vec{v})$$

$\Rightarrow$ Faraday's law gives

$$\mathcal{E} = \oint_C \vec{E}'\cdot d\vec{l} = -k\int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, da - k\oint_C d\vec{l}\cdot(\vec{B}\times\vec{v})$$

$$\Rightarrow \oint_C d\vec{l}\cdot\left(\vec{E}' - k(\vec{v}\times\vec{B})\right) = -k\int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, da$$

This is Faraday's law for a moving circuit.

If we consider the case such that the circuit is at rest at the same position as the moving one, in this situation we have

$$\oint_C \vec{E}\cdot d\vec{l} = -k\int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, da$$

$$\Rightarrow \boxed{\vec{E}' = \vec{E} + k\,\vec{v}\times\vec{B}}$$

~ electric field in one frame becomes magnetic induction in another

moving frame ($\vec{E}'$); lab frame ($\vec{E}$, $\vec{B}$)

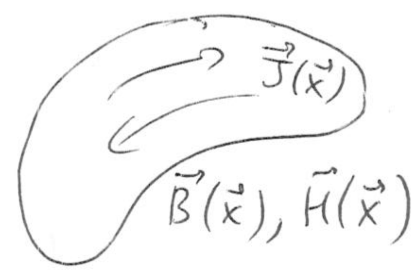
as $F = qE$ and $F = qvB \Rightarrow k = 1$

$$\oint_C \vec{E}\cdot d\vec{l} = \int_S (\vec{\nabla}\times\vec{E})\cdot\hat{n}\, da = -\int_S \frac{\partial \vec{B}}{\partial t}\cdot\hat{n}\, da$$

$$\Rightarrow \boxed{\vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}}$$

differential form of Faraday's law (generalizes $\vec{\nabla}\times\vec{E}=0$ of electrostatics).

Energy in the Magnetic Field



Energy change rate is (for a point charge)

$$\frac{dW}{dt} = \vec{V}\cdot\vec{F} = q\,\vec{V}\cdot\vec{E}$$

(point charge q moving with velocity $\vec{v}$)

$\vec{\nabla}\cdot\vec{J}=0 \Rightarrow$ can break the current into small loops.

$\leftarrow$ as $\vec{J} = q\vec{v}$

$\Rightarrow \delta W = -\delta t \cdot \vec{J}\cdot\vec{E}$. ~ work done to bring in a small current loop $\Rightarrow$ a

Let's change $\vec{B}$ by $\delta\vec{B}(\vec{x})$: $\left(\vec{\nabla}\times\vec{E} = -\frac{\delta \vec{B}}{\delta t}\right)$

$$\delta W = -\delta t \int d^3x\, \vec{J}\cdot\vec{E} = -\delta t \int d^3x\, (\vec{\nabla}\times\vec{H})\cdot\vec{E}$$

As $\vec{\nabla}(\vec{E}\times\vec{H}) = \vec{H}\cdot(\vec{\nabla}\times\vec{E}) - \vec{E}\cdot\vec{\nabla}\times\vec{H} \Rightarrow$

$$\delta W = +\delta t \int d^3x \left\{ \vec{\nabla}(\vec{E}\times\vec{H}) - \vec{H}\cdot\vec{\nabla}\times\vec{E} \right\}$$

surface integral

$$= -\delta t \int d^3x\, \vec{H}\cdot(\vec{\nabla}\times\vec{E}) \Rightarrow \text{as } \vec{\nabla}\times\vec{E} = -\frac{\partial \vec{B}}{\partial t}$$

$$\Rightarrow \boxed{\delta W = \int d^3x \; \vec{H} \cdot \delta \vec{B}}$$

If $\vec{B} = \mu \vec{H} \Rightarrow \boxed{W = \frac{1}{2} \int d^3x \; \vec{H} \cdot \vec{B}}$

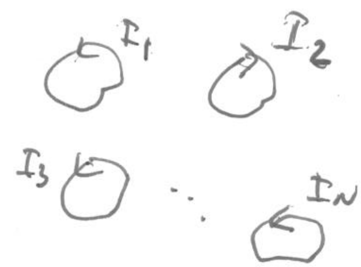
Alternatively, as $\vec{B} = \vec{\nabla} \times \vec{A} \Rightarrow$

$$W = \frac{1}{2} \int d^3x \; \vec{H} \cdot (\vec{\nabla} \times \vec{A}) = \frac{1}{2} \int d^3x \; \vec{A} \cdot \underbrace{(\vec{\nabla} \times \vec{H})}_{= \vec{J}}$$

$$\Rightarrow \boxed{W = \frac{1}{2} \int d^3x \; \vec{A} \cdot \vec{J}}$$

Self- and Mutual Inductances.

$$\vec{A}(\vec{x}) = \frac{\mu_0}{4\pi} \int d^3x' \; \frac{\vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|} \Rightarrow$$



$$\Rightarrow W = \frac{\mu_0}{8\pi} \int d^3x \, d^3x' \; \frac{\vec{J}(\vec{x}) \cdot \vec{J}(\vec{x}')}{|\vec{x} - \vec{x}'|}$$

If we have N circuits with currents $I_1, \ldots, I_N$:

$$W = \frac{1}{2} \sum_{i=1}^{N} L_i I_i^2 + \frac{1}{2} \sum_{i \neq j} M_{ij} I_i I_j$$

Definition L_i ~ self-inductance

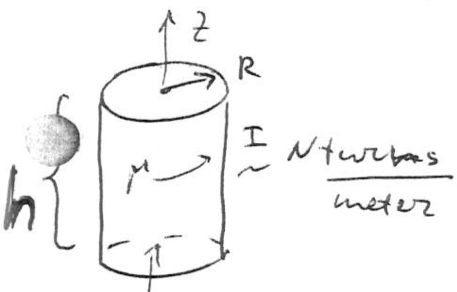
M_{ij} ~ mutual inductance between i & j

as $W = \frac{\mu_0}{8\pi} \sum_{i=1}^{N} \sum_{j=1}^{N} \int_{V_i} d^3x_i \int_{V_j} d^3x_j' \frac{\vec{J}(\vec{x}_i)\cdot\vec{J}(\vec{x}_j')}{|\vec{x}_i - \vec{x}_j'|}$

$$\Rightarrow \quad L_i = \frac{\mu_0}{4\pi I_i^2} \int_{V_i} d^3x_i \int_{V_i} d^3x_i' \frac{\vec{J}(\vec{x}_i)\cdot\vec{J}(\vec{x}_i')}{|\vec{x}_i - \vec{x}_i'|}$$

$$M_{ij} = \frac{\mu_0}{4\pi I_i I_j} \int_{V_i} d^3x_i \int_{V_j} d^3x_j' \frac{\vec{J}(\vec{x}_i)\cdot\vec{J}(\vec{x}_j')}{|\vec{x}_i - \vec{x}_j'|}$$

<u>Example</u>: self-inductance of a solenoid:



filled with material with permeability μ.

$\vec{\nabla}\times\vec{H} = \vec{J} \Rightarrow \vec{H}_{in} = N I \hat{z}, \ \vec{H}_{out} = 0$

$\vec{B}_{in} = \mu \vec{H}_{in} = \mu N I \hat{z}$

$\Rightarrow W = \frac{1}{2}\int d^3x \ \vec{B}\cdot\vec{H} = \frac{1}{2}\mu N^2 I^2 \cdot V$

$V = \pi R^2 h$ ~ volume

$\Rightarrow W = \frac{1}{2} L I^2 = \frac{1}{2}\mu N^2 I^2 \pi R^2 h$

$$\Rightarrow \boxed{L = \mu N^2 \pi R^2 h}$$

Linear currents: $\vec{J}\, d^3x = I\, d\vec{l} \Rightarrow W = \frac{1}{2}\int d^3x \ \vec{J}\cdot\vec{A} = \frac{1}{2}\oint \vec{A}\cdot d\vec{l}\cdot I =$

$= I \cdot \frac{1}{2}\int da \cdot \hat{n}\underbrace{(\vec{\nabla}\times\vec{A})}_{=\vec{B}} = \frac{1}{2} I \Phi = \frac{1}{2} L I^2 \Rightarrow \boxed{L = \frac{\Phi}{I}}$

magnetic flux

formula known from undergraduate E&M.

For simple linear loops:

$$\vec{J}\, d^3x \rightarrow I\, d\vec{\ell}$$

$$\Rightarrow \boxed{\begin{aligned} L_i &= \frac{\mu_0}{4\pi} \oint_{C_i} d\vec{\ell}_i \cdot \oint_{C_i} d\vec{\ell}_i' \, \frac{1}{|\vec{x}_i - \vec{x}_i'|} \\ M_{ij} &= \frac{\mu_0}{4\pi} \oint_{C_i} d\vec{\ell}_i \cdot \oint_{C_j} d\vec{\ell}_j \, \frac{1}{|\vec{x}_i - \vec{x}_j|} \end{aligned}}$$