Last time | Faraday's Law:

$$\boxed{\vec{\nabla} \times \vec{E} = - \frac{\partial \vec{B}}{\partial t}}$$

Energy in Magnetic Field:

$$\boxed{W = \int d^3x \; \frac{1}{2} \vec{H} \cdot \vec{B}} = \int d^3x \; \frac{1}{2} \vec{A} \cdot \vec{J}$$

Self- and Mutual Inductances

$$W = \frac{1}{2} \sum_{i=1}^{N} L_i I_i^2 + \frac{1}{2} \sum_{i \neq j} M_{ij} I_i I_j$$

self-inductance

mutual inductance

Maxwell Equations

Up to now we've derived the following relations

$\vec{\nabla} \cdot \vec{D} = \rho$ Coulomb's Law $(\vec{D} = \varepsilon_0 \vec{E} + \vec{P})$

$\vec{\nabla} \times \vec{H} = \vec{J}$ Ampere's Law $(\vec{H} = \frac{1}{\mu_0} \vec{B} - \vec{M})$

$\vec{\nabla} \times \vec{E} + \frac{\partial \vec{B}}{\partial t} = 0$ Faraday's Law

$\vec{\nabla} \cdot \vec{B} = 0$ Absense of magnetic monopoles.

$\Rightarrow$ only Faraday's Law was derived for time-dependent case, the rest is a priori for static case only.

ρ, $\vec{J}$ are charge and current densities

continuity equation $\quad \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = 0$

However, if we use Ampere's Law we'd get

$\vec{\nabla} \cdot \vec{J} = \vec{\nabla} \cdot (\vec{\nabla} \times \vec{H}) = 0$ ~ contradiction.

How do we fix this? Use Coulomb's Law

to write $0 = \frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{J} = \frac{\partial}{\partial t} \vec{\nabla} \cdot \vec{D} + \vec{\nabla} \cdot \vec{J} =$

$$= \vec{\nabla} \cdot \left[\vec{J} + \frac{\partial \vec{D}}{\partial t} \right] = 0$$

$\Rightarrow$ Maxwell suggested to replace $\vec{J} \to \underbrace{\vec{J} + \frac{\partial \vec{D}}{\partial t}}_{\text{displacement current}}$ on the r.h.s. of Ampere's law:

$$\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}$$

reduces to Ampere's Law in the static case.

Maxwell Equations (ca. 1865):

$$\boxed{\begin{array}{ll} \vec{\nabla} \cdot \vec{D} = \rho & \vec{\nabla} \times \vec{H} = \vec{J} + \dfrac{\partial \vec{D}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} = 0 & \vec{\nabla} \times \vec{E} + \dfrac{\partial \vec{B}}{\partial t} = 0 \end{array}}$$

the cornerstone of electrodynamics!

Vector and Scalar Potentials

$\vec{\nabla} \cdot \vec{B} = 0 \Rightarrow \vec{B} = \vec{\nabla} \times \vec{A}$, $\vec{A}$ is vector-potential

$\Rightarrow$ Faraday's Law gives $\vec{\nabla} \times \left[\vec{E} + \frac{\partial \vec{A}}{\partial t} \right] = 0$

$$\Rightarrow \vec{E} + \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} \Phi \Rightarrow \vec{E} = -\vec{\nabla} \Phi - \frac{\partial \vec{A}}{\partial t}$$

Φ is scalar potential.

(cf. $\vec{E} = -\vec{\nabla} \Phi$ in electrostatics)

Remember the derivation of Ampere's Law.

We got:
$$\vec{\nabla} \times \vec{B} = \mu_0 \vec{J} + \mu_0 \varepsilon_0 \frac{\partial \vec{E}}{\partial t}$$

as $\vec{H} = \frac{\vec{B}}{\mu_0}$ in vacuum and $\vec{D} = \varepsilon_0 \vec{E}$ (also vacuum)

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{H} = \vec{J} + \frac{\partial \vec{D}}{\partial t}}$$

In vacuum $\vec{D} = \varepsilon_0 \vec{E}$, $\vec{B} = \mu_0 \vec{H}$ and

Maxwell equations read

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \rho/\varepsilon_0 & \vec{\nabla}\times\vec{B} = \mu_0 \vec{J} + \frac{1}{c^2}\frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla}\cdot\vec{B} = 0 & \vec{\nabla}\times\vec{E} + \frac{\partial \vec{B}}{\partial t} = 0 \end{array}}$$

(Vacuum!)

Here $c^2 = \frac{1}{\mu_0 \varepsilon_0}$, $c \sim$ speed of light

$\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$ satisfy the last two equations. Plug into the first two:

$$\begin{cases} \nabla^2 \Phi + \frac{\partial}{\partial t}(\vec{\nabla}\cdot\vec{A}) = -\frac{\rho}{\varepsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} - \vec{\nabla}\cdot\left[\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t}\right] = -\mu_0 \vec{J} \end{cases}$$

A somewhat more compact way of writing Maxwell equations in vacuum.

Now, $\vec{A}$ and Φ are <u>not</u> uniquely defined: one can always redefine

$$\boxed{\begin{array}{l} \vec{A} \to \vec{A}' = \vec{A} + \vec{\nabla}\Lambda \\ \Phi \to \Phi' = \Phi - \frac{\partial \Lambda}{\partial t} \end{array}}$$

gauge transformations.

without changing $\vec{B} = \vec{\nabla}\times\vec{A}$ and $\vec{E} = -\vec{\nabla}\Phi - \frac{\partial \vec{A}}{\partial t}$

Electrodynamics is <u>the first</u> gauge theory known to people.

$\Rightarrow$ By choosing various scalar functions $\Lambda(\vec{x},t)$ can satisfy different <u>gauge conditions:</u>

I Lorentz gauge: $\vec{\nabla}\cdot\vec{A} + \frac{1}{c^2}\frac{\partial \Phi}{\partial t} = 0$

Does it exist? Start from some random $\Phi, \vec{A}$.

Is there $\Lambda(\vec{x},t)$ such that $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c^2}\frac{\partial \Phi'}{\partial t} = 0$?

$$\vec{\nabla}\cdot\left[\vec{A} + \vec{\nabla}\Lambda\right] + \frac{1}{c^2}\frac{\partial}{\partial t}\left[\Phi - \frac{\partial \Lambda}{\partial t}\right] = 0$$

$$\Rightarrow \nabla^2\Lambda - \frac{1}{c^2}\frac{\partial^2\Lambda}{\partial t^2} = -\vec{\nabla}\cdot\vec{A} - \frac{1}{c^2}\frac{\partial\Phi}{\partial t}$$

$\Rightarrow$ can always solve this to find $\Lambda(\vec{x},t)$

Λ is not unique: can alwayse shift $\Lambda \to \Lambda + \tilde{\Lambda}$

where $\nabla^2\tilde{\Lambda} - \frac{1}{c^2}\frac{\partial^2\tilde{\Lambda}}{\partial t^2} = 0$ $\Rightarrow$ wave in empty space $\Rightarrow \tilde{\Lambda} \neq 0$ (does not have to be $\emptyset$). $\Rightarrow$ Lorentz gauge condition is not fully restrictive, there is a residual gauge freedom left, but that's Ok, it happens with most gauges.

Using Lorentz gauge condition we rewrite Maxwell equations as

$$\boxed{\begin{aligned} \nabla^2 \Phi - \frac{1}{c^2}\frac{\partial^2 \Phi}{\partial t^2} &= -\frac{\rho}{\varepsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J} \end{aligned}}$$

Maxwell equns in Lorentz gauge

II) Another interesting gauge is Coulomb gauge:

$$\vec{\nabla}\cdot\vec{A} = 0$$

Coulomb gauge condition

Maxwell equations become:

$$\boxed{\begin{aligned} \nabla^2 \Phi &= -\frac{\rho}{\varepsilon_0} \\ \nabla^2 \vec{A} - \frac{1}{c^2}\frac{\partial^2 \vec{A}}{\partial t^2} &= -\mu_0 \vec{J} + \frac{1}{c^2}\vec{\nabla}\frac{\partial \Phi}{\partial t} \end{aligned}}$$

Maxwell equns in Coulomb gauge

$$\Rightarrow \Phi(\vec{x},t) = \frac{1}{4\pi\varepsilon_0}\int d^3x' \, \frac{\rho(\vec{x}',t)}{|\vec{x}-\vec{x}'|}$$

just like in statics

Note: time is the same on both sides, interaction is "instantaneous" $\Rightarrow$ hence the name Coulomb gauge.

III Other notable gauges: $x_\mu A^\mu = 0$ (Schwinger gauge),

$A_0 + A_z = 0 \Rightarrow \Phi + c A_z = 0$ (light cone gauge)

$A_0 = \Phi = 0$ (axial gauge)

Green Function for Wave Equation.

In solving Maxwell equations in, say, Lorentz gauge one often encounters equations of the type:

$$\boxed{\nabla^2 \Psi - \frac{1}{c^2}\frac{\partial^2 \Psi}{\partial t^2} = -4\pi f(\vec{x},t)}$$ Inhomogeneous wave eqn.

with $f(\vec{x},t)$ some known function (source).
The strategy for solving those is the same as in electrostatics: find the Green function.

$$\left(\nabla^2 - \frac{1}{c^2}\frac{\partial^2}{\partial t^2}\right) G(\vec{x},t;\vec{x}',t') = -4\pi\, \delta^3(\vec{x}-\vec{x}')\,\delta(t-t')$$

Then $$\Psi(\vec{x},t) = \int d^3x'\,dt'\; G(\vec{x},t;\vec{x}',t')\, f(\vec{x}',t')$$

is a solution of the inhomogeneous wave equation. (in unlimited space-time).